4 The elements of relativistic mechanics

4.1 Newtonian theory

Before discussing relativistic mechanics, we shall review some basic ideas of Newtonian theory. We have met Newton's first law in § 2.4, and it states that a body not acted upon by a force moves in a straight line with uniform velocity. The second law describes what happens if an object changes its velocity. In this case, something is causing it to change its velocity and this something is called a **force**. For the moment, let us think of a force as something tangible like a push or a pull. Now, we know from experience that it is more difficult to push a more massive body and get it moving than it is to push a less massive body. This resistance of a body to motion, or rather change in motion, is called its **inertia**. To every body, we can ascribe, at least at one particular time, a number measuring its inertia, which (again for the moment) we shall call its **mass** m. If a body is moving with velocity $\boldsymbol{v}$, we define its **linear momentum** $\boldsymbol{p}$ to be the product of its mass and velocity. Then Newton's second law (N2) states that the force acting on a body is equal to the rate of change of linear momentum. The third law (N3) is less general and talks about a restricted class of forces called **internal** forces, namely, forces acting on a body due to the influence of other bodies in a system. The third law states that the force acting on a body due to the influence of the other bodies, the so-called **action**, is equal and opposite to the force acting on these other bodies due to the influence of the first body, the so-called **reaction**. We state the two laws below.

N2: The rate of change of momentum of a body is equal to the force acting on it, and is in the direction of the force.

N3: To every action there is an equal and opposite reaction.

Then, for a body of mass m with a force $\boldsymbol{F}$ acting on it, Newton's second law states

$$\boldsymbol{F}=\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t}. \tag{4.1}$$

If, in particular, the mass is a constant, then

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a} \tag{4.2}$$

where $\boldsymbol{a}$ is the acceleration.

Now, strictly speaking, in Newtonian theory, all observable quantities should be defined in terms of their measurement. We have seen how an observer equipped with a frame of reference, ruler, and clock can map the events of the universe, and hence measure such quantities as position, velocity, and acceleration. However, Newton's laws introduce the new concepts of force and mass, and so we should give a prescription for their measurement. Unfortunately, any experiment designed to measure these quantities involves Newton's laws themselves in its interpretation. Thus, Newtonian mechanics has the rather unexpected property that the operational definitions of force and mass which are required to make the laws physically significant are actually contained in the laws themselves.

To make this more precise, let us discuss how we might use the laws to measure the mass of a body. We consider two bodies isolated from all other influences other than the force acting on one due to the influence of the other and vice versa (Fig. 4.1). Since the masses are assumed to be constant, we have, by Newton's second law in the form (4.2),

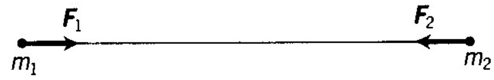

Fig. 4.1 Measuring mass by mutually induced accelerations.

$$\boldsymbol{F}_1 = m_1\boldsymbol{a}_1 \quad \text{and} \quad \boldsymbol{F}_2 = m_2\boldsymbol{a}_2.$$

In addition, by Newton's third law, $\boldsymbol{F}_1 = -\boldsymbol{F}_2$. Hence, we have

$$m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2. \tag{4.3}$$

Therefore, if we take one standard body and define it to have **unit** mass, then we can find the mass of the other body, by using (4.3). We can keep doing this with any other body and in this way we can calibrate masses. In fact, this method is commonly used for comparing the masses of elementary particles. Of course, in practice, we cannot remove all other influences, but it may be possible to keep them almost constant and so neglect them.

We have described how to use Newton's laws to measure mass. How do we measure force? One approach is simply to use Newton's second law, work out $m\boldsymbol{a}$ for a body and then read off from the law the force acting on m. This is consistent, although rather circular, especially since a force has independent properties of its own. For example, Newton has provided us with a way for working out the force in the case of gravitation in his **universal law of gravitation** (UG).

UG: Two particles attract each other with a force directly proportional to their masses and inversely proportional to the distance between them.

If we denote the constant of proportionality by G (with value 6.67×10^{-11} in m.k.s. units), the so-called Newtonian constant, then the law is (see Fig. 4.2)

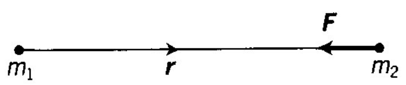

Fig. 4.2 Newton's universal law of gravitation.

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}, \tag{4.4}$$

where a hat denotes a unit vector. There are other force laws which can be stated separately. Again, another independent property which holds for certain forces is contained in Newton's third law. The standard approach to defining force is to consider it as being **fundamental**, in which case force laws can be stated separately or they can be worked out from other considerations. We postpone a more detailed critique of Newton's laws until Part C of the book.

Special relativity is concerned with the behaviour of material bodies and light rays **in the absence of gravitation**. So we shall also postpone a detailed consideration of gravitation until we discuss general relativity in Part C of the book. However, since we have stated Newton's universal laws of gravitation in (4.4), we should, for completeness, include a statement of Newtonian gravitation for a distribution of matter. A distribution of matter of mass density $\rho = \rho(x, y, z, t)$ gives rise to a gravitational potential ϕ which satisfies **Poisson's equation**

$$\nabla^2 \phi = 4\pi G \rho \tag{4.5}$$

at points inside the distribution, where the Laplacian operator ∇^2 is given in Cartesian coordinates by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

At points external to the distribution, this reduces to **Laplace's equation**

$$\nabla^2 \phi = 0. \tag{4.6}$$

We assume that the reader is familiar with this background to Newtonian theory.

4.2 Isolated systems of particles in Newtonian mechanics

In this section, we shall, for completeness, derive the conservation of linear momentum in Newtonian mechanics for a system of n particles. Let the ith particle have constant mass m_i and position vector $\boldsymbol{r}_i$ relative to some arbitrary origin. Then the ith particle possesses linear momentum $\boldsymbol{p}_i$ defined by $\boldsymbol{p}_i = m_i \dot{\boldsymbol{r}}_i$, where the dot denotes differentiation with respect to time t. If $\boldsymbol{F}_i$ is the total force on m_i, then, by Newton's second law, we have

$$\boldsymbol{F}_i = \dot{\boldsymbol{p}}_i = m_i \ddot{\boldsymbol{r}}_i. \tag{4.7}$$

The total force $\boldsymbol{F}_i$ on the ith particle can be divided into an external force $\boldsymbol{F}_i^{\text{ext}}$ due to any external fields present and to the resultant of the internal forces. We write

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{ext}} + \sum_{j=1}^{n} \boldsymbol{F}_{ij},$$

where $\boldsymbol{F}_{ij}$ is the force or the ith particle due to the jth particle and where, for convenience, we define $\boldsymbol{F}_{ii} = \mathbf{0}$. If we sum over i in (4.7), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} \boldsymbol{p}_i = \sum_{i=1}^{n} \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} + \sum_{i,j=1}^{n} \boldsymbol{F}_{ij}.$$

Using Newton's third law, namely, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, then the last term is zero and we obtain $\dot{\boldsymbol{P}} = \boldsymbol{F}^{\mathrm{ext}}$, where $\boldsymbol{P} = \sum_{i=1}^{n} \boldsymbol{p}_i$ is termed the **total linear momentum** of the system and $\boldsymbol{F}^{\mathrm{ext}} = \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}}$ is the **total external force** on the system. If, in particular, the system of particles is **isolated**, then

$$\boldsymbol{F}^{\mathrm{ext}} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{P} = \boldsymbol{c},$$

where $\boldsymbol{c}$ is a constant vector. This leads to the law of the **conservation of linear momentum** of the system, namely,

$$\boldsymbol{P}_{\mathrm{initial}} = \boldsymbol{P}_{\mathrm{final}}. \tag{4.8}$$

4.3 Relativistic mass

The transition from Newtonian to relativistic mechanics is not, in fact, completely straightforward, because it involves at some point or another the introduction of *ad hoc* assumptions about the behaviour of particles in relativistic situations. We shall adopt the approach of trying to keep as close to the non-relativistic definition of energy and momentum as we can. This leads to results which in the end must be confronted with experiment. The ultimate justification of the formulae we shall derive resides in the fact that they have been repeatedly confirmed in numerous laboratory experiments in particle physics. We shall only derive them in a simple case and state that the arguments can be extended to a more general situation.

It would seem plausible that, since length and time measurements are dependent on the observer, then mass should also be an observer-dependent quantity. We thus assume that a particle which is moving with a velocity $\boldsymbol{u}$ relative to an inertial observer has a mass, which we shall term its **relativistic mass**, which is some function of $\boldsymbol{u}$, that is,

$$m = m(\boldsymbol{u}), \tag{4.9}$$

where the problem is to find the explicit dependence of m on $\boldsymbol{u}$. We restrict attention to motion along a straight line and consider the special case of two equal particles colliding **inelastically** (in which case they stick together), and look at the collision from the point of view of two inertial observers S and S' (see Fig. 4.3). Let one of the particles be at rest in the frame S and the other possess a velocity u before they collide. We then assume that they coalesce and that the combined object moves with velocity U. The masses of the two particles are respectively $m(0)$ and $m(u)$ by (4.9). We denote $m(0)$ by m_0 and term it the **rest mass** of the particle. In addition, we denote the mass of the combined object by $M(U)$. If we take S' to be the **centre-of-mass frame**, then it should be clear that, relative to S', the two equal particles collide with equal and opposite speeds, leaving the combined object with mass M_0 at rest. It follows that S' must have velocity U relative to S.

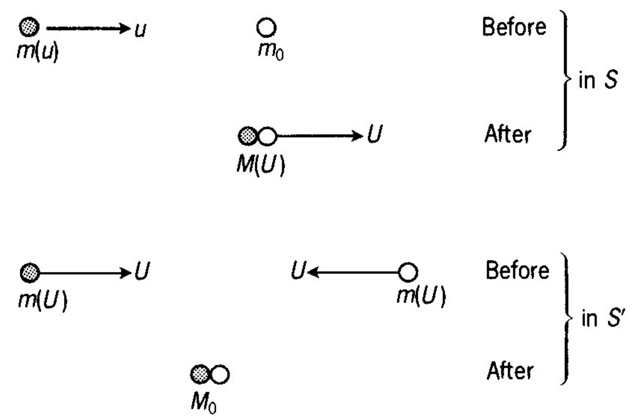

Fig. 4.3 The inelastic collision in the frames S and S'.

We shall assume both conservation of relativistic mass and conservation of linear momentum and see what this leads to. In the frame S, we obtain

$$m(u) + m_0 = M(U), \qquad m(u)u + 0 = M(U)U,$$

from which we get, eliminating $M(U)$,

$$m(u) = m_0\left(\frac{U}{u - U}\right). \tag{4.10}$$

The left-hand particle has a velocity U relative to S', which in turn has a velocity U relative to S. Hence, using the composition of velocities law, we can compose these two velocities and the resultant velocity must be identical with the velocity u of the left-hand particle in S. Thus, by (2.6) in non-relativistic units,

$$u = \frac{2U}{(1 + U^2/c^2)}.$$

Solving for U in terms of u, we obtain the quadratic

$$U^2 - \left(\frac{2c^2}{u}\right)U + c^2 = 0,$$

which has roots

$$U = \frac{c^2}{u} \pm \left[\left(\frac{c^2}{u}\right)^2 - c^2\right]^{\frac{1}{2}} = \frac{c^2}{u}\left[1 \pm \left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}\right].$$

In the limit $u \to 0$, this must produce a finite result, so we must take the negative sign (check), and, substituting in (4.10), we find finally

$$m(u) = \gamma m_0, \tag{4.11}$$

where

$$\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}. \tag{4.12}$$

This is the basic result which relates the relativistic mass of a moving particle to its rest mass. Note that this is the same in structure as the time dilation formula (3.16), i.e. $T = \beta T_0$, where $\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$, except that time dilation involves the factor β which depends on the velocity v of the frame S' relative to S, whereas γ depends on the velocity u of the particle relative to S. If we plot m against u, we see that relativistic mass increases without bound as u approaches c (Fig. 4.4).

It is possible to extend the above argument to establish (4.11) in more general situations. However, we emphasize that it is not possible to derive the result a priori, but only with the help of extra assumptions. However it is produced, the only real test of the validity of the result is in the experimental arena and here it has been extensively confirmed.

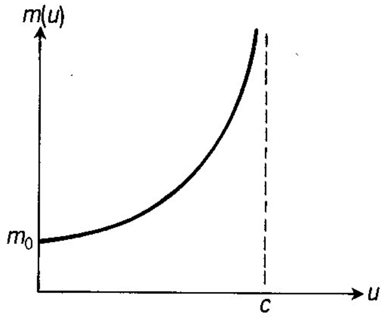

Fig. 4.4 Relativistic mass as a function of velocity.

4.4 Relativistic energy

Let us expand the expression for the relativistic mass, namely,

$$m(u) = \gamma m_0 = m_0(1 - u^2/c^2)^{-\frac{1}{2}},$$

in the case when the velocity u is small compared with the speed of light c. Then we get

$$m(u) = m_0 + \frac{1}{c^2}(\tfrac{1}{2}m_0u^2) + O\left(\frac{u^4}{c^4}\right), \tag{4.13}$$

where the final term stands for all terms of order $(u/c)^4$ and higher. If we multiply both sides by c^2, then, apart from the constant m_0c^2, the right-hand side is to first approximation the classical kinetic energy (k.e.), that is,

$$mc^2 = m_0c^2 + \tfrac{1}{2}m_0u^2 + \cdots \simeq \text{constant} + \text{k.e.} \tag{4.14}$$

We have seen that relativistic mass contains within it the expression for classical kinetic energy. In fact, it can be shown that the conservation of relativistic mass leads to the conservation of kinetic energy in the Newtonian approximation. As a simple example, consider the collision of two particles with rest mass m_0 and $\bar{m}_0$, initial velocities v_1 and $\bar{v}_1$, and final velocities v_2 and $\bar{v}_2$, respectively (Fig. 4.5). Conservation of relativistic mass gives

$$m_0(1 - v_1^2/c^2)^{-\frac{1}{2}} + \bar{m}_0(1 - \bar{v}_1^2/c^2)^{-\frac{1}{2}} = m_0(1 - v_2^2/c^2)^{-\frac{1}{2}} + \bar{m}_0(1 - \bar{v}_2^2/c^2)^{-\frac{1}{2}}. \tag{4.15}$$

If we now assume that v_1, v_2, $\bar{v}_1$, and $\bar{v}_2$ are all small compared with c, then we find (exercise) that the leading terms in the expansion of (4.15) give

$$\tfrac{1}{2}m_0v_1^2 + \tfrac{1}{2}\bar{m}_0\bar{v}_1^2 = \tfrac{1}{2}m_0v_2^2 + \tfrac{1}{2}\bar{m}_0\bar{v}_2^2, \tag{4.16}$$

which is the usual conservation of energy equation. Thus, in this sense, conservation of relativistic mass includes within it conservation of energy. Now, since energy is only defined up to the addition of a constant, the result

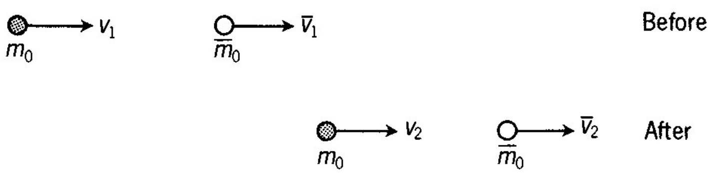

Fig. 4.5 Two colliding particles.

(4.14) suggest that we regard the **energy** E of a particle as given by

$$E = mc^2. \tag{4.17}$$

This is one of the most famous equations in physics. However, it is not just a mathematical relationship between two different quantities, namely energy and mass, but rather states that energy and mass are **equivalent** concepts. Because of the arbitrariness in the actual value of E, a better way of stating the relationship is to say that a change in energy is equal to a change in relativistic mass, namely,

$$\Delta E = \Delta mc^2$$

Using conventional units, c^2 is a large number and indicates that a small change in mass is equivalent to an enormous change in energy. As is well known, this relationship and the deep implications it carries with it for peace and war, have been amply verified. For obvious reasons, the term m_0c^2 is termed the **rest energy** of the particle. Finally, we point out that conservation of linear momentum, using relativistic mass, leads to the usual conservation law in the Newtonian approximation. For example (exercise), the collision problem considered above leads to the usual conservation of linear momentum equation for slow-moving particles:

$$m_0v_1 + \bar{m}_0\bar{v}_1 = m_0v_2 + \bar{m}_0\bar{v}_2. \tag{4.18}$$

Extending these ideas to three spatial dimensions, then a particle moving with velocity $\boldsymbol{u}$ relative to an inertial frame S has relativistic mass m, energy E, and linear momentum $\boldsymbol{p}$ given by

$$m = \gamma m_0, \quad E = mc^2, \quad \boldsymbol{p} = m\boldsymbol{u}. \tag{4.19}$$

Some straightforward algebra (exercise) reveals that

$$(E/c)^2 - p_x^2 - p_y^2 - p_z^2 = (m_0c)^2, \tag{4.20}$$

where m_0c is an invariant, since it is the same for all inertial observers. If we compare this with the invariant (3.13), i.e.

$$(ct)^2 - x^2 - y^2 - z^2 = s^2,$$

then it suggests that the quantities $(E/c, p_x, p_y, p_z)$ transform under a Lorentz transformation in the same way as the quantities (ct, x, y, z). We shall see in Part C that the language of tensors provides a better framework for discussing transformation laws. For the moment, we shall assume that energy and momentum transform in an identical manner and quote the results. Thus, in a frame S' moving in standard configuration with velocity v relative to S, the transformation equations are (see (3.12))

$$E' = \beta(E - vp_x), \quad p'_x = \beta(p_x - vE/c^2), \quad p'_y = p_y, \quad p'_z = p_z. \tag{4.21}$$

The inverse transformations are obtained in the usual way, namely, by interchanging primes and unprimes and replacing v by $-v$, which gives

$$E = \beta(E' + vp'_x), \quad p_x = \beta(p'_x + vE'/c^2), \quad p_y = p'_y, \quad p_z = p'_z. \tag{4.22}$$

If, in particular, we take S' to be the instantaneous rest frame of the particle, then $\boldsymbol{p}' = \boldsymbol{0}$ and $E' = E_0 = m_0c^2$. Substituting in (4.22), we find

$$E = \beta E' = \frac{m_0c^2}{(1 - v^2/c^2)^{\frac{1}{2}}} = mc^2,$$

where $m = m_0(1 - v^2/c^2)^{-\frac{1}{2}}$ and $\boldsymbol{p} = (\beta vE'/c^2, 0, 0) = (mv, 0, 0) = m\boldsymbol{v}$, which are precisely the values of the energy, mass, and momentum arrived at in (4.19) with $\boldsymbol{u}$ replaced by $\boldsymbol{v}$.

4.5 Photons

At the end of the last century, there was considerable conflict between theory and experiment in the investigation of radiation in enclosed volumes. In an attempt to resolve the difficulties, Max Planck proposed that light and other electromagnetic radiation consisted of individual 'packets' of energy, which he called **quanta**. He suggested that the energy E of each quantum was to depend on its frequency ν, and proposed the simple law, called **Planck's hypothesis**,

$$E = h\nu, \tag{4.23}$$

where h is a universal constant known now as **Planck's constant**. The idea of the quantum was developed further by Einstein, especially in attempting to explain the photoelectric effect. The effect is to do with the ejection of electrons from a metal surface by incident light (especially ultraviolet) and is strongly in support of Planck's quantum hypothesis. Nowadays, the quantum theory is well established and applications of it to explain properties of molecules, atoms, and fundamental particles are at the heart of modern physics. Theories of light now give it a dual wave–particle nature. Some properties, such as diffraction and interference, are wavelike in nature, while the photoelectric effect and other cases of the interaction of light and atoms are best described on a particle basis.

The particle description of light consists in treating it as a stream of quanta called **photons**. Using equation (4.19) and substituting in the speed of light, $u = c$, we find

$$m_0 = \gamma^{-1}m = (1 - u^2/c^2)^{\frac{1}{2}}m = 0, \tag{4.24}$$

that is, the rest mass of a photon must be zero! This is not so bizarre as it first seems, since no inertial observer ever sees a photon at rest — its speed is always c — and so the rest mass of a photon is merely a notional quantity. If we let $\hat{\boldsymbol{n}}$ be a unit vector denoting the direction of travel of the photon, then

$$\boldsymbol{p} = (p_x, p_y, p_z) = p\hat{\boldsymbol{n}},$$

and equation (4.20) becomes

$$(E/c)^2 - p^2 = 0.$$

Taking square roots (and remembering c and p are positive), we find that the energy E of a photon is related to the magnitude p of its momentum by

$$E = pc. \tag{4.25}$$

Finally, using the energy–mass relationship $E = mc^2$, we find that the relativistic mass of a photon is non-zero and is given by

$$m = p/c. \tag{4.26}$$

Combining these results with Planck's hypothesis, we obtain the following formulae for the energy E, relativistic mass m, and linear momentum $\boldsymbol{p}$ of the photon:

$$E = h\nu, \quad m = h\nu/c^2, \quad \boldsymbol{p} = (h\nu/c)\hat{\boldsymbol{n}}. \tag{4.27}$$

It is gratifying to discover that special relativity, which was born to reconcile conflicts in the kinematical properties of light and matter, also includes their mechanical properties in a single all-inclusive system.

We finish this section with an argument which shows that Planck's hypothesis can be derived directly within the framework of special relativity. We have already seen in the last chapter that the radial Doppler effect for a moving source is given by (3.27), namely

$$\frac{\lambda}{\lambda_0} = \left(\frac{1 + v/c}{1 - v/c}\right)^{\frac{1}{2}},$$

where λ_0 is the wavelength in the frame of the source and λ is the wavelength in the frame of the observer. We write this result, instead, in terms of frequency, using the fundamental relationships $c = \lambda\nu$ and $c = \lambda_0\nu_0$, to obtain

$$\frac{\nu_0}{\nu} = \left(\frac{1 + v/c}{1 - v/c}\right)^{\frac{1}{2}}. \tag{4.28}$$

Now, suppose that the source emits a light flash of total energy E_0. Let us use the equations (4.22) to find the energy received in the frame of the observer S. Since, recalling Fig. 3.11, the light flash is travelling along the negative x-direction of both frames, the relationship (4.25) leads to the result $p'_x = -E_0/c$, with the other primed components of momentum zero. Substituting in the first equation of (4.22), namely,

$$E = \beta(E' + vp'_x),$$

we get

$$E = \beta(E_0 - vE_0/c) = \frac{E_0(1 - v/c)}{(1 - v^2/c^2)^{\frac{1}{2}}} = E_0\left(\frac{1 - v/c}{1 + v/c}\right)^{\frac{1}{2}},$$

or

$$\frac{E_0}{E} = \left(\frac{1 + v/c}{1 - v/c}\right)^{\frac{1}{2}}. \tag{4.29}$$

Combining this with equation (4.28), we obtain

$$\frac{E_0}{\nu_0} = \frac{E}{\nu}.$$

Since this relationship holds for **any** pair of inertial observers, it follows that the ratio must be a universal constant, which we call h. Thus, we have derived Planck's hypothesis $E = h\nu$.

We leave our considerations of special relativity at this point and turn our attention to the formalism of tensors. This will enable us to reformulate special relativity in a way which will aid our transition to general relativity, that is, to a theory of gravitation consistent with special relativity.

Exercises

4.1 (**§4.1**) Discuss the possibility of using force rather than mass as the basic quantity, taking, for example, a standard weight at a given latitude as the unit of force. How should one then define and measure the mass of a body?

4.2 (**§4.3**) Show that, in the inelastic collision considered in §4.3, the rest mass of the combined object is greater than the sum of the original rest masses. Where does this increase derive from?

4.3 (**§4.3**) A particle of rest mass $\bar{m}_0$ and speed u strikes a stationary particle of rest mass m_0. If the collision is perfectly inelastic, then find the rest mass of the composite particle.

4.4 (**§4.4**) (i) Establish the transition from equation (4.15) to (4.16).
(ii) Establish the Newtonian approximation equation (4.18).

4.5 (**§4.4**) Show that (4.19) leads to (4.20). Deduce (4.21).

4.6 (**§4.4**) Newton's second law for a particle of relativistic mass m is

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{u}).$$

Define the work done $\mathrm{d}E$ in moving the particle from $\boldsymbol{r}$ to $\boldsymbol{r} + \mathrm{d}\boldsymbol{r}$. Show that the rate of doing work is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{u})}{\mathrm{d}t}\cdot\boldsymbol{u}.$$

Use the definition of relativistic mass to obtain the result

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{m_0}{(1 - u^2/c^2)^{3/2}}u\frac{\mathrm{d}u}{\mathrm{d}t} \qquad \left[\text{Hint: } \boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = u\frac{\mathrm{d}u}{\mathrm{d}t}\right].$$

Express this last result in terms of $\mathrm{d}m/\mathrm{d}t$ and integrate to obtain

$$E = mc^2 + \text{constant}.$$

4.7 (**§4.4**) Two particles whose rest masses are m_1 and m_2 move along a straight line with velocities u_1 and u_2, measured in the same direction. They collide inelastically to form a new particle. Show that the rest mass and velocity of the new particle are m_3 and u_3, respectively, where

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2(1 - u_1u_2/c^2),$$

$$u_3 = \frac{m_1\gamma_1u_1 + m_2\gamma_2u_2}{m_1\gamma_1 + m_2\gamma_2},$$

with

$$\gamma_1 = (1 - u_1^2/c^2)^{-\frac{1}{2}}, \qquad \gamma_2 = (1 - u_2^2/c^2)^{-\frac{1}{2}}.$$

4.8 (**§4.4**) A particle of rest mass m_0, energy e_0, and momentum p_0 suffers a head on elastic collision (i.e. masses of particles unaltered) with a stationary mass M. In the collision, M is knocked straight forward, with energy E and momentum P, leaving the first particle with energy e and p. Prove that

$$P = \frac{2p_0M(e_0 + Mc^2)}{2Me_0 + M^2c^2 + m_0^2c^2}$$

and

$$p = \frac{p_0(m^2c^2 - M^2c^2)}{2Me_0 + M^2c^2 + m_0^2c^2}.$$

What do these formulae become in the classical limit?

4.9 (**§4.4**) Assume that the formulae (4.19) hold for a tachyon, which travels with speed $v > c$. Taking the energy to be a measurable quantity, then deduce that the rest mass of a tachyon is imaginary and define the real quantity μ_0 by $m_0 = \mathrm{i}\mu_0$.

If the tachyon moves along the x-axis and if we assume that the x-component of the momentum is a real positive quantity, then deduce

$$m = \frac{v}{|v|}\alpha\mu_0, \qquad p = \mu_0|v|\alpha, \qquad E = mc^2,$$

where $\alpha = (v^2/c^2 - 1)^{-\frac{1}{2}}$.

Plot E/m_0c^2 against v/c for both tachyons and sub-luminal particles.

4.10 (**§4.5**) Two light rays in the (x, y)-plane of an inertial observer, making angles θ and $-\theta$, respectively, with the positive x axis, collide at the origin. What is the velocity v of the inertial observer (travelling in standard configuration) who sees the light rays collide head on?

4.11 (§**4.5**) An atom of rest mass m_0 is at rest in a laboratory and absorbs a photon of frequency ν. Find the velocity and mass of the recoiling particle.

4.12 (§**4.5**) An atom at rest in a laboratory emits a photon and recoils. If its initial mass is m_0 and it loses the rest energy e in the emission, prove that the frequency of the emitted photon is given by

$$\nu = \frac{e}{h}(1 - e/2m_0c^2).$$