SILESIAN UNIVERSITY
SCHOOL OF BUSINESS ADMINISTRATION IN KARVINA

Mathematics in economics

Lecture 4

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Mathematics in Economics/PMAT

Differential calculus of two real variables

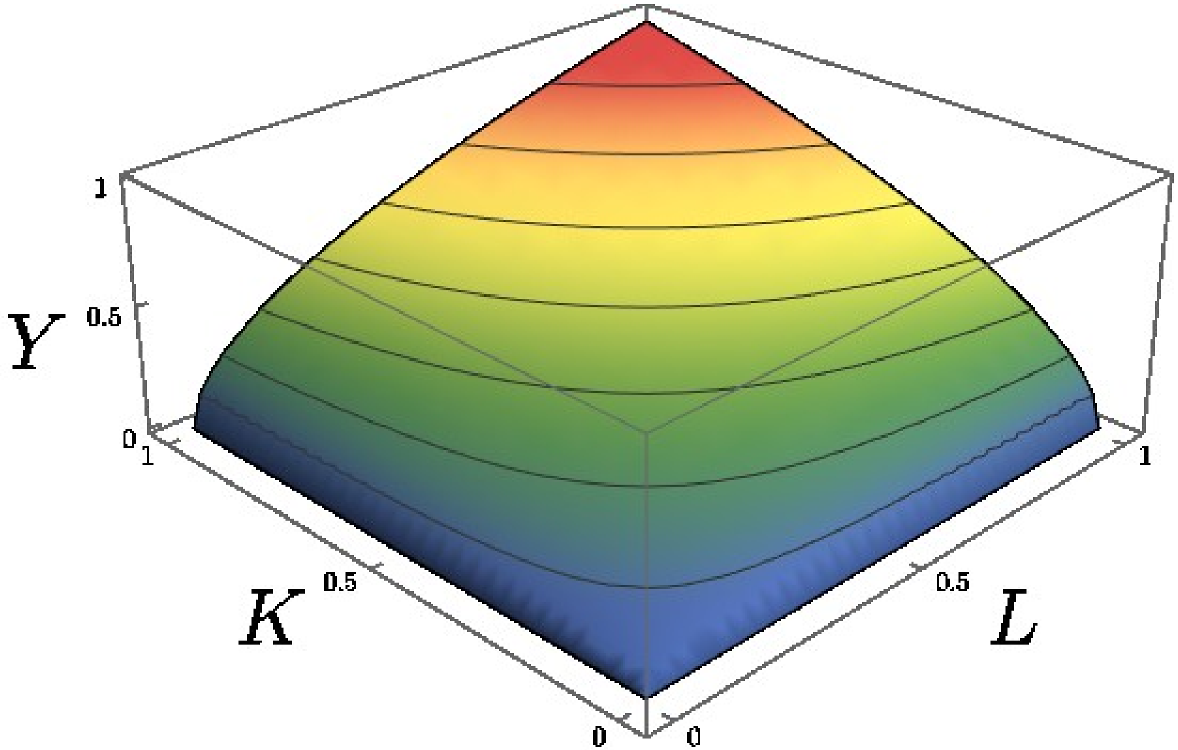
Many economic functions contain more then one variable.

For example, **Cobb-Douglas function** includes labour L and capital K as well as the technological parameter A:

$$Q(K,L) = AK^{\alpha}L^{\beta}$$

We will limit ourselved to functions of two variables. A graph of a function of two real variables is a plane in 3D space, See the next slide.

A graph of Cobb-Douglas function



The domain of a function of two real variables

By the domain of a function $f(x, y)$ we mean all ordered pairs $[x, y] \in R^2$ for which the function f is defined.

The domain is usually a subspace of Cartesian coordinate system (CCS).

In the problems solved later, we will depict the domain by shadowing an appropriate part of CCS.

Finding a domain of a function – Problem 1

Find the domain of the function $f(x,y)=\sqrt{x+y-2}$.

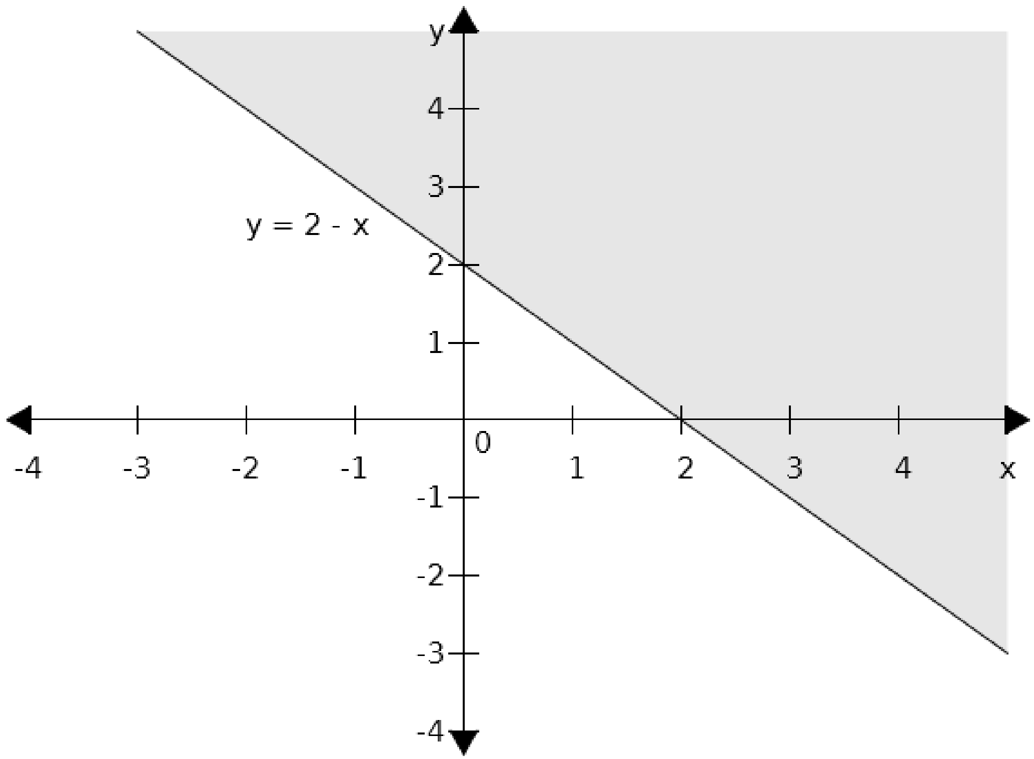
Solution: $x+y-2\geq 0$, (1)

Now we change the inequality into the equality:

$$x+y-2=0$$

This is the quation of a linear function (a line), which we draw in CCS (see the picture on the next slied). Then, we must find whether the domain lies on the left or right hand side to the line via one suitable point. We know that point [0,0] is on the left. We substitute this point into (1). Because (1) is untrue, the domain is on the right side of the line.

Problem 1 – cont.



Finding a domain of a function – Problem 2

Find the domain of the function: $f(x,y)=\sqrt{x^2+y^2-9}$.

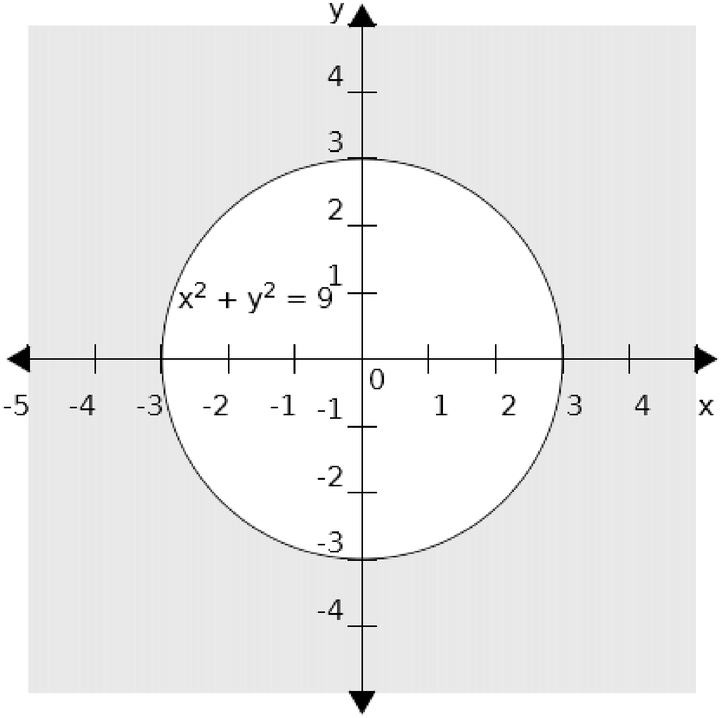
Solution:
From the square root we get: $x^2+y^2-9\geq 0$ (1).

Transforming the inequality into equality yields:

$$x^2+y^2-9=0$$

This is the equation of the circle with a radius r = 3 and a centre [0,0]. Again, one point will decide whether the domain is inside or outside of the circle. We can use [0,0] again: $0^2+0^2-9\geq 0$ is untrue, so the domain is outside of the circle, including the circle.

Finding a domain of a function – Problem 2



A derivative of a function of two real variables

Let f (x,y) be a function of two real varibles. Then, this function can be differentiated with regard to x and/or y. These derivatives are called *partial derivatives* and are denoted as follows:

$$\frac{\partial f(x,y)}{\partial y}, \quad f_y'(x,y), \quad f_y{}' \qquad\qquad \frac{\partial f(x,y)}{\partial x}, \quad f_x'(x,y), \quad f_x{}'$$

Definition: $$f_x'(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y'(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Partial derivatives – Problem 3

Find partial derivatives of the function: $f(x, y) = x^2 y + 2y^3$.

Solution: when differentiating with regard to x, we consider x to be a variable while y is considered a constant, and vice versa.

$$\frac{\partial f(x, y)}{\partial x} = 2xy$$

$$\frac{\partial f(x, y)}{\partial y} = x^2 + 6y^2$$

Second derivatives of a function of two real variables

If first derivatives are differentiable again, we obtain second derivatives:

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial^2 f}{\partial y \partial x}$$

The latter two derivatives are called mixed derivatives. It can be prooved they are equal:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Second derivatives – Problem 4

Find all four second derivatives of a function: $f(x, y) = 2x^3 - 6xy + 5$

Solution:
We start with the first derivatives:

$$\frac{\partial f}{\partial x} = 6x^2 - 6y \qquad \frac{\partial f}{\partial y} = -6x$$

Then, we compute the second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 12x \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = -6 \qquad \frac{\partial^2 f}{\partial y \partial x} = -6$$

Cobb – Douglas function

C-D function: $Q = AK^a L^b$.
Usually, we assume that $a + b = 1$.
Then, C-D: $Q = AK^a L^{1-a}$

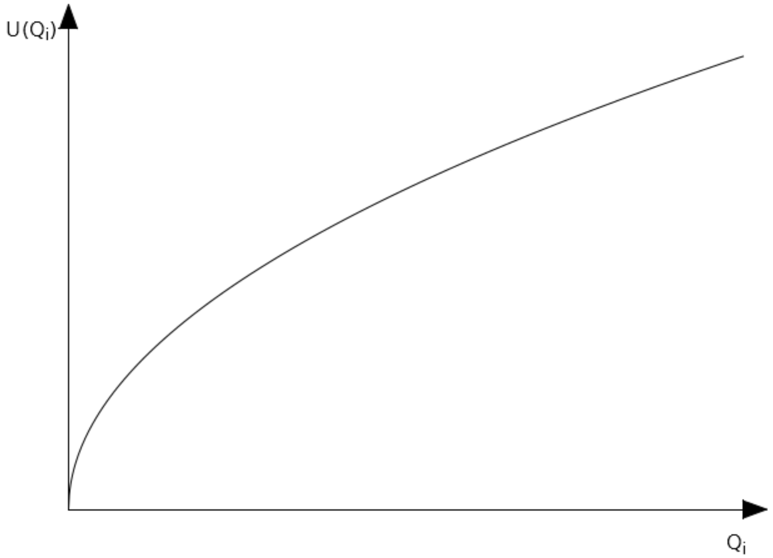
Marginal product of labour: $MP_L = \dfrac{\partial Q}{\partial L} = AK^a(1-a)L^{-a} = \dfrac{A}{1-a}\left(\dfrac{K}{L}\right)^a$

Marginal product of capital: $MP_K = \dfrac{\partial Q}{\partial K} = AaK^{a-1}L^{1-a} = Aa\left(\dfrac{K}{L}\right)^{a-1}$

A utility function

Let n be the number of different types of good. Let Q_1, Q_2, …. Be the amount of the good 1, 2, etc.

Then a function $U(Q_1, Q_2, ..., Q_n)$ is called the utility function. Typically, a utility function is concave:



Marginal utility

Marginal utility is defined as follows:

$$MU_1 = \frac{\partial U(Q_1, Q_2)}{\partial Q_1} \qquad MU_2 = \frac{\partial U(Q_1, Q_2)}{\partial Q_2} \qquad \text{etc.}$$

Example:

Find marginal utilities of the utility function $U = Q_1^{0,5} \cdot Q_1^{0,2}$.

Solution:

$$MU_1 = \frac{\partial U}{\partial Q_1} = 0{,}5Q_1^{-0,5} \cdot Q_1^{0,2}$$

$$MU_2 = \frac{\partial U}{\partial Q_2} = 0,2Q_1^{0,5} \cdot Q_1^{-0,8}$$

A tangent plane

$f(x,y)$ be a differentiable function at a point $C[x_0, y_0, z_0]$.
Then the tangent plane to the function f at the point C is:

$$z = z_0 + \frac{\partial f}{\partial x}(C)\cdot\left(x - x_0\right) + \frac{\partial f}{\partial y}(C)\cdot\left(y - y_0\right)$$

The norm vector is given as follows:

$$\vec{n} = \left(\frac{\partial f}{\partial x}(C), \frac{\partial f}{\partial y}(C), -1\right)$$

A tangent plane – Problem 5

Find the tangent plane to the function $f(x,y) = x^3 + xy^2$ at the point *C* [2,1,10].

Solution:
We compute the first derivatives:

$$\frac{\partial f(x,y)}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial f(x,y)}{\partial y} = 2xy$$

Then, we substitute point C:

$$\frac{\partial f}{\partial x}(C) = 3 \cdot 2^2 + 1^2 = 13 \qquad \frac{\partial f}{\partial y}(C) = 2 \cdot 2 \cdot 1 = 4$$

Therefore, we obtain: $z = 10 + 13 \cdot (x-2) + 4 \cdot (y-1)$
And finally: $13x + 4y - z - 20 = 0$

A total differential

Let $f(x, y)$ be differentiable at a point $C = [c_1, c_2, c_3]$.

Then, the total differential is given as:

$$df(C) = \frac{\partial f}{\partial x}(C)dx + \frac{\partial f}{\partial y}(C)dy$$

The total differential expresses (approximately) an increment of a function, which depends on increments of x (dx) and y (dy).

A total differential – Problem 6

Find the total differential of the function $f(x,y)=3x^2+5xy+y$, at the point *C* [1,1,9] and for *dx* = 0,1, *dy* = 0,2 .

Solution:
We start with the first derivatives:

$$\frac{\partial f}{\partial x}=6x+5,\frac{\partial f}{\partial y}=5x+1$$

Substituting into general form yields: $df=(6x+5y)dx+(5x+1)dy$

Substituting *C* and differentials:

$$df(C)=(6+5)dx+(5+1)dy=11dx+6dy=1,1+1,2=2,3$$

Problems to solve 1 (Assignment 6)

Find the domain of the following functions:

$$f(x,y) = \sqrt{2x - y + 3}$$

$$f(x,y) = \sqrt{x^2 + y^2 - 4} + \frac{1}{x}$$

$$f(x,y) = \arccos(3 - x)$$

$$f(x,y) = \log(y^2 + x)$$

$$f(x,y) = \sqrt{x - y + 1} + \sqrt{x + y + 1}$$

Problems to solve 2 (Assignment 6)

Find the partial derivatives of the first and second order:

$$f(x,y) = x^2 + y^2$$

$$f(x,y) = x^2 y^3 + 5x + y - 1$$

$$f(x,y) = e^{x+y}$$

$$f(x,y) = \ln(xy) + \frac{5}{x}$$

$$f(x,y) = \ln\left(xy + y^4\right)$$

Problems to solve 3 (Assignment 6)

Find marginal products of labour and capital:

$$Q = 6K^{0,4}L^{0,6}$$

$$Q = 10K^{0,5}L^{0,5}$$

$$Q(K,L) = 20K^{0,5} \cdot L^{0,5}$$

Find the total differential of the function $f(x,y) = x^3 + 5xy$.

Problems to solve 4 (skip it)

Find the tangent plane of the function $f(x,y)=x^3y^2$ at the point *C* [–2,2, ?].

Find the marginal utilities of the function $U=\sqrt{Q_1}\cdot\sqrt[3]{Q_1}$ at the point $Q_1 = 9$ and $Q_2 = 8$.

Problems to solve 5 (skip it)

A mathematical model of revenue R based on a price P and advertisement costs A has the following form:

$$R = \frac{54\sqrt{A}}{\sqrt{p}}$$

Find:

a) change of R with regard to the change of p,

b) change of R with regard to the change of A,

c) write a total differential of R.

Thank you for your attention