6.6 The Conservation of Momentum and of Energy

Momentum was defined in such a manner so that, in an interaction of particles, both the mass and the momentum are conserved. Given that the total energy of a free body, not acted on by external forces, is equal to $E = mc^2$, the fact that in an interaction of n particles the conservation of relativistic mass applies,

$$\sum_{i=1}^{n} m_i = \text{const.} \tag{6.40}$$

means that the energy is also conserved:

$$\sum_{i=1}^{n} E_i = \text{const.} \tag{6.41}$$

This law is used, together with the law of conservation of momentum

$$\sum_{i=1}^{n} \mathbf{p}_i = \text{const.} \tag{6.42}$$

in the solution of problems in which particles interact in any way.

For n particles, the components of the total momentum and the energy are defined as

$$P_{\text{tot},x} \equiv \sum_{i=1}^{n} p_{ix} \quad P_{\text{tot},y} \equiv \sum_{i=1}^{n} p_{iy} \quad P_{\text{tot},z} \equiv \sum_{i=1}^{n} p_{iz} \quad E_{\text{tot}} \equiv \sum_{i=1}^{n} E_i \tag{6.43}$$

for frame S, with similar expressions for frame S′.

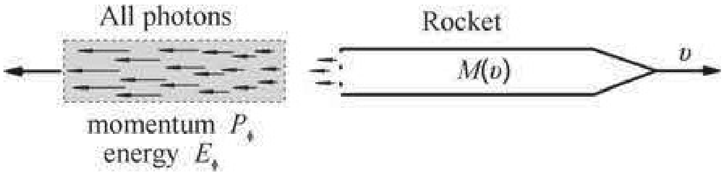
If, in frame S, during the interaction of n particles, the momentum and the energy are conserved, then, for the total values of the components of momentum and of energy, the following relations are true:

$$\begin{gathered}(P_{\text{tot},x})_{\text{initially}} = (P_{\text{tot},x})_{\text{finally}} \quad (P_{\text{tot},y})_{\text{initially}} = (P_{\text{tot},y})_{\text{finally}} \quad (P_{\text{tot},z})_{\text{initially}} = (P_{\text{tot},z})_{\text{finally}} \\ (E_{\text{tot}})_{\text{initially}} = (E_{\text{tot}})_{\text{finally}}\end{gathered} \tag{6.44}$$

The conservation of these quantities is valid in general, in every physical process. The subject will be further discussed in the next section.

Example 6.3 The Photon Rocket

The use of photons in the propulsion of rockets has been proposed. While for chemical fuels there is a limit in the speed of ejection of mass of the order of 10 km/s, for photons this speed is 30 000 times greater. For this reason, the emission of photons by the rocket, with their high speed, was considered that it would give the rocket a greater speed per unit mass ejected. Find the speed attained by such a rocket, as a function of the fraction κ of its mass that has been ejected as photons.



Let the initial rest mass of the rocket be M_0 and, at some time, a fraction κ of this has been emitted as photons. All the photons are considered to be emitted in the same direction. The emitted photons carry, in total, an energy E_ϕ and momentum $P_\phi = E_\phi/c$, as measured by an observer on the Earth. If at that moment the rocket is moving with a speed equal to υ and has a relativistic mass $M(\upsilon) = (1-\kappa)M_0\gamma$, where $\gamma = 1\Big/\sqrt{1-\upsilon^2/c^2}$, the laws of conservation give:

Conservation of energy: $E_{\text{total}} = M_0c^2 = M(\upsilon)c^2 + E_\phi = (1-\kappa)M_0c^2\gamma + E_\phi$
Conservation of momentum: $P_{\text{total}} = 0 = M(\upsilon)\upsilon - E_\phi/c = (1-\kappa)M_0\gamma\upsilon - E_\phi/c$
Eliminating E_ϕ between these two relations, it follows that

$$M_0c^2 = (1-\kappa)M_0c^2\gamma(1+\beta) \text{ and } (1-\kappa)\gamma(1+\beta) = 1, \quad (1-\kappa)\sqrt{\frac{1+\beta}{1-\beta}} = 1,$$

$$\frac{1-\beta}{1+\beta} = (1-\kappa)^2, \text{ from which we find } \beta = \frac{1-(1-\kappa)^2}{1+(1-\kappa)^2} \text{ and}$$

$$\gamma = \frac{1}{2}\left(1-\kappa+\frac{1}{1-\kappa}\right).$$

Some numerical values are:

β	0.5	0.9	0.95	0.99	0.995	0.999
γ	1.16	2.94	3.20	7.09	10	22.4
κ	0.42	0.77	0.84	0.93	0.95	0.98

We see that, in order to achieve a speed of $0.995c$, which corresponds to $\gamma = 10$, a proportion of 95 % of the rocket's total mass must be emitted as photons. The useful load is, therefore, only 5 % of the spaceship. Of course, if we wish to decelerate the spaceship in order to stop it, then accelerate it towards its starting point and finally decelerate it to a halt at its starting point, the useful load would be only a fraction equal to $(1-\kappa)^4$, which has a value smaller than 10^{-5} for maximum γ equal to 10. This makes the photon rockets of doubtful usefulness, without the situation being very much better for chemical fuels of course. The problem is examined in greater detail in Sect. 7.6.1.

6.7 The Equivalence of Mass and Energy

The relation $E = mc^2$ describes an equivalence of mass and energy which is extremely important for Physics. Einstein considered this to be one of the most important achievements of the Theory of Relativity. The conserved magnitude E includes energy which is possessed by a body even when it is at rest and is due to its rest mass. Conversely, the inertial properties of a body, as these are expressed by its mass, vary when energy is given to the body and its speed increases. An equivalence between mass and energy seems to exist, which is verified experimentally in many ways. This equivalence makes possible the creation or annihilation of particles in a reaction, with the assumption of the conservation of mass-energy, m or E, as well as of other physical quantities (momentum, angular momentum, charge, hypercharge, lepton numbers, baryon number, taste, color, charm and others, in addition to some that are still unknown!).

The relation $E = mc^2$ has a twofold meaning: it states that mass may be changed into energy, but also that to every form of energy E there corresponds a mass equal to $m = E/c^2$. Although the mass appearing in the equation $E = mc^2$ is the inertial mass, in the General Theory of Relativity the inertial mass is equal to the gravitational, and, hence, the relation also attributes a gravitational mass to every form of energy. We must, however, be careful. If the motion of the gravitational mass of a photon in a gravitational field is examined, a deflection of light will be predicted, as demonstrated experimentally. The angle of deflection will, however, be wrong by a factor of 2, because the calculation does not take into account the geometrical distortion or curving of space by the mass to which the gravitational field is due.

In general, a change of Δm in the mass, corresponds to a change ΔE in the energy, and vice versa, where the two quantities are related by

$$\Delta E = \Delta m\, c^2. \tag{6.45}$$

The experimental evidence for the validity of this relation is indisputable.

It is worth making a comment, here, regarding this equation. The opinion is sometimes heard that Einstein discovered the equation $E = mc^2$, which showed that great amounts of energy may be released when mass is transformed into energy, something that led to the construction of the atomic bomb, initially, and the nuclear bomb subsequently. This point of view is wrong, both as regards History as well as Physics. It must be understood that this equation applies in all physical and chemical changes, not only in processes in nuclear Physics. It is valid when a spring is compressed or stretched (Problem 6.12) and in every chemical reaction (Example 6.4). It applies when we light a match, a fire and in all the chemical reactions happening in our bodies. Man has used fire for many millennia before the discovery of the equation. He did not need to know the origin of the energy he used for war purposes when he used incendiary bombs. The same was true in the construction of the atom bomb. From the moment the phenomenon of nuclear fission was discovered in the laboratory, between 1934 and 1939, and the enormous amount of energy released was measured, the construction of the atomic bomb was a matter of time. The only difference with the past was that, now, the amount of energy released is so great that the change in the nuclear masses is measurable. Neither Einstein nor the equation $E = mc^2$ should be incriminated. The equation simply gives us the ability to interpret the origin of the released energy, both in nuclear reactions and in many others.

6.7.1 The Validity of the Conservation of Momentum and Energy During the Transmutation of Nuclei and the Annihilation and Creation of Particles

The equation of equivalence of mass and energy is valid in all changes a system suffers and not only to 'classical' processes in which the reacting bodies retain their form unchanged. In the processes of nuclear Physics and the Physics of Elementary Particles, the nuclei may decay or suffer fission or fusion and the elementary particles may be created or annihilated. In these processes, the reacting bodies are not necessarily the same before and after the reaction. It has been established experimentally that, in all these changes, momentum and energy, as defined by the Special Theory of Relativity, are conserved.

It was found that relativistic definitions of mass and momentum make it possible to conserve both momentum and energy in all inertial frames of reference in the case of a decaying particle. This is shown in Fig. 6.6a and may be symbolized by $(1) \to (2) + (3)$.

If we reverse time, we have the creation of one body from the collision of two particles, as seen in Fig. 6.6b, and is symbolized by $(1) + (2) \to (3)$. The conservation of momentum and energy in case (a) makes it possible to conserve the two quantities in case (b) too. The laws of conservation also hold in more complex cases, such as the one shown in Fig. 6.6c where two bodies react to create two other bodies but with different rest masses. We may assume that, intermediately, a single body is formed (circle A), which then decays into two bodies. Thus, we have a process (b) followed by a process (a), again safeguarding the possibility of the conservation of momentum and energy at all stages. Symbolically, we may write the equation of the reaction as $(1) + (2) \to (A) \to (3) + (4)$. By the same method, more complex processes may be analyzed, such as the interaction of two bodies for the creation of three, as shown in Fig. 6.6d, with the equation of the reaction being $(1) + (2) \to (A) \to (3) + (B) \to (3) + (4) + (5)$. In this way it is possible to

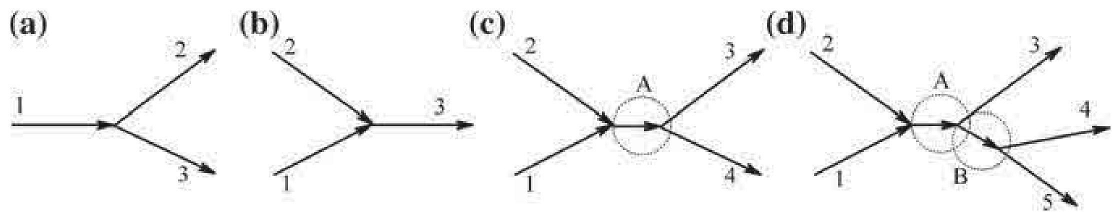


Fig. 6.6 Various collisions of point masses in order to form a single body or other bodies with different rest masses

conserve relativistic momentum and energy during the interaction of particles for the creation of other, possibly different, particles. It must be stressed that in these cases, the rest masses and the velocities of the particles created do not have uniquely determined values, as may be seen, for example, in the elastic collision of two identical particles which is examined in Sect. 6.11. Other laws determine which particles may be created and limit further the range of values their energies and momenta may have. There exist many such examples which have been studied experimentally, three of which will be discussed below.

The equivalence of mass and energy and the validity of the laws of conservation of momentum and energy are demonstrated in a dramatic way in the annihilation of matter-antimatter such as an electron-positron pair, in the creation of particles from a photon as in the phenomenon of pair production, as well as in the fission or fusion of atomic nuclei:

(a) *Annihilation of an electron-positron pair*. An electron and its antiparticle, the positron, annihilate completely producing two photons,

$$\mathrm{e}^- + \mathrm{e}^+ \rightarrow 2\gamma.$$

Since for 'thermal' particles the momentum of the pair of particles is initially almost zero, the two photons have the same energy (511 keV) and move in opposite directions, in order to conserve momentum. The energy available, which is entirely due to the rest masses of the two particles, is $2m_\mathrm{e}c^2 = 1.022$ MeV, which is shared equally among the two photons.

(b) *Production of an electron-positron pair*. A photon may produce an electron-positron pair, provided it has an energy higher than $2m_\mathrm{e}c^2 = 1.022$ MeV. For both momentum and energy to be conserved, the process can only take place in the vicinity of another body, such as the nucleus of an atom. The *threshold* of $2m_\mathrm{e}c^2 = 1.022$ MeV in energy exists because this is the energy equivalent of the two electron masses. If the photon possesses just this amount of energy in a certain frame of reference, then the two particles will have zero kinetic energies, in this frame of reference, after their creation. Photons of higher energy also impart kinetic energy to the products of pair production.

(c) *Fission and fusion of atomic nuclei*. Another example of the conversion of mass to energy is observed during the phenomenon of nuclear fission, which will be examined in the next chapter. During fission, a small proportion of the rest mass of the nucleus is transformed into kinetic energy of the fission products. The same phenomenon is observed during the fusion of nuclei.

In the final analysis, the validity of the laws of conservation of momentum and of energy in the creation or annihilation of particles may be considered to be the conclusion from the results of countless experiments in modern Physics. The theoretical proof of the validity of these laws may be based on symmetry arguments [2]. The conservation of energy is derived from the invariance of the laws of Physics during a displacement in time and the law of the conservation of (vectorial) momentum follows from the invariance of the laws of Physics on the transportation along the three dimensions of space. If this invariance holds then the laws of conservation follow. Modern Physics in general and Nuclear Physics and the Physics of Elementary Particles in particular, are based to a large extent on the validity of these laws. If nothing else, therefore, we have to admit that the validity of the laws of the conservation of momentum and of energy in all physical processes are based on indisputable experimental evidence. In the examples and the problems that follow, the usefulness of these laws in the study of physical processes and phenomena will be demonstrated.

Example 6.4 Conversion of Mass into Energy in a Chemical Reaction
During the combination of 1 kg of hydrogen with 8 kg of oxygen for the production of 9 kg of water, an energy of approximately 10^8 J is released. Is it possible to detect the conversion of mass into energy in this case, using a balance which is capable in measuring a proportional change in mass equal to 1 part in 10^7?

The mass equivalent of the energy released is $\Delta m = \Delta E/c^2 = 10^8 \Big/ (3 \times 10^8)^2 \approx 10^{-9}$ kg. This is approximately 1 part in 10^{10} of the mass involved in the chemical reaction and cannot be detected with the balance we have or with any other balance available today.

Example 6.5 The De-excitation of a Nucleus
A stationary excited nucleus returns to its ground state by liberating an energy ΔE and emitting a photon. The rest mass of the de-excited nucleus is M. What is the energy E_γ of the photon? What fraction of the excitation energy is given to the nucleus as kinetic energy?

The rest energy $M^* c^2$ of the excited nucleus is equal to the rest energy Mc^2 of the de-excited nucleus plus the excitation energy ΔE.
The laws of conservation give:
 Conservation of energy $M^* c^2 = Mc^2 + \Delta E = E + E_\gamma$, where $E = Mc^2 \gamma$.
 Conservation of momentum $\dfrac{E_\gamma}{c} = Mc\gamma\beta$.

Therefore, $Mc^2 + \Delta E = Mc^2 \gamma (1+\beta) = Mc^2 \sqrt{\dfrac{1+\beta}{1-\beta}}$,

from which we get $\sqrt{\dfrac{1+\beta}{1-\beta}} = 1 + \dfrac{\Delta E}{Mc^2}$.

This relation may be written in terms of the ratio $\alpha \equiv \dfrac{\Delta E}{Mc^2}$ as $\dfrac{1+\beta}{1-\beta} = (1+\alpha)^2$.

Solving, $\beta = \dfrac{(1+\alpha)^2 - 1}{(1+\alpha)^2 + 1}$, $\gamma = \dfrac{1 + (1+\alpha)^2}{2(1+\alpha)}$.

It follows that $E_\gamma = Mc^2\beta\gamma = Mc^2\left[\dfrac{(1+\alpha)^2-1}{(1+\alpha)^2+1}\right]\left[\dfrac{1+(1+\alpha)^2}{2(1+\alpha)}\right]$

or $E_\gamma = \dfrac{Mc^2}{2}\left(\dfrac{(1+\alpha)^2-1}{1+\alpha}\right) = \dfrac{Mc^2}{2}\left(\dfrac{\alpha^2+2\alpha}{1+\alpha}\right)$

and $E_\gamma = \dfrac{\Delta E}{2}\dfrac{\dfrac{\Delta E}{Mc^2}+2}{\dfrac{\Delta E}{Mc^2}+1}$, $\quad E_\gamma = \Delta E\left(\dfrac{Mc^2+\frac{1}{2}\Delta E}{Mc^2+\Delta E}\right)$.

The energy given to the de-excited nucleus as kinetic energy is

$$K = \Delta E - E_\gamma = \Delta E\left(1-\frac{Mc^2+\frac{1}{2}\Delta E}{Mc^2+\Delta E}\right) = \frac{\Delta E}{2}\left(\frac{\Delta E}{Mc^2+\Delta E}\right).$$

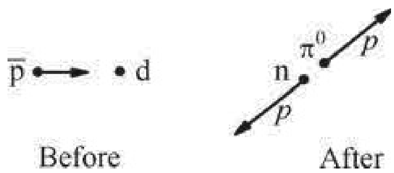
The fraction of the excitation energy which is given to the de-excited nucleus as kinetic energy is

$$\frac{K}{\Delta E} = \frac{1}{2}\frac{\Delta E}{Mc^2+\Delta E}.$$

If it is $\Delta E \ll Mc^2$, then the fraction is approximately equal to $\dfrac{K}{\Delta E} \approx \dfrac{\Delta E}{2Mc^2}$.

Example 6.6 The Creation of a Neutral Pion

What is the energy of the π^0 produced during the capture of an antiproton, which is moving with a low speed, by a stationary deuterium nucleus ($\overline{\mathrm{p}} + \mathrm{d} \to \mathrm{n} + \pi^0$)? The rest energies of the particles $\overline{\mathrm{p}}$, d, n and π^0 are, respectively, $E_{\mathrm{p}0} = 938.2$ MeV, $E_{\mathrm{d}0} = 1875.5$ MeV, $E_{\mathrm{n}0} = 939.5$ MeV and $E_{\pi 0} = 135.0$ MeV.



Since the initial momentum of the system is negligible, the momenta of n and π^0 must be equal and opposite. Let the momenta of the two particles have magnitude equal to p. The initial kinetic energy is also negligible. The conservation of energy gives

$$E_{\mathrm{p}0} + E_{\mathrm{d}0} = E_{\mathrm{n}} + E_\pi \tag{1}$$

where E_{n} and E_π are the energies of n and π^0, respectively, while, from the relation $E^2 = (m_0c^2)^2 + (pc)^2$ we have, for the particles n and π^0

$$(pc)^2 = E_n^2 - E_{n0}^2 = E_\pi^2 - E_{\pi 0}^2. \tag{2}$$

Equation (1) gives $\quad E_n^2 = (E_{p0} + E_{d0} - E_\pi)^2$

and Eq. (2) $\quad E_n^2 = E_\pi^2 + E_{n0}^2 - E_{\pi 0}^2$.

Equating, $$\left(E_{0\mathrm{p}} + E_{0\mathrm{d}} - E_\pi\right)^2 = E_\pi^2 + E_{0\mathrm{n}}^2 - E_{0\pi}^2$$

or $$\left(E_{\mathrm{p}0} + E_{\mathrm{d}0}\right)^2 - 2E_\pi\left(E_{\mathrm{p}0} + E_{\mathrm{d}0}\right) + \cancel{E_\pi^2} = \cancel{E_\pi^2} + E_{\mathrm{n}0}^2 - E_{\pi 0}^2$$

and, finally, $$E_\pi = \frac{\left(E_{\mathrm{p}0} + E_{\mathrm{d}0}\right)^2 + E_{\pi 0}^2 - E_{\mathrm{n}0}^2}{2\left(E_{\mathrm{p}0} + E_{\mathrm{d}0}\right)}.$$

Substituting,

$$E_\pi = \frac{(938.2 + 1875.5)^2 + 135.0^2 - 939.5^2}{2(938.2 + 1875.5)} = 1253\ \mathrm{MeV} = 1.25\ \mathrm{GeV}.$$

Example 6.7 The Rest Mass of a Spherical Distribution of Charge due to Its Electrostatic Energy

Verify Eddington's statement that, if a quantity of electrons of mass 1 g could be compressed uniformly into a sphere of a radius of 10 cm, the rest mass of this sphere, which would correspond to its electrostatic energy, would be of the order of magnitude of 10 million tons.

We evaluate the electric charge of the spherical distribution. If m_e is the mass of the electron and e its charge, a mass of electrons equal to m will have a charge equal to $Q = em/m_\mathrm{e}$. From theory we know that the electrostatic energy of a charge Q distributed uniformly inside a sphere of radius R is $E_0 = \dfrac{3Q^2}{20\pi\varepsilon_0 R}$.

Therefore, $E_0 = \dfrac{3}{20\pi\varepsilon_0}\dfrac{m^2 e^2}{m_\mathrm{e}^2 R}$.

This energy corresponds to a rest mass $M_0 = \dfrac{E_0}{c^2} = \dfrac{3}{20\pi\varepsilon_0 c^2}\dfrac{m^2 e^2}{m_\mathrm{e}^2 R}$.

Substituting $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m, $m_\mathrm{e} = 9.11 \times 10^{-31}$ kg, $e = -1.602 \times 10^{-19}$ C and $m = 0.001$ kg, $R = 0.1$ m, we find $M_0 = 1.86 \times 10^{10}$ kg or 20 million tons, approximately.

Problems

6.12 A spring has a constant equal to $k = 2 \times 10^4$ N/m. What is the increase in the mass of the spring when it is compressed by $x = 5$ cm from its natural length?
Ans.: $\Delta m = 2.8 \times 10^{-16}$ kg

6.13 A particle with rest mass m, is stationary in the laboratory. The particle splits into two others: one with rest mass m_1 which moves with a speed $V_1 = (3/5)c$, and another with rest mass m_2 which moves with a speed $V_2 = (4/5)c$.

(a) Find m_1 and m_2 as fractions of m. Ans.: $m_1 = (16/35)m$, $m_2 = (9/35)m$
(b) What are the kinetic energies, K_1 and K_2, of the two particles, in terms of m? Ans.: $K_1 = (4/35)mc^2$, $K_2 = (6/35)mc^2$

6.14 A particle with rest mass m moves in the laboratory with speed $V = (3/5)c$. The particle disintegrates into two others: one with rest mass m_1 which remains stationary and another with rest mass m_2 which moves with a speed of $\upsilon = (4/5)c$. Find the masses m_1 and m_2 in terms of m. Ans.: $m_1 = (5/16)m$, $m_2 = (9/16)m$

6.15 A particle of rest mass $m_1 = 1$ GeV/c^2 moves with a speed of $\upsilon_1 = (4/5)c$ and collides with another particle with rest mass $m_2 = 10$ GeV/c^2 which is stationary. After the collision, the two particles form a single body with rest mass M. Find:

(a) The total energy of the system, in GeV. Ans.: 11.7 GeV
(b) The total momentum of the system, in GeV/c. Ans.: 1.33 GeV/c
(c) The mass M, in GeV/c^2. Ans.: 11.6 GeV/c^2

6.16 Show that, if a photon could disintegrate into two photons, both the photons produced would have to be moving in the same direction as the original photon.

6.17 A neutral pion disintegrates into two photons, which move on the same straight line. The energy of the one photon is twice that of the other. Show that the speed of the pion was $c/3$.

6.18 A π particle, which is at rest in the laboratory frame of reference, disintegrates into a muon and a neutrino: $\pi \to \mu + \nu$. Show that the energy of μ is $E_\mu = \dfrac{c^2}{2m_\pi}\left(m_\pi^2 + m_\mu^2 - m_\upsilon^2\right)$, where m_π, m_μ and m_ν are the rest masses of the three particles, respectively. What is the energy of ν? Ans.: $E_\nu = \dfrac{c^2}{2m_\pi}\left(m_\nu^2 + m_\pi^2 - m_\mu^2\right)$

6.19 A stationary particle of rest mass M, disintegrates into a new particle of rest mass m and a photon. Find the energies of the new particle and the photon.

Ans.: $E = \dfrac{M^2 + m^2}{2M}c^2$, $E_\gamma = \dfrac{M^2 - m^2}{2M}c^2$

6.20 A particle decays, producing a π^+ and a π^-. Both pions have momenta equal to 530 MeV/c and move in directions which are normal to each other. Find the rest mass of the original particle. The rest mass of $\pi^\pm$ is $m_\pi = 140$ MeV/c^2.

Ans.: 799 MeV/c^2

6.21 A neutral kaon decays into two pions: $K^0 \to \pi^+ + \pi^-$. If the negative pion produced is at rest, what is the energy of the positive pion? What was the energy of the kaon? Given are the rest masses of K^0, $m_K = 498$ MeV/c^2 and the $\pi^\pm$, $m_\pi = 140$ MeV/c^2. Ans.: $E_\pi = 753$ MeV, $E_K = 886$ MeV

6.22 A photon which has an energy equal to E, collides with an electron which is moving in the opposite direction to that of the photon. After the collision, the photon still has an energy E and reverses direction of motion. Show that, for this to happen, the electron must initially have a momentum of magnitude E/c. Also show that the final speed of the electron is $\upsilon = c\Big/\sqrt{1 + (m_0c^2/E)^2}$, where m_0c^2 is its rest energy.

6.23 A particle which has rest mass m_1 and speed υ_1, collides with a particle at rest, which has a rest mass equal to m_2. The two particles are united into one body which has a rest mass M and moves with speed υ. Show that $\upsilon = (m_1\gamma_1\upsilon_1)/(m_1\gamma_1 + m_2)$ and $M^2 = m_1^2 + m_2^2 + 2\gamma_1 m_1 m_2$, where $\gamma_1 = 1\Big/\sqrt{1 - \upsilon_1^2/c^2}$.

6.24 Two identical particles with rest mass m_0 each move towards each other, in the laboratory frame of reference, with the same speed βc. Find the energy of the one particle in the frame of reference of the other. Ans.: $E_{BA} = E_{AB} = m_0 c^2 \dfrac{1+\beta^2}{1-\beta^2}$

6.25 A particle with rest mass M, moves with velocity $\mathbf{\upsilon}$. The particle disintegrates into two others, 1 and 2, with rest masses m_1 and m_2 respectively. If particle 2 moves in a direction perpendicular to $\mathbf{\upsilon}$, find the angle θ that the direction of motion of particle 1 makes with $\mathbf{\upsilon}$. Ans.: $\tan\theta = \dfrac{1}{2\beta\gamma^2}\sqrt{\left[1-\left(\dfrac{m_1}{M}\right)^2+\left(\dfrac{m_2}{M}\right)^2\right]^2-\left(2\gamma\dfrac{m_2}{M}\right)^2}$

6.26 A Σ particle disintegrates, in motion, into three charged pions, A, B and C. The rest mass of each pion is 140 MeV/c^2. Their kinetic energies are, respectively, $K_A = 190$ MeV, $K_B = 321$ MeV and $K_C = 58$ MeV. The velocities of the pions form with the x-axis angles equal to $\theta_A = 22.4^\circ$, $\theta_B = 0^\circ$ and $\theta_C = -12.25^\circ$, respectively. Evaluate the rest mass of the original particle and the direction of its motion. Ans.: $M_\Sigma = 495$ MeV/c^2, $\theta = 5.6^\circ$

6.27 A particle with rest mass M, disintegrates, in motion, into two other particles, 1 and 2. Particle 1 has rest mass m_1, momentum p_1 and energy E_1, while particle 2 has rest mass m_2, momentum p_2 and energy E_2. The angle formed by their directions of motion is θ. Show that $E_1E_2 - p_1p_2c^2\cos\theta = \frac{1}{2}\left(M^2 - m_1^2 - m_2^2\right)c^4$, a quantity that is an invariant.

6.28 A neutral pion with energy E disintegrates into two photons, $\pi^0 \to 2\gamma$, with energies E_1 and E_2. Express the angle between the directions of motion of the two photons, θ, in terms of E, the rest mass m_π of the pion and the variable $\varepsilon = (E/2) - E_1 = E_2 - (E/2) = (E_2 - E_1)/2$. What is the minimum value of this angle for $E = 10$ GeV? It is given that $m_\pi c^2 = 0.140$ GeV. Ans.: $\sin\dfrac{\theta}{2} = \dfrac{m_\pi c^2}{\sqrt{E^2 - 4\varepsilon^2}}$, $\theta_{min} = 1.6^\circ$

6.29 A Λ hyperon disintegrates in motion into a proton and a pion ($\Lambda \to p + \pi$). These have, in the laboratory frame of reference, momenta p_p and p_π, and energies E_p and E_π, respectively. The angle between p_p and p_π is equal to θ. Show that the energy released in this disintegration is $Q = \sqrt{m_p^2c^4 + m_\pi^2c^4 + 2E_pE_\pi - 2p_pp_\pi c^2\cos\theta} - \left(m_p + m_\pi\right)c^2$, where m_p and m_π are the rest masses of the proton and the pion, respectively.

6.30 In the disintegration $K^0 \to \pi^+ + \pi^-$, both the momenta of the particles produced both have magnitude equal to $p_\pi = 360$ MeV/c and form an angle of 70° between them. What is the rest mass m_K of K^0 in MeV/c^2? For the pions, it is $m_\pi c^2 = 140$ MeV. Ans.: $m_K = 499$ MeV/c^2

6.31 A K^0 particle, having a rest energy $m_K c^2 = 498$ MeV, disintegrates into two mesons, π^+ and π^-, which have rest masses equal to m_π. In the frame of reference of K^0 both the mesons move with speed $0.83\,c$.

(a) Find the ratio m_π/m_K of the rest masses and the rest mass m_π of the π particles. Ans.: $m_\pi/m_K = 0.28$, $m_\pi = 139\ \text{MeV}/c^2$

(b) Let the K^0 particle move with a speed of $0.83\,c$ in the laboratory frame of reference and the two mesons move on the initial direction of motion of the K^0 particle. Find the kinetic energies of the mesons in the laboratory frame of reference. Ans.: 0 and 616 MeV

6.32 *Symmetric Elastic Collision*. A particle with rest mass m and kinetic energy K collides elastically with a particle at rest that has the same rest mass. After the collision, the two particles move in directions which form equal and opposite angles, $\pm\theta$, with the direction of motion of the initially moving particle. Find angle θ in terms of m and K. *Note:* In the Special Theory of Relativity, the term *elastic collision* implies that the reacting particles remain the same, with the same rest masses, respectively, before and after the collision. Ans.: $\tan\theta = \pm 1\Big/\sqrt{1+K/2mc^2}$

6.8 The Transformation of Momentum and Energy

The momentum and the energy of a particle in a frame of reference S is $\mathbf{p} = m_0\gamma_P\boldsymbol{\upsilon}$ and $E = m_0\gamma_P c^2$, where $\gamma_P = \frac{1}{\sqrt{1-\upsilon^2/c^2}}$ is the Lorentz factor for the particle in the frame of reference in which it is being observed. Therefore,

$$p_x = m_0\gamma_P\frac{dx}{dt}, \qquad p_y = m_0\gamma_P\frac{dy}{dt}, \qquad p_z = m_0\gamma_P\frac{dz}{dt}, \qquad \frac{E}{c^2} = m_0\gamma_P. \tag{6.46}$$

If time dt elapses in the laboratory, the corresponding time interval in the frame of reference of the particle (the particle's rest time or proper time), is $d\tau = dt/\gamma_P$. So, equivalently, Eq. (6.46) may be written as

$$p_x = m_0\frac{dx}{d\tau}, \qquad p_y = m_0\frac{dy}{d\tau}, \qquad p_z = m_0\frac{dz}{d\tau}, \qquad \frac{E}{c^2} = m_0\frac{dt}{d\tau}. \tag{6.47}$$

In frame S′,

$$p'_x = m_0\frac{dx'}{d\tau}, \qquad p'_y = m_0\frac{dy'}{d\tau}, \qquad p'_z = m_0\frac{dz'}{d\tau}, \qquad \frac{E'}{c^2} = m_0\frac{dt'}{d\tau}. \tag{6.48}$$

Given that the magnitude $d\tau$ remains invariant under a transformation from one inertial frame of reference to another and the same is true for m_0, these relations show that the magnitudes $p_x,\ p_y,\ p_z$ and E/c^2 transform like the magnitudes x, y, z and t, respectively.

Alternatively, we may use the differential form of the Lorentz transformation for x, y, z and t from frame S to frame S′