The key attributes of special relativity 3

3.1 Standard derivation of the Lorentz transformations

We start this chapter by deriving again the Lorentz transformations, but this time by using a more standard approach. We shall work in non-relativistic units in which the speed of light is denoted by c. We restrict attention to two inertial observers S and S' in standard configuration. As before, we shall show that the Lorentz transformations follow from the two postulates, namely, the principle of special relativity and the constancy of the velocity of light.

Now, by the first postulate, if the observer S sees a **free** particle, that is, a particle with no forces acting on it, travelling in a straight line with constant velocity, then so will S'. Thus, using vector notation, it follows that under a transformation connecting the two frames

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{u}t \quad \Leftrightarrow \quad \boldsymbol{r}' = \boldsymbol{r}'_0 + \boldsymbol{u}'t'.$$

Since straight lines get mapped into straight lines, it suggests that the transformation between the frames is **linear** and so we shall assume that the transformation from S to S' can be written in matrix form

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = L \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}, \tag{3.1}$$

where L is a 4×4 matrix of quantities which can only depend on the speed of separation v. Using exactly the same argument as we used at the end of §2.12, the assumption that space is isotropic leads to the transformations of y and z being

$$y' = y \quad \text{and} \quad z' = z. \tag{3.2}$$

We next use the second postulate. Let us assume that, when the origins of S and S' are coincident, they zero their clocks, i.e. $t = t' = 0$, and emit a flash of light. Then, according to S, the light flash moves out radially from the origin with speed c. The wave front of light will constitute a sphere. If we define the quantity I by

$$I(t, x, y, z) = x^2 + y^2 + z^2 - c^2t^2,$$

then the events comprising this sphere must satisfy $I = 0$. By the second postulate, S' must also see the light move out in a spherical wave front with speed c and satisfy

$$I' = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0.$$

Thus it follows that, under a transformation connecting S and S',

$$I = 0 \quad \Leftrightarrow \quad I' = 0, \tag{3.3}$$

and since the transformation is linear by (3.1), we may conclude

$$I = nI', \tag{3.4}$$

where n is a quantity which can only depend on v. Using the same argument as we did in §2.12, we can reverse the role of S and S' and so by the relativity principle we must also have

$$I' = nI. \tag{3.5}$$

Combining the last two equations we find

$$n^2 = 1 \quad \Rightarrow \quad n = \pm 1.$$

In the limit as $v \to 0$, the two frames coincide and $I' \to I$, from which we conclude that we must take $n = 1$.

Substituting $n = 1$ in (3.4), this becomes

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2,$$

and, using (3.2), this reduces to

$$x^2 - c^2t^2 = x'^2 - c^2t'^2. \tag{3.6}$$

We next introduce imaginary time coordinates T and T' defined by

$$T = \mathrm{i}ct, \tag{3.7}$$

$$T' = \mathrm{i}ct', \tag{3.8}$$

in which case equation (3.6) becomes

$$x^2 + T^2 = x'^2 + T'^2.$$

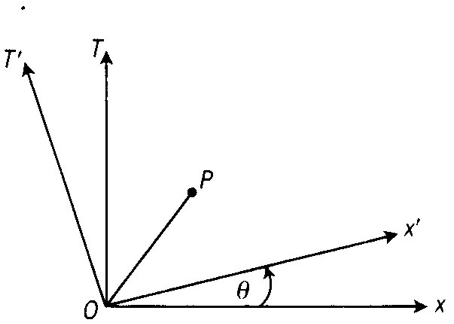

Fig. 3.1 A rotation in (x, T)-space.

In a two-dimensional (x, T)-space, the quantity $x^2 + T^2$ represents the distance of a point P from the origin. This will only remain invariant under a **rotation** in (x, T)-space (Fig. 3.1). If we denote the angle of rotation by θ, then a rotation is given by

$$x' = x\cos\theta + T\sin\theta, \tag{3.9}$$

$$T' = -x\sin\theta + T\cos\theta. \tag{3.10}$$

Now, the origin of S' ($x' = 0$), as seen by S, moves along the positive x-axis of S with speed v and so must satisfy $x = vt$. Thus, we require

$$x' = 0 \quad \Leftrightarrow \quad x = vt \quad \Leftrightarrow \quad x = vT/\mathrm{i}c,$$

using (3.7). Substituting this into (3.9) gives

$$\tan\theta = \mathrm{i}v/c, \tag{3.11}$$

from which we see that the angle θ is imaginary as well. We can obtain an expression for $\cos\theta$, using

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{(1 + \tan^2\theta)^{\frac{1}{2}}} = \frac{1}{(1 - v^2/c^2)^{\frac{1}{2}}}.$$

If we use the conventional symbol β for this last expression, i.e.

$$\beta \equiv \frac{1}{(1 - v^2/c^2)^{\frac{1}{2}}},$$

where the symbol $\equiv$ here means 'is defined to be', then (3.9) gives

$$x' = \cos\theta(x + T\tan\theta) = \beta[x + ict(iv/c)] = \beta(x - vt).$$

Similarly, (3.10) gives

$$T' = ict' = \cos\theta(-x\tan\theta + T) = \beta[-x(iv/c) + ict],$$

from which we find

$$t' = \beta(t - vx/c^2).$$

Thus, collecting the results together, we have rederived the special Lorentz transformation or boost (in non-relativistic units):

$$t' = \beta(t - vx/c^2), \quad x' = \beta(x - vt), \quad y' = y, \quad z' = z. \tag{3.12}$$

If we put $c = 1$, this takes the same form as we found in §2.13.

3.2 Mathematical properties of Lorentz transformations

From the results of the last section, we find the following properties of a special Lorentz transformation or boost.

1. Using the imaginary time coordinate T, a boost along the x-axis of speed v is equivalent to an imaginary rotation in (x, T)-space through an angle θ given by $\tan\theta = iv/c$.

2. If we consider v to be very small compared with c, for which we use the notation $v \ll c$, and neglect terms of order v^2/c^2, then we regain a Galilean transformation

$$t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z.$$

We can obtain this result formally by taking the limit $c \to \infty$ in (3.12).

3. If we solve (3.12) for the unprimed coordinates, we get

$$t = \beta(t' + vx'/c^2), \quad x = \beta(x' + vt'), \quad y = y', \quad z = z'.$$

This can be obtained formally from (3.12) by interchanging primed and unprimed coordinates and replacing v by $-v$. This we should expect from physical reasons, since, if S' moves along the positive x-axis of S with speed v, then S moves along the negative x'-axis of S' with speed v, or, equivalently, S moves along the positive x'-axis of S' with speed $-v$.

4. Special Lorentz transformations form a **group**:
(a) The identity element is given by $v = 0$.
(b) The inverse element is given by $-v$ (as in 3 above).

(c) The product of two boosts with velocities v and v' is another boost with velocity v''. Since v and v' correspond to rotations in (x, T)-space of θ and θ', where

$$\tan\theta = iv/c \quad \text{and} \quad \tan\theta' = iv'/c,$$

then their resultant is a rotation of $\theta'' = \theta + \theta'$, where

$$iv''/c = \tan\theta'' = \tan(\theta + \theta') = \frac{\tan\theta + \tan\theta'}{1 - \tan\theta\tan\theta'},$$

from which we find

$$v'' = \frac{v + v'}{1 + vv'/c^2}.$$

Compare this with equation (2.6) in relativistic units.

(d) Associativity is left as an exercise.

5. The square of the infinitesimal interval between infinitesimally separated events (see (2.13)),

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{3.13}$$

is invariant under a Lorentz transformation.

We now turn to the key physical attributes of Lorentz transformations. Throughout the remaining sections, we shall assume that S and S' are in standard configuration with non-zero relative velocity v.

3.3 Length contraction

Consider a rod fixed in S' with endpoints x'_A and x'_B, as shown in Fig. 3.2. In S, the ends have coordinates x_A and x_B (which, of course, vary in time) given by the Lorentz transformations

$$x'_A = \beta(x_A - vt_A), \qquad x'_B = \beta(x_B - vt_B). \tag{3.14}$$

In order to measure the lengths of the rod according to S, we have to find the x-coordinates of the end points at the same time according to S. If we denote the **rest length**, namely, the length in S', by

$$l_0 = x'_A - x'_B$$

and the length in S at time $t = t_A = t_B$ by

$$l = x_B - x_A,$$

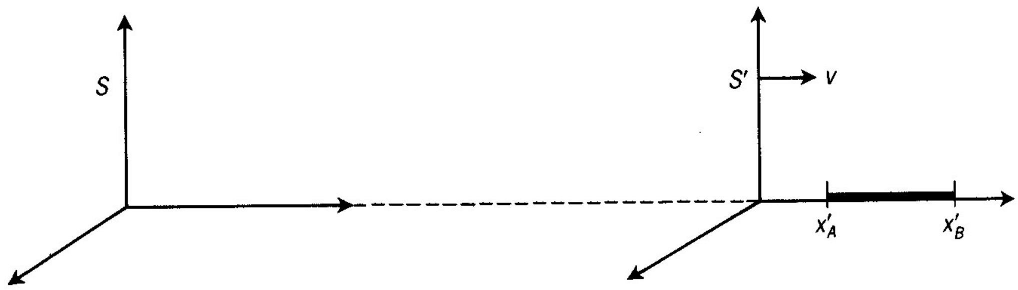

Fig. 3.2 A rod moving with velocity v relative to S.

then, subtracting the formulae in (3.14), we find the result

$$l = \beta^{-1} l_0. \tag{3.15}$$

Since

$$|v| < c \quad \Leftrightarrow \quad \beta > 1 \quad \Leftrightarrow \quad l < l_0,$$

the result shows that the length of a body in the direction of its motion with uniform velocity v is **reduced by a factor $(1 - v^2/c^2)^{\frac{1}{2}}$**. This phenomenon is called **length contraction**. Clearly, the body will have greatest length in its rest frame, in which case it is called the rest length or **proper length**. Note also that the length approaches zero as the velocity approaches the velocity of light.

In an attempt to explain the null result of the Michelson–Morley experiment, Fitzgerald had suggested the apparent shortening of a body in motion relative to the ether. This is rather different from the length contraction of special relativity, which is not to be regarded as illusory but is a very real effect. It is closely connected with the relativity of simultaneity and indeed can be deduced as a direct consequence of it. Unlike the Fitzgerald contraction, the effect is **relative**, i.e. a rod fixed in S appears contracted in S'. Note also that there are no contraction effects in directions transverse to the direction of motion.

3.4 Time dilation

Let a clock fixed at $x' = x'_A$ in S' record two successive events separated by an interval of time T_0 (Fig. 3.3). The successive events in S' are (x'_A, t'_1) and $(x'_A, t'_1 + T_0)$, say. Using the Lorentz transformation, we have in S

$$t_1 = \beta(t'_1 + vx'_A/c^2), \qquad t_2 = \beta(t'_1 + T_0 + vx'_A/c^2).$$

On subtracting, we find the time interval in S defined by

$$T = t_2 - t_1$$

is given by

$$T = \beta T_0. \tag{3.16}$$

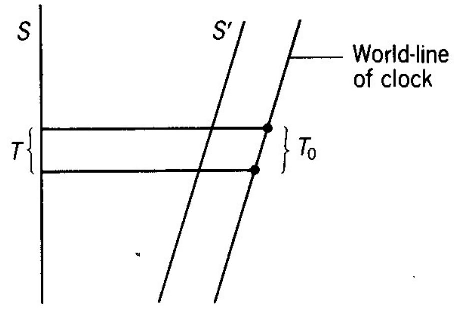

Fig. 3.3 Successive events recorded by a clock fixed in S'.

Thus, **moving clocks go slow by a factor $(1 - v^2/c^2)^{-\frac{1}{2}}$**. This phenomenon is called **time dilation.** The fastest rate of a clock is in its rest frame and is called its **proper rate**. Again, the effect has a reciprocal nature.

Let us now consider an accelerated clock. We define an **ideal clock** to be one unaffected by its acceleration; in other words, its instantaneous rate depends only on its instantaneous speed v, in accordance with the above phenomenon of time dilation. This is often referred to as the **clock hypothesis.** The time recorded by an ideal clock is called the **proper time** τ (Fig. 3.4). Thus, the proper time of an ideal clock between t_0 and t_1 is given by

$$\tau = \int_{t_0}^{t_1} \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} dt. \tag{3.17}$$

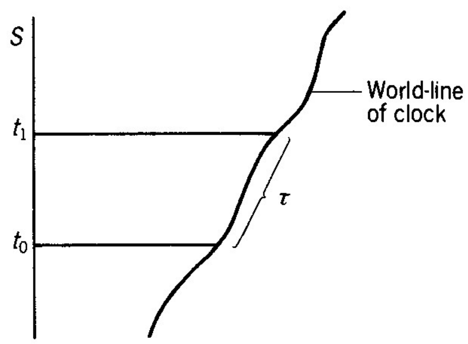

Fig. 3.4 Proper time recorded by an accelerated clock.

The general question of what constitutes a clock or an ideal clock is a non-trivial one. However, an experiment has been performed where an atomic clock was flown round the world and then compared with an identical clock left back on the ground. The travelling clock was found on return to be running slow by precisely the amount predicted by time dilation. Another instance occurs in the study of cosmic rays. Certain mesons reaching us from the top of the Earth's atmosphere are so short-lived that, even had they been travelling at the speed of light, their travel time in the absence of time dilation would exceed their known proper lifetimes by factors of the order of 10. However, these particles are in fact detected at the Earth's surface because their very high velocities keep them young, as it were. Of course, whether or not time dilation affects the human clock, that is, biological ageing, is still an open question. But the fact that we are ultimately made up of atoms, which do appear to suffer time dilation, would suggest that there is no reason by which we should be an exception.

3.5 Transformation of velocities

Consider a particle in motion (Fig. 3.5) with its Cartesian components of velocity being

$$(u_1, u_2, u_3) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \quad \text{in } S$$

and

$$(u'_1, u'_2, u'_3) = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right) \quad \text{in } S'.$$

Taking differentials of a Lorentz transformation

$$t' = \beta(t - vx/c^2), \qquad x' = \beta(x - vt), \qquad y' = y, \qquad z' = z,$$

we get

$$dt' = \beta(dt - v\,dx/c^2), \qquad dx' = \beta(dx - v\,dt), \qquad dy' = dy, \qquad dz' = dz,$$

and hence

$$u'_1 = \frac{dx'}{dt'} = \frac{\beta(dx - v\,dt)}{\beta(dt - v\,dx/c^2)} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{1}{c^2}\left(v\dfrac{dx}{dt}\right)} = \frac{u_1 - v}{1 - u_1 v/c^2}, \tag{3.18}$$

$$u'_2 = \frac{dy'}{dt'} = \frac{dy}{\beta(dt - v\,dx/c^2)} = \frac{\dfrac{dy}{dt}}{\beta\left[1 - \dfrac{1}{c^2}\left(v\dfrac{dx}{dt}\right)\right]} = \frac{u_2}{\beta(1 - u_1 v/c^2)}, \tag{3.19}$$

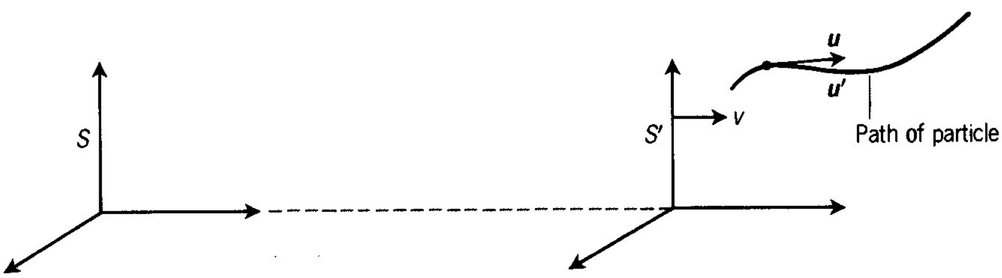

Fig. 3.5 Particle in motion relative to S and S'

$$u_3' = \frac{dz'}{dt'} = \frac{dz}{\beta(dt - v\,dx/c^2)} = \frac{\dfrac{dz}{dt}}{\beta\left[1 - \dfrac{1}{c^2}\left(v\dfrac{dx}{dt}\right)\right]} = \frac{u_3}{\beta(1 - u_1 v/c^2)}. \quad (3.20)$$

Notice that the velocity components u_2 and u_3 transverse to the direction of motion of the frame S' are affected by the transformation. This is due to the time difference in the two frames. To obtain the inverse transformations, simply interchange primes and unprimes and replace v by $-v$.

3.6 Relationship between space-time diagrams of inertial observers

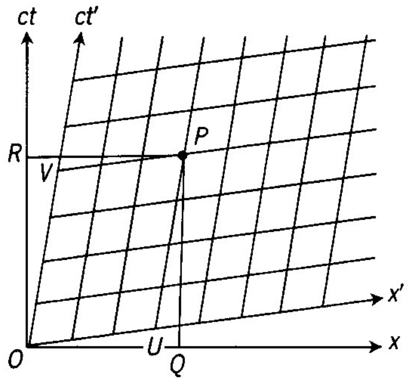

Fig. 3.6 The world-lines in S of the fixed points and simultaneity lines of S'.

We now show how to relate the space-time diagrams of S and S' (see Fig. 3.6). We start by taking ct and x as the coordinate axes of S, so that a light ray has slope $\frac{1}{4}\pi$ (as in relativistic units). Then, to draw the ct'- and x'-axes of S', we note from the Lorentz transformation equations (3.12)

$$ct' = 0 \quad \Leftrightarrow \quad ct = (v/c)x,$$

that is, the x'-axis, $ct' = 0$, is the straight line $ct = (v/c)x$ with slope $v/c < 1$. Similarly,

$$x' = 0 \quad \Leftrightarrow \quad ct = (c/v)x,$$

that is, the ct'-axis, $x' = 0$, is the straight line $ct = (c/v)x$ with slope $c/v > 1$. The lines parallel to $O(ct')$ are the world-lines of fixed points in S'. The lines parallel to Ox' are the lines connecting points at a fixed time according to S' and are called **lines of simultaneity in S'**. The coordinates of a general event P are $(ct, x) = (OR, OQ)$ relative to S and $(ct', x') = (OV, OU)$ relative to S'. However, the diagram is somewhat misleading because the length scales along the axes are not the same. To relate them, we draw in the hyperbolae

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 = \pm 1,$$

as shown in Fig. 3.7. Then, if we first consider the positive sign, setting $ct' = 0$, we get $x' = \pm 1$. It follows that OA is a unit distance on Ox'. Similarly, taking the negative sign and setting $x' = 0$ we get $ct' = \pm 1$ and so OB is the unit measure on Oct'. Then the coordinates of P in the frame S' are given by

$$(ct', x') = \left(\frac{OU}{OA}, \frac{OV}{OB}\right).$$

Note the following properties from Fig. 3.7.

1. A boost can be thought of as a rotation through an imaginary angle in the (x, T)-plane, where T is imaginary time. We have seen that this is equivalent, in the real (x, ct)-plane, to a skewing of the coordinate axes inwards through the same angle. (This was not appreciated by some past opponents of special relativity, who gave some erroneous counter-arguments based on the mistaken idea that a boost could be represented by a real rotation in the (x, ct)-plane.)
2. The hyperbolae are the same for all frames and so we can draw in any number of frames in the same diagram and use the hyperbolae to calibrate them.

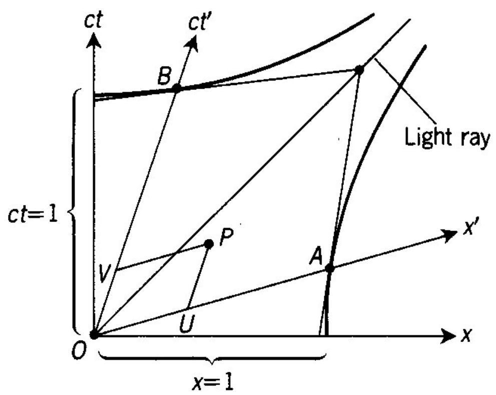

Fig. 3.7 Length scales in S and S'.

3. The length contraction and time dilation effects can be read off directly from the diagram. For example, the world-lines of the endpoints of a unit rod OA in S', namely $x' = 0$ and $x' = 1$, cut Ox in less than unit distance. Similarly world-lines $x = 0$ and $x = 1$ in S cut Ox' inside OE, from which the reciprocal nature of length contraction is evident.
4. Even A has coordinates $(ct', x') = (0, 1)$ relative to S', and hence by a Lorentz transformation coordinates $(ct, x) = (\beta v/c, \beta)$ relative to S. The quantity OA defined by

$$OA = (c^2t^2 + x^2)^{\frac{1}{2}} = \beta(1 + v^2/c^2)^{\frac{1}{2}}$$

is a measure of the calibration factor

$$\left(\frac{1 + v^2/c^2}{1 - v^2/c^2}\right)^{\frac{1}{2}}.$$

3.7 Acceleration in special relativity

We start with the inverse transformation of (3.18), namely,

$$u_1 = \frac{u_1' + v}{1 + u_1'v/c^2},$$

from which we find the differential

$$\begin{aligned} du_1 &= \frac{du_1'}{1 + u_1'v/c^2} - \left(\frac{u_1' + v}{(1 + u_1'v/c^2)^2}\right)\frac{v}{c^2}\,du_1' \\ &= \frac{1}{\beta^2}\frac{du_1'}{(1 + u_1'v/c^2)^2}. \end{aligned}$$

Similarly, from the inverse Lorentz transformation

$$t = \beta(t' + x'v/c^2),$$

we find the differential

$$dt = \beta(dt' + dx'v/c^2) = \beta(1 + u_1'v/c^2)\,dt'.$$

Combining these results, we find that the x-component of the acceleration transforms according to

$$\frac{du_1}{dt} = \frac{1}{\beta^3(1 + u_1'v/c^2)^3}\frac{du_1'}{dt'}. \tag{3.21}$$

Similarly, we find

$$\frac{du_2}{dt} = \frac{1}{\beta^2(1 + u_1'v/c^2)^2}\frac{du_2'}{dt'} - \frac{vu_2'}{c^2\beta^2(1 + u_1'v/c^2)^3}\frac{du_1'}{dt'}, \tag{3.22}$$

$$\frac{du_3}{dt} = \frac{1}{\beta^2(1 + u_1'v/c^2)^3}\frac{du_3'}{dt'} - \frac{vu_3'}{c^2\beta^2(1 + u_1'v/c^2)^3}\frac{du_1'}{dt'}. \tag{3.23}$$

The inverse transformations can be found in the usual way.

It follows from the transformation formulae that acceleration is not an invariant in special relativity. However, it is clear from the formulae that acceleration is an **absolute** quantity, that is, all observers agree whether a body is accelerating or not. Put another way, if the acceleration is zero in one frame, then it is necessarily zero in any other frame. We shall see that this is

Table 3.1

Theory	Position	Velocity	Time	Acceleration
Newtonian	Relative	Relative	Absolute	Absolute
Special relativity	Relative	Relative	Relative	Absolute
General relativity	Relative	Relative	Relative	Relative

no longer the case in general relativity. We summarize the situation in Table 3.1, which indicates why the subject matter of the book is 'relativity' theory.

3.8 Uniform acceleration

The Newtonian definition of a particle moving under uniform acceleration is

$$\frac{du}{dt} = \text{constant}.$$

This turns out to be inappropriate in special relativity since it would imply that $u \to \infty$ as $t \to \infty$, which we know is impossible. We therefore adopt a different definition. Acceleration is said to be **uniform** in special relativity if it has the same value in any **co-moving frame**, that is, at each instant, the acceleration in an inertial frame travelling with the same velocity as the particle has the same value. This is analogous to the idea in Newtonian theory of motion under a constant force. For example, a spaceship whose motor is set at a constant emission rate would be uniformly accelerated in this sense. Taking the velocity of the particle to be $u = u(t)$ relative to an inertial frame S, then at any instant in a co-moving frame S', it follows that $v = u$, the velocity relative to S' is zero, i.e. $u' = 0$, and the acceleration is a constant, a say, i.e. $du'/dt' = a$. Using (3.21), we find

$$\frac{du}{dt} = \frac{1}{\beta^3}a = \left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}} a.$$

We can solve this differential equation by separating the variables

$$\frac{du}{(1 - u^2/c^2)^{\frac{3}{2}}} = a\,dt$$

and integrating both sides. Assuming that the particle starts from rest at $t = t_0$, we find

$$\frac{u}{(1 - u^2/c^2)^{\frac{1}{2}}} = a(t - t_0).$$

Solving for u, we get

$$u = \frac{dx}{dt} = \frac{a(t - t_0)}{[1 + a^2(t - t_0)^2/c^2]^{\frac{1}{2}}}.$$

Next, integrating with respect to t, and setting $x = x_0$ at $t = t_0$, produces

$$(x - x_0) = \frac{c}{a}[c^2 + a^2(t - t_0)^2]^{\frac{1}{2}} - \frac{c^2}{a}.$$

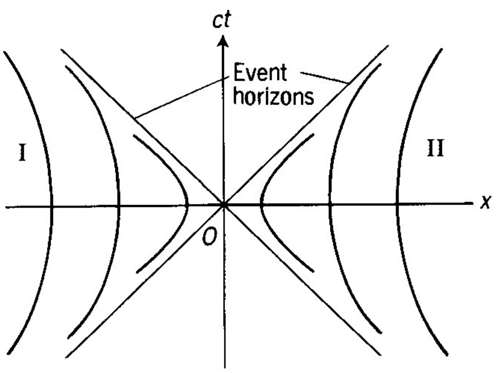

Fig. 3.8 Hyperbolic motions.

This can be rewritten in the form

$$\frac{(x - x_0 + c^2/a)^2}{(c^2/a)^2} - \frac{(ct - ct_0)^2}{(c^2/a)^2} = 1, \tag{3.24}$$

which is a hyperbola in (x, ct)-space. If, in particular, we take $x_0 - c^2/a = t_0 = 0$, then we obtain a family of hyperbolae for different values of a (Fig. 3.8). These world-lines are known as **hyperbolic motions** and, as we shall see in Chapter 23, they have significance in cosmology. It can be shown that the radar distance between the world-lines is a constant. Moreover, consider the regions I and II bounded by the light rays passing through O, and a system of particles undergoing hyperbolic motions as shown in Fig. 3.8 (in some cosmological models, the particles would be galaxies). Then, remembering that light rays emanating from any point in the diagram do so at 45°, no particle in region I can communicate with another particle in region II, and vice versa. The light rays are called **event horizons** and act as barriers beyond which no knowledge can ever be gained. We shall see that event horizons will play an important role later in this book.

3.9 The twin paradox

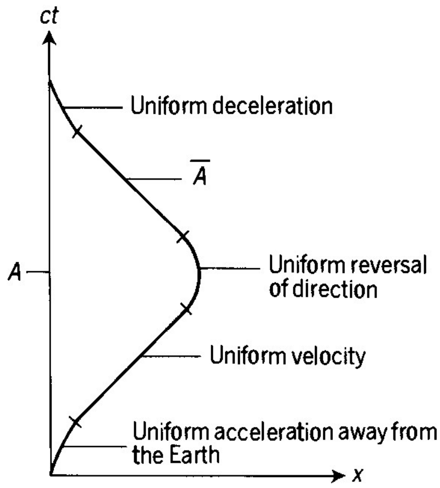

Fig. 3.9 The twin paradox.

This is a form of the clock paradox which has caused the most controversy — a controversy which raged on and off for over 50 years. The paradox concerns two twins whom we shall call A and $\bar{A}$. The twin $\bar{A}$ takes off in a spaceship for a return trip to some distant star. The assumption is that $\bar{A}$ is uniformly accelerated to some given velocity which is retained until the star is reached, whereupon the motion is uniformly reversed, as shown in Fig. 3.9. According to A, $\bar{A}$'s clock records slowly on the outward and return journeys and so, on return, $\bar{A}$ will be younger than A. If the periods of acceleration are negligible compared with the periods of uniform velocity, then could not $\bar{A}$ reverse the argument and conclude that it is A who should appear to be the younger? This is the basis of the paradox.

The resolution rests on the fact that the accelerations, however brief, have immediate and finite effects on $\bar{A}$ but not on A who remains inertial throughout. One striking way of seeing this effect is to draw in the simultaneity lines of $\bar{A}$ for the periods of uniform velocity, as in Fig. 3.10. Clearly, the period of uniform reversal has a marked effect on the simultaneity lines. Another way of looking at it is to see the effect that the periods of acceleration have on shortening the length of the journey as viewed by $\bar{A}$. Let us be specific: we assume that the periods of acceleration are T_1, T_2, and T_3, and that, after the period T_1, $\bar{A}$ has attained a speed $v = \sqrt{3}c/2$. Then, from A's viewpoint, during the period T_1, $\bar{A}$ finds that more than half the outward journey has been accomplished, in that $\bar{A}$ has transferred to a frame in which the distance between the Earth and the star is more than halved by length contraction. Thus, $\bar{A}$ accomplishes the outward trip in about half the time which A ascribes to it, and the same applies to the return trip. In fact, we could use the machinery of previous sections to calculate the time elapsed in both the periods of uniform acceleration and uniform velocity, and we would again reach the conclusion that on return $\bar{A}$ will be younger than A. As we have said before, this points out the fact that in special relativity time is a route-dependent quantity. The fact that in Fig. 3.9 $\bar{A}$'s world-line is longer

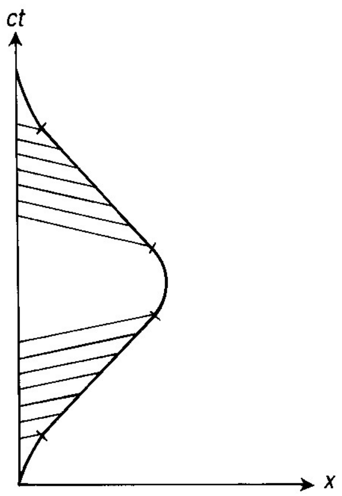

Fig. 3.10 Simultaneity lines of $\bar{A}$ on the outward and return journeys.

than A's, and yet takes **less** time to travel, is connected with the Minkowskian metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

and the **negative** signs which appear in it compared with the positive signs occurring in the usual three-dimensional Euclidean metric.

3.10 The Doppler effect

All kinds of waves appear lengthened when the source recedes from the observer: sounds are deepened, light is reddened. Exactly the opposite occurs when the source, instead, approaches the observer. We first of all calculate the **classical** Doppler effect.

Consider a source of light emitting radiation whose wavelength in its rest frame is λ_0. Consider an observer S relative to whose frame the source is in motion with radial velocity u_r. Then, if two successive pulses are emitted at time differing by dt' as measured by S', the distance these pulses have to travel will differ by an amount $u_r dt'$ (see Fig. 3.11). Since the pulses travel with speed c, it follows that they arrive at S with a time difference

$$\Delta t = dt' + u_r dt'/c,$$

giving

$$\Delta t/dt' = 1 + u_r/c.$$

Now, using the fundamental relationship between wavelength and velocity, set

$$\lambda = c\Delta t \quad \text{and} \quad \lambda_0 = c\,dt'.$$

We then obtain the **classical Doppler formula**

$$\lambda/\lambda_0 = 1 + u_r/c. \tag{3.25}$$

Let us now consider the special relativistic formula. Because of time dilation (see Fig. 3.3), the time interval between successive pulses according to S is $\beta\,dt'$ (Fig. 3.12). Hence, by the same argument, the pulses arrive at S with a time difference

$$\Delta t = \beta\,dt' + u_r \beta\,dt'/c$$

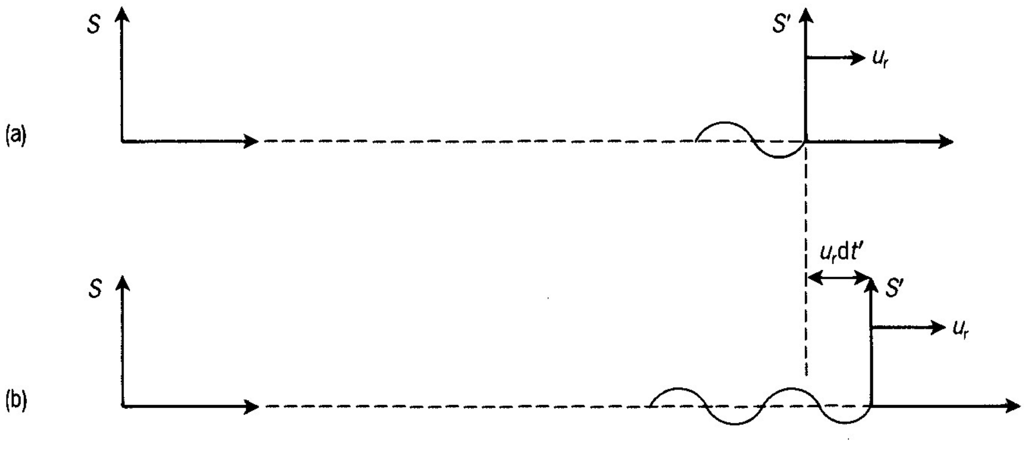

Fig. 3.11 The Doppler effect: (a) first pulse; (b) second pulse.

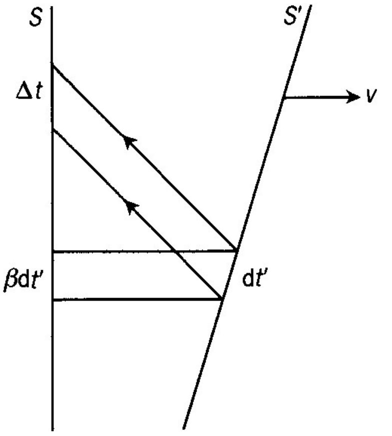

Fig. 3.12 The special relativistic Doppler shift.

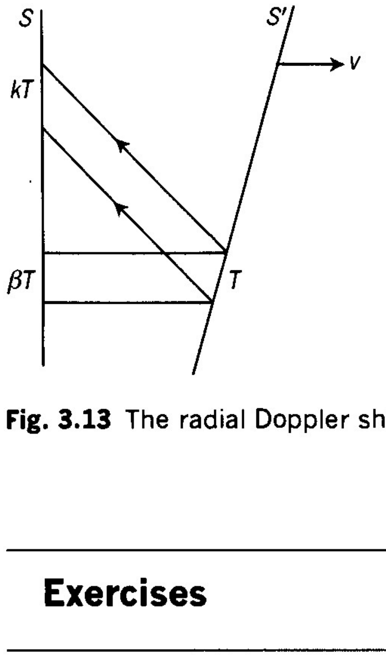

Fig. 3.13 The radial Doppler shift k.

and so this time we find that the **special relativistic Doppler formula** is

$$\frac{\lambda}{\lambda_0} = \frac{1 + u_r/c}{(1 - v^2/c^2)^{\frac{1}{2}}}. \tag{3.26}$$

If the velocity of the source is purely radial, then $u_r = v$ and (3.26) reduces to

$$\frac{\lambda}{\lambda_0} = \left(\frac{1 + v/c}{1 - v/c}\right)^{\frac{1}{2}}. \tag{3.27}$$

This is the **radial Doppler shift**, and, if we set $c = 1$, we obtain (2.4), which is the formula for the ***k*-factor.** Combining Figs. 2.7 and 3.12, the radial Doppler shift is illustrated in Fig. 3.13, where dt' is replaced by T. From equation (3.26), we see that there is also a change in wavelength, even when the radial velocity of the source is zero. For example, if the source is moving in a circle about the origin of S with speed v (as measured by an instantaneous co-moving frame), then the **transverse Doppler shift** is given by

$$\frac{\lambda}{\lambda_0} = \frac{1}{(1 - v^2/c^2)^{\frac{1}{2}}}. \tag{3.28}$$

This is a purely relativistic effect due to the time dilation of the moving source. Experiments with revolving apparatus using the so-called 'Mössbauer effect' have directly confirmed the transverse Doppler shift in full agreement with the relativistic formula, thus providing another striking verification of the phenomenon of time dilation.

Exercises

3.1 (§**3.1**) S and S' are in standard configuration with $v = \alpha c$ $(0 < \alpha < 1)$. If a rod at rest in S' makes an angle of 45° with Ox in S and 30° with $O'x$ in S', then find α.

3.2 (§**3.1**) Note from the previous question that perpendicular lines in one frame need not be perpendicular in another frame. This shows that there is no obvious meaning to the phrase 'two inertial frames are parallel', unless their relative velocity is along a common axis, because the axes of either frame need not appear rectangular in the other. Verify that the Lorentz transformation between frames in standard configuration with relative velocity $\boldsymbol{v} = (v, 0, 0)$ may be written in vector form

$$\boldsymbol{r}' = \boldsymbol{r} + \left(\frac{\boldsymbol{v}\cdot\boldsymbol{r}}{v^2}(\beta - 1) - \beta t\right)\boldsymbol{v}, \qquad t' = \beta\left(t - \frac{\boldsymbol{v}\cdot\boldsymbol{r}}{c^2}\right),$$

where $\boldsymbol{r} = (x, y, z)$. The formulae are said to comprise the 'Lorentz transformation without relative rotation'. Justify this name by showing that the formulae remain valid when the frames are not in standard configuration, but are parallel in the sense that the same rotation must be applied to each frame to bring the two into standard configuration (in which case $\boldsymbol{v}$ is the velocity of S' relative to S, but $\boldsymbol{v} = (v, 0, 0)$ no longer applies).

3.3 (§**3.1**) Prove that the first two equations of the special Lorentz transformation can be written in the form

$$ct' = -x\sinh\phi + ct\cosh\phi, \qquad x' = x\cosh\phi - ct\sinh\phi,$$

where the **rapidity** ϕ is defined by $\phi = \tanh^{-1}(v/c)$. Establish also the following version of these equations:

$$ct' + x' = e^{-\phi}(ct + x),$$
$$ct' - x' = e^{\phi}(ct - x),$$
$$e^{2\phi} = (1 + v/c)/(1 - v/c).$$

What relation does ϕ have to θ in equation (3.11)?

3.4 (§**3.1**) **Aberration** refers to the fact that the direction of travel of a light ray depends on the motion of the observer. Hence, if a telescope observes a star at an inclination θ' to the horizontal, then show that **classically** the 'true' inclination θ of the star is related to θ' by

$$\tan\theta' = \frac{\sin\theta}{\cos\theta + v/c},$$

where v is the velocity of the telescope relative to the star. Show that the corresponding relativistic formula is

$$\tan\theta' = \frac{\sin\theta}{\beta(\cos\theta + v/c)}.$$

3.5 (§**3.2**) Show that special Lorentz transformations are associative, that is, if $O(v_1)$ represents the transformation from observer S to S', then show that

$$\big(O(v_1)O(v_2)\big)O(v_3) = O(v_1)\big(O(v_2)O(v_3)\big).$$

3.6 (§**3.3**) An athlete carrying a horizontal 20-ft-long pole runs at a speed v such that $(1 - v^2/c^2)^{-\frac{1}{2}} = 2$ into a 10-ft-long room and closes the door. Explain, in the athlete's frame, in which the room is only 5 ft long, how this is possible. [Hint: no effect travels faster than light.] Show that the minimum length of the room for the performance of this trick is $20/(\sqrt{3} + 2)$ ft. Draw a space-time diagram to indicate what is going on in the rest frame of the athlete.

3.7 (§**3.5**) A particle has velocity $\boldsymbol{u} = (u_1, u_2, u_3)$ in S and $\boldsymbol{u}' = (u'_1, u'_2, u'_3)$ in S'. Prove from the velocity transformation formulae that

$$c^2 - u^2 = \frac{c^2(c^2 - u'^2)(c^2 - v^2)}{(c^2 + u'_1 v)^2}.$$

Deduce that, if the speed of a particle is less than c in any one inertial frame, then it is less than c in every inertial frame.

3.8 (§**3.7**) Check the transformation formulae for the components of acceleration (3.21)–(3.23). Deduce that acceleration is an absolute quantity in special relativity.

3.9 (§**3.8**) A particle moves from rest at the origin of a frame S along the x-axis, with constant acceleration α (as measured in an instantaneous rest frame). Show that the equation of motion is

$$\alpha x^2 + 2c^2 x - \alpha c^2 t^2 = 0,$$

and prove that the light signals emitted after time $t = c/\alpha$ at the origin will never reach the receding particle. A standard clock carried along with the particle is set to read zero at the beginning of the motion and reads τ at time t in S. Using the clock hypothesis, prove the following relationships:

$$\frac{u}{c} = \tanh\frac{\alpha\tau}{c}, \quad \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = \cosh\frac{\alpha\tau}{c},$$

$$\frac{\alpha t}{c} = \sinh\frac{\alpha\tau}{c}, \quad x = \frac{c^2}{\alpha}\left(\cosh\frac{\alpha\tau}{c} - 1\right).$$

Show that, if $T^2 \ll c^2/\alpha^2$, then, during an elapsed time T in the inertial system, the particle clock will record approximately the time $T(1 - \alpha^2 T^2/6c^2)$.

If $\alpha = 3g$, find the difference in recorded times by the spaceship clock and those of the inertial system

(a) after 1 hour;
(b) after 10 days.

3.10 (§**3.9**) A space traveller $\bar{A}$ travels through space with uniform acceleration g (to ensure maximum comfort). Find the distance covered in 22 years of $\bar{A}$'s time. [Hint: using years and light years as time and distance units, respectively, then $g = 1.03$]. If on the other hand, $\bar{A}$ describes a straight double path $XYZYX$, with acceleration g on XY and ZY, and deceleration on YZ and YX, for 6 years each, then draw a space-time diagram as seen from the Earth and find by how much the Earth would have aged in 24 years of $\bar{A}$'s time.

3.11 (§**3.10**) Let the relative velocity between a source of light and an observer be u, and establish the **classical** Doppler formulae for the frequency shift:

$$\text{source moving, observer at rest:} \quad \nu = \frac{\nu_0}{1 + u/c},$$

$$\text{observer moving, source at rest:} \quad \nu = (1 - u/c)\nu_0,$$

where ν_0 is the frequency in the rest frame of the source. What are the corresponding relativistic results?

3.12 (§**3.10**) How fast would you need to drive towards a red traffic light for the light to appear green? [Hint: $\lambda_{\text{red}} \approx 7 \times 10^{-5}$ cm, $\lambda_{\text{green}} \approx 5 \times 10^{-5}$ cm.]