6.3 The Relationship Between Momentum and Energy

From the relation $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we have $\gamma^2 - \beta^2\gamma^2 = 1$. Multiplying throughout by $m_0^2c^4$, it follows that

$$\gamma^2 m_0^2 c^4 - \beta^2\gamma^2 m_0^2 c^4 = m_0^2 c^4. \tag{6.28}$$

Because it is $E = \gamma m_0 c^2$ and $p = \beta\gamma m_0 c$, the last equation may also be written as

$$E^2 - p^2c^2 = m_0^2c^4. \tag{6.29}$$

The magnitude $m_0^2c^4$ is an invariant, because, evaluated in any inertial frame of reference, takes the same value. It follows that the quantity $E^2 - p^2c^2$ is also invariant.

The relation

$$E^2 = m_0^2c^4 + p^2c^2 \tag{6.30}$$

connects the magnitudes of the rest mass, the momentum and the energy of a body. Geometrically, it may be considered as an application of the Pythagorean theorem on a four-dimensional[1] 'right-angled triangle' with sides $E_0 = m_0c^2$, p_xc, p_yc and p_zc. The triangle of Fig. 6.5 may serve as a memory aid for relation (6.30).

The energy E may also be expressed as

$$E = \sqrt{m_0^2c^4 + p^2c^2}. \tag{6.31}$$

In addition, since it is $E = mc^2$ and $\mathbf{p} = m\boldsymbol{\upsilon}$, it follows that

$$\mathbf{p} = \frac{E}{c^2}\boldsymbol{\upsilon}. \tag{6.32}$$

6.3.1 Units of Energy, Mass and Momentum

In Atomic Physics, Nuclear Physics and the Physics of Elementary Particles, we use as the unit of energy the electron-volt (eV), defined as the change in the kinetic

[1]In fact, the quantity m_0c^2 is the modulus of the four-vector of energy-momentum, which has as components the quantities E, p_xc, p_yc and p_zc. The norm of this four-vector is, according to the definition, equal to $m_0^2c^4 = E^2 - p_x^2c^2 - p_y^2c^2 - p_z^2c^2 = E^2 - p^2c^2$ and is invariant. The difference between the magnitude of this and the magnitude of a vector in a four-dimensional Euclidean space is in the negative signs which appear. For more on four-vectors, see Chap. 8.

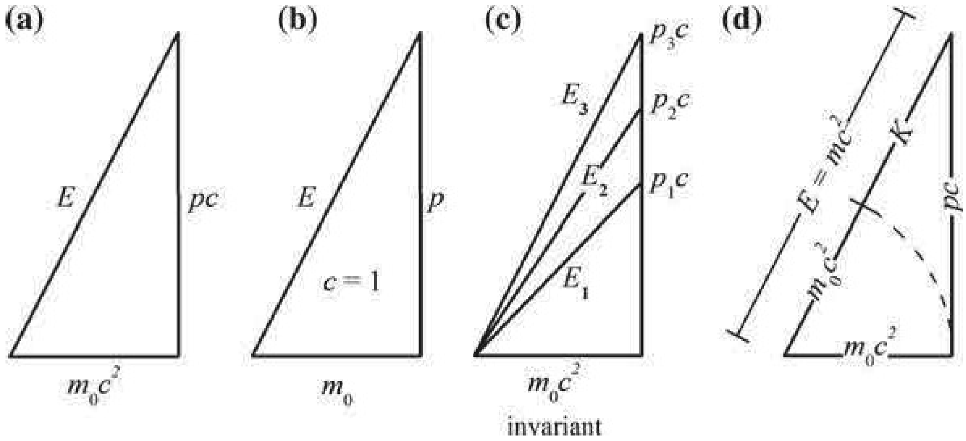


Fig. 6.5 Geometrical memory aid for the relationship $E^2 = m_0^2c^4 + p^2c^2$. Pythagoras' theorem, applied to the right-angle triangle of figure (**a**), with sides equal to m_0c^2 and pc and the total energy E as its hypotenuse, gives the relationship. In figure (**b**), a system of units is used in which $c = 1$. Figure (**c**) shows the variation of the body's total energy with its momentum. Figure (**d**) also shows the relationships $E = mc^2 = m_0c^2 + K$, where K is the body's kinetic energy

energy of a particle with a charge equal to that of the proton, $|e|$, on moving between two points among which there exists a potential difference of one volt. It is $1\ \text{eV} = |e|\Delta V = (1.602 \times 10^{-19}\ \text{C}) \times (1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$. Multiples of this unit which are used are the keV $(10^3\ \text{eV})$, the MeV $\left(10^6\ \text{eV}\right)$, the GeV $(10^9\ \text{eV})$ or even the TeV $(10^{12}\ \text{eV})$.

Given that the magnitude m_0c^2 has the dimensions of energy and, for relativistic particles, is measured in MeV or GeV, it is customary to use MeV/c^2 and GeV/c^2 as units of mass. So, if we find, for example, the numerical result $mc^2 = \text{X MeV}$, we express the mass as $m = \text{X MeV}/c^2$.

Similarly, for momentum, if we evaluate the quantity pc, which has the dimensions of energy, and we find the numerical result $pc = \text{X MeV}$, we express the momentum as $p = \text{X MeV}/c$.

Inversely, a body with mass $m = \text{X MeV}/c^2$ has $mc^2 = \text{X MeV}$. A body with momentum $p = \text{X MeV}/c$ has $pc = \text{X MeV}$.

The advantage in the use of these units is that we have simple numbers for the energy, the mass and the momentum of relativistic particles and the numerical values of these quantities for a certain particle are of the same order of magnitude. For example, for a very energetic particle, which moves with a speed very close to c, if its energy is $mc^2 = \text{X MeV}$, its momentum will be $p = m\upsilon \approx mc = \text{X MeV/c}$. The particle's mass expressed in MeV/c^2 has the same numerical value as its energy expressed in MeV. For a high-energy particle, its momentum, expressed in MeV/c, will almost have the same numerical value as well. A proton with a total energy of 10 GeV, has a relativistic mass of 10 GeV/c^2, a Lorentz factor of about 10, which corresponds to a reduced speed of $\beta = 0.995$, and a momentum p for which it is $pc = m\upsilon c = m\beta c^2 = 0.995mc^2 = 9.95\ \text{GeV}$. It is, therefore, $p = 9.95\ \text{GeV/c} \approx 10\ \text{GeV/c}$.

In the Theory of Relativity, especially the General, a system of units is frequently used which takes the speed of light in vacuum as equal to unity, $c = 1$. This makes the symbol c 'disappear' from all equations, leading (according to some) to a simplification. This system will not be used in this book.

6.4 Classical Approximations

The relativistic equations must reduce to the classical ones for $c \to \infty$ or, equivalently, for $\beta \to 0$ and $\gamma \to 1$. The first approximations to the relativistic cases are found for $\gamma = \frac{1}{\sqrt{1-\upsilon^2/c^2}} \approx 1 + \frac{1}{2}\frac{\upsilon^2}{c^2}$. In this way, the kinetic energy is approximately equal to

$$K = \frac{m_0 c^2}{\sqrt{1-\upsilon^2/c^2}} - m_0 c^2 = m_0 c^2(\gamma - 1) \approx \frac{1}{2}\frac{\upsilon^2}{c^2} m_0 c^2 = \frac{1}{2} m_0 \upsilon^2, \tag{6.33}$$

the kinetic energy according to classical Mechanics. The total energy is, for a free body,

$$E = mc^2 = \gamma m_0 c^2 \approx \left(1 + \frac{1}{2}\frac{\upsilon^2}{c^2}\right) m_0 c^2 = m_0 c^2 + \frac{1}{2} m_0 \upsilon^2, \tag{6.34}$$

i.e. approximately equal to the rest energy of the body plus its classical kinetic energy.

The momentum of a body is [Eq. (6.11)]

$$\mathbf{p} = m\boldsymbol{\upsilon} = \frac{m_0 \boldsymbol{\upsilon}}{\sqrt{1-\upsilon^2/c^2}} \approx m_0 \boldsymbol{\upsilon}\left(1 + \frac{1}{2}\frac{\upsilon^2}{c^2}\right) \quad \text{or} \quad \mathbf{p} = m_0 \boldsymbol{\upsilon}, \tag{6.35}$$

to the approximation involving only powers lower than the second in the ratio υ/c.

It must be stressed that it is wrong to use these relations as exact in relativistic Mechanics!

Example 6.1 Speeds of Electrons and Protons From Modern Accelerators
The energy to which electrons are accelerated by modern synchrotrons has exceeded the value of 25 GeV, while protons are accelerated to 7 TeV at CERN, with the accelerator LHC. What are the speeds of the particles at these energies?

The rest energies of the electron and the proton are, respectively, $m_{e0}c^2 = 0.511$ MeV and $m_{p0}c^2 = 938$ MeV. At energies as high as the ones given, the Lorentz factor is so large that the kinetic energy may be taken as equal to the total. It is, therefore, $K = m_0 c^2(\gamma - 1) \approx \gamma m_0 c^2$ and thus $\gamma \approx K/m_0 c^2$, from which we have $\beta = \sqrt{1 - 1/\gamma^2} \approx 1 - 1/2\gamma^2 \approx 1 - (m_0 c^2)^2/2K^2$.

For an electron with $K_e = 25$ GeV $= 2.5 \times 10^{10}$ eV, it is

$$\gamma_e \approx K_e/m_{e0}c^2 = 2.5 \times 10^{10}/0.511 \times 10^6 = 49\,000.$$

We see that the approximation we made is justified. The reduced speed of the electron is

$$\beta_e \approx 1 - 1/2\gamma_e^2 = 1 - 2.1 \times 10^{-10} = 0.999\,999\,999\,8.$$

For a proton with $K_p = 7$ TeV $= 7 \times 10^{12}$ eV, it is

$$\gamma_p \approx K_p/m_{p0}c^2 = 7 \times 10^{12}/0.938 \times 10^9 = 7560.$$

We see that the approximation we made is, here, also justified. The reduced speed of the proton is

$$\beta_p \approx 1 - 1/2\gamma_p^2 = 1 - 8.7 \times 10^{-9} = 0.999\,999\,991\,3.$$

Example 6.2 Kinetic Energy in One-Dimensional Motion
Show that the kinetic energy of a body moving on a straight line is given by $K = mc^2 - m_0c^2$.

It is

$$K = \int_{\upsilon=0}^{\upsilon} \mathbf{F} \cdot d\mathbf{s} = \int_{\upsilon=0}^{\upsilon} F dx = \int_{\upsilon=0}^{\upsilon} \frac{d}{dt}(m\upsilon)dx = \int_{\upsilon=0}^{\upsilon} d(m\upsilon)\frac{dx}{dt} =$$
$$= \int_{\upsilon=0}^{\upsilon} (md\upsilon + \upsilon dm)\upsilon = \int_{\upsilon=0}^{\upsilon} \left(m\upsilon d\upsilon + \upsilon^2 dm\right)$$

From $m = \dfrac{m_0}{\sqrt{1 - \upsilon^2/c^2}}$, it follows that $m^2c^2 - m^2\upsilon^2 = m_0c^2$.

Taking differentials, we find $2mc^2dm - m^2 2\upsilon d\upsilon - \upsilon^2 2mdm = 0$ or $m\upsilon d\upsilon + \upsilon^2 dm = c^2 dm$. Substituting in K,

$$K \int_{\upsilon=0}^{\upsilon} \left(m\upsilon d\upsilon + \upsilon^2 dm\right) = \int_{m=m_0}^{m} c^2 dm = c^2(m - m_0).$$

Problems

6.1 What is the speed of an electron whose kinetic energy is 2 MeV? What is the ratio of its relativistic mass to its rest mass? The rest mass of the electron is $m_0 = 0.511$ MeV/c^2. Ans.: $\upsilon = 0.98c$, $m/m_0 = 5$

6.2 The extremely rare event of the indirect observation, in cosmic radiation, of a particle with an energy of the order of 10^{20} eV (16 J!) occurred a few years ago [see J. Linsey, *Phys. Rev. Lett.* **10**, 146 (1963)]. Assuming the particle was a proton, which has a rest energy of 1 GeV, approximately, evaluate its speed relative to the Earth. How much time would this proton need, in its own frame of reference, in order to cross our galaxy, whose diameter is 10^5 light years? What is the diameter of the Galaxy as seen by the proton? Ans.: $\upsilon = 0.999\,999\,999\,999\,999\,999\,999\,95\ c = c - 1.5 \times 10^{-14}$ m/s, 32 s, 10^{10} m approx.

6.3 The kinetic energy and the momentum of a particle were measured and found equal to 250 MeV and 368 MeV$/c$, respectively. Find the rest mass of the particle in MeV$/c^2$. Ans.: 270 MeV/c^2

6.4 At what value of the speed of a particle is its kinetic energy equal to (a) its rest energy and (b) 10 times its rest energy? Ans.: (a) $0.866c$, (b) $0.996c$

6.5 An electron with an energy of 100 MeV moves along a tube which is 5 m long. What is the length of the tube in the frame of reference of the electron? The rest energy of the electron is $E_0 = m_0c^2 = 0.511$ MeV, where m_0 is its rest mass. Ans.: 26 mm

6.6 A beam of identical particles with the same speed is produced by an accelerator. The particles of the beam travel, inside a tube, its full length of $l = 2400$ m in a time $\Delta t = 10\,\mu$s, as measured by an observer in the laboratory frame of reference.

(a) Find β and γ for the particles and the duration $\Delta t'$ of the trip as measured in their own frame of reference. Ans.: $\beta = 4/5$, $\gamma = 5/3$, $\Delta t' = 6\ \mu$s
(b) If the particles are unstable, with a mean lifetime of $\tau = 10^{-6}$ s, what proportion of the particles is statistically expected to reach the end of the tube? (Use: $e^3 \approx 20$). Ans.: $1/400$
(c) If the rest energy of each particle is $m_0c^2 = 3$ GeV, find their kinetic energy. Ans.: $K = 2$ GeV
(d) Determine the quantity pc for a particle of the beam, where p is its momentum, and express p in units of GeV$/c$. Ans.: $p = 4$ GeV$/c$

6.7 Two particles, A and B, each having a rest mass of $m_0 = 1$ GeV/c^2, move, in the frame of reference of an accelerator, on the x-axis and in opposite directions, approaching each other. In this frame, particle A moves with a velocity of $\upsilon_{Ax} = -0.6\ c$ and particle B with a velocity of $\upsilon_{Bx} = 0.6\ c$. In the frame of reference of particle A,

(a) what is the speed of particle B? Ans.: $\upsilon'_{Bx} = 0.88c$
(b) what is the energy (in GeV) and the momentum (in GeV$/c$) of particle B? Ans.: $E'_B = 2.11$ GeV, $p'_{Bx} = 1.86$ GeV$/c$

6.8 A beam of π^+ particles, each with an energy of 1 GeV, has a total flow rate of 10^6 particles/s at the start of a trip that has a length of 10 m in the laboratory frame of reference. What is the flow rate of particles at the end of the trip? π^+ has a rest mass of $m_\pi = 140$ MeV$/c^2$ and a mean lifetime $\tau_\pi = 2.56 \times 10^{-8}$ s. Ans.: $0.83 \times 10^6\ \pi^+/$s

6.9 Show that, for a body of rest mass m_0, which moves with a speed υ and has momentum p and kinetic energy K, it is $\dfrac{p\upsilon}{K} = 1 + \dfrac{1}{1 + K/m_0c^2}$.

6.5 Particles with Zero Rest Mass

There exist in nature particles with zero rest mass. The better known of these is the photon, the carrier of the electromagnetic field. The graviton, the carrier of the gravitational field also has zero rest mass. The neutrino possibly has zero rest mass, although at present we can only set an upper limit to it, which is 0.3 eV/c^2. Strictly speaking, that is the situation with the other particles as well. The upper limit for the rest mass of the photon is known today, from measurements of Luo [1] and his co-workers, to be 1.2×10^{-54} kg or 7×10^{-21} eV/c^2.

If we put $m_0 = 0$ in the relativistic energy relations, we have for the energy, from Eq. (6.30)

$$E = pc. \tag{6.36}$$

Inversely, the momentum of a particle with zero rest mass which has an energy E is

$$p = E/c. \tag{6.37}$$

Since in general it is $p = \frac{E}{c^2}\upsilon$ [Eq. (6.32)], substituting in Eq. (6.37), we find that $\upsilon = c$. The same result is derived by letting $m_0 \to 0$ in the equation $E = \frac{m_0\, c^2}{\sqrt{1-\upsilon^2/c^2}}$, in which case it follows that $\upsilon \to c$. This important result tells us that *all particles having zero rest mass move with the maximum possible speed, which is the speed of light in vacuum, c*.

Also valid for the photon are the relations between its energy, its frequency f and its wavelength λ,

$$E = hf = \frac{hc}{\lambda}, \tag{6.38}$$

where h is Planck's constant $\left(h = 6.626 \times 10^{-34}\ \text{J}\cdot\text{s} = 4.136 \times 10^{-15}\text{eV}\cdot\text{s}\right)$. It follows that the momentum of a photon is also given by

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}. \tag{6.39}$$

We have to be careful in the use of this expression, since, given that the energy of the photon is always positive, Eq. (6.39) only gives the magnitude of the momentum and not its algebraic value, which may, in some cases, be negative.

Problems

6.10 What is the mass m_ϕ which corresponds to the energy E_ϕ of a photon of wavelength 500 nm? Ans.: $m_\phi = 4.42 \times 10^{-36}$ kg

6.11 The wavelength of the photons from a laser is 633 nm. What is the momentum of one such photon? Ans.: $p = 1.96$ eV/c

6.6 The Conservation of Momentum and of Energy

Momentum was defined in such a manner so that, in an interaction of particles, both the mass and the momentum are conserved. Given that the total energy of a free body, not acted on by external forces, is equal to $E = mc^2$, the fact that in an interaction of n particles the conservation of relativistic mass applies,

$$\sum_{i=1}^{n} m_i = \text{const.} \tag{6.40}$$

means that the energy is also conserved:

$$\sum_{i=1}^{n} E_i = \text{const.} \tag{6.41}$$

This law is used, together with the law of conservation of momentum

$$\sum_{i=1}^{n} \mathbf{p}_i = \text{const.} \tag{6.42}$$

in the solution of problems in which particles interact in any way.

For n particles, the components of the total momentum and the energy are defined as

$$P_{\text{tot},x} \equiv \sum_{i=1}^{n} p_{ix} \quad P_{\text{tot},y} \equiv \sum_{i=1}^{n} p_{iy} \quad P_{\text{tot},z} \equiv \sum_{i=1}^{n} p_{iz} \quad E_{\text{tot}} \equiv \sum_{i=1}^{n} E_i \tag{6.43}$$

for frame S, with similar expressions for frame S′.

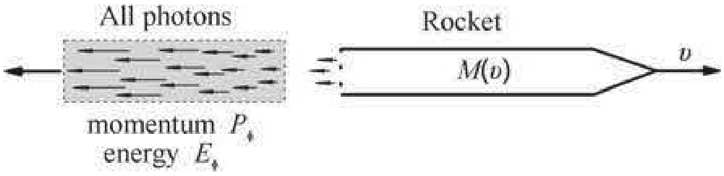
If, in frame S, during the interaction of n particles, the momentum and the energy are conserved, then, for the total values of the components of momentum and of energy, the following relations are true:

$$\begin{gathered}\left(P_{\text{tot},x}\right)_{\text{initially}} = \left(P_{\text{tot},x}\right)_{\text{finally}} \quad \left(P_{\text{tot},y}\right)_{\text{initially}} = \left(P_{\text{tot},y}\right)_{\text{finally}} \quad \left(P_{\text{tot},z}\right)_{\text{initially}} = \left(P_{\text{tot},z}\right)_{\text{finally}} \\ \left(E_{\text{tot}}\right)_{\text{initially}} = \left(E_{\text{tot}}\right)_{\text{finally}}\end{gathered} \tag{6.44}$$

The conservation of these quantities is valid in general, in every physical process. The subject will be further discussed in the next section.

Example 6.3 The Photon Rocket

The use of photons in the propulsion of rockets has been proposed. While for chemical fuels there is a limit in the speed of ejection of mass of the order of 10 km/s, for photons this speed is 30 000 times greater. For this reason, the emission of photons by the rocket, with their high speed, was considered that it would give the rocket a greater speed per unit mass ejected. Find the speed attained by such a rocket, as a function of the fraction κ of its mass that has been ejected as photons.



Let the initial rest mass of the rocket be M_0 and, at some time, a fraction κ of this has been emitted as photons. All the photons are considered to be emitted in the same direction. The emitted photons carry, in total, an energy E_ϕ and momentum $P_\phi = E_\phi/c$, as measured by an observer on the Earth. If at that moment the rocket is moving with a speed equal to υ and has a relativistic mass $M(\upsilon) = (1-\kappa)M_0\gamma$, where $\gamma = 1\Big/\sqrt{1-\upsilon^2/c^2}$, the laws of conservation give:

Conservation of energy: $E_{\text{total}} = M_0c^2 = M(\upsilon)c^2 + E_\phi = (1-\kappa)M_0c^2\gamma + E_\phi$

Conservation of momentum: $P_{\text{total}} = 0 = M(\upsilon)\upsilon - E_\phi/c = (1-\kappa)M_0\gamma\upsilon - E_\phi/c$

Eliminating E_ϕ between these two relations, it follows that

$$M_0c^2 = (1-\kappa)M_0c^2\gamma(1+\beta) \text{ and } (1-\kappa)\gamma(1+\beta) = 1, \quad (1-\kappa)\sqrt{\frac{1+\beta}{1-\beta}} = 1,$$

$\dfrac{1-\beta}{1+\beta} = (1-\kappa)^2$, from which we find $\beta = \dfrac{1-(1-\kappa)^2}{1+(1-\kappa)^2}$ and

$$\gamma = \frac{1}{2}\left(1-\kappa+\frac{1}{1-\kappa}\right).$$

Some numerical values are:

β	0.5	0.9	0.95	0.99	0.995	0.999
γ	1.16	2.94	3.20	7.09	10	22.4
κ	0.42	0.77	0.84	0.93	0.95	0.98

We see that, in order to achieve a speed of $0.995c$, which corresponds to $\gamma = 10$, a proportion of 95 % of the rocket's total mass must be emitted as photons. The useful load is, therefore, only 5 % of the spaceship. Of course, if we wish to decelerate the spaceship in order to stop it, then accelerate it towards its starting point and finally decelerate it to a halt at its starting point, the useful load would be only a fraction equal to $(1-\kappa)^4$, which has a value smaller than 10^{-5} for maximum γ equal to 10. This makes the photon rockets of doubtful usefulness, without the situation being very much better for chemical fuels of course. The problem is examined in greater detail in Sect. 7.6.1.