

Statistical Methods for Economists

Lecture 9

Full Factorial Design of Experiments



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Outline of the lecture



- Motivation & Introduction
 - Full Factorial Experiments
 - Graphical assessment of factor / interaction significance
 - Graphs of interaction effects
-

Full Factorial Experiments: Motivation



- We consider an observed variable Y , such as the quality of a product.
- We also consider several factors $A, B, C, D, \dots$, such as the conditions during the production.
- And we conjecture that the levels of the factors (and the interactions between the factors) influence the response variable Y , which we wish to maximize, say.
- The purpose is to identify the significant factors that influence the response variable Y most, and to find their optimal levels.
- We study the effect of the factors $A, B, C, D, \dots$ on the response variable Y

Full Factorial Experiments: Motivation & Introduction

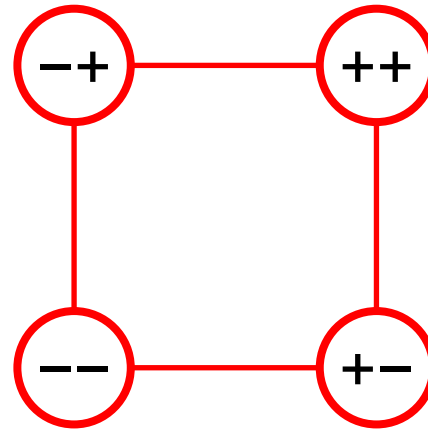


- We study the effect of the factors $A, B, C, D, \dots$ on the response variable Y by means of experiment.
- We consider only a few (a finite number) of distinct values of each of the factors.
- We usually consider only two distinct values of each of the factors.
- The first value is "low" and is denoted by " $-$ ".
- The second value is "high" and is denoted by " $+$ ".
- If n factors are considered,
then there are 2^n various combinations of the levels of the factors.

Full Factorial Experiments: Motivation & Introduction



- If we consider $n = 2$ factors A and B, then the arrangement of the factor levels is called a 2×2 or 2^2 factorial design.
- There are $2 \times 2 = 4$ factorial points, which constitute a square:

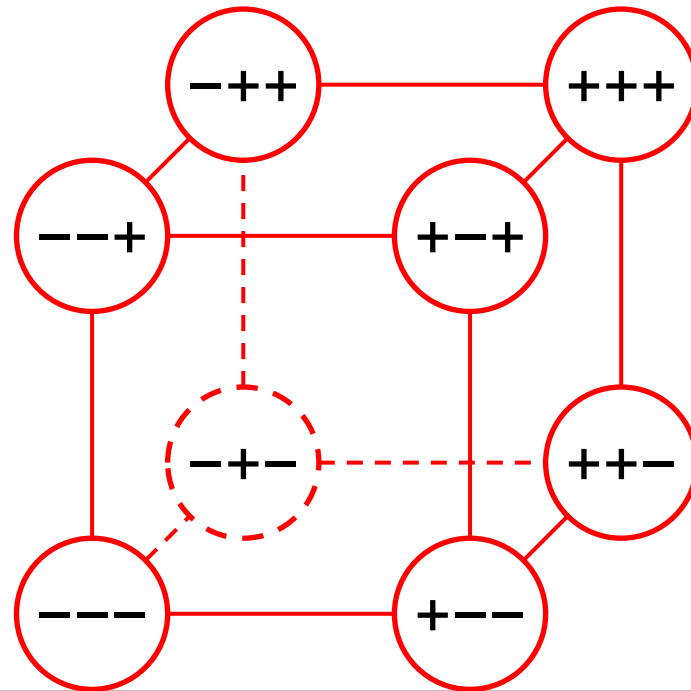


Factor	
A	B
-	-
+	-
-	+
+	+

Full Factorial Experiments: Motivation & Introduction



- If we consider $n = 3$ factors A, B, C, then the arrangement of the factor levels is called a $2 \times 2 \times 2$ or 2^3 factorial design.
- There are $2 \times 2 \times 2 = 8$ factorial points, which constitute a cube:



Factor		
A	B	C
-	-	-
+	-	-
-	+	-
+	+	-
-	-	+
+	-	+
-	+	+
+	+	+

Full Factorial Experiments: Motivation & Introduction



- The theory of the Full Factorial Experiments is based upon the theory of Multiple Linear Regression (again).
- The theory and all the calculations are simplified significantly if we assume that the number of the experiments carried out in each case is the same.
- In other words, considering n factors and assuming that the experiment is repeated K -times for each combination of the factors, we have

$$2^n \times K$$

observations of the experiment in total.

Full Factorial Experiments: Motivation & Introduction



- In other words, considering n factors and assuming that the experiment is repeated K -times for each combination of the factors, we have

$$2^n \times K$$

numerical outcomes of the mutually independent random variables

$$Y_{sk} \sim \mathcal{N}(\mu_s, \sigma^2) \quad \text{for } s \in \mathcal{S} \quad \text{and for } k = 1, 2, \dots, K$$

where

$$\mathcal{S} = \{\pm\}^n = \{+, -\}^n$$

is the index set of all the 2^n possible combinations of the levels of the factors.

Full Factorial Experiments: Motivation & Introduction



We thus have a sample

$$y_{s1}, y_{s2}, \dots, y_{sK} \quad \text{for } s \in \mathcal{S}$$

of the observations of the mutually independent random variables

$$Y_{s1}, Y_{s2}, \dots, Y_{sK} \sim \mathcal{N}(\mu_s, \sigma^2) \quad \text{for } s \in \mathcal{S}$$

where $\mathcal{S}$ is the index set of all the 2^n combinations of the levels of the factors.

(Here, the expected values μ_s and the variance σ^2 are unknown.

The variance σ^2 is assumed to be the same – homoskedasticity.)

Full Factorial Experiments: Example

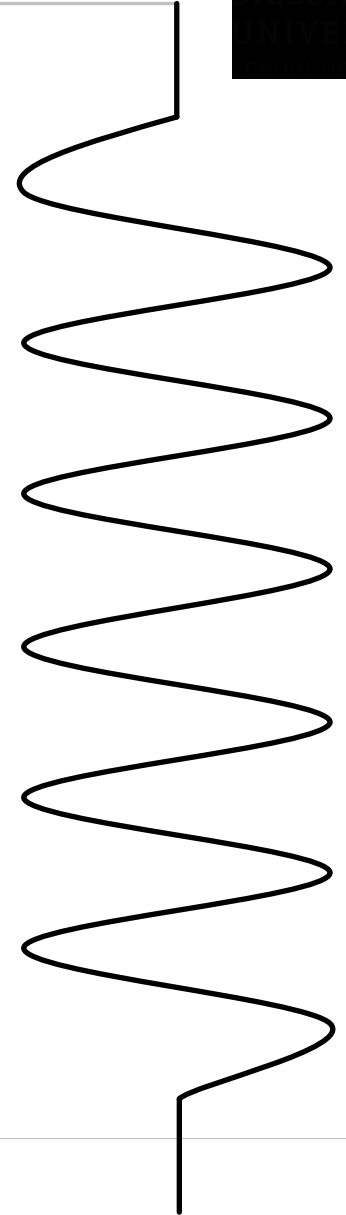


Consider a *spring*.

The observed variable Y is the lifespan of the spring,
i.e. the number of pressures of the spring until it cracks.

We consider three factors:

- Factor L = the length of the spring
- Factor G = the thickness of the wire of the spring
- Factor T = the material of (the wire of) the spring

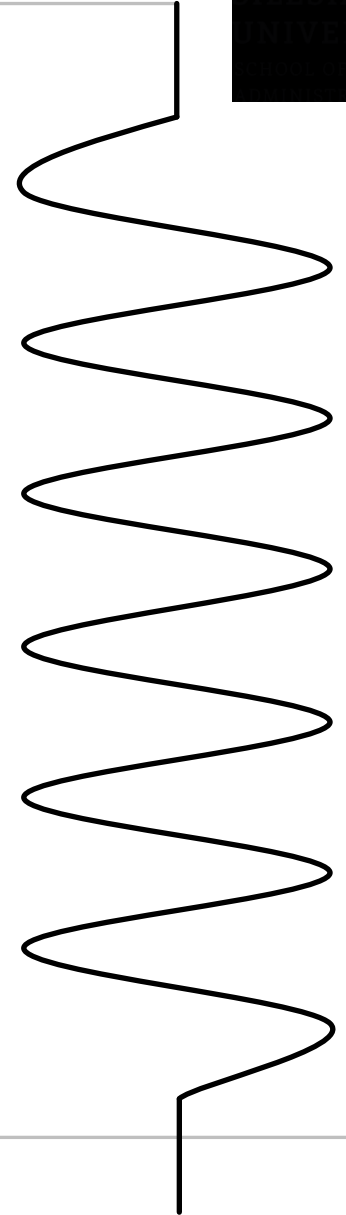


Full Factorial Experiments: Example



We consider the following levels of the three factors:

- Factor L = the length of the spring
 - “−” or “−1” = 10 cm
 - “+” or “+1” = 15 cm
- Factor G = the thickness of the wire of the spring
 - “−” or “−1” = 5 mm
 - “+” or “+1” = 7 mm
- Factor T = the material of (the wire of) the spring
 - “−” or “−1” = Material / alloy “A”
 - “+” or “+1” = Material / alloy “B”

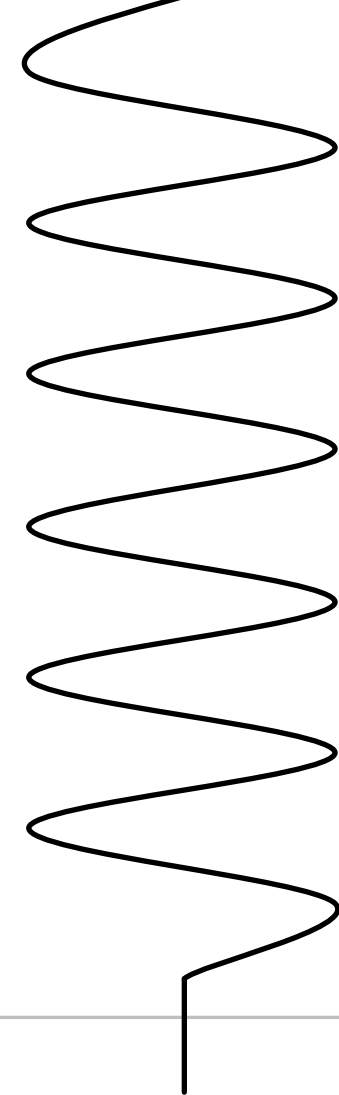


Full Factorial Experiments: Example



The experiment was carried out 2× for each combination of the factors:

Factor			Observed value	
L	G	T	$Y_{\pm\pm\pm 1}$	$Y_{\pm\pm\pm 2}$
–	–	–	77	81
+	–	–	98	96
–	+	–	76	74
+	+	–	90	94
–	–	+	63	65
+	–	+	82	86
–	+	+	72	74
+	+	+	92	88



Full Factorial Experiments: Example



Now, the goal is to identify the factors, including their interactions, with significant effect on the lifespan of the spring, i.e. the observed values of the variable Y .

We shall use the theory of Multiple Linear Regression to this end.

We assume the model with the intercept term:

$$Y = 1\beta_0 + x_L\beta_L + x_G\beta_G + x_T\beta_T + x_{LG}\beta_{LG} + x_{LT}\beta_{LT} + x_{GT}\beta_{GT} + x_{LGT}\beta_{LGT}$$

Full Factorial Experiments: Example



We assume the model with the intercept term:

$$Y = 1\beta_0 + x_L\beta_L + x_G\beta_G + x_T\beta_T + x_{LG}\beta_{LG} + x_{LT}\beta_{LT} + x_{GT}\beta_{GT} + x_{LGT}\beta_{LGT}$$

The effect of the factor or the interaction is **significant** if and only if the respective coefficient β is non-zero.

→ We shall use the t -test for the respective parameter β (see Theorem 6) to this end.

Full Factorial Experiments: Example



We set up the matrix X as the first step:

“−” = “−1” & “+” = “+1”

Interaction = the product of the Factors.

Intercept	Factor			Interaction			
0	L	G	T	LG	LT	GT	LGT
1	−	−	−	+	+	+	−
1	+	−	−	−	−	+	+
1	−	+	−	−	+	−	+
1	+	+	−	+	−	−	−
1	−	−	+	+	−	−	+
1	+	−	+	−	+	−	−
1	−	+	+	−	−	+	−
1	+	+	+	+	+	+	+

This is a $2^n \times 2^n$ matrix in general.



Full Factorial Experiments: Example

Since we carried out each experiment 2× for each combination of the factors, we double each row of the table:

Interaction = the product of the Factors.

X:

Intercept		Factor				Interaction			y:	Observed
0	L	G	T	LG	LT	GT	LGT	$Y_{\pm\pm\pm k}$		
1	-1	-1	-1	+1	+1	+1	-1	77		
1	-1	-1	-1	+1	+1	+1	-1	81		
1	+1	-1	-1	-1	-1	+1	+1	98		
1	+1	-1	-1	-1	-1	+1	+1	96		
1	-1	+1	-1	-1	+1	-1	+1	76		
1	-1	+1	-1	-1	+1	-1	+1	74		
1	+1	+1	-1	+1	-1	-1	-1	90		
1	+1	+1	-1	+1	-1	-1	-1	94		
1	-1	-1	+1	+1	-1	-1	+1	63		
1	-1	-1	+1	+1	-1	-1	+1	65		
1	+1	-1	+1	-1	+1	-1	-1	82		
1	+1	-1	+1	-1	+1	-1	-1	86		
1	-1	+1	+1	-1	-1	+1	-1	72		
1	-1	+1	+1	-1	-1	+1	-1	74		
1	+1	+1	+1	+1	+1	+1	+1	92		
1	+1	+1	+1	+1	+1	+1	+1	88		

y:

This is a $(2^n \times K) \times (2^n)$ matrix in general.

Full Factorial Experiments



We shall perform the t -test for each individual parameter β .

Recall the Corollary of Theorem 6:

If $\beta_j = 0$, then

$$T = \frac{b_j}{\sqrt{s^2} \sqrt{c_{jj}}} \sim t_{N - \text{rank}(X)}$$

where

- b_j is the estimate of the parameter β_j
- s^2 is the residual variance = mean square error
- c_{jj} the j -th element on the diagonal of the matrix $C = (X^T X)^{-1}$
- N is the total number of observations, we have $N = 2^n \times K$ here

Full Factorial Experiments



Having recalled the Corollary of Theorem 6:

$$\text{if } \beta_j = 0 \quad \text{then} \quad T = \frac{b_j}{\sqrt{s^2} \sqrt{c_{jj}}} \sim t_{N-\text{rank}(X)}$$

— the only task is to calculate everything in this special case, i.e. for this matrix X .

Calculating, we obtain that:

$$X^T X = \begin{pmatrix} N & & & \\ & N & & \\ & & \ddots & \\ & & & N \end{pmatrix}$$

i.e. the $2^n \times 2^n$ diagonal matrix with the number $N = 2^n \times K$ on the diagonal.

Full Factorial Experiments: Example



In our example, we have $n = 3$ factors (L, G, T),
and each experiment is replicated $K = 2$ times. We thus have

$$N = 2^n \times K = 16$$

Moreover, the result $X^T X$ is a $2^n \times 2^n$, i.e. 8×8 , matrix
with the number 16 on its diagonal:

$$X^T X = \begin{pmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}$$

Full Factorial Experiments



Calculating, we obtain that:

$$\mathbf{C} = (\mathbf{X}^T \mathbf{X})^{-1} = \begin{pmatrix} 1/N & & & \\ & 1/N & & \\ & & \ddots & \\ & & & 1/N \end{pmatrix}$$

i.e. the $2^n \times 2^n$ diagonal matrix with the number $c_{jj} = 1/N = 1/(2^n \times K)$ on the diagonal.

Moreover, the estimates are

$$\mathbf{b} = \mathbf{C} \mathbf{X}^T \mathbf{y} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} = \frac{1}{N} \mathbf{X}^T \mathbf{y}$$

Full Factorial Experiments: Example



In our example, we

$$C = (X^T X)^{-1} = \begin{pmatrix} 1/16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/16 \end{pmatrix}$$

And the estimates are:

— the intercept term = the sample mean of the observed values:

$$\begin{aligned} b_0 &= \frac{1}{N} (X_0)^T y = \frac{1}{16} (+77 + 98 + 76 + 90 + 63 + 82 + 72 + 92 + 81 + 96 + 74 + 94 + 65 + 86 + 74 + 88) = \\ &= 1308/16 = 81.75 \end{aligned}$$

Full Factorial Experiments: Example



The estimates of the effects of the factors are:

$$\begin{aligned} b_L &= \frac{1}{N} (X_L)^T y = \frac{1}{16} (-77 - 98 + 76 + 90 - 63 - 82 + 72 + 92 - 81 - 96 + 74 + 94 - 65 - 86 + 74 + 88) = \\ &= 144/16 = 9 \end{aligned}$$

$$\begin{aligned} b_G &= \frac{1}{N} (X_G)^T y = \frac{1}{16} (-77 - 98 - 76 - 90 + 63 + 82 + 72 + 92 - 81 - 96 - 74 - 94 + 65 + 86 + 74 + 88) = \\ &= 12/16 = 0.75 \end{aligned}$$

$$\begin{aligned} b_T &= \frac{1}{N} (X_T)^T y = \frac{1}{16} (-77 - 98 - 76 - 90 - 63 - 82 - 72 - 92 + 81 + 96 + 74 + 94 + 65 + 86 + 74 + 88) = \\ &= -64/16 = -4 \end{aligned}$$

Full Factorial Experiments: Example



The estimates of the effects of the interactions between / among the factors are:

$$\begin{aligned} b_{LG} &= \frac{1}{N} (X_{LG})^T \mathbf{y} = \frac{1}{16} (+77 + 98 - 76 - 90 - 63 - 82 + 72 + 92 + 81 + 96 - 74 - 94 - 65 - 86 + 74 + 88) = \\ &= -8/16 = -0.5 \end{aligned}$$

$$\begin{aligned} b_{LT} &= \frac{1}{N} (X_{LT})^T \mathbf{y} = \frac{1}{16} (+77 + 98 - 76 - 90 + 63 + 82 - 72 - 92 - 81 - 96 + 74 + 94 - 65 - 86 + 74 + 88) = \\ &= 4/16 = 0.25 \end{aligned}$$

$$\begin{aligned} b_{GT} &= \frac{1}{N} (X_{GT})^T \mathbf{y} = \frac{1}{16} (+77 + 98 + 76 + 90 - 63 - 82 - 72 - 92 - 81 - 96 - 74 - 94 + 65 + 86 + 74 + 88) = \\ &= 48/16 = 3 \end{aligned}$$

$$b_{LGT} = \frac{1}{N} (X_{LGT})^T \mathbf{y} = \frac{1}{16} (+77 + 98 + 76 + 90 - 63 - 82 - 72 - 92 - 81 - 96 - 74 - 94 + 65 + 86 + 74 + 88)$$

Full Factorial Experiments



We further have to calculate the residual variance = mean square error:

$$s^2 = \frac{\text{RSS}}{N - \text{rank}(X)} = \frac{\mathbf{e}^T \mathbf{e}}{N - \text{rank}(X)} = \frac{(\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}})}{N - \text{rank}(X)}$$

where

$$N = 2^n \times K$$

and it is easy to see that

$$\text{rank}(X) = 2^n$$

Moreover, the predicted values are

$$\hat{\mathbf{y}} = X\mathbf{b} = X\mathbf{C}X^T\mathbf{y} = \frac{1}{N}XX^T\mathbf{y}$$

Full Factorial Experiments



Calculating, we obtain that:

$$\mathbf{X}\mathbf{X}^T = \begin{pmatrix} 2^n \mathbf{E} & & & \\ & 2^n \mathbf{E} & & \\ & & \ddots & \\ & & & 2^n \mathbf{E} \end{pmatrix}$$

i.e. the $(2^n \times K) \times (2^n \times K)$ matrix

with the $K \times K$ matrix $2^n \mathbf{E}$ repeated 2^n -times on its diagonal.

The symbol

$\mathbf{E}$ is the $K \times K$ matrix of all ones

and

$2^n \mathbf{E}$ is the $K \times K$ matrix of the numbers 2^n

Full Factorial Experiments



Substituting into the above equation

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{b} = \mathbf{X}\mathbf{C}\mathbf{X}^T\mathbf{y} = \frac{1}{N}\mathbf{X}\mathbf{X}^T\mathbf{y} = \frac{1}{2^n \times K}\mathbf{X}\mathbf{X}^T\mathbf{y}$$

and calculating, we obtain the predicted values:

$$\hat{y}_{sk} = \frac{1}{K}(y_{s1} + y_{s2} + \dots + y_{sK}) \quad \text{for } s \in \mathcal{S} \quad \text{for } k = 1, 2, \dots, K$$

where – recall –

$$\mathcal{S} = \{\pm\}^n = \{+, -\}^n$$

is the index set of all the 2^n possible combinations of the levels of the factors
and K is the number of the replications of the experiment for the given

Full Factorial Experiments



We have:

$$s^2 = \frac{\text{RSS}}{N - \text{rank}(\mathbf{X})} = \frac{\mathbf{e}^T \mathbf{e}}{N - \text{rank}(\mathbf{X})} = \frac{(\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}})}{N - \text{rank}(\mathbf{X})} = \frac{(\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}})}{2^n \times K - 2^n}$$

Substituting, we have:

$$\begin{aligned} s^2 &= \frac{(\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}})}{(K - 1)2^n} = \frac{1}{(K - 1)2^n} \sum_{s \in \mathcal{S}} \sum_{k=1}^K (y_{sk} - \hat{y}_{sk})^2 = \\ &= \frac{1}{(K - 1)2^n} \sum_{s \in \mathcal{S}} \sum_{k=1}^K \left(y_{sk} - \frac{y_{s1} + y_{s2} + \dots + y_{sK}}{K} \right)^2 \end{aligned}$$

Full Factorial Experiments: Example



In our example, we have:

Factor	L	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+
	G	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+
	T	-	-	-	-	-	-	-	-	+	+	+	+	+	+	+	+
Observed	y_{++k}	77	81	98	96	76	74	90	94	63	65	82	86	72	74	92	88
Sample means	$\hat{y}_{++}$	79		97		75		92		64		84		73		90	

$$\begin{aligned}
 \text{RSS} = & (77 - 79)^2 + (81 - 79)^2 + (98 - 97)^2 + (96 - 97)^2 + (76 - 75)^2 + \\
 & + (74 - 75)^2 + (90 - 92)^2 + (94 - 92)^2 + (63 - 64)^2 + (65 - 64)^2 + \\
 & + (82 - 84)^2 + (86 - 84)^2 + (72 - 73)^2 + (74 - 73)^2 + (92 - 90)^2 + \\
 & + (88 - 90)^2 = 40
 \end{aligned}$$

and

$$s^2 = \frac{\text{RSS}}{(K - 1)2^n} = \frac{40}{(2 - 1) \times 2^3} = \frac{40}{8} = 5$$

Finally, we calculate the statistic

$$\begin{aligned} T &= \frac{b_j}{\sqrt{s^2} \sqrt{c_{jj}}} = \\ &= \frac{\frac{1}{N} (\mathbf{X}_j)^T \mathbf{y}}{\sqrt{\frac{\text{RSS}}{(K-1)2^n}} \sqrt{\frac{1}{N}}} = \sqrt{\frac{1}{K2^n}} (\mathbf{X}_j)^T \mathbf{y} \sqrt{\frac{(K-1)2^n}{\text{RSS}}} = \sqrt{\frac{(K-1)/K}{\text{RSS}}} (\mathbf{X}_j)^T \mathbf{y} \end{aligned}$$

Full Factorial Experiments



Finally, we perform the t -test of the significance of the factor or the interaction. To this end, we compare the value of the above statistic T with the quantile

$$t_{N-2^n} \left(1 - \frac{\alpha}{2} \right)$$

- If $|T| \geq t_{N-2^n}(1 - \alpha/2)$, the critical region, then **reject** the null hypothesis that $b_j = 0$, that is, consider the factor / interaction significant.
- If $|T| < t_{N-2^n}(1 - \alpha/2)$, then **fall to reject** the null hypothesis $b_j = 0$, that is, consider the factor / interaction insignificant – **neglect it**.

Full Factorial Experiments: Example



In our example – choosing significance level $\alpha = 5\%$ – we have the quantile

$$q = t_{N-2^n} \left(1 - \frac{\alpha}{2} \right) = t_{16-8} \left(1 - \frac{0.05}{2} \right) = t_8(0.975) \doteq 2.306004$$

and

b_L :

$$T = \frac{b_L}{\sqrt{s^2} \sqrt{c_{LL}}} = \frac{9}{\sqrt{5} \sqrt{1/16}} \doteq 16.100 \geq q \quad \Rightarrow \text{significant}$$

b_G :

$$T = \frac{b_G}{\sqrt{s^2} \sqrt{c_{GG}}} = \frac{0.75}{\sqrt{5} \sqrt{1/16}} \doteq 1.342 < q \quad \Rightarrow \text{neglect}$$

Full Factorial Experiments: Example



b_T :

$$T = \frac{b_T}{\sqrt{s^2} \sqrt{c_{TT}}} = \frac{-4}{\sqrt{5} \sqrt{1/16}} \doteq -7.155 \leq -q \quad \Rightarrow \text{significant}$$

b_{LG} :

$$T = \frac{b_{LG}}{\sqrt{s^2} \sqrt{c_{LGLG}}} = \frac{-0.5}{\sqrt{5} \sqrt{1/16}} \doteq -0.894 > -q \quad \Rightarrow \text{neglect}$$

b_{LT} :

$$T = \frac{b_{LT}}{\sqrt{s^2} \sqrt{c_{LTLT}}} = \frac{0.25}{\sqrt{5} \sqrt{1/16}} \doteq 0.447 < q \quad \Rightarrow \text{neglect}$$

b_{GT} :

$$T = \frac{b_{GT}}{\sqrt{s^2} \sqrt{c_{GTGT}}} = \frac{3}{\sqrt{5} \sqrt{1/16}} \doteq 5.367 > q \quad \Rightarrow \text{significant}$$

Full Factorial Experiments: Example



b_{LGT} :

$$T = \frac{b_{LGT}}{\sqrt{s^2} \sqrt{c_{LGT LGT}}} = \frac{-0.25}{\sqrt{5} \sqrt{1/16}} \doteq -0.447 > -q \quad \Rightarrow \text{neglect}$$

CONCLUSION:

the factors L T and the interaction GT

have significant effect on the lifespan of the spring.

Moreover, we conclude that

$$Y \approx b_0 + x_L b_L + x_T b_T + x_{GT} b_{GT}$$

$$\approx 81.75 + 9x_L - 4x_T + 3x_{GT} \quad \text{where } x_L, x_T, x_{GT} \in \{-1, +1\}$$

Graphical assessment of factor / interaction significance

Full Factorial Experiments



If the number of the replications of each experiment is

$$K = 1$$

then the above computational procedure cannot be used
because

$$s^2 = \frac{RSS}{(K - 1)2^n}$$

that is, we cannot perform the calculations because we would divide by zero,
or we would always calculate $T = 0$.

We perform the graphical assessment of the significance then.

Full Factorial Experiments



Assuming $K = 1$, so that $N = 2^n$,

the graphical assessment proceeds as follows:

- Sort all the estimates b_s , for $s \in S$, in ascending order.
- Denote the i -th least value by $b_{(i)}$ for $i = 1, 2, \dots, 2^n - 1$ so that we have

$$b_{(1)} \leq b_{(2)} \leq \dots \leq b_{(2^n-1)}$$

- Calculate the values

$$P_i = \frac{i - \frac{1}{2}}{2^n - 1} \quad \text{for } i = 1, 2, \dots, 2^n - 1$$

- Plot the points

$$[b_{(i)}, P_i] \quad \text{for } i = 1, 2, \dots, 2^n - 1$$

Full Factorial Experiments: Example



In our example, we have:

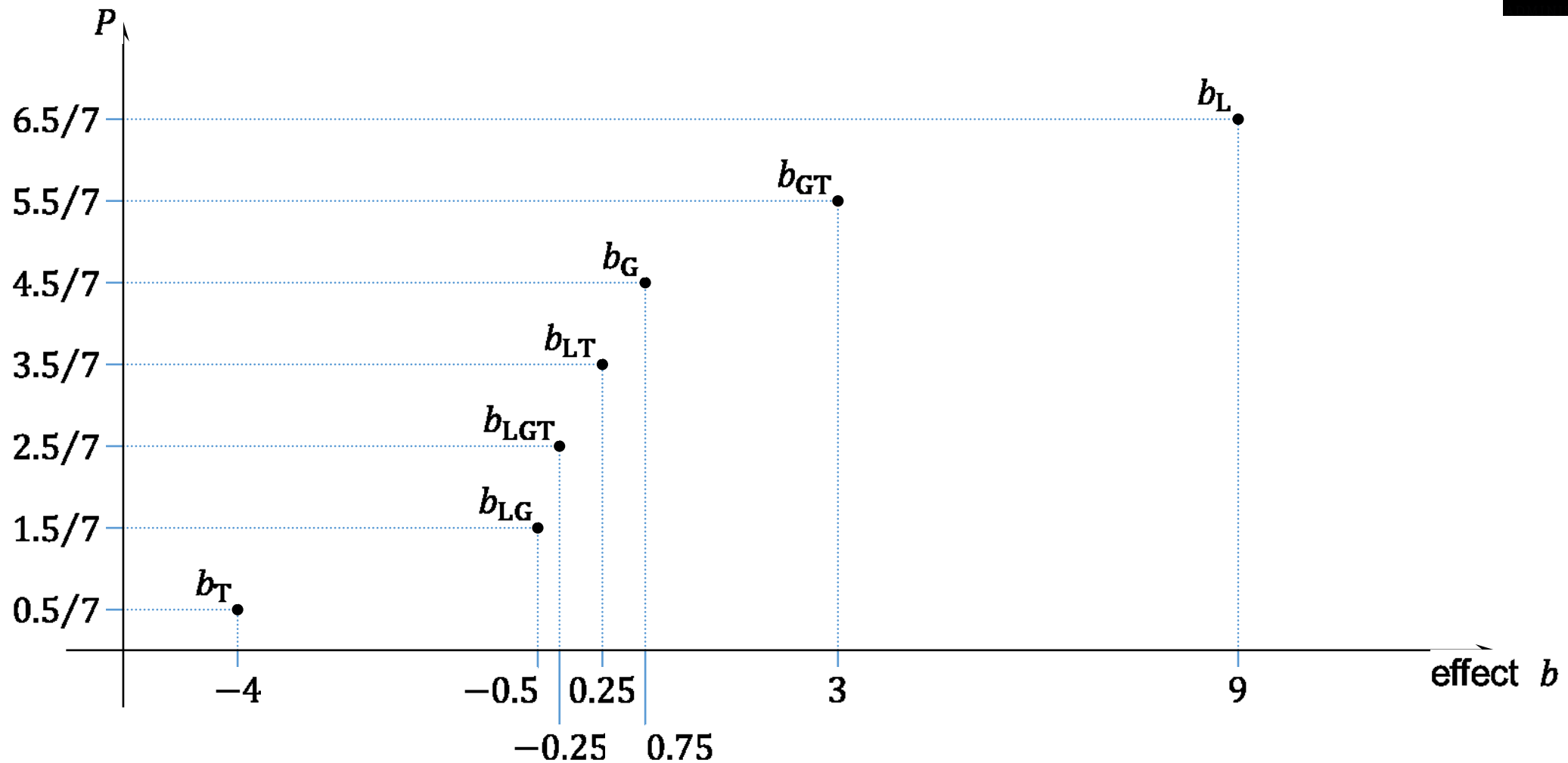
$$b_T < b_{LG} < b_{LGT} < b_{LT} < b_G < b_{GT} < b_L$$

$$-4 < -0.5 < -0.25 < 0.25 < 0.75 < 3 < 9$$

and P_i :

$$\frac{0.5}{7} < \frac{1.5}{7} < \frac{2.5}{7} < \frac{3.5}{7} < \frac{4.5}{7} < \frac{5.5}{7} < \frac{6.5}{7}$$

Full Factorial Experiments: Example



Full Factorial Experiments: Graphical assessment



The points $[b_{(i)}, P_i]$ take the shape of an S-curve sometimes.

Those points that lie out of the approximately linear middle part of the S-curve indicate the significant factors or interactions.

In our example: the points

$$b_T \quad b_{GT} \quad b_L$$

lie out of the approximately linear middle part of the S-curve,
which is the same result that we obtained by using the t -test above.

Graphs of interaction effects

Full Factorial Experiments: Interaction effect



There is an **interaction effect** between two factors in a full factorial experiment when the effect of one independent variable (factor) depends on the level of another independent variable (factor).

Considering two factors A and B, take

- all the observations when factors A and B were at the levels --, respectively
- all the observations when factors A and B were at the levels -+, respectively
- all the observations when factors A and B were at the levels +- , respectively
- all the observations when factors A and B were at the levels ++, respectively

and calculate the average (sample mean) for each of the four groups

Full Factorial Experiments: Interaction effect

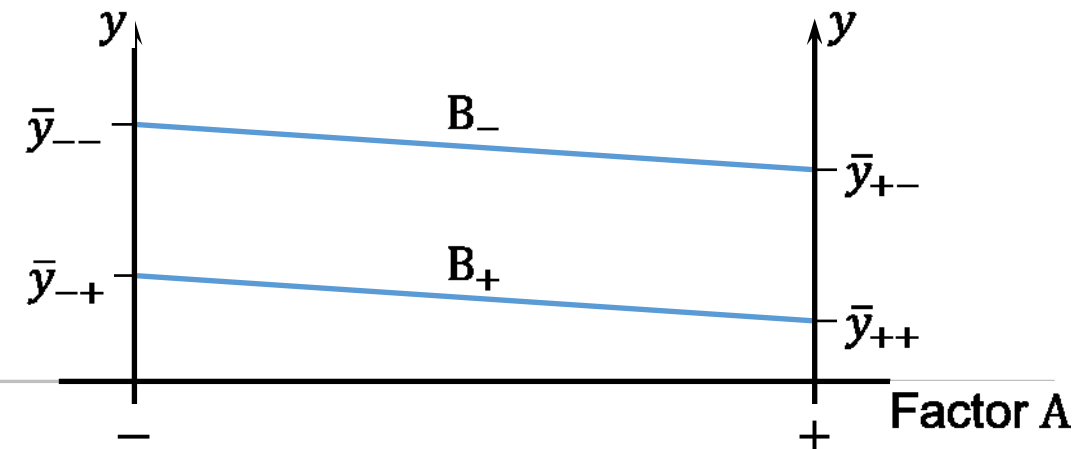


We then have:

Factor B	+	$\bar{y}_{-+}$	$\bar{y}_{++}$
	-	$\bar{y}_{--}$	$\bar{y}_{+-}$
		-	+
		Factor A	

We then depict the values in the form of a chart
(here as the dependence of Factor B on Factor A):

- If the lines are $\approx$ parallel,
then the interaction is not significant.
- If the lines are not parallel,
then the interaction is significant.



Full Factorial Experiments: Example



In our example, we have:

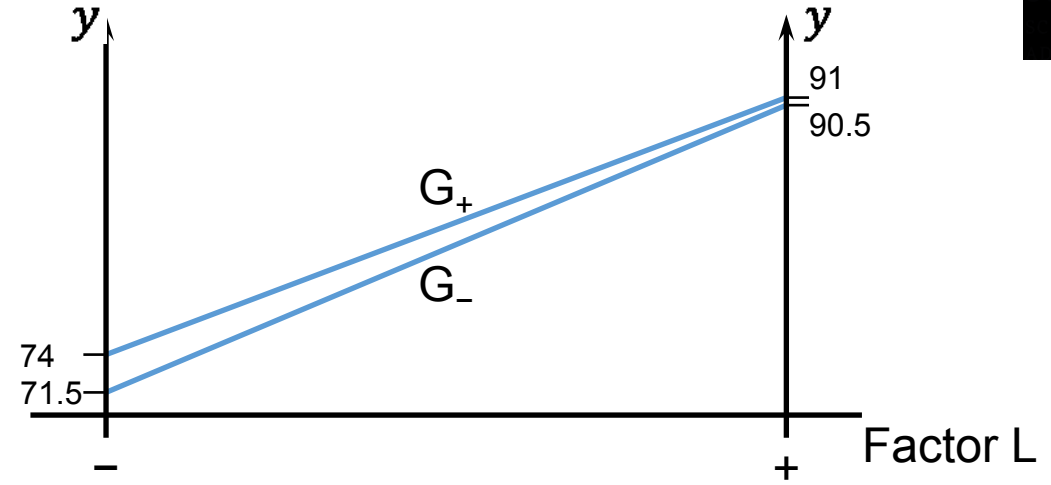
Factor	L	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+
	G	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+
	T	-	-	-	-	-	-	-	-	+	+	+	+	+	+	+	+
	Observed y_{ttkk}	77	81	98	96	76	74	90	94	63	65	82	86	72	74	92	88

Factor G	+	76, 74, 72, 790, 94, 92, 8	
		74	91
	-	77, 81, 63, 698, 96, 82, 8	
		71.5	90.5
		-	+
Factor L			

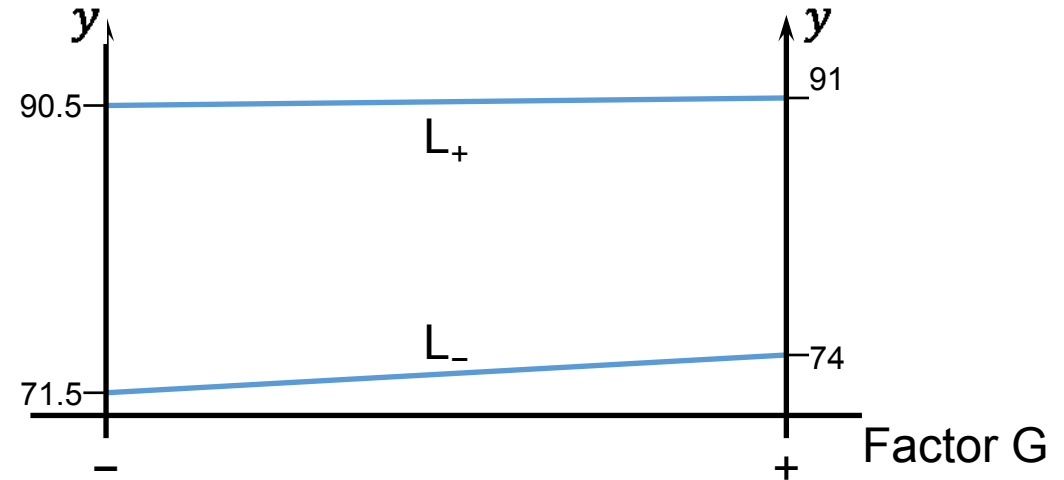
Factor T	+	63, 65, 72, 782, 86, 92, 8	
		68.5	87
	-	77, 81, 76, 798, 96, 90, 9	
		77	94.5
		-	+
Factor L			

Full Factorial Experiments: Example

G	+	74	91
	-	71.5	90.5
		-	+
		L	



L	+	90.5	91
	-	71.5	74
		-	+
		G	



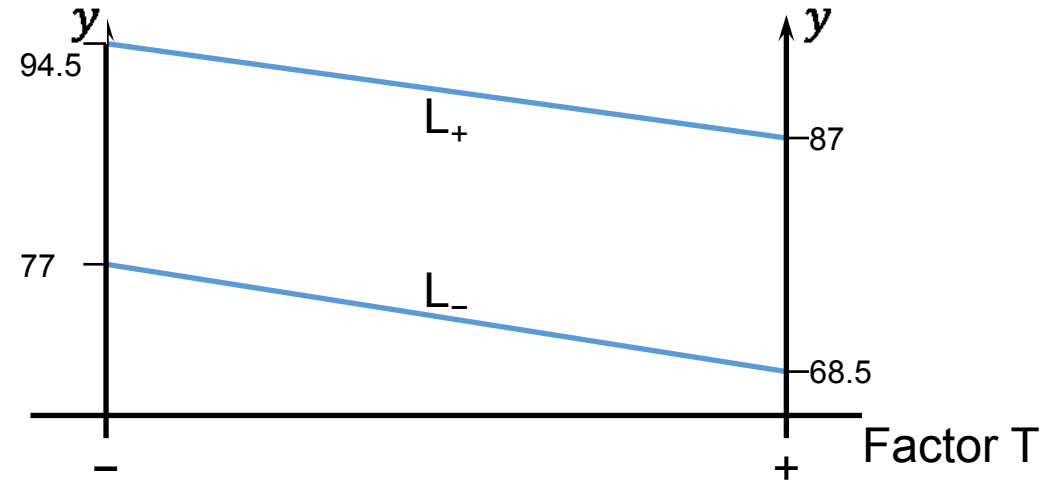
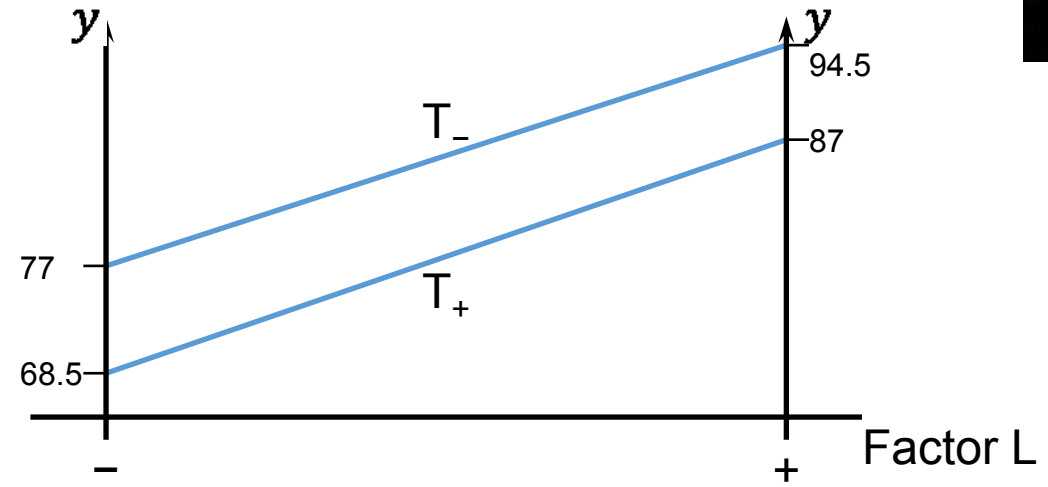
$\approx$ parallel $\Rightarrow$ the interaction is not significant

Full Factorial Experiments: Example



T	+	68.5	87
	-	77	94.5
		-	+
		L	

L	+	94.5	87
	-	77	68.5
		-	+
		T	



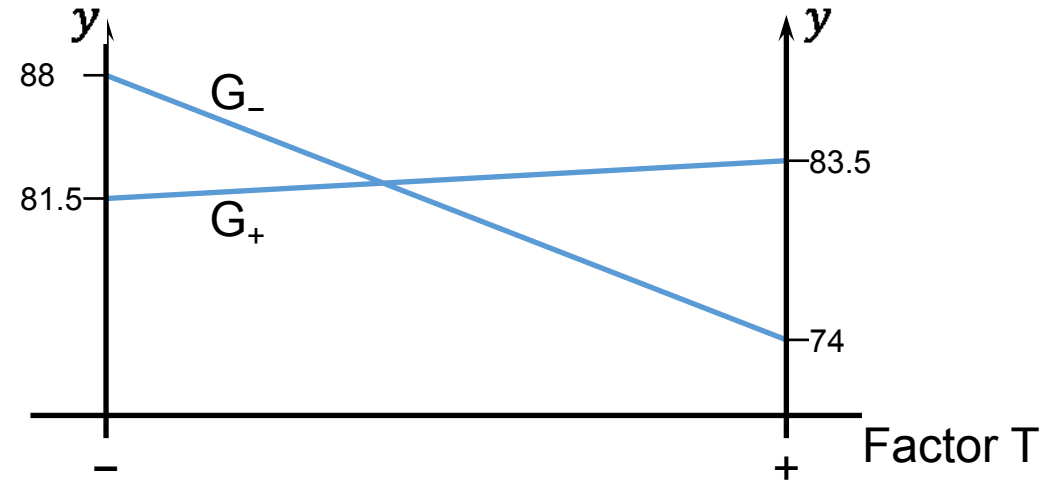
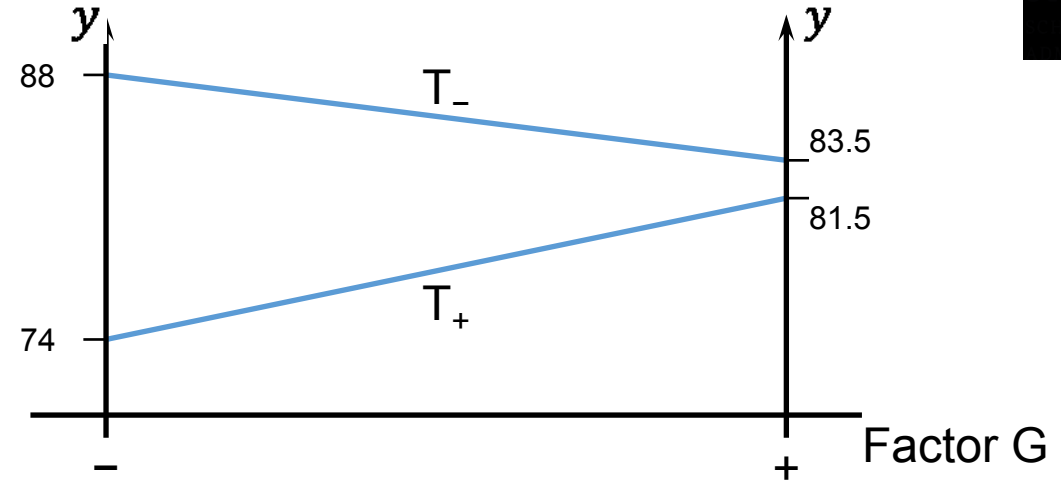
$\approx$ parallel $\Rightarrow$ the interaction is not significant

Full Factorial Experiments: Example



T	+	74	83.5
	-	88	81.5
		-	+
		G	

G	+	81.5	83.5
	-	88	74
		-	+
		T	



NOT parallel $\Rightarrow$ the interaction is SIGNIFICANT