Statistical Methods for Economists

Lecture 2 & 3

Hypothesis testing:
Parametric and Non-parametric tests
(in marketing and elsewhere)

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Outline of the lecture

- About statistical hypothesis testing
- PARAMETRIC TESTS
- *t*-tests for the means
- Two sample *F*-test for the equality of variances
- NON-PARAMETRIC TESTS
- Sign test for the median
- Pearson's χ^2-test for the goodness of fit
- χ^2-test of independence of qualitative data items

About statistical hypothesis testing

- The general outline
  of a statistical hypothesis test
- The *p*-value of a test
- Parametric and Non-parametric tests

The general outline of a statistical hypothesis test

- A statistical test consists in the study of the outcomes of a random experiment.
- We put down a hypothesis about the probability distribution of the outcomes of the random experiment.
- We also make up a statistic S – a formula, i.e. a mathematical expression – and we **prove** (!) as a mathematical Theorem that, under our hypotheses, the statistic S follows a certain probability distribution.
- We carry out the random experiment several (or many) times.
- We put down the results of the experiment, i.e., count positive results, count the negative results, and so on.

The general outline of a statistical hypothesis test

- We substitute the results (the counts, and so on) into the mathematical expression-statistic S, which is a random variable thus (its value depends on the results of the random experiment).
- We then choose the **significance level** α – a small probability – such as **α = 5 % = 0.05** (i.e. "one error per twenty trials").
  (Other popular choices include α = 10 % = 0.1 or α = 1 % = 0.01.)
- By using the mathematical Theorem, which we proved (see above), we find the **critical region** $C \subseteq \mathbb{R}$ so that – if our hypotheses are true – then **the probability of the event that** $S \in C$ is **$\leq \alpha$.**

The general outline of a statistical hypothesis test

- The critical region C is usually a closed interval or the union of two closed intervals.
- Finally, make a statistical conclusion:

- If $S \in C$, then **reject** the hypothesis.
- If the hypotheses are true, then it is quite improbable that $S \in C$;
  the probability is $\leq \alpha$. So we are making a mistake – type I error,
  i.e. rejecting a hypothesis which is true – about once per twenty trials,
  if $\alpha = 5$ %.

The general outline of a statistical hypothesis test

- If $S \notin C$, then **do not reject** (or **fail to reject**) the hypothesis.
- The fact that we fail to reject the hypothesis is not a confirmation that the hypothesis is true!
- Since the statistic S is a random variable, it may happen by chance that $S \notin C$ even if the hypothesis is false.
- This situation – failing to reject a false hypothesis – is a type II error. The probability of type II error is β, and this probability is difficult to calculate… If $\alpha = 5$ %, then the probability β should be ≤ 20 %. (Is it ≤ 20 %?) The probability $1 - \beta$ is the **power of the test.**

The *p*-value of the test

The above outline of the test is as follows:

- Choose the significance level α (such as α = 5 %).
- Depending upon the α, find the critical region $C_\alpha \subseteq \mathbb{R}$ so that – if the hypothesis is true – then the probability that $(S \in C_\alpha)$ is $\leq \alpha$.
- Carry out the experiment, enumerate the expression S, and see if $S \in C_\alpha$.

<u>Another procedure:</u>

- Carry out the experiment and enumerate the expression S.
- Find the least number $p \in (0, 1)$ such that $S \in C_p$.
- This value p is the ***p*-value of the test.**

hypothesis.)

Parametric and Non-parametric tests

There are two large classes of statistical tests: **parametric** and **non-parametric**.

- The **parametric** tests make assumptions about the probability distributions of the random variables that are subject to the test. It is often assumed that the underlying distribution is normal (Gaussian).
- The **non-parametric** tests do not make such assumptions. The non-parametric tests can be used if the random variables are not normally distributed.

PARAMETRIC TESTS

- *t*-tests for the means
- Two sample *F*-test

  for the equality of variances

t-tests for the means

- One-sample *t*-test for the population mean
- Paired-sample *t*-test for the difference of the population means
- Two-sample *t*-test for the difference of the population means

One-sample *t*-test for the population mean

Theorem:

If $X_1, X_2, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent, then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$

where

- $\bar{X} = \sum_{i=1}^{n} X_i/n$ is the sample mean of the random variables
- $\sigma = \sqrt{\sigma^2}$ is the standard deviation of the random variables
- $\mathcal{N}(0,1)$ is the standard normal distribution

One-sample *t*-test for the population mean

<u>**Theorem:**</u>

If $X_1, X_2, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent, then $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

where

- $s^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ is the sample variance of the random variables
- χ^2_{n-1} is Pearson's χ^2-distribution with $n-1$ degrees of freedom

One-sample t-test for the population mean

Theorem – Corollary:

If $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent, then $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$

where

- $\bar{X} = \sum_{i=1}^{n} X_i / n$ is the sample mean of the random variables
- $s = \sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1)}$ is the sample standard deviation of the random variables
- t_{n-1} is Student's t-distribution with $n-1$ degrees of freedom

Theorem – Corollary – Proof:

If $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent, then

$$\frac{\bar{X}-\mu}{s/\sqrt{n}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \times \frac{\sigma}{s} = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \times \frac{\sqrt{n-1}\,\sigma/s}{\sqrt{n-1}} = \frac{\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{\dfrac{(n-1)s^2}{\sigma^2}}{n-1}}} \sim t_{n-1}$$

by the definition of Student's *t*-distribution

$$\frac{Z}{\sqrt{\dfrac{X_{n-1}^2}{n-1}}} \sim t_{n-1} \quad \text{if} \quad Z \sim \mathcal{N}(0,1) \text{ and } X_{n-1}^2 \sim \chi_{n-1}^2$$

(having used the preceding two Theorems before).

One-sample *t*-test for the population mean

<ins>**Theorem – Corollary:**</ins>

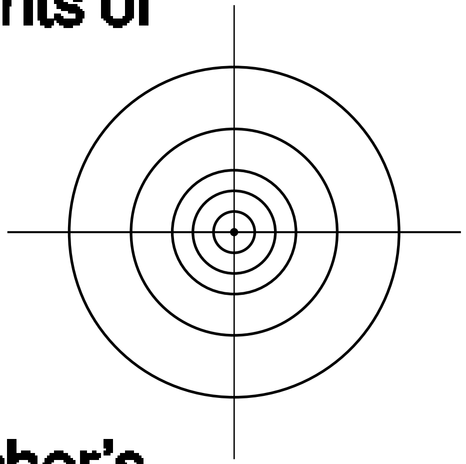
If $X_1, X_2, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent, then $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$

<ins>**Example or motivation:**</ins>

Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$ is a random variable following the normal probability distribution with some mean $\mu \in \mathbb{R}$ and with some variance $\sigma^2 \in \mathbb{R}_0^+$.

Knowing <ins>neither the variance</ins> σ^2 <ins>nor the true value of the population mean</ins> $\mu \in \mathbb{R}$, we conjecture / we assume / we … / that the population mean $\mu = \mu_0$, i.e. the (unknown) population mean μ is equal to some prescribed value $\mu_0 \in \mathbb{R}$.

One-sample *t*-test for the population mean



<u>Example:</u> An archer shoots an arrow against the plane.

The sample space $\Omega = \mathbb{R}^2 = \{[x,y] : x,y \in \mathbb{R}\}$ is the set of all the points of the plane. The random variable X is the x-coordinate of the hit, i.e.

$X(\omega) = X([x,y]) = x.$

We do not know the archer's variance σ^2 and we do not know the archer's intention, i.e. we do not know the point which the archer intends to hit,

i.e. we do not know the archer's mean μ.

We conjecture that the archer's intention is to hit the origin, i.e. $\mu = 0$.

One-sample *t*-test for the population mean

Let $x_1 = X(\omega_1),\ x_2 = X(\omega_2),\ \ldots,\ x_n = X(\omega_n)$ be the numerical results of n trials of a random experiment, where $X \sim \mathcal{N}(\mu, \sigma^2)$, such as the x-coordinates of the archer's n hits.

We do not know the variance σ^2 and we do not know the mean μ.

We state the **null hypothesis** (about the mean):

$$H_0: \quad \mu = \mu_0$$

where $\mu_0 \in \mathbb{R}$ is some number such that we conjecture that the true mean could equal the μ_0.

One-sample *t*-test for the population mean

The meaning of the null hypothesis (such as $H_0\colon \mu = \mu_0$ in our example) is that

- the observed distinct values are caused by the randomness only
  (according to the assumed distribution, such as $X \sim \mathcal{N}(\mu, \sigma^2)$ in our example)
- there are no other factors causing the distinct values
- everything is all right, no need to reconfigure anything
- all factors under the consideration are equivalent (have the same effect)

One-sample *t*-test for the population mean

Having stated the **null hypothesis**

$$H_0: \quad \mu = \mu_0$$

we also state the **alternative hypothesis**:

- two-sided: $H_1: \quad \mu \neq \mu_0$
- one-sided: $H_1: \quad \mu < \mu_0$
- one-sided: $H_1: \quad \mu > \mu_0$

One-sample *t*-test for the population mean

Which alternative hypothesis ($\mu \neq \mu_0$ or $\mu < \mu_0$ or $\mu > \mu_0$) do we choose?

→ That depends upon our knowledge of the situation.

In our example:

- If we suspect that the archer's intention is to hit a point different from the given point (such as the origin), we choose H_1: $\mu \neq \mu_0$.
- If we conjecture that the archer's intention is to hit a point to the left of the given point (such as the origin), we choose H_1: $\mu < \mu_0$.
- If we conjecture that the archer's intention is to hit a point to the right of the given point (such as the origin), we choose H_1: $\mu > \mu_0$.

One-sample *t*-test for the population mean

Under our assumptions ($x_1, \dots, x_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent and $\mu = \mu_0$), it follows by the Theorem that

$$\frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

Thus, having the n measurements $x_1, x_2, \dots, x_n$, we calculate the statistic

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

We know (or assume) that $T \sim t_{n-1}$.

One-sample *t*-test for the population mean

We have $T = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \sim t_{n-1}$.

Then, if $-\infty \leq a < b \leq +\infty$, the probability that $a < T < b$ is

$$P(a < T < b) = \int_a^b f(x)\,\mathrm{d}x$$

where

$$f(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi(n-1)}}\left(1+\frac{x^2}{n-1}\right)^{-\frac{k}{2}}$$

is the density of Student's *t*-distribution with $n-1$ degrees of freedom.

The gamma function

$$\Gamma(z) = \int_0^{+\infty} x^z e^{-x}\, dx \qquad \text{for} \quad z \in \mathbb{C} \ \text{ such that } \ \mathrm{Re}(z) > 0$$

It is easy to calculate:

$$\Gamma(1) = 1$$

$$\Gamma(z+1) = z\Gamma(z)$$

Therefore:

$$\Gamma(n+1) = n! \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

The gamma function – another definition (due to Euler)

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}} \qquad \text{for} \qquad z \in \mathbb{C} \setminus \{0,-1,-2,-3,\dots\}$$

One-sample *t*-test for the population mean

Consider the first case (H_1: $\mu \neq \mu_0$) first. We have:

$$H_0: \quad \mu = \mu_0$$

$$H_1: \quad \mu \neq \mu_0$$

Knowing that $T = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \sim t_{n-1}$, the probability

$$P(-c < T < +c)$$

is quite high,

so having $-c < T < +c$ accords with H_0, if $c > 0$ is large enough.

One-sample *t*-test for the population mean

On the other hand, if H_0 is true, then it is quite improbable that $T \notin (-c, +c)$.

Therefore, **if we observe that**

$$T \leq -c \qquad \text{or} \qquad +c \leq T$$

then we may conclude that H_0 **is probably not true,**

i.e. **we reject the null hypothesis** H_0.

Therefore, the statistical test proceeds as follows:

(see below)

One-sample *t*-test for the population mean

Statistical one-sample *t*-test with two-sided alternative hypothesis ($\mu \neq \mu_0$):

- choose **the level of significance**, a small number $\alpha > 0$, a very popular value is $\alpha = 5\ \%$, other popular values are $10\ \%$ or $1\ \%$ or $0.1\ \%$ etc.
- find **the critical value** $c > 0$ so that

$$\int_{-\infty}^{-c} f(x)\,\mathrm{d}x + \int_{+c}^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $n-1$ degrees of freedom
- if $T \in (-\infty, -c] \cup [+c, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-c, +c)$, then **do not reject** (or fail to reject) the null hypothesis

Type I and Type II error

There are exactly four possibilities when testing the null hypothesis H_0:

- the null hypothesis (H_0) is actually true & we do not reject it — OK
- the null hypothesis (H_0) is actually true & we reject it — type I error
- the null hypothesis (H_0) is actually not true & we do not reject it — type II error
- the null hypothesis (H_0) is actually not true & we reject it — OK

The purpose is that the probability of the type I error and that of the type II error is as little as possible.

Type I and Type II error

What is the probability of type I error

(the null hypothesis (H_0) is actually true & we reject it)?

The probability is equal to the significance level α, usually $\alpha = 5\ \%$.

<u>Recall:</u> The null hypothesis H_0 is rejected if and only if

$T \in (-\infty, -c] \cup [+c, +\infty)$, i.e. if and only if $|T| \geq c$.

The critical value c is such that – if H_0 holds true – then $P(|T| \geq c) = \alpha$,

i.e. the probability of the type I error (rejecting H_0 when it is true) is α.

Type I and Type II error

What is the probability of type II error

(the null hypothesis (H_0) is actually false & we fail to reject it)?

The probability of type II error is denoted by β.

The power of the test is the probability $1-\beta$

It is much more difficult to calculate the probability β of type II error.

It must be calculated for each test separately.

Type I and Type II error

To calculate the probability β of type II error, consider that the null hypothesis

H_0 is not true $(\mu \neq \mu_0)$ and we fail to reject it $\left(|T| = \left|\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\right| < c\right)$.

By the Theorem then, we have $\frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{x-\mu_0+(\mu_0-\mu)}{s/\sqrt{n}} \sim t_{n-1}$.

Then the probability of the type II error (H_0 not true & fail to reject it) is:

$$P\left(-c < \frac{\bar{x}-\mu_0}{s/\sqrt{n}} < +c\right) = P\left(-c < \frac{\bar{x}-\mu+(\mu_0-\mu)}{s/\sqrt{n}} < +c\right) =$$

$$= P\left(\frac{\mu-\mu_0}{s/\sqrt{n}} - c < \frac{\bar{x}-\mu}{s/\sqrt{n}} < \frac{\mu-\mu_0}{s/\sqrt{n}} + c\right) =$$

$$= \beta = \int_{(\mu-\mu_0)/(s/\sqrt{n})-c}^{(\mu-\mu_0)/(s/\sqrt{n})+c} f(x)\,\mathrm{d}x$$

Type I and Type II error

Notice that, if the true μ is close to the hypothesized μ_0 $(\mu \approx \mu_0)$, then $\frac{\mu-\mu_0}{s/\sqrt{n}} \approx 0$, hence

$$\beta = \int_{(\mu-\mu_0)/(s/\sqrt{n})-c}^{(\mu-\mu_0)/(s/\sqrt{n})+c} f(x)\,\mathrm{d}x \approx \int_{-c}^{+c} f(x)\,\mathrm{d}x = 1-\alpha = 95\ \%$$

if $\alpha = 5\ \%$, say.

It is recommended that β should be $\leq 20\ \%$.

Therefore, if we wish to have $\beta \approx 20\ \%$ or $\beta \leq 20\ \%$,

then we must not consider the true μ close to the hypothesized μ_0.

Type I and Type II error: Summary

There are exactly four possibilities when testing the null hypothesis H_0:

- the null hypothesis (H_0) is actually true & we do not reject it — OK
- the null hypothesis (H_0) is actually true & we reject it — type I error
- the null hypothesis (H_0) is actually not true & we do not reject it — type II error
- the null hypothesis (H_0) is actually not true & we reject it — OK

The probability of the type I error is the **significance level** α

The probability of the type II error is β

The **power of the test** is the probability $1-\beta$

One-sample *t*-test for the population mean

Consider now the second case (H_1: $\mu < \mu_0$). We have:

$$H_0: \quad \mu = \mu_0$$

$$H_1: \quad \mu < \mu_0$$

Knowing that $T = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \sim t_{n-1}$ and assuming that $c > 0$ is large enough,

what is the probability that $-c < T$?

If H_0 is true, then it is quite improbable that $T \notin (-c, +\infty)$.

Therefore, if we observe that $T \leq -c$, then we may conclude that

H_0 is <u>probably not true</u>, i.e. **we <u>reject the null hypothesis</u> H_0.**

One-sample *t*-test for the population mean

Statistical one-sample *t*-test with one-sided alternative hypothesis ($\mu < \mu_0$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$, other popular values are $\alpha = 10\,\%$ or $\alpha = 1\,\%$ or $\alpha = 0.1\,\%$ etc.
- find **the critical value** $c > 0$ so that

$$\int_{-\infty}^{-c} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $n-1$ degrees of freedom
- if $T \in (-\infty, -c]$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-c, +\infty)$, then **do not reject** (or fail to reject) the null hypothesis

One-sample *t*-test for the population mean

Consider finally the third case (H_1: $\mu > \mu_0$). We have:

$$H_0: \quad \mu = \mu_0$$

$$H_1: \quad \mu > \mu_0$$

Knowing that $T = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \sim t_{n-1}$ and assuming that $c > 0$ is large enough,

what is the probability that $T < +c$?

If H_0 is true, then it is quite improbable that $T \notin (-\infty, +c)$.

Therefore, if we observe that $+c \leq T$, then we may conclude that

H_0 **is <u>probably not true</u>**, i.e. **we <u>reject the null hypothesis</u>** H_0.

One-sample *t*-test for the population mean

Statistical one-sample *t*-test with one-sided alternative hypothesis ($\mu > \mu_0$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\ \%$, other popular values are $\alpha = 10\ \%$ or $\alpha = 1\ \%$ or $\alpha = 0.1\ \%$ etc.
- find the **critical value** $c > 0$ so that

$$\int_{+c}^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $n - 1$ degrees of freedom
- if $T \in [+c, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-\infty, +c)$, then **do not reject** (or fail to reject) the null hypothesis

Paired-sample *t*-test for the difference of the pop.means

<u>Example or motivation:</u>

Let us have a sample of n objects, e.g. n patients.

We do two measurements with each of the objects (patients)

- before some treatment
- after the treatment

The purpose it to learn whether the treatment has any effect.

(Hence the null hypothesis: "The treatment has no effect.")

Let $x_1, x_2, \dots, x_n$ be the values measured before the treatment, and

let $y_1, y_2, \dots, y_n$ be the values measured after the treatment.

Paired-sample *t*-test for the difference of the pop.means

That is, the measurement x_i and y_i is done with the i-th object (patient) before and after the treatment for $i = 1, 2, \ldots, n$.

We assume that $X \sim \mathcal{N}(\mu^{\text{before}}, \sigma_X^2)$, i.e. the random variable of the measurement before the treatment follows the normal distribution, and that $Y \sim \mathcal{N}(\mu^{\text{after}}, \sigma_Y^2)$, i.e. the random variable of the measurement after the treatment also follows the normal distribution, for some $\mu^{\text{before}}, \mu^{\text{after}} \in \mathbb{R}$ and for some $\sigma_X^2, \sigma_Y^2 \in \mathbb{R}_0^+$.
We do not know the true values of the population means μ^{before} and μ^{after}, and we do not know the true values of the variances σ_X^2 and σ_Y^2.

Paired-sample *t*-test for the difference of the pop.means

We formulate the **null hypothesis**:

the treatment has no effect, i.e. the population means are the same

$$H_0: \quad \mu^{\text{before}} = \mu^{\text{after}}$$

Recall that we do not know the true population means μ^{before} and μ^{after}. We only test the hypothesis by having done a sample of n pairs of measurements.

Formulate the alternative hypothesis:

- two-sided: $H_1: \quad \mu^{\text{before}} \neq \mu^{\text{afer}}$ (the treatment has some effect)
- one-sided: $H_1: \quad \mu^{\text{before}} < \mu^{\text{after}}$ (the treatment increases / …
- one-sided: $H_1: \quad \mu^{\text{before}} > \mu^{\text{after}}$ … / decreases the quantity)

Paired-sample *t*-test for the difference of the pop.means

<u>Recall the theorem:</u>

If $X_1, X_2, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent, then $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$

<u>Notice also:</u>

If $X_1, X_2, \dots, X_n \sim \mathcal{N}(\mu^{\text{before}}, \sigma_X^2)$ and $Y_1, Y_2, \dots, Y_n \sim \mathcal{N}(\mu^{\text{after}}, \sigma_Y^2)$ are independent,

then the differences

$$X_1 - Y_1,\ X_2 - Y_2,\ \dots,\ X_n - Y_n \sim \mathcal{N}\left(\mu^{\text{before}} - \mu^{\text{after}},\ \sigma_X^2 + \sigma_Y^2\right)$$

Now, the hypothesis $\mu^{\text{before}} = \mu^{\text{after}}$ is equivalent to that

the mean of the difference $X - Y$ is $\mu = \mu_0 = 0$.

Paired-sample *t*-test for the difference of the pop.means

We have thus

reduced

the paired-sample *t*-test for the difference of the population means

to

the one-sample *t*-test for the population mean,

which we already know.

Paired-sample *t*-test for the difference of the pop.means

Having the n pairs of the measurements $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$, calculate the statistic

$$T = \frac{\bar{x} - \bar{y}}{\sqrt{s^2}/\sqrt{n}} \qquad \text{or} \qquad T = \frac{\bar{x} - \bar{y} - \mu_0}{\sqrt{s^2}/\sqrt{n}}$$

where

- $\bar{x} = (\sum_{i=1}^{n} x_i)/n$ is the sample mean of the measurements <u>before</u> the treatment
- $\bar{y} = (\sum_{i=1}^{n} y_i)/n$ is the sample mean of the measurements <u>after</u> the treatment
- $\mu_0 = 0$ for no difference of the means $(\mu^{\text{before}} = \mu^{\text{after}})$
- $\mu_0 = \text{const.}$ for a general difference of the means $(\mu^{\text{before}} = \mu^{\text{after}} + \text{const.})$
- $s^2 = (\sum_{i=1}^{n} (x_i - y_i - \bar{x} + \bar{y})^2)/(n-1)$ is the sample variance of the differences

Paired-sample *t*-test for the difference of the pop.means

In the first case (H_1: $\mu^{\text{before}} \neq \mu^{\text{after}}$), we have:

$$H_0: \quad \mu^{\text{before}} = \mu^{\text{after}}$$

$$H_1: \quad \mu^{\text{before}} \neq \mu^{\text{after}}$$

Knowing that

$$T = \frac{\bar{x} - \bar{y}}{\sqrt{s^2}/\sqrt{n}} \sim t_{n-1}$$

we <u>fail to reject</u> H_0 iff

$$-c < T < +c$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that

$P(-c < T < +c) = 1 - \alpha$ where the probability α of type I error is small.

Paired-sample *t*-test for the difference of the pop.means

Statistical paired-sample *t*-test for the difference of the population means with two-sided alternative hypothesis ($\mu^{\text{before}} \neq \mu^{\text{after}}$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find the **critical value** $c > 0$ so that

$$\int_{-\infty}^{-c} f(x)\,\mathrm{d}x + \int_{+c}^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $n - 1$ degrees of freedom
- if $T \in (-\infty, -c] \cup [+c, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-c, +c)$, then **do not reject** (or fail to reject) the null hypothesis

Paired-sample *t*-test for the difference of the pop.means

In the second case $(H_1\colon\ \mu^{\text{before}} < \mu^{\text{after}})$, we have:

$$H_0\colon \quad \mu^{\text{before}} = \mu^{\text{after}}$$

$$H_1\colon \quad \mu^{\text{before}} < \mu^{\text{after}}$$

Knowing that

$$T = \frac{\bar{x} - \bar{y}}{\sqrt{s^2}/\sqrt{n}} \sim t_{n-1}$$

we fail to reject H_0 iff

$$-c < T$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that $P(-c < T) = 1 - \alpha$ where the probability α of type I error is small.

Paired-sample *t*-test for the difference of the pop.means

Statistical paired-sample *t*-test for the difference of the population means with one-sided alternative hypothesis ($\mu^{\text{before}} < \mu^{\text{after}}$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find **the critical value** $c > 0$ so that

$$\int_{-\infty}^{-c} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $n - 1$ degrees of freedom
- if $T \in (-\infty, -c]$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-c, +\infty)$, then **do not reject** (or fail to reject) the null hypothesis

Paired-sample *t*-test for the difference of the pop.means

In the third case (H_1: $\mu^{\text{before}} > \mu^{\text{after}}$), we have:

$$H_0: \quad \mu^{\text{before}} = \mu^{\text{after}}$$

$$H_1: \quad \mu^{\text{before}} > \mu^{\text{after}}$$

Knowing that

$$T = \frac{\bar{x} - \bar{y}}{\sqrt{s^2}/\sqrt{n}} \sim t_{n-1}$$

we <u>fail to reject</u> H_0 iff

$$T < +c$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that

$P(T < +c) = 1 - \alpha$ where the probability α of type I error is small.

Paired-sample *t*-test for the difference of the pop.means

Statistical paired-sample *t*-test for the difference of the population means with one-sided alternative hypothesis ($\mu^{\text{before}} > \mu^{\text{after}}$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find **the critical value** $c > 0$ so that

$$\int_{+c}^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $n - 1$ degrees of freedom
- if $T \in [+c, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-\infty, +c)$, then **do not reject** (or fail to reject) the null hypothesis

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

<u>Motivation:</u>

We have two unknown random variables X and Y. We ask (test the hypothesis) whether the population means of both random variables are the same.

We assume that both random variables are normal, i.e. $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, for some $\mu_X, \mu_Y \in \mathbb{R}$ and for some $\sigma_X^2, \sigma_Y^2 \in \mathbb{R}_0^+$.

Although we do not know the means μ_X, μ_Y nor the variances σ_X^2, σ_Y^2, we assume that

¡¡¡ ¡¡¡ ¡¡¡ $\sigma_X^2 = \sigma_Y^2$!!! !!! !!!

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Having the m samples $x_1, x_2, \dots, x_m$ of the random variable $X \sim \mathcal{N}(\mu_X, \sigma^2)$ and having the n samples $y_1, y_2, \dots, y_n$ of the random variable $Y \sim \mathcal{N}(\mu_Y, \sigma^2)$, we formulate the **null hypothesis**:

both samples come from the same population:
the values of the population means are the same

$$H_0: \quad \mu_X = \mu_Y$$

Recall that we do not know the true population means μ_X and μ_Y. **We only test the hypothesis** by the means of two samples of m and n measurements **with the same variance.**

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Having the m samples $x_1, x_2, \dots, x_m$ of the random variable $X \sim \mathcal{N}(\mu_X, \sigma^2)$,

the n samples $y_1, y_2, \dots, y_n$ of the random variable $Y \sim \mathcal{N}(\mu_Y, \sigma^2)$, and

$$H_0: \quad \mu_X = \mu_Y$$

formulate the alternative hypothesis:

- two-sided: $H_1: \quad \mu_X \neq \mu_Y$ (the means are different)
- one-sided: $H_1: \quad \mu_X < \mu_Y$ (the first mean < the second mean)
- one-sided: $H_1: \quad \mu_X > \mu_Y$ (the first mean > the second mean)

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X=\sigma_Y$

By the Theorem:

If $X_1, X_2, \dots, X_m \sim \mathcal{N}(\mu_X, \sigma^2)$ and $Y_1, Y_2, \dots, Y_n \sim \mathcal{N}(\mu_Y, \sigma^2)$ are independent, then

$$\frac{\bar{X}-\mu_X}{\sigma/\sqrt{m}} \sim \mathcal{N}(0,1) \quad \text{and} \quad \frac{\bar{Y}-\mu_Y}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$$

equivalently

$$\bar{X} \sim \mathcal{N}\left(\mu_X, \frac{\sigma^2}{m}\right) \quad \text{and} \quad \bar{Y} \sim \mathcal{N}\left(\mu_Y, \frac{\sigma^2}{n}\right)$$

Therefore:

$$\bar{X}-\bar{Y} \sim \mathcal{N}\left(\mu_X-\mu_Y, \frac{\sigma^2}{m}+\frac{\sigma^2}{n}\right)$$

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

We have shown:

If $X_1, X_2, \dots, X_m \sim \mathcal{N}(\mu_X, \sigma^2)$ and $Y_1, Y_2, \dots, Y_n \sim \mathcal{N}(\mu_Y, \sigma^2)$ are independent, then

$$\bar{X} - \bar{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma^2}{m} + \frac{\sigma^2}{n}\right)$$

equivalently

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim \mathcal{N}(0,1)$$

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

The above Theorem says:

If $X_1, X_2, \ldots, X_m \sim \mathcal{N}(\mu_X, \sigma^2)$ and $Y_1, Y_2, \ldots, Y_n \sim \mathcal{N}(\mu_Y, \sigma^2)$ are independent, then

$$\frac{(m-1)s_X^2}{\sigma^2} \sim \chi^2_{m-1} \quad \text{and} \quad \frac{(n-1)s_Y^2}{\sigma^2} \sim \chi^2_{n-1}$$

Therefore:

$$\frac{(m-1)s_X^2 + (n-1)s_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}$$

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Recall also that, if

$$Z = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim \mathcal{N}(0,1)$$

and

$$Y = \frac{(m-1)s_X^2 + (n-1)s_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}$$

then

$$T = \frac{Z}{\sqrt{\frac{Y}{m+n-2}}} \sim t_{m+n-2}$$

by the definition of Student's *t*-distribution.

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Therefore:

$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\frac{(m-1)s_X^2+(n-1)s_Y^2}{m+n-2}}\sqrt{\frac{1}{m}+\frac{1}{n}}} = \frac{\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}}}{\sqrt{\frac{\frac{(m-1)s_X^2+(n-1)s_Y^2}{\sigma^2}}{m+n-2}}} \sim t_{m+n-2}$$

where

$$s_X^2 = \frac{\sum_{i=1}^{m}(X_i-\bar{X})^2}{m-1} \quad \text{and} \quad s_Y^2 = \frac{\sum_{j=1}^{n}(Y_i-\bar{Y})^2}{n-1}$$

are the sample variances.

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Having the m measurements $x_1, x_2, \dots, x_m$ and n measurements $y_1, y_2, \dots, y_n$, recall that

- $\bar{x} = \sum_{i=1}^{m} x_i / m$ is the sample mean of the <u>first</u> sample
- $\bar{y} = \sum_{j=1}^{n} y_j / n$ is the sample mean of the <u>second</u> sample
- $s_x^2 = \sum_{i=1}^{m} (x_i - \bar{x})^2 / (m-1)$ is the sample variance of the <u>first</u> sample
- $s_y^2 = \sum_{j=1}^{n} (y_j - \bar{y})^2 / (n-1)$ is the sample variance of the <u>second</u> sample
- m is the size of the <u>first</u> sample
- n is the size of the <u>second</u> sample

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Having the m measurements $x_1, x_2, \dots, x_m$ and n measurements $y_1, y_2, \dots, y_n$, calculate the statistic

$$T = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{(m-1)s_x^2 + (n-1)s_y^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{1}{n}}}$$

for no difference of the means $(\mu_X = \mu_Y)$

We know (or assume) that $T \sim t_{m+n-2}$

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Or, having the m measurements $x_1, x_2, \dots, x_m$ and n measurements $y_1, y_2, \dots, y_n$,

calculate the statistic

$$T = \frac{\bar{x} - \bar{y} - \mu_0}{\sqrt{\frac{(m-1)s_x^2 + (n-1)s_y^2}{m+n-2}}\sqrt{\frac{1}{m} + \frac{1}{n}}}$$

for a general difference of the means $(\mu_X = \mu_Y + \mu_0)$

We know (or assume) that $T \sim t_{m+n-2}$

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X=\sigma_Y$

In the first case $(H_1\!: \ \mu_X \neq \mu_Y)$, we have:

$$H_0\!: \quad \mu_X = \mu_Y$$

$$H_1\!: \quad \mu_X \neq \mu_Y$$

Knowing that

$$T \sim t_{m+n-2}$$

we <u>fail to reject</u> H_0 iff

$$-c < T < +c$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that

$P(-c < T < +c) = 1 - \alpha$ where the probability α of type I error is small.

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Statistical two-sample *t*-test for the difference of the population means with two-sided alternative hypothesis ($\mu_X \neq \mu_Y$) and with the same variances:

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find the **critical value** $c > 0$ so that

$$\int_{-\infty}^{-c} f(x)\,\mathrm{d}x + \int_{+c}^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $m + n - 2$ degrees of freedom

- if $T \in (-\infty, -c] \cup [+c, +\infty)$, **the critical region**, then <u>**reject**</u> the null hypothesis
- if $T \in (-c, +c)$, then <u>**do not reject**</u> (or <u>fail to reject</u>) the null hypothesis

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

In the second case $(H_1\colon\ \mu_X < \mu_Y)$, we have:

$$H_0\colon \quad \mu_X = \mu_Y$$

$$H_1\colon \quad \mu_X < \mu_Y$$

Knowing that

$$T \sim t_{m+n-2}$$

we fail to reject H_0 iff

$$-c < T$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that

$P(-c < T) = 1 - \alpha$ where the probability α of type I error is small.

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

Statistical two-sample *t*-test for the difference of the population means with one-sided alternative hypothesis ($\mu_X < \mu_Y$) and with the same variances :

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find **the critical value** $c > 0$ so that

$$\int_{-\infty}^{-c} f(x)\,\mathrm{d}x = \alpha$$

where f is the density of the *t*-distribution with $m + n - 2$ degrees of freedom

- if $T \in (-\infty, -c]$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-c, +\infty)$, then **do not reject** (or fail to reject) the null hypothesis

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X = \sigma_Y$

In the third case $(H_1\colon\ \mu_X > \mu_Y)$, we have:

$$H_0\colon \quad \mu_X = \mu_Y$$

$$H_1\colon \quad \mu_X > \mu_Y$$

Knowing that

$$T \sim t_{m+n-2}$$

we <u>fail to reject</u> H_0 iff

$$T < +c$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that

$P(T < +c) = 1 - \alpha$ where the probability α of type I error is small.

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X=\sigma_Y$

Statistical two-sample *t*-test for the difference of the population means with one-sided alternative hypothesis ($\mu_X > \mu_Y$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find the **critical value** $c > 0$ so that

$$\int_{+c}^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *t*-distribution with $m + n - 2$ degrees of freedom
- if $T \in [+c, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $T \in (-\infty, +c)$, then **do not reject** (or fail to reject) the null hypothesis

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X \neq \sigma_Y$

Consider two normal random variables $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, for some $\mu_X, \mu_Y \in \mathbb{R}$ and for some $\sigma_X^2, \sigma_Y^2 \in \mathbb{R}_0^+$.

We ask (test the hypothesis) whether the population means of both random variables are the same.

Once $\sigma_X^2 = \sigma_Y^2$ is not assumed, the things get complicated.

We have an approximate result only.

Theorem (Satterthwaite's approximation):

If the random variables $X_1, X_2, \dots, X_m \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y_1, Y_2, \dots, Y_n \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent, then the statistic

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{s_X^2}{m} + \frac{s_Y^2}{n}}} \sim t_\nu \qquad \textit{approximately}$$

where

$$\nu = \frac{\left(\frac{s_X^2}{m} + \frac{s_Y^2}{n}\right)^2}{\frac{s_X^4}{m^2(m-1)} + \frac{s_Y^4}{n^2(n-1)}}$$

Two-sample *t*-test for the diff. of the pop. means // $\sigma_X \neq \sigma_Y$

Exercise:

Use the last Theorem (Satterthwaite's approximation) to formulate a statistical two-sample *t*-test for the difference of the population means with two-sided / one-sided alternative hypothesis (not assuming the same variance).

Two sample *F*-test for the equality of variances

Motivation

We have two unknown random variables X and Y. We ask (test the hypothesis) whether the population variances of both random variables are the same.

We assume that both random variables are normal, i.e. $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, for some $\mu_X, \mu_Y \in \mathbb{R}$ and for some $\sigma_X^2, \sigma_Y^2 \in \mathbb{R}_0^+$.

We ask (test the hypothesis) whether

$$¿ \quad \sigma_X^2 = \sigma_Y^2 \quad ?$$

Two sample *F*-test for the equality of variances

Having the m samples $x_1, x_2, \dots, x_m$ of the random variable $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and having the n samples $y_1, y_2, \dots, y_n$ of the random variable $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$,

we formulate the **null hypothesis**:

both samples come from populations with the same variances:

$$H_0: \quad \sigma_X^2 = \sigma_Y^2$$

Recall that we do not know the true population variances σ_X^2 and σ_Y^2.

We only test the hypothesis by the means of the two samples of m and n independent measurements.

Two sample *F*-test for the equality of variances

The **meaning of the null hypothesis** (such as $H_0\colon \sigma_X^2 = \sigma_Y^2$ in our example) is that

- the observed distinct values are caused by the randomness only (according to the assumed distribution, such as $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 = \sigma_X^2 = \sigma_Y^2$, in our example)
- there are no other factors causing the distinct values
- everything is all right, no need to reconfigure anything
- all factors under the consideration are equivalent (have the same effect)

Two sample *F*-test for the equality of variances

Having stated the **null hypothesis**

$$H_0: \quad \sigma_X^2 = \sigma_Y^2$$

we also state the **alternative hypothesis**:

- two-sided: $H_1: \quad \sigma_X^2 \neq \sigma_Y^2$
- one-sided: $H_1: \quad \sigma_X^2 < \sigma_Y^2$
- one-sided: $H_1: \quad \sigma_X^2 > \sigma_Y^2$

Theorem

If the random variables $X_1, X_2, \dots, X_m \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y_1, Y_2, \dots, Y_n \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent and

$$\sigma_X^2 = \sigma_Y^2$$

then

$$\frac{s_X^2}{s_Y^2} \sim F_{m-1,n-1}$$

where

$F_{m-1,n-1}$ is Fisher's *F*-distribution with $m-1$ and $n-1$ degrees of freedom

$s_X^2 = \sum_{i=1}^{m}(X_i - \bar{X})^2/(m-1)$ is the sample variance of the first sample

$s_Y^2 = \sum_{j=1}^{n}\left(Y_j - \bar{Y}\right)^2/(n-1)$ is the sample variance of the second sample

Two-sample *F*-test for the equality of variances

Having the m measurements $x_1, x_2, \dots, x_m$ and n measurements $y_1, y_2, \dots, y_n$, calculate the statistic

$$F = \frac{s_x^2}{s_y^2}$$

where

- $s_x^2 = \sum_{i=1}^{m}(x_i - \bar{x})^2/(m-1)$ is the sample variance of the first sample
- $s_y^2 = \sum_{j=1}^{n}(y_j - \bar{y})^2/(n-1)$ is the sample variance of the second sample
- $\bar{x} = \sum_{i=1}^{m} x_i/m$ is the sample mean of the first sample
- $\bar{y} = \sum_{j=1}^{n} y_j/n$ is the sample mean of the second sample
- m and n is the size of the first and second, respectively, sample

Two-sample *F*-test for the equality of variances

In the first case $(H_1\colon\ \sigma_X^2 \neq \sigma_Y^2)$, we have:

$$H_0\colon \quad \sigma_X^2 = \sigma_Y^2$$

$$H_1\colon \quad \sigma_X^2 \neq \sigma_Y^2$$

Knowing that

$$F = \frac{s_x^2}{s_y^2} \sim F_{m-1,n-1}$$

we fail to reject H_0 iff

$$c < F < d$$

where the critical value $d > c > 0$, under the assumption that H_0 is true, are such that $P(c < F < d) = 1 - \alpha$ where the probability α of type I error is small.

Two-sample *F*-test for the equality of variances

Statistical two-sample *F*-test for the equality of the population variances with two-sided alternative hypothesis ($\sigma_X^2 \neq \sigma_Y^2$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find the **critical values** $0 < c < d$ so that

$$\int_0^c f(x)\,\mathrm{d}x = \frac{\alpha}{2} \qquad \text{and} \qquad \int_d^{+\infty} f(x)\,\mathrm{d}x = \frac{\alpha}{2}$$

  where f is the density of the *F*-distribution with $m-1$ and $n-1$ d.f.

- if $F \in [0, c] \cup [d, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $F \in (c, d)$, then **do not reject** (or fail to reject) the null hypothesis

Two-sample *F*-test for the equality of variances

In the second case $(H_1\colon\ \sigma_X^2 < \sigma_Y^2)$, we have:

$$H_0\colon \quad \sigma_X^2 = \sigma_Y^2$$

$$H_1\colon \quad \sigma_X^2 < \sigma_Y^2$$

Knowing that

$$F = \frac{s_x^2}{s_y^2} \sim F_{m-1,n-1}$$

we <u>fail to reject</u> H_0 iff

$$c < F$$

where the critical value $c > 0$, under the assumption that H_0 is true, is such that $P(c < F < +\infty) = 1 - \alpha$ where the probability α of type I error is small.

Two-sample *F*-test for the equality of variances

Statistical two-sample *F*-test for the equality of the population variances with one-sided alternative hypothesis ($\sigma_X^2 < \sigma_Y^2$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find **the critical value** $c > 0$ so that

$$\int_0^c f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the *F*-distribution with $m-1$ and $n-1$ d.f.

- if $F \in [0, c]$, **the critical region**, then **reject** the null hypothesis
- if $F \in (c, +\infty)$, then **do not reject** (or fail to reject) the null hypothesis

Two-sample *F*-test for the equality of variances

In the third case $(H_1\colon\ \sigma_X^2 > \sigma_Y^2)$, we have:

$$H_0\colon \quad \sigma_X^2 = \sigma_Y^2$$

$$H_1\colon \quad \sigma_X^2 > \sigma_Y^2$$

Knowing that

$$F = \frac{s_x^2}{s_y^2} \sim F_{m-1,n-1}$$

we <u>fail to reject</u> H_0 iff

$$F < d$$

where the critical value $d > 0$, under the assumption that H_0 is true, is such that $P(0 \leq F < d) = 1 - \alpha$ where the probability α of type I error is small.

Two-sample *F*-test for the equality of variances

Statistical two-sample *F*-test for the difference of the population variances with one-sided alternative hypothesis ($\sigma_X^2 > \sigma_Y^2$):

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\ \%$
- find the **critical value** $d > 0$ so that

$$\int_d^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

where f is the density of the *F*-distribution with $m-1$ and $n-1$ d.f.

- if $F \in [d, +\infty)$, **the critical region**, then **reject** the null hypothesis
- if $Z \in [0, d)$, then **do not reject** (or fail to reject) the null hypothesis

NON-PARAMETRIC TESTS

- Sign test for the median
- Pearson's χ^2-test for the goodness of fit
- χ^2-test of independence of qualitative data items

Sign test for the median

- Sign test for the median
- Paired sign test for

  the difference of the medians

Sign test for the median

Motivation:

Let X be a random variable (of any distribution), but assume that its cumulative distribution function F is continuous.

Recall that the median $\tilde{x}$ of the random variable X is the value such that

$$P(X < \tilde{x}) = \frac{1}{2} = P(\tilde{x} < X)$$

We conjecture / we assume / we speculate / we … / that the mean $\tilde{x}$ of the random variable X is equal to some given value $\tilde{x}_0 \in \mathbb{R}$.

We thus formulate the null hypothesis: $H_0\colon \quad \tilde{x} = \tilde{x}_0$

Sign test for the median

The sign test proceeds as follows:

- Let us have n samples $x_1, x_2, \dots, x_n$ of the random variable X, whose cumulative distribution function F is continuous.
- Considering the null hypothesis $(H_0: \tilde{x} = \tilde{x}_0)$ about the median, calculate the n differences

$$x_1 - \tilde{x}_0, \quad x_2 - \tilde{x}_0, \quad \dots, \quad x_n - \tilde{x}_0$$

- Drop any zero differences (i.e., if $x_i - \tilde{x}_0 = 0$, then drop x_i from the sample).
- We have a sample of m non-zero differences

$$x_{j_1} - \tilde{x}_0, \quad x_{j_2} - \tilde{x}_0, \quad \dots, \quad x_{j_m} - \tilde{x}_0$$

Sign test for the median

- Let

$$Z = \left|\left\{ i : x_{j_i} - \tilde{x}_0 < 0 \right\}\right|$$

be the number of the negative differences.

__Theorem:__

Under the null hypothesis $(H_0: \tilde{x} = \tilde{x}_0)$ that the median $\tilde{x}$ of the random variable X is $\tilde{x}_0$

$$Z \sim \mathrm{Bi}\left(m, \tfrac{1}{2}\right)$$

i.e. the random variable Z follows the binomial probability distribution.

Sign test for the median

Remark: We actually test the hypothesis that the probability

$$P(X < \tilde{x}_0) = P(X \leq \tilde{x}_0) = \frac{1}{2}$$

(We have $P(X < \tilde{x}_0) = P(X \leq \tilde{x}_0)$ because we assume that the cumulative distribution function F is continuous at $\tilde{x}_0$.)

Therefore, we could test in the same manner the null hypothesis that

$\tilde{x}_0$ is the first quartile $(P(X < \tilde{x}_0) = P(X \leq \tilde{x}_0) = \frac{1}{4}$, whence $Z \sim \mathrm{Bi}\left(m, \frac{1}{4}\right))$, or that

$\tilde{x}_0$ is the third decile $(P(X < \tilde{x}_0) = P(X \leq \tilde{x}_0) = \frac{3}{10}$, whence $Z \sim \mathrm{Bi}\left(m, \frac{3}{10}\right))$, etc.

Sign test for the median

Having stated the **null hypothesis** about the median

$$H_0: \quad \tilde{x} = \tilde{x}_0 \qquad \text{or} \qquad H_0: \quad P(X < \tilde{x}_0) = p_0 = \frac{1}{2}$$

we also state the **alternative hypothesis:**

- two-sided: $H_1: \quad \tilde{x} \neq \tilde{x}_0$ or $H_1: \quad P(X < \tilde{x}_0) \neq p_0$
- one-sided: $H_1: \quad \tilde{x} > \tilde{x}_0$ or $H_1: \quad P(X < \tilde{x}_0) < p_0$
- one-sided: $H_1: \quad \tilde{x} < \tilde{x}_0$ or $H_1: \quad P(X < \tilde{x}_0) > p_0$

The test then proceeds as the binomial test (or z-test approximately) for the

Sign (binomial) test for the median

Consider the first case $(H_1\colon \tilde{x} \neq \tilde{x}_0)$ first. We have:

$H_0\colon\ P(X < \tilde{x}_0) = p_0 = 1/2$
$H_1\colon\ P(X < \tilde{x}_0) \neq p_0$

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find **the critical values** $K, L \in \{0, 1, \dots, m\}$ so that

  K is the largest number and L is the least number such that

$$\sum_{k=0}^{K} \binom{m}{k} p_0^k q_0^{m-k} = \sum_{k=0}^{K} \binom{m}{k} \frac{1}{2^m} \leq \frac{\alpha}{2} \qquad \text{and} \qquad \sum_{k=L}^{m} \binom{m}{k} p_0^k q_0^{n-k} = \sum_{k=L}^{m} \binom{m}{k} \frac{1}{2^m} \leq \frac{\alpha}{2}$$

- if $Z \in \{0, \dots, K\} \cup \{L, \dots, n\}$, **the critical region**, then **reject** the null hypothesis
- if $Z \in \{K+1, \dots, L-1\}$, then **do not reject** (or fail to reject) the null hypothesis

Sign (binomial) test for the median

Consider now the second case $(H_1\colon \tilde{x} > \tilde{x}_0)$. We have: $H_0\colon\; P(X < \tilde{x}_0) = p_0 = 1/2$

$H_1\colon\; P(X < \tilde{x}_0) < p_0$

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find the **critical value** $K \in \{0,1,\dots,m\}$ so that K is the largest number such that

$$\sum_{k=0}^{K} \binom{m}{k} p_0^k q_0^{m-k} = \sum_{k=0}^{K} \binom{m}{k} \frac{1}{2^m} \leq \alpha$$

- if $Z \in \{0,\dots,K\}$, **the critical region**, then **reject** the null hypothesis
- if $Z \in \{K+1,\dots,m\}$, then **do not reject** (or **fail to reject**) the null hypothesis

Sign (binomial) test for the median

Consider finally the third case $(H_1\colon \tilde{x} < \tilde{x}_0)$. We have: $H_0\colon\ P(X < \tilde{x}_0) = p_0 = 1/2$
$H_1\colon\ P(X < \tilde{x}_0) > p_0$

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find the **critical value** $L \in \{0, 1, \dots, m\}$ so that L is the least number such that

$$\sum_{k=L}^{m} \binom{m}{k} p_0^k q_0^{m-k} = \sum_{k=L}^{m} \binom{m}{k} \frac{1}{2^m} \leq \alpha$$

- if $Z \in \{L, \dots, m\}$, **the critical region**, then **<u>reject</u>** the null hypothesis
- if $Z \in \{0, \dots, L-1\}$, then **<u>do not reject</u>** (or <u>fail to reject</u>) the null hypothesis

Sign (z-) test for the median

It is inconvenient to calculate the sums $\sum_{k=0}^{K}\binom{m}{k}\frac{1}{2^m}$ and $\sum_{k=L}^{m}\binom{m}{k}\frac{1}{2^m}$ if m is large. It is more convenient then to approximate the sums by using the de Moivre-Laplace Central Limit Theorem (for $p = q = 1/2$):

It holds, whenever $-\infty \leq a < b \leq +\infty$, that

$$\frac{\sum_{k=A_m}^{B_m}\binom{m}{k}\frac{2}{2^m} - n}{\sqrt{m}} \to \underbrace{\int_a^b \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t}_{\Phi(b)-\Phi(a)} \qquad \text{as} \quad m \to \infty$$

where $A_m = \lceil (m + a\sqrt{m})/2 \rceil \geq 0$ and $B_m = \lfloor (m + b\sqrt{m})/2 \rfloor \leq m$ if $m \geq \max(a^2, b^2)$.

Moreover, the convergence is uniform with respect to a and b.

Sign (z-) test for the median

De Moivre-Laplace Central Limit Theorem (reformulated):

If $X \sim \mathrm{Bi}(m, 1/2)$, whenever $-\infty \leq a < b \leq +\infty$, it then holds

$$P\left(a < \frac{2X - m}{\sqrt{m}} < b\right) \to \underbrace{\int_a^b \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t}_{\Phi(b) - \Phi(a)} \qquad \text{as} \quad m \to \infty$$

and the convergence is uniform with respect to a and b.

Sign (z-) test for the median

Consider the first case $(H_1\colon \tilde{x} \neq \tilde{x}_0)$ first. We have:

$$H_0\colon\ P(X < \tilde{x}_0) = p_0 = 1/2$$
$$H_1\colon\ P(X < \tilde{x}_0) \neq p_0$$

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find $c > 0$ so that

$$\int_{-\infty}^{-c} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt = \frac{\alpha}{2} \qquad \text{and} \qquad \int_{+c}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt = \frac{\alpha}{2}$$

- if $Z \leq (m - c\sqrt{m})/2$ or $(m + c\sqrt{m})/2 \leq Z$, **the critical region**, then **reject** the null hypothesis
- if $(m - c\sqrt{m})/2 < Z < (m + c\sqrt{m})/2$, then **do not reject** (or fail to reject) the null hypothesis

Sign (z-) test for the median

Consider now the second case ($H_1: \tilde{x} > \tilde{x}_0$). We have: $H_0: \; P(X < \tilde{x}_0) = p_0 = 1/2$

$H_1: \; P(X < \tilde{x}_0) < p_0$

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find $c > 0$ so that

$$\int_{-\infty}^{-c} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}} \, \mathrm{d}t = \alpha$$

- if $Z \leq (m - c\sqrt{m})/2$, **the critical region**, then **reject** the null hypothesis
- if $(m - c\sqrt{m})/2 < Z$, then **do not reject** (or fail to reject) the null hypothesis

Sign (z-) test for the median

Consider finally the third case $(H_1\colon \tilde{x} < \tilde{x}_0)$. We have: $H_0\colon\ P(X < \tilde{x}_0) = p_0 = 1/2$
$H_1\colon\ P(X < \tilde{x}_0) > p_0$

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find $c > 0$ so that

$$\int_{+c}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t = \alpha$$

- if $(m + c\sqrt{m})/2 \le Z$, **the critical region**, then **reject** the null hypothesis
- if $Z < (m + c\sqrt{m})/2$, then **do not reject** (or fail to reject) the null hypothesis

Sign test for the median

Remarks:

- By using another probability (such as $p_0 = 0.25$, $p_0 = 0.3$, etc.) we can test the null hypothesis that $\tilde{x}_0$ is, e.g., the first quartile, the third decile, etc.

- If we know that the distribution of X is symmetric $(F(x) = 1 - F(-x))$, then the mean $\mu = \mathrm{E}[X]$ and the median $\tilde{x}$ of the random variable X coincide $(\tilde{x} = \mu)$. Then the sign test for the median can also be used as another test for the mean $(H_0\colon\ \mu = \tilde{x}_0)$.

Sign test for the median

Remarks:

- More generally, if we know that the mean $\mu = \mathrm{E}[X]$ is the p_0-quantile $(0 < p_0 < 1)$ of the distribution of the random variable X with a continuous cumulative distribution function, then the sign test can also be used as another test for the mean $(H_0\colon \mu = \tilde{x}_0$ with $Z = \left|\left\{ i : x_{j_i} < \tilde{x}_0 \right\}\right| \sim \mathrm{Bi}(m, p_0))$.

- Exercise: Apply the procedure of the sign test to determine the confidence interval for the median, i.e. the interval of values $\tilde{x}_0$ such that the null hypothesis is not rejected for them.

Paired sign test for the difference of the medians

<u>Motivation:</u>

Let us have a sample of n objects, e.g. n patients.

We do two measurements with each of the objects (patients)

- before some treatment
- after the treatment

The purpose it to learn whether the treatment has any effect.

(Hence the null hypothesis: "The treatment has no effect.")

Let $x_1, x_2, \dots, x_n$ be the values measured before the treatment, and

let $y_1, y_2, \dots, y_n$ be the values measured after the treatment.

Paired sign test for the difference of the medians

That is, the measurement x_i and y_i is done with the i-th object (patient) before and after the treatment for $i = 1, 2, \dots, n$.

FIRST, assume that only two outcomes are possible:

- $x_i < y_i$ (improvement)
- $x_i > y_i$ (worsening)

Objects with $x_i = y_i$ are dropped from the sample.

We then can test the null hypothesis that the treatment has no effect, i.e.

$$Z = |\{i : x_i < y_i\}| \sim \mathrm{Bi}\left(m, \tfrac{1}{2}\right)$$

etc. (Finish the details of the test analogously as above as an exercise.)

Paired sign test for the difference of the medians

That is, the measurement x_i and y_i is done with the i-th object (patient) before and after the treatment for $i = 1, 2, \ldots, n$.

SECOND, assume that $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are the numerical outcomes of the random variable X and Y, respectively, with a continuous cumulative distribution function F_X and F_Y, respectively.

<u>Theorem:</u> The median $\tilde{x}_0$ of the difference $X - Y$ of the random variables is

$$\tilde{x}_0 = \tilde{x} - \tilde{y}$$

Paired sign test for the difference of the medians

Thus, we can test the null hypothesis that the median $\tilde{x}$ of the random variable X (before the treatment) is the same as the median $\tilde{y}$ of the random variable Y (after the treatment), i.e. their difference is $\tilde{x}_0 = \tilde{x} - \tilde{y} = 0$.

(More generally, we can test that the difference $\tilde{x} - \tilde{y}$ is equal to some prescribed value $\tilde{x}_0 \in \mathbb{R}$.)

(Complete the details of the test analogously as above as an exercise.)

χ^2-test for goodness of fit

- Pearson's χ^2-test for the goodness of fit

Pearson's χ^2-test for the goodness of fit

Let X be a random variable (discrete or continuous) and

let F be the cumulative distribution function of the random variable X.

We do not know the cumulative distribution function F.

We have the numerical results $x_1 = X(\omega_1),\ x_2 = X(\omega_2),\ \ldots,\ x_N = X(\omega_N)$

of N trials of the corresponding random experiment.

Let F_0 be some cumulative distribution function. We conjecture / we assume /

we speculate / we … / that $F = F_0$, i.e. the random variable X follows the

probability distribution with the cumulative distribution function $F = F_0$.

Pearson's χ^2-test for the goodness of fit

More generally, let $\mathcal{F}_0$ be a class of cumulative distribution functions (c.d.f.'s) of a certain type, such as

- the collection of all c.d.f.'s of $\mathcal{U}(a, b)$ for various $a, b \in \mathbb{R}$, $a < b$
- the collection of all c.d.f.'s of $\mathcal{N}(\mu, \sigma^2)$ for various $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_0^+$
- the collection of all c.d.f.'s of $\mathrm{Exp}(\lambda)$ for various $\lambda \in \mathbb{R}^+$
- etc.

Having the numerical results $x_1 = X(\omega_1)$, $x_2 = X(\omega_2)$, …, $x_N = X(\omega_N)$ of N trials of a random experiment, we conjecture / we assume / we speculate / we … / that $F \in \mathcal{F}_0$, i.e. the random variable X follows the probability distribution

Pearson's χ^2-test for the goodness of fit

Having the numerical results $x_1 = X(\omega_1),\ x_2 = X(\omega_2),\ \ldots,\ x_N = X(\omega_N)$ of the N trials of the random experiment and having the class $\mathcal{F}_0$ of the cumulative distribution functions – <u>first of all</u> – find the cumulative distribution function $F_0 \in \mathcal{F}_0$ that best fits the experimental data:

- if $\mathcal{F}_0 = \{F_0\}$, then the c.d.f. F_0 is given; the number of parameters is $\nu = 0$
- if $\mathcal{F}_0$ is the collection of all c.d.f.'s of $\mathcal{N}(\mu, \sigma^2)$, then put

$$\mu = \bar{x} \qquad \text{and} \qquad \sigma^2 = s^2$$

(the sample mean and the sample variance); the number of parameters is $\nu = 2$

Pearson's χ^2-test for the goodness of fit

- if $\mathcal{F}_0$ is the collection of all c.d.f.'s of $\mathrm{Exp}(\lambda)$, then put

$$\text{either} \quad \lambda = \frac{1}{\bar{x}} \qquad \text{or} \quad \lambda = \sqrt{\frac{1}{s^2}}$$

  the number of parameters is $\nu = 1$

  (recall: if $X \sim \mathrm{Exp}(\lambda)$, then $\mathrm{E}[X] = 1/\lambda$ and $\mathrm{Var}(X) = 1/\lambda^2$)

- if $\mathcal{F}_0$ is the collection of all c.d.f.'s of $\mathcal{U}(a, b)$, then consider the German Tank Problem (see previous lectures); the number of parameters is $\nu = 2$
- etc.

Pearson's χ^2-test for the goodness of fit

Having the sample data $x_1, x_2, \dots, x_N$ of the random variable X and the cumulative distribution function $F_0 \in \mathcal{F}_0$ that best fits the sample.

Now – as the second step – choose n intervals

$$(t_0, t_1], \quad (t_1, t_2], \quad (t_2, t_3], \quad \dots, \quad (t_{n-2}, t_{n-1}], \quad (t_{n-1}, t_n]$$

with

$$t_0 < t_1 < t_2 < t_3 < \dots < t_{n-2} < t_{n-1} < t_n$$

as well as

$$t_0 < \min\{x_1, \dots, x_N\} \quad \text{and} \quad \max\{x_1, \dots, x_N\} \le t_n$$

so that

— there are at least 5 outcomes in each of the intervals

Pearson's χ^2-test for the goodness of fit

Formulate <u>the null hypothesis</u>: The random variable X follows the probability distribution with the cumulative distribution function $F = F_0$:

$$H_0: \quad F = F_0$$

Next – <u>as the third step</u> – assume the null hypothesis H_0 and calculate the theoretical probability that $t_{i-1} < X \leq t_i$, i.e.

$$p_i = P(t_{i-1} < X \leq t_i) =$$

$$= F_0(t_i) - F_0(t_{i-1}) \qquad \text{for} \quad i = 1, 2, \dots, n$$

Pearson's χ^2-test for the goodness of fit

Since p_i is the expected probability (under the null hypothesis H_0) that $X \in (t_{i-1}, t_i]$ and we have a sample $x_1, x_2, \dots, x_N$ of N observations, we should find about

$$E_i = N \times p_i$$

observations in the interval $(t_{i-1}, t_i]$ for $i = 1, 2, \dots, n$.

Let

$$O_i = \left|\{j : x_j \in (t_{i-1}, t_i]\}\right|$$

be the true number of the observations found in the interval $(t_{i-1}, t_i]$ for $i = 1, 2, \dots, n$.

Pearson's χ^2-test for the goodness of fit

Theorem: If the null hypothesis $H_0: F = F_0$ is true, then the statistic

$$X^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_{n-\nu-1} \quad \textit{approximately} \quad \text{as} \quad N \to \infty$$

where

- n is the number of the intervals $(t_{i-1}, t_i]$
- ν is the number of the parameters that have been determined when finding the cumulative distribution function F_0 $(\nu = 0, 1, 2, \ldots)$
- O_i is the number of the results found (observed) in the i-th interval $(t_{i-1}, t_i]$
- E_i is the number of the results expected (if H_0 is true) in the interval $(t_{i-1}, t_i]$

Pearson's χ^2-test for the goodness of fit

Now, finish Pearson's χ^2-test for the goodness of fit $(H_0: F = F_0)$ as follows:

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\ \%$, other popular values are $\alpha = 10\ \%$ or $\alpha = 1\ \%$ or $\alpha = 0.1\ \%$ etc.
- find the **critical value** $c > 0$ so that

$$\int_c^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the χ^2-distribution with $n - \nu - 1$ degrees of freedom
- if $X^2 \geq c$, **the critical region**, then **reject** the null hypothesis
- if $X^2 < c$, then **do not reject** (or fail to reject) the null hypothesis

Example: Tests for population proportion

Tossing a coin repeatedly, we ask whether the coin is fair.

More generally, we consider a Bernoulli trial, with the probability of the success being $p \in (0,1)$, and with the probability of the failure being $q = 1 - p$.

We do not know the true probability p.

We conjecture / We assume / We … / that the probability $p = p_0$, i.e. the (unknown) probability p is equal to some prescribed value $p_0 \in (0,1)$, e.g., in the case of the coin, conjecture that $p_0 = 50\ \%$ (meaning the coin is fair).

Example: Tests for population proportion

We now know three statistical tests to test the null hypothesis that $p = p_0$:

- the binomial test for the population proportion
- the *z*-test for the population proportion
- Pearson's χ^2-test for the goodness of fit

The binomial test is exact and the *z*-test is an approximation of it.

Both binomial test and *z*-test allow one-sided or two-sided alternative hypothesis.

Pearson's χ^2-test for the goodness of fit allows two-sided alternative hypothesis $(H_1\colon F \neq F_0)$ only.

Example: Tests for population proportion

Pearson's χ^2-test for the goodness of fit proceeds as follows:

- there are two intervals (1 = "success" and 0 = "failure")
- having N observations of the random variable X, we expect (under the null hypothesis that $p = p_0$) that $E_1 = N \times p_0$ and $E_0 = N \times (1 - p_0)$
- let O_1 and O_0 be the observed number of successes and failures, respectively
- the statistic

$$X^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_0 - E_0)^2}{E_0} \sim \chi_1^2 \quad \textit{approximately} \quad \text{as} \quad N \to \infty$$

(we have $n = 2$ and $\nu = 0$, therefore $n - \nu - 1 = 1$)

Pearson's χ^2-test for the goodness of fit

Remark: In Pearson's χ^2-test for the goodness of fit, we have

$$X^2 \sim \chi^2_{n-\nu-1}$$

where

- n is the number of the intervals $(t_{i-1}, t_i]$
- ν is the number of the parameters that have been determined when finding the cumulative distribution function F_0 $(\nu = 0, 1, 2, \ldots)$

Notice that one degree of freedom ("−1") must always be subtracted because the observed counts $O_1, O_2, \ldots, O_n$ are bound by the equation

$$O_1 + O_2 + \cdots + O_n = N$$

therefore only $n-1$ of the counts (such as $O_1, O_2, \ldots, O_{n-1}$, say) are free,

χ^2-test of independence of qualitative data items

- χ^2-test of independence of qualitative data items

χ^2-test of independence of qualitative data items

Consider a dataset where each data unit has two qualitative data items (i.e. two qualitative variables).

Let the qualitative variables under the consideration be denoted by A and B.

Let the variable A can attain up to r ("rows") distinct categories

$$A_1, \quad A_2, \quad \ldots, \quad A_r$$

Let the variable B can attain up to s ("columns") distinct categories

$$B_1, \quad B_2, \quad \ldots, \quad B_s$$

The counts of the occurrences of all the $r \times s$ combinations of the categories are easily summarized by a contingency table.

Contingency table

the observed counts of the combinations of the categories A_i&B_j for i=1,…,r & j=1,…,s

A \ B	B_1	B_2	…	B_s	TOTAL
A_1	n_{11}	n_{12}	…	n_{1s}	$n_{1.}$
A_2	n_{21}	n_{22}	…	n_{2s}	$n_{2.}$
…	⋮	⋮	…	⋮	⋮
A_r	n_{r1}	n_{r2}	…	n_{rs}	$n_{r.}$
TOTAL	$n_{.1}$	$n_{.2}$	…	$n_{.s}$	n

marginal totals

marginal totals

the grand total

2   2 contingency table

The 2   2 contingency table is popular.

It is a contingency table with *r*=2 rows and *s*=2 columns.

the observed counts of the combinations of the categories A_i&B_j for *i*=1,2 & *j*=1,2

$A \setminus B$	B_1	B_2	TOTAL
A_1	n_{11}	n_{12}	$n_{1.}$
A_2	n_{21}	n_{22}	$n_{2.}$
TOTAL	$n_{.1}$	$n_{.2}$	n

marginal totals

marginal totals

the grand total

χ^2-test of independence of qualitative data items

Having all the observed counts of the combinations of the categories A_i & B_j summarized in the contingency table for $i=1,\ldots,r$ and for $j=1,\ldots,s$, we ask whether the category of the data item (variable) ***B*** depends upon the category of the data item (variable) ***A***, or whether the categories of both data items (variables) ***A*** and ***B*** are independent of each other.

Assume therefore the null hypothesis H_0:

the categories of both data items (variables) ***A*** and ***B*** are independent of each other

χ^2-test of independence of qualitative data items

Having all the observed counts of the combinations of the categories A_i & B_j summarized in the contingency table for $i=1,\ldots,r$ and for $j=1,\ldots,s$, assume the null hypothesis H_0 that the categories of both data items (variables) ***A*** and ***B*** are independent of each other.

Now – if we choose a data unit randomly:

- What is the probability that the data item ***A*** of the chosen data unit is of category A_i for some $i=1,\ldots,r$?
- What is the probability that the data item ***B*** of the chosen data unit is of category B_j for some $j=1,\ldots,s$?

χ^2-test of independence of qualitative data items

The total number of all data units is n.

The count of the data units of category A_i is $n_{i\cdot}$

Therefore, the probability that a randomly selected data unit is of category A_i is

$$p_{i\cdot} = \frac{n_{i\cdot}}{n}$$

The count of the data units of category B_j is $n_{\cdot j}$

Therefore, the probability that a randomly selected data unit is of category B_j is

$$p_{\cdot j} = \frac{n_{\cdot j}}{n}$$

χ^2-test of independence of qualitative data items

Recall that the probability that a randomly selected data unit is of category A_i and B_j is

$$p_{i\cdot} = \frac{n_{i\cdot}}{n} \qquad \text{and} \qquad p_{\cdot j} = \frac{n_{\cdot j}}{n}$$

respectively. If the null hypothesis H_0 (that the categories of A and B are independent of each other) is true, then the (cumulative) probability that a randomly selected data unit is of category A_i and B_j should be

$$p_{ij} = p_{i\cdot} \times p_{\cdot j} = \frac{n_{i\cdot} n_{\cdot j}}{n^2}$$

for $i = 1, 2, \dots, r$ and for $j = 1, 2, \dots, s$.

χ^2-test of independence of qualitative data items

Once the probability that a randomly selected data unit is of category A_i and B_j is

$$p_{ij} = p_{i\cdot} \times p_{\cdot j} = \frac{n_{i\cdot} n_{\cdot j}}{n^2}$$

then we should expect

$$E_{ij} = p_{ij} \times n = \frac{n_{i\cdot n} \times n_{\cdot j}}{n}$$

data units of category A_i and B_j for $i = 1, 2, \dots, r$ and for $j = 1, 2, \dots, s$

if the null hypothesis H_0 (that the categories of A and B are independent of each other) is true.

χ^2-test of independence of qualitative data items

Expecting

$$E_{ij} = p_{ij} \times n = \frac{n_{i \cdot n} \times n_{\cdot j}}{n}$$

and observing

$$O_{ij} = n_{ij}$$

data units of category A_i and B_j for $i = 1, 2, \ldots, r$ and for $j = 1, 2, \ldots, s$,

we apply Pearson's χ^2-test for the goodness of fit to see if the observed counts agree with the expected counts, i.e. if the null hypothesis H_0 (that the categories of A and B are independent of each other) is true.

χ^2-test of independence of qualitative data items

Calculate

$$X^2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\frac{(O_{ij} - E_{ij})^2}{E_{ij}} = \sum_{i=1}^{r}\sum_{j=1}^{s}\frac{(n \times n_{ij} - n_{i.} \times n_{.j})^2}{n_{i.} \times n_{.j}}$$

<u>Theorem:</u>

If the null hypothesis is true, then

$$X^2 \sim \chi^2_{(r-1)(s-1)} \quad \textit{approximately} \quad \text{as} \quad n \to \infty$$

Notice the number of the degrees of freedom

(see below)

χ^2-test of independence of qualitative data items

The number of the degrees of freedom:

The observed counts O_{ij} for $i = 1, \dots, r$ and for $j = 1, \dots, s$

are bound by the system of $r + s$ equations:

$$\sum_{j=1}^{s} O_{ij} = \sum_{j=1}^{s} n_{ij} = n_{i\cdot} \qquad \text{for} \quad i = 1, 2, \dots, r$$

$$\sum_{i=1}^{r} O_{ij} = \sum_{i=1}^{r} n_{ij} = n_{\cdot j} \qquad \text{for} \quad j = 1, 2, \dots, s$$

of which only $r + s - 1$ are linearly independent, i.e. one of the equations depends on the others.

χ^2-test of independence of qualitative data items

<u>The number of the degrees of freedom:</u>

We thus have $r \times s$ observed counts O_{ij} for $i = 1, \ldots, r$ and for $j = 1, \ldots, s$

bound by $r + s - 1$ linearly independent equations, i.e. only

$$r \times s - r - s + 1 \;=\; (r-1) \times (s-1)$$

of the observed counts are free.

Therefore, the number of the degrees of freedom is

$$(r-1)(s-1)$$

χ^2-test of independence of qualitative data items

Now, finish the χ^2-test of independence of qualitative data items

(H_0: the categories of A and B are independent of each other) as follows:

- choose **the level of significance**, a small number $\alpha > 0$, such as $\alpha = 5\,\%$
- find **the critical value** $c > 0$ so that

$$\int_c^{+\infty} f(x)\,\mathrm{d}x = \alpha$$

  where f is the density of the χ^2-distribution with $(r-1)(s-1)$ d.f.

- if $X^2 \geq c$, **the critical region**, then **reject** the null hypothesis
- if $X^2 < c$, then **do not reject** (or fail to reject) the null hypothesis