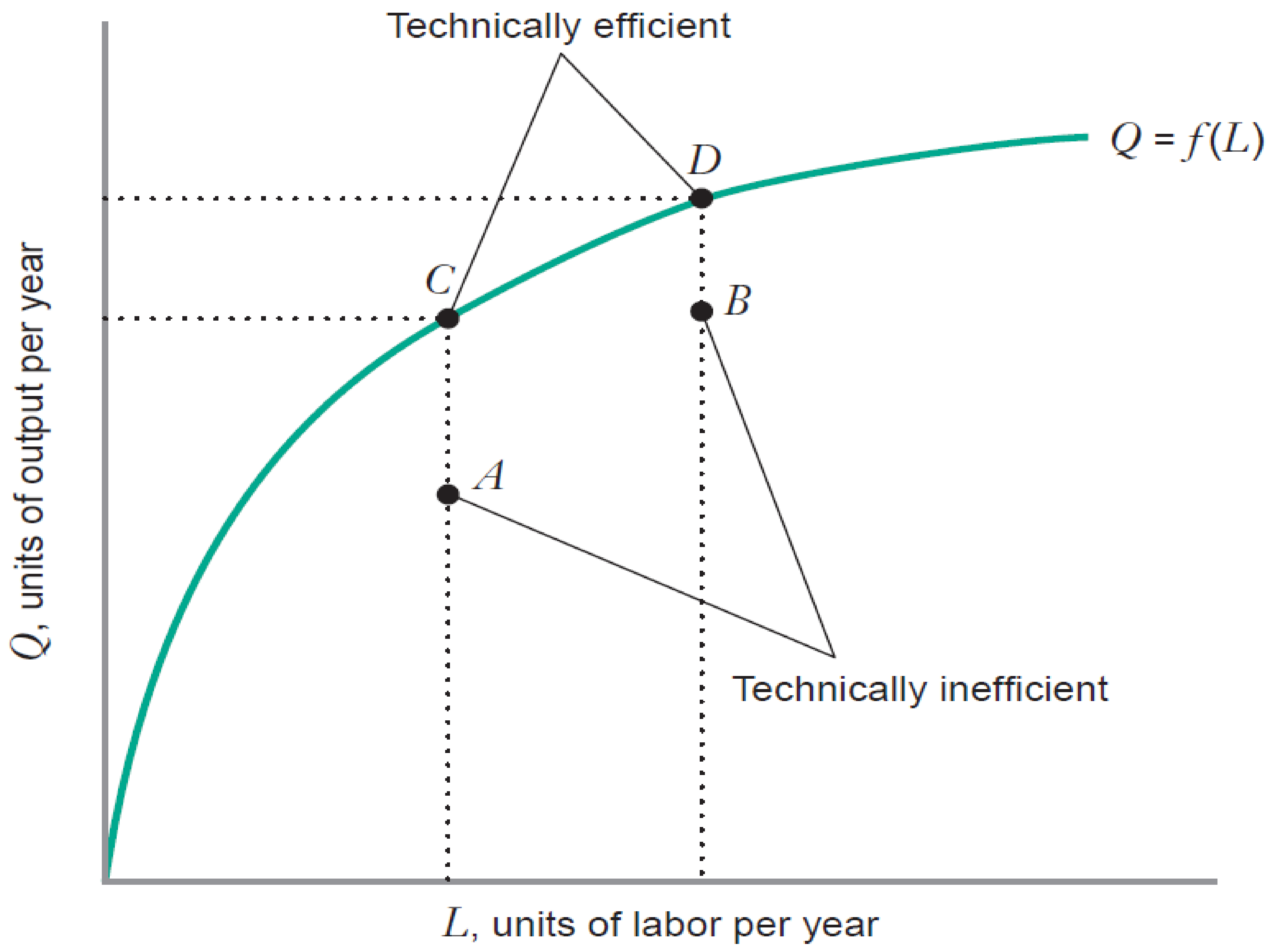
The Technology of Production

- **Production Function**:
  - A mathematical representation that Indicates the highest output that a firm can produce for every specified combination of inputs given the state of technology (during time these conditions may change).
  - Shows what is *technically feasible* when the firm operates *efficiently*.
  - **Inputs**= resources such labour, capital equipment, raw materials

- The production function for two inputs, for a given technology:

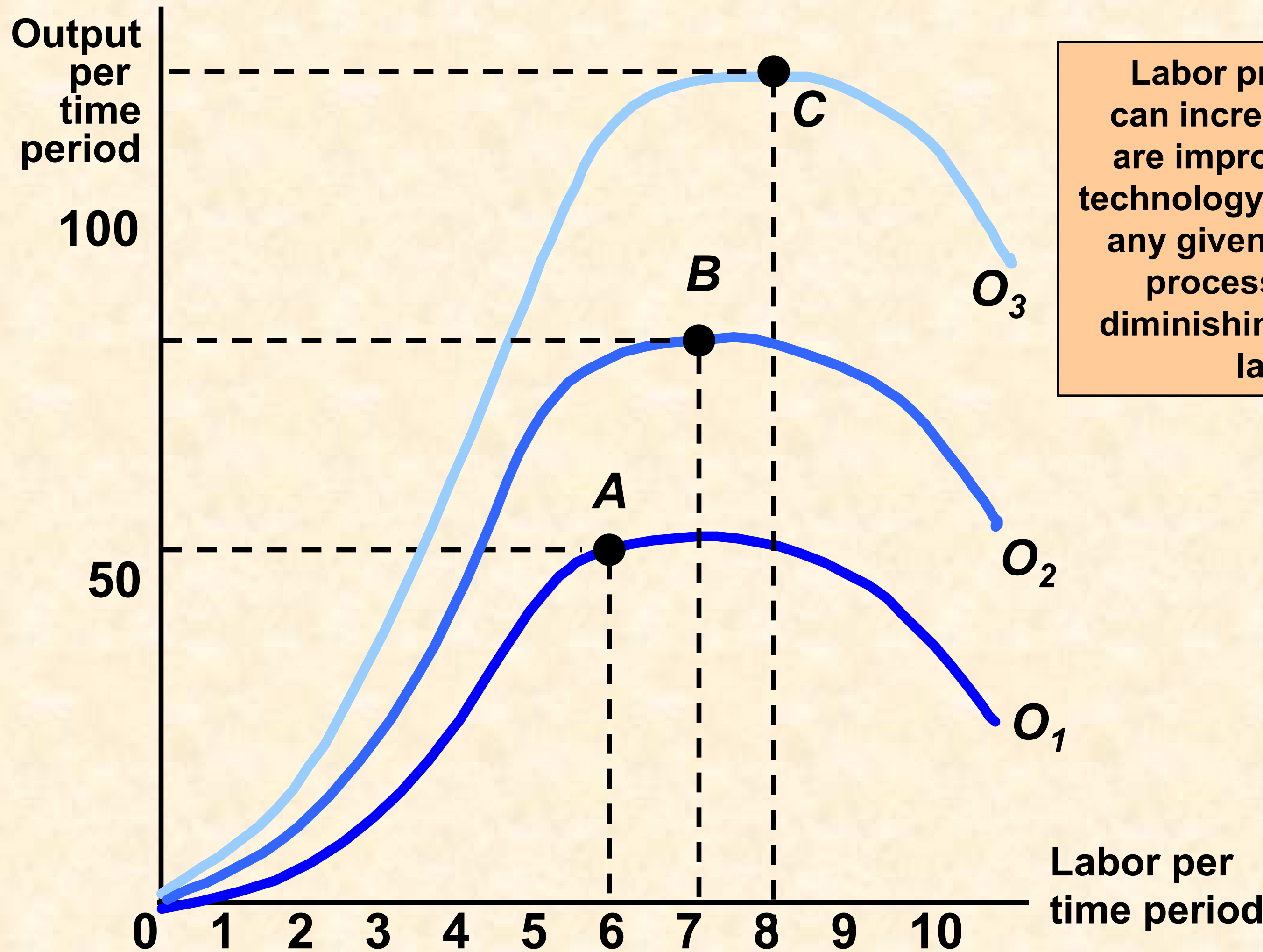
$$Q = f(K,L) \quad or \; Q \leq f(K,L)$$

Q = *quantity of* output, *K* = *quantity of* capital employed, *L* = quantity of labour used



Technically efficient
Q = f(L)
D
C
B
A
Technically inefficient
Q, units of output per year
L, units of labor per year

The Effect of Technological Improvement



Labor productivity can increase if there are improvements in technology, even though any given production process exhibits diminishing returns to labor.

The Technology of Production

- Input requirements function:
  - Shows the minimum amount of input (e.g. Labour) required to produce a given amount of output Q

- Example:

$$Q = L^{1/2} \text{ then } L=Q^2$$

to produce 7 units of output we need 49 units of labour

Production with One Variable Input (Labor)

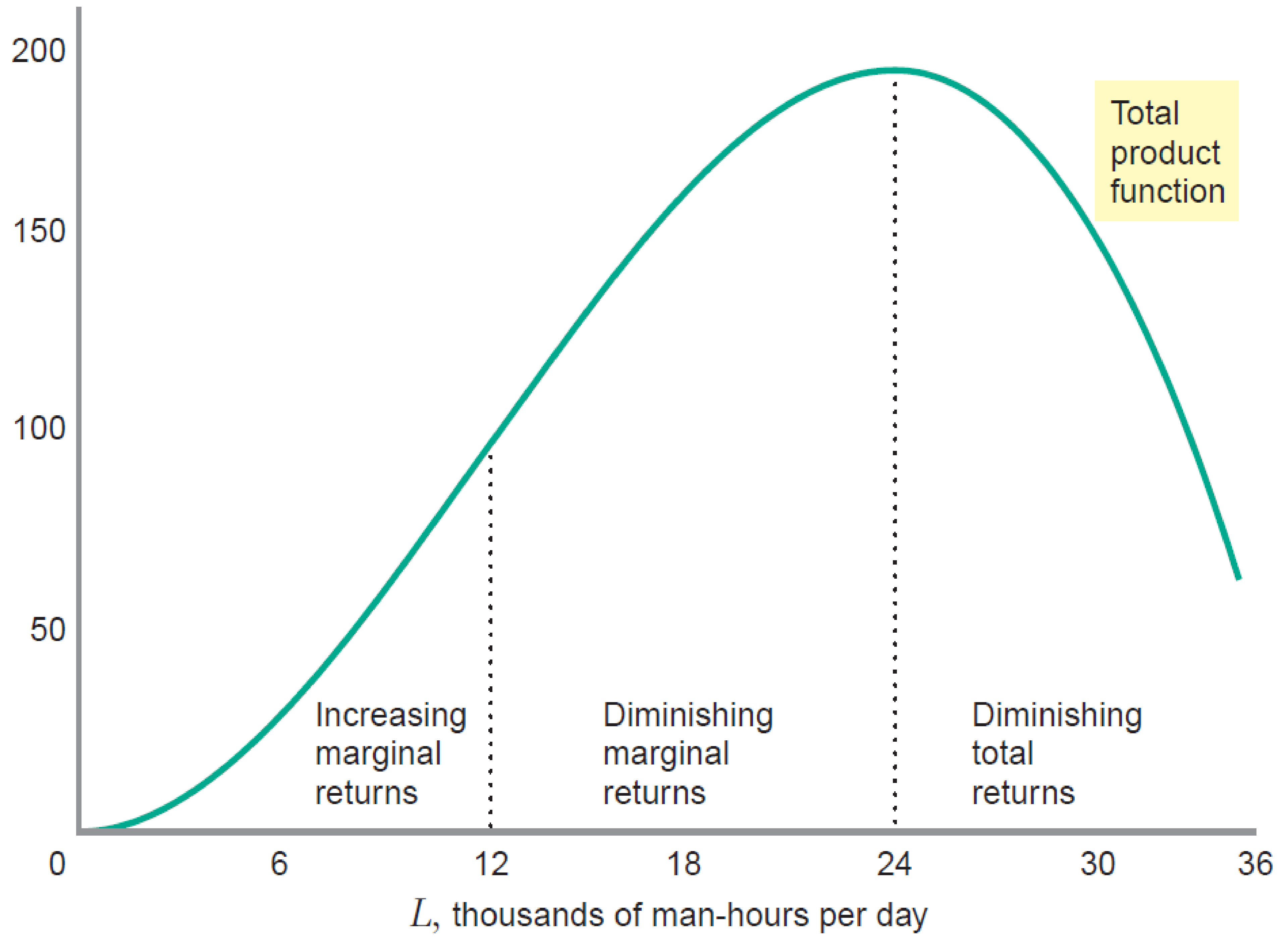
- Labor Productivity: AP = Total Output/Total Labor Input

- Labor Productivity and the Standard of Living: Consumption can increase only if productivity increases.

- Determinants of Productivity
  - Stock of capital
  - Technological change

Production with One Variable Input (Labor)

Amount of Labor (*L*)	Amount of Capital (*K*)	Total Output (*Q*)	Average Product	Marginal Product
0	10	0	---	---
1	10	10	10	10
2	10	30	15	20
3	10	60	20	30
4	10	80	20	20
5	10	95	19	15
6	10	108	18	13
7	10	112	16	4
8	10	112	14	0
9	10	108	12	-4
10	10	100	10	-8


200
150
100
50
0
Total product function
Increasing marginal returns
Diminishing marginal returns
Diminishing total returns
6
12
18
24
30
36
L, thousands of man-hours per day

Total product functions

- Single-intput production function (e.g. labour)
- **Increasing marginal returns to labour**: the region along the total product function where output rises with additional labour at an increasing rate
- **Diminishing marginal returns to labour**: the region along the total product function where output rises with additional labour at an increasing rate
- **Diminishing total returns to labour**: the region along the total product function where output rises with additional labour at an increasing rate

Production with One Variable Input (Labor)

1) With additional workers, output (Q) increases, reaches a maximum, and then decreases.

2) The average product of labor (AP), or output per worker, increases and then decreases.

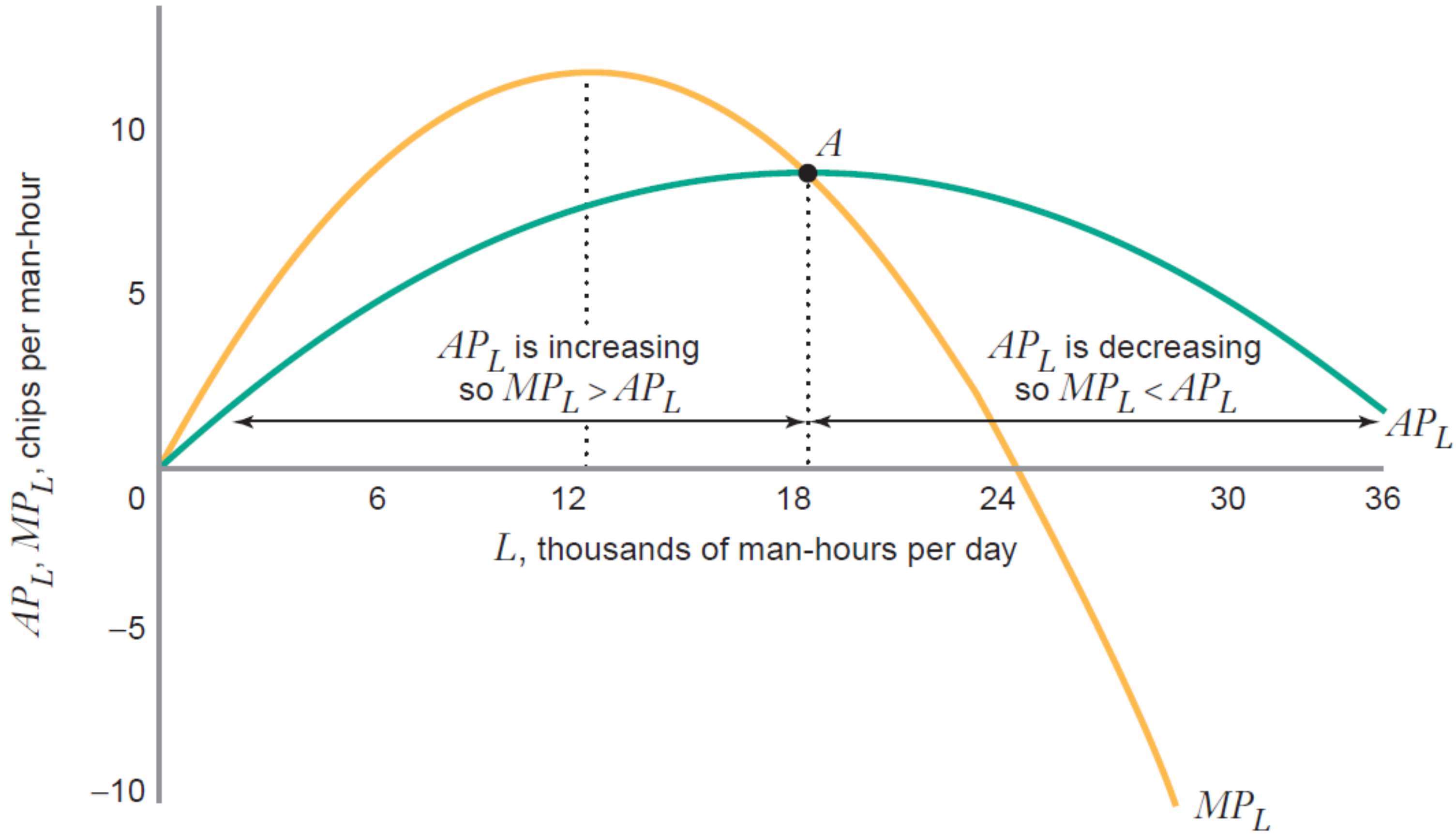
$$AP = \frac{Output}{Labor\ Input}$$

L	Q	$AP_L = \frac{Q}{L}$
6	30	5
12	96	8
18	162	9
24	192	8
30	150	5

Production with One Variable Input (Labor)

3) The marginal product (MP) of labor or output of the additional worker increases rapidly initially and then decreases and becomes negative..

$$MP = \frac{\Delta Output}{\Delta Input}$$



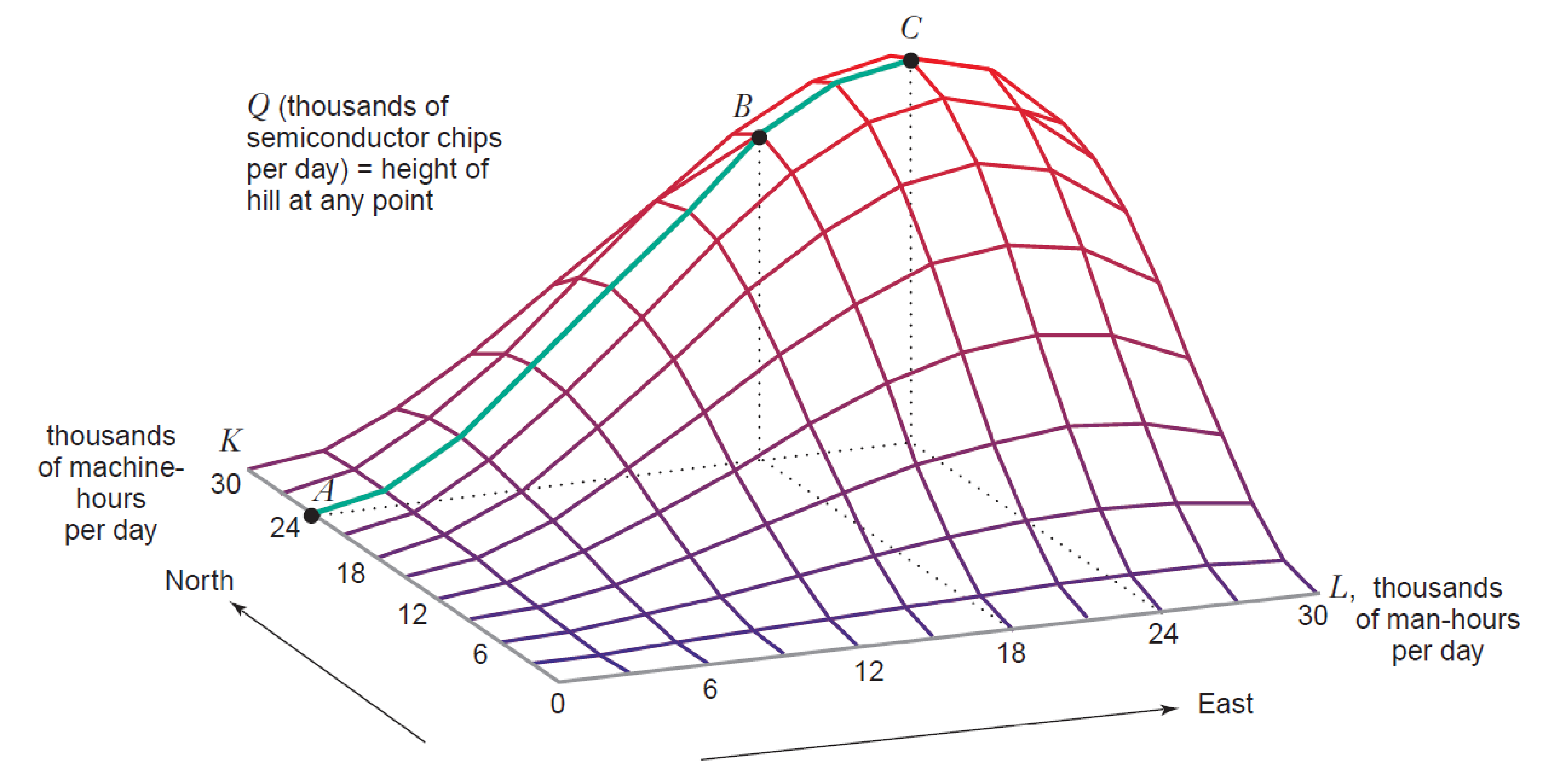
AP_L, MP_L, chips per man-hour
10
5
0
–5
–10
A
AP_L is increasing so $MP_L > AP_L$
AP_L is decreasing so $MP_L < AP_L$
AP_L
MP_L
6
12
18
24
30
36
L, thousands of man-hours per day

Production function with more than one input

- how easily a firm can substitute among the inputs within its production
- two inputs: labor and capital
- the quantity of output Q depends on the quantity of labor L and the quantity of capital K employed
- **total product hill** - a three-dimensional graph that shows the relationship between the quantity of output and the quantity of the two inputs employed by the firm.

Production Function for Food

	Labor Input				
Capital Input	**1**	**2**	**3**	**4**	**5**
1	20	40	55	65	75
2	40	60	75	85	90
3	55	75	90	100	105
4	65	85	100	110	115
5	75	90	105	115	120


C
B
Q (thousands of semiconductor chips per day) = height of hill at any point
K
thousands of machine-hours per day
30
A
24
18
12
6
0
North
6
12
18
24
30
L, thousands of man-hours per day
East

Isoquants

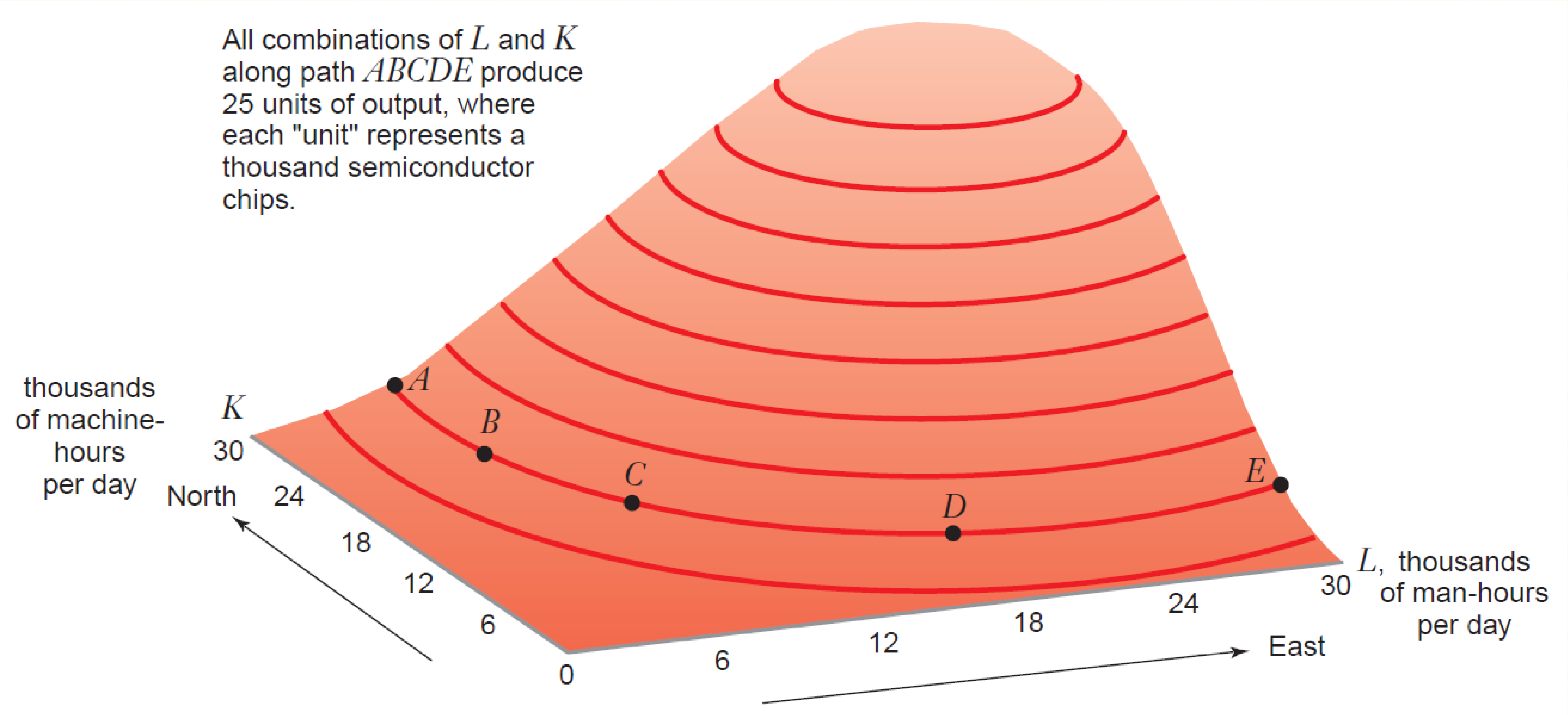
- To illustrate economic trade-offs, it helps to reduce the three-dimensional graph of the production function (the total product hill) to two dimensions
- Observations:

  1) For any level of K, output increases with more L.

  2) For any level of L, output increases with more K.

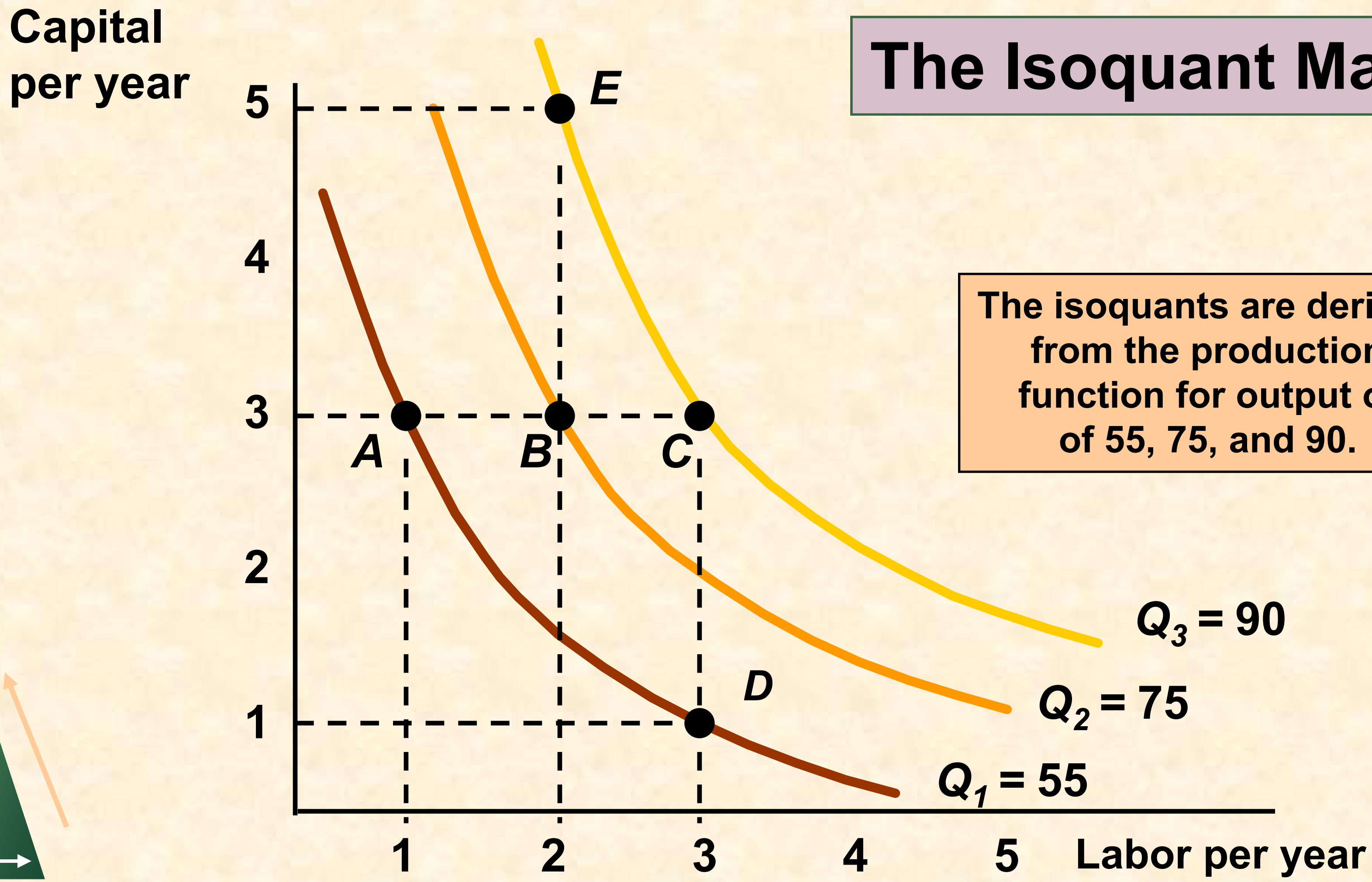
  3) Various combinations of inputs produce the same output.

- Isoquants: Curves showing all possible combinations of inputs that yield the **same output**


All combinations of L and K along path $ABCDE$ produce 25 units of output, where each "unit" represents a thousand semiconductor chips.
thousands of machine-hours per day
K
30
North
24
18
12
6
0
6
12
18
24
30
East
L, thousands of man-hours per day
A
B
C
D
E

Production with Two Variable Inputs (*L*,*K*)

The Isoquant Map



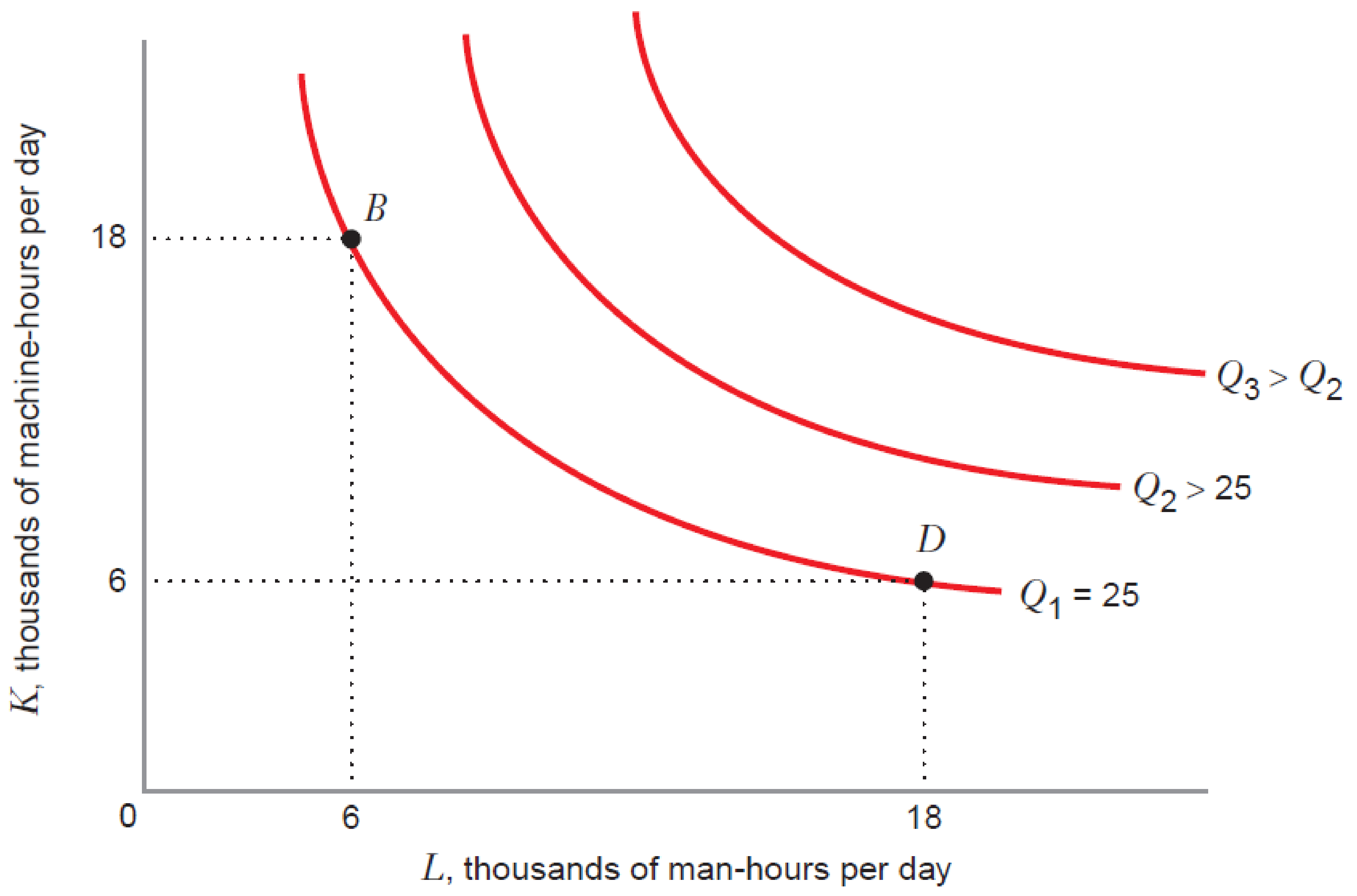
The isoquants are derived from the production function for output of of 55, 75, and 90.

Isoquants - exercise

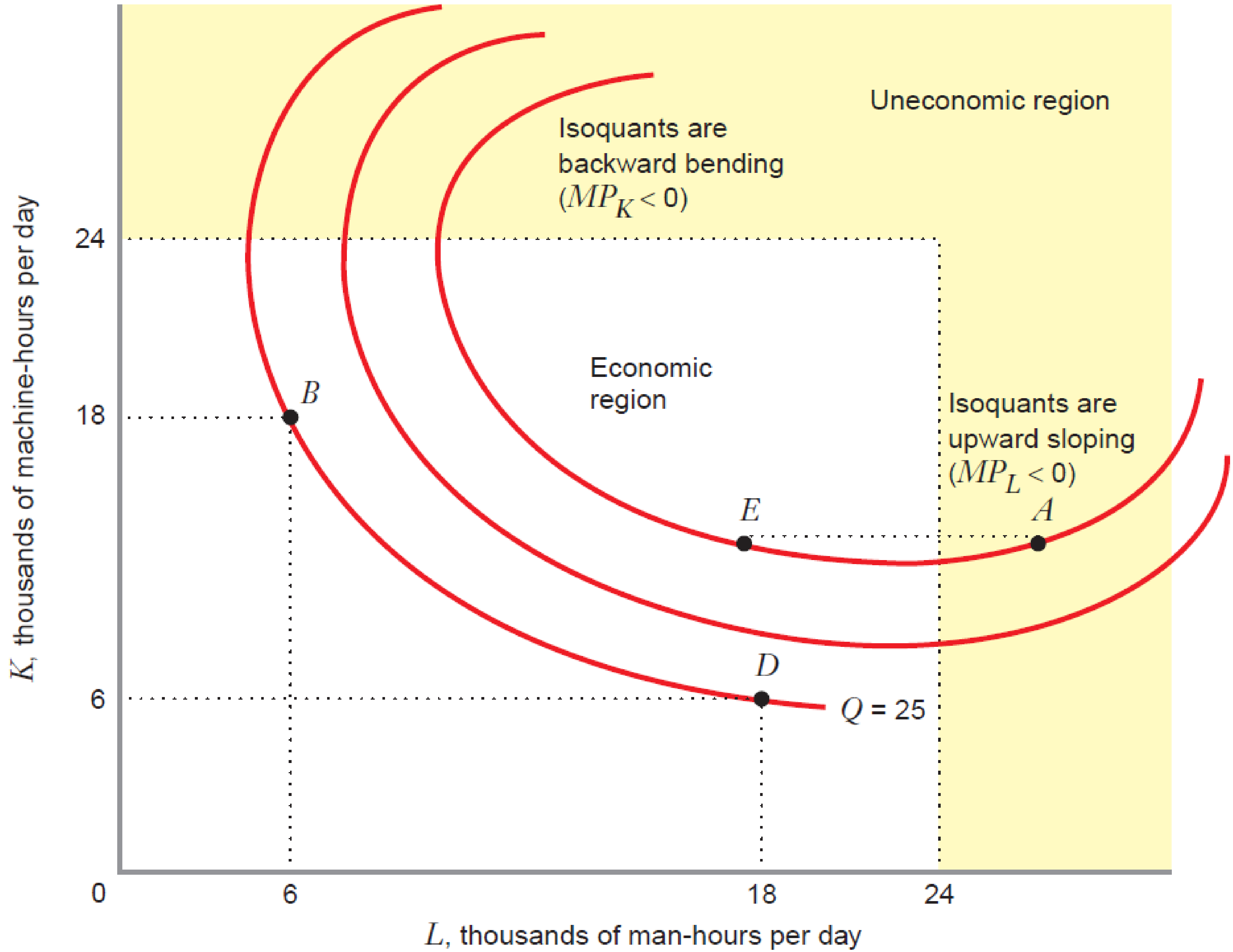
Consider the production function whose equation is given by the formula $Q = \sqrt{KL}$.

(a) What is the equation of the isoquant corresponding to *Q* 20?

(b) For the same production function, what is the general equation of an isoquant, corresponding to any level of output *Q*?



K, thousands of machine-hours per day
18
6
0
6
18
L, thousands of man-hours per day
B
D
Q3 > Q2
Q2 > 25
Q1 = 25



Uneconomic region
Isoquants are
backward bending
($MP_K < 0$)
Economic
region
Isoquants are
upward sloping
($MP_L < 0$)
B
E
A
D
Q = 25
24
18
6
0
6
18
24
K, thousands of machine-hours per day
L, thousands of man-hours per day

Isoquants

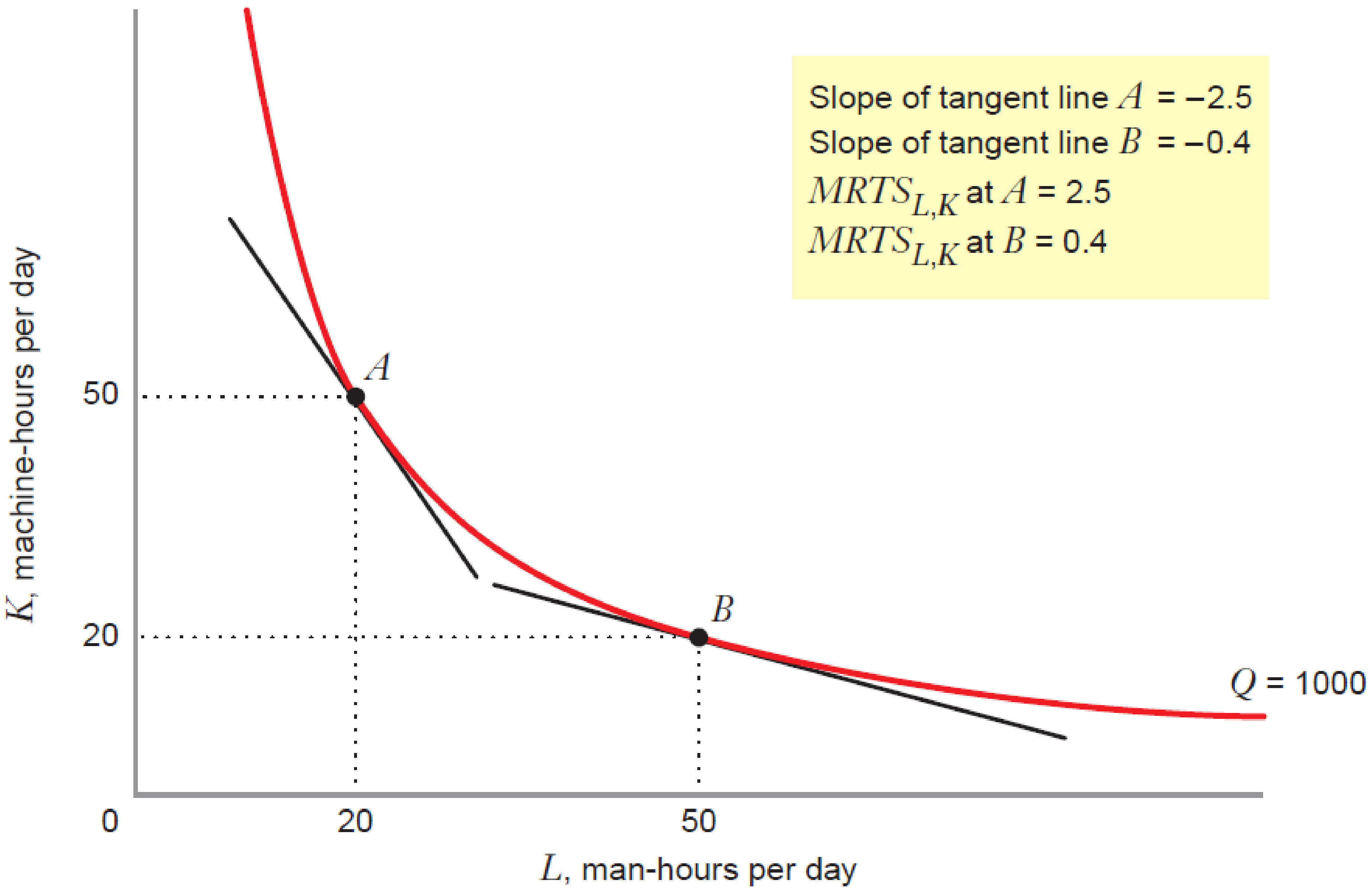
- The backward-bending and upward-sloping regions of the isoquants make up the uneconomic region of production. In this region, the marginal product of one of the inputs is negative (diminishing total returns).
- A cost-minimizing firm would never produce in the uneconomic region.

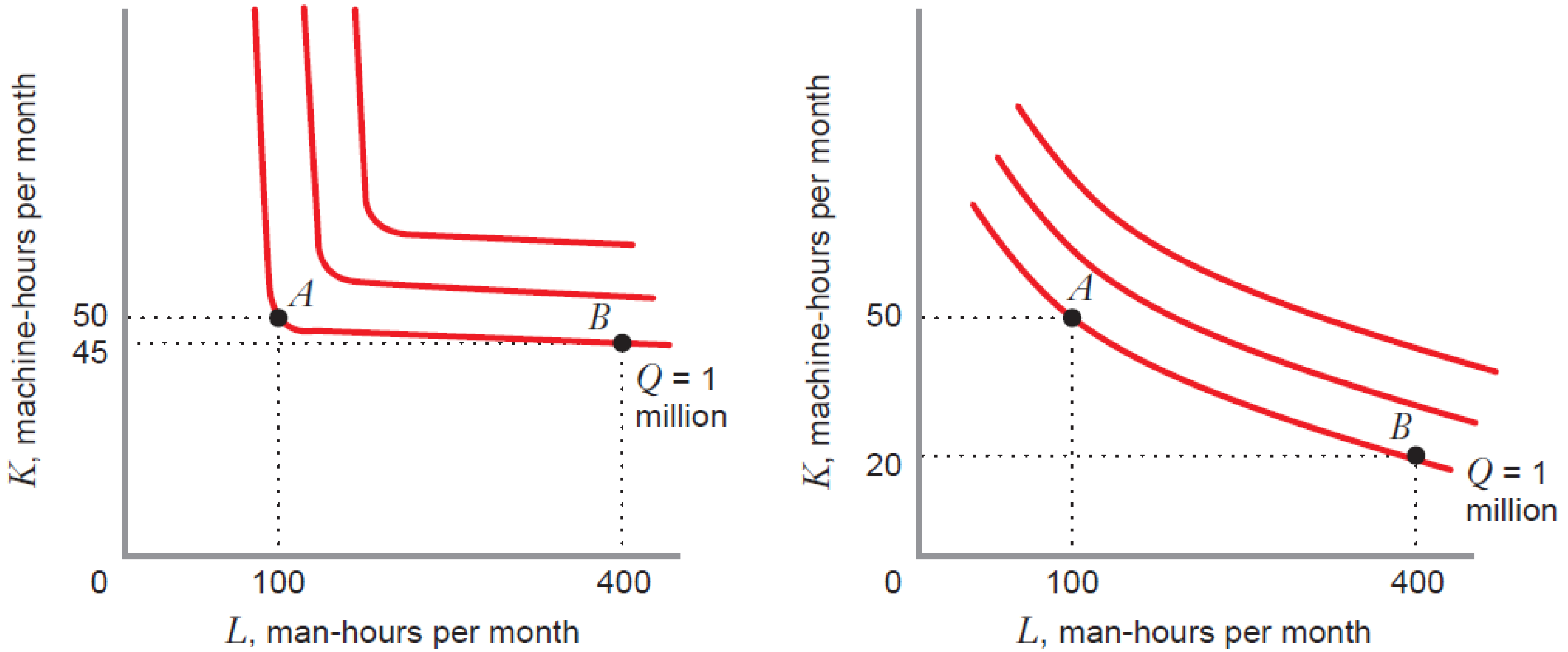
Marginal rate of technical substitution (MRTS)

- If we would like to invest in sophisticated robotics we would naturally be interested in the extent to which it can replace humans with robots
- How many robots will it need to invest in to replace the labor power of one worker?
- The “steepness” of an isoquant determines the rate at which the firm can substitute between labor and capital in its production process= MRTS

Marginal rate of technical substitution (MRTS)

- The rate at which the quantity of capital can be **decreased** for every one-unit **increase** in the quantity of labor, holding the quantity of output constant, or
- The rate at which the quantity of capital must be **increased** for every one-unit **decrease** in the quantity of labor, holding the quantity of output constant
- As we move down along the isoquant, the slope of the isoquant increases (becomes less negative), which means that the $MRTS_{L,K}$ gets smaller and smaller => **diminishing marginal rate of technical substitution**


Slope of tangent line A = –2.5
Slope of tangent line B = –0.4
$MRTS_{L,K}$ at A = 2.5
$MRTS_{L,K}$ at B = 0.4
K, machine-hours per day
50
20
0
20
50
L, man-hours per day
A
B
Q = 1000



(a) Production Function with Limited Input Substitution Opportunities

(b) Production Function with Abundant Input Substitution Opportunities

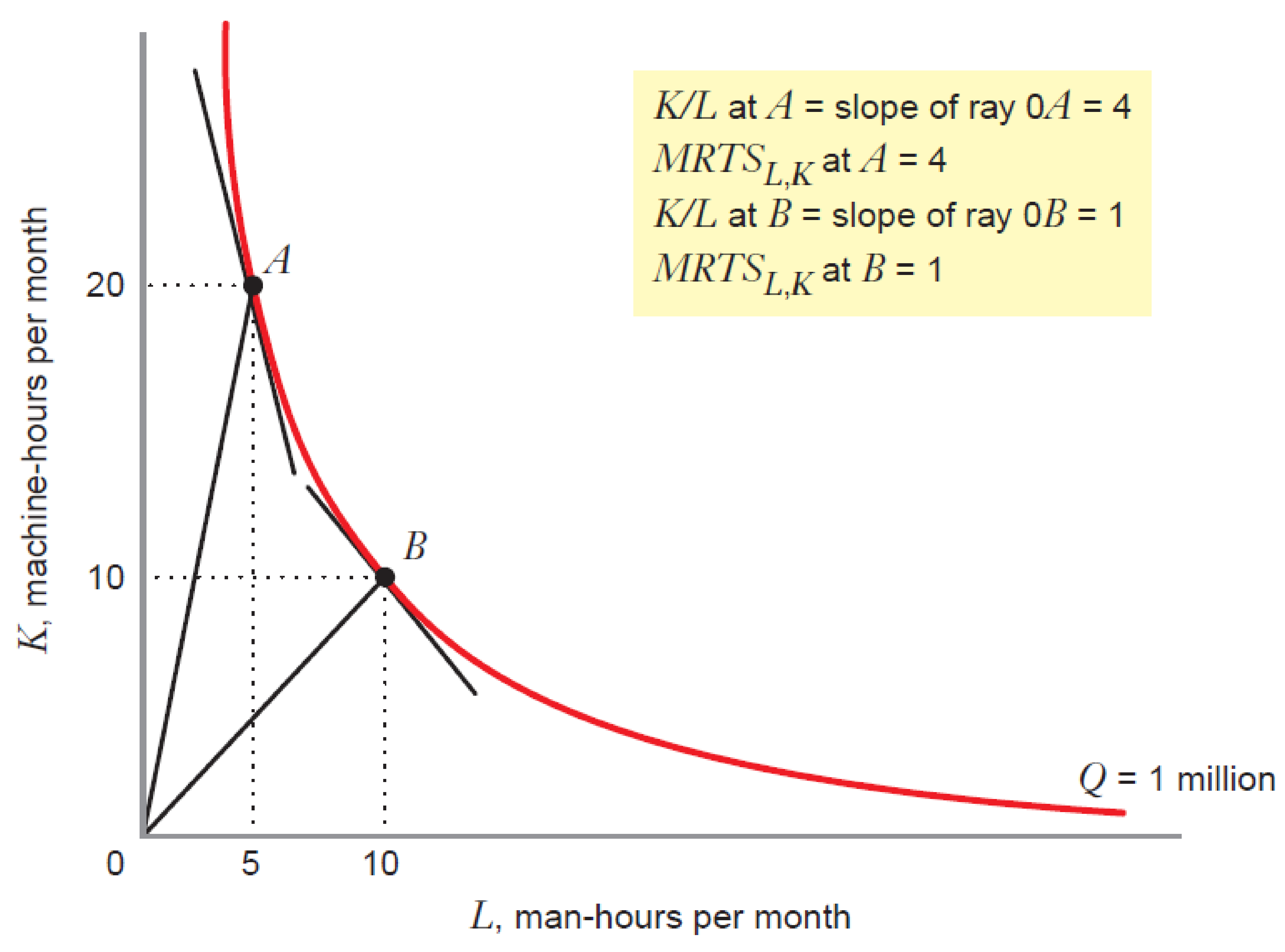
Marginal rate of technical substitution (MRTS)

- When the production function offers limited input substitution opportunities, the $MRTS_{L,K}$ changes **substantially** as we move along an isoquant. In this case, the isoquants are nearly **L-shaped**,
- When the production function offers abundant input substitution opportunities, the $\mathrm{MRTS}_{L,K}$ changes **gradually** as we move along an isoquant. In this case, the isoquants are nearly **straight lines**

Elasticity of substitution

- a numerical measure that describe the firm’s input substitution opportunities
- measures how quickly the marginal rate of technical substitution of labor for capital changes as we move along an isoquant = the percentage change in the capital–labor ratio for each 1 percent change in $MRTS_{L,K}$ as we move along an isoquant:

$$\sigma = \frac{\textit{percentage change} \text{ in capital–labor ratio}}{\textit{percentage change} \text{ in } MRTS_{L,K}} = \frac{\%\,\Delta\left(\frac{K}{L}\right)}{\%\,\Delta MRTS_{L,K}}$$



K/L at A = slope of ray 0A = 4
MRTS L,K at A = 4
K/L at B = slope of ray 0B = 1
MRTS L,K at B = 1
K, machine-hours per month
20
10
A
B
0
5
10
Q = 1 million
L, man-hours per month

Elasticity of substitution

- **Consider a production function whose equation is given by the formula:**

$$Q = \sqrt{KL},$$

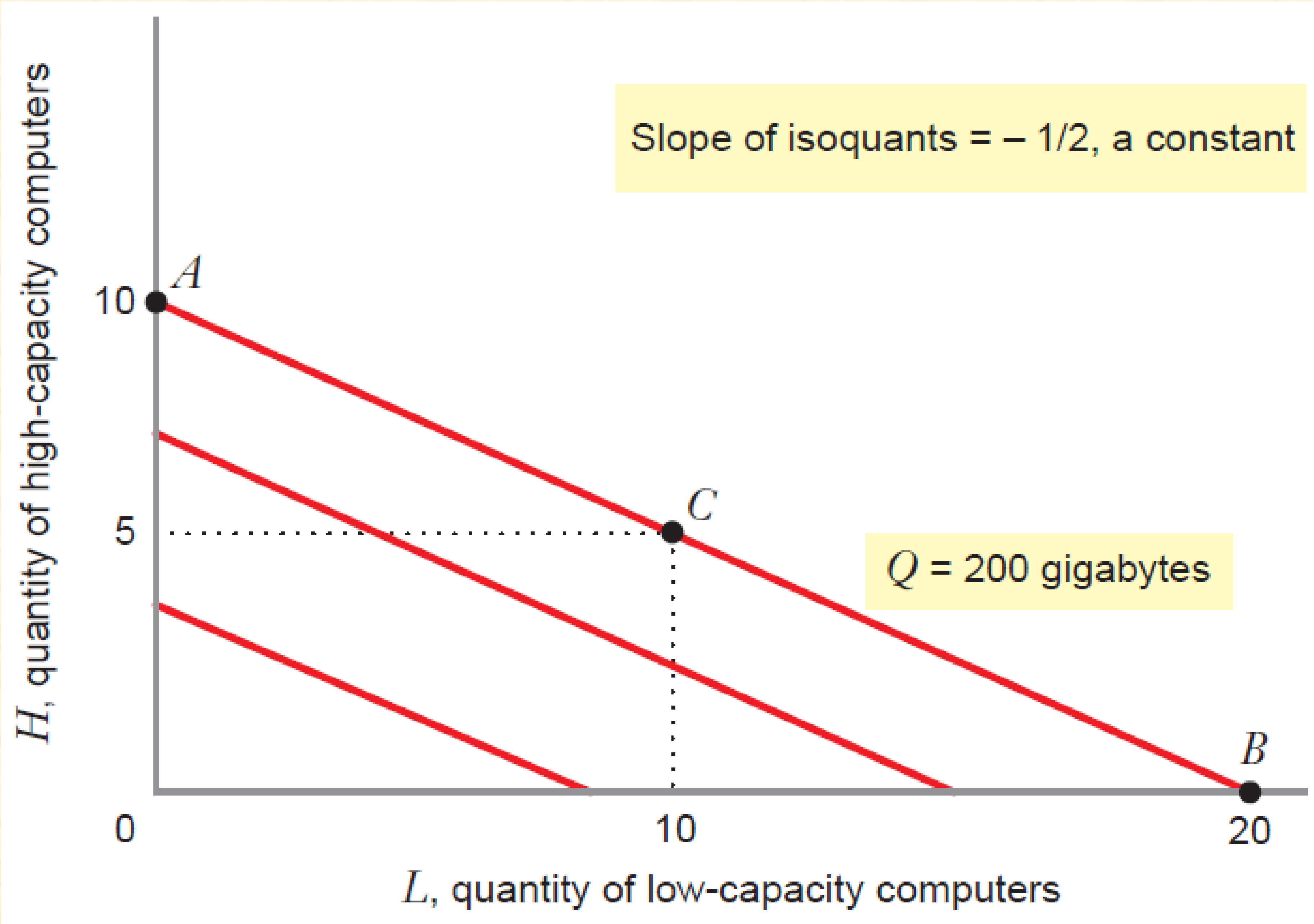
a) Show that the elasticity of substitution for this production function is exactly equal to 1, no matter what the values of *K* and *L* are.

Special production functions

- LINEAR PRODUCTION FUNCTION (PERFECT SUBSTITUTES)
- FIXED-PROPORTIONS PRODUCTION FUNCTION (PERFECT COMPLEMENTS)
- COBB-DOUGLAS PRODUCTION FUNCTION
- CONSTANT ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTION

Linear Production Function

- the marginal rate of technical substitution of one input for another may be constant
- A production function of the form **Q=aL + bK**, where a and b are positive constants.
- A linear production function is a production function whose isoquants are straight lines
- the slope of any isoquant is constant, and the marginal rate of technical substitution does not change as we move along the isoquant.
- $MRTS_{L,H}=0$, this means that the elasticity of substitution for a linear production function must be infinite



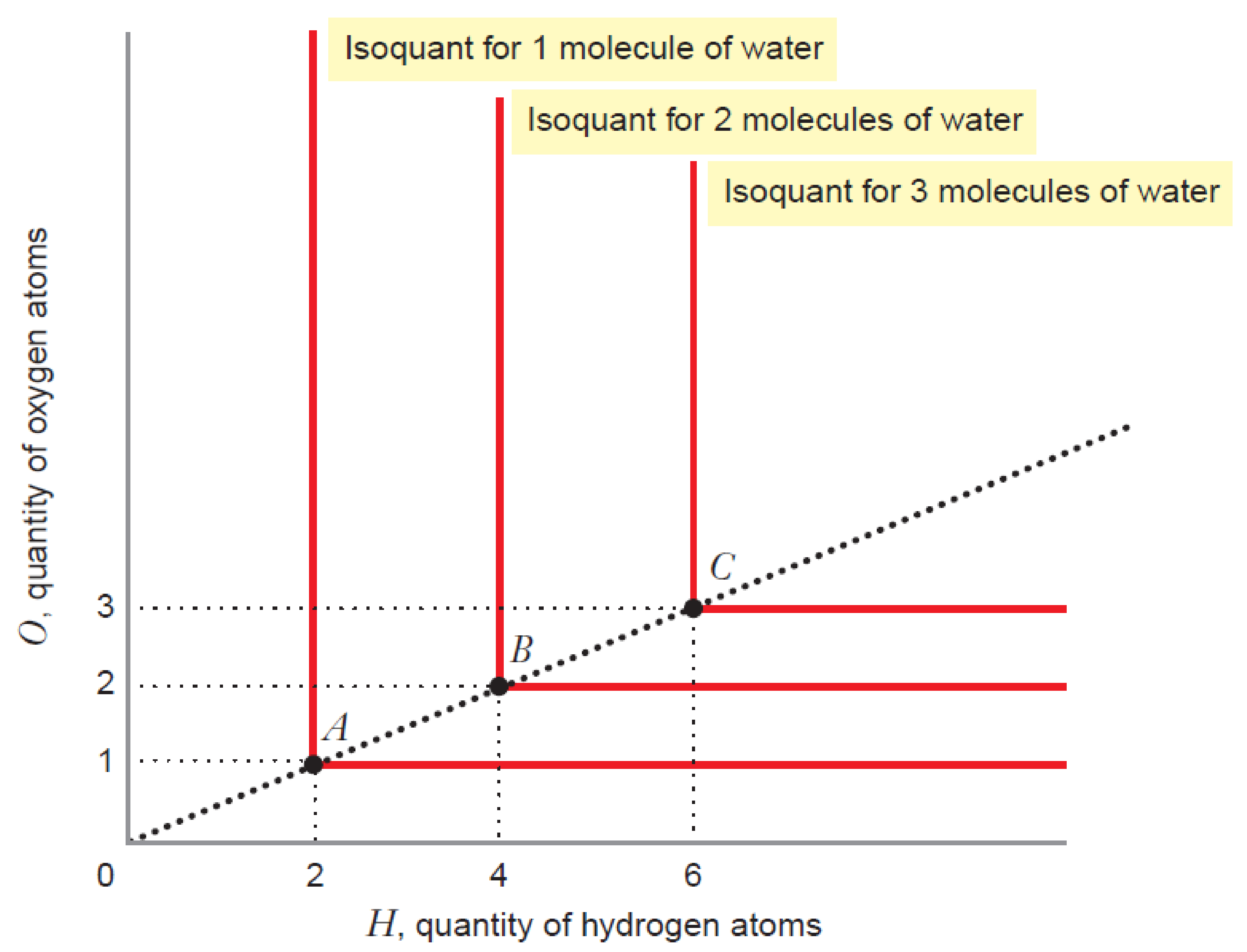
Slope of isoquants = – 1/2, a constant
A
10
C
5
Q = 200 gigabytes
B
0
10
20
H, quantity of high-capacity computers
L, quantity of low-capacity computers

Fixed-proportions production function

- A production function where the inputs must be combined in fixed proportions e.g. isoquants for the production of water, where the inputs are atoms of hydrogen (H) and atoms of oxygen (O).
- Adding more hydrogen to a fixed number of oxygen atoms gives us no additional water molecules; neither does adding more oxygen to a fixed number of hydrogen atoms.
- Thus, the quantity Q of water molecules that we get is given by:

$$Q = \min\left(\frac{H}{2}, O\right)$$

- the elasticity of substitution is zero


Isoquant for 1 molecule of water
Isoquant for 2 molecules of water
Isoquant for 3 molecules of water
O, quantity of oxygen atoms
H, quantity of hydrogen atoms
A
B
C
0
1
2
3
2
4
6

Cobb-Douglas Production Function

- intermediate between a linear production function and a fixed-proportions production function.
- A production function of the form $Q=AL^{\alpha}K^{\beta}$ where Q is the quantity of output from L units of labor and K units of capital and where A, α and ß are positive constants.
- With the Cobb–Douglas production function, capital and labor can be substituted for each other.
- the elasticity of substitution along a Cobb–Douglas production function is always equal to 1

Production Function	Elasticity of Substitution (σ)	Other Characteristics
Linear production function	$\sigma = \infty$	Inputs are *perfect substitutes* Isoquants are straight lines
Fixed-proportions production function	$\sigma = 0$	Inputs are *perfect complements* Isoquants are L-shaped
Cobb–Douglas production function	$\sigma = 1$	Isoquants are curves
CES production function	$0 \leq \sigma \leq \infty$	Includes other three production functions as special cases Shape of isoquants varies

Returns to scale

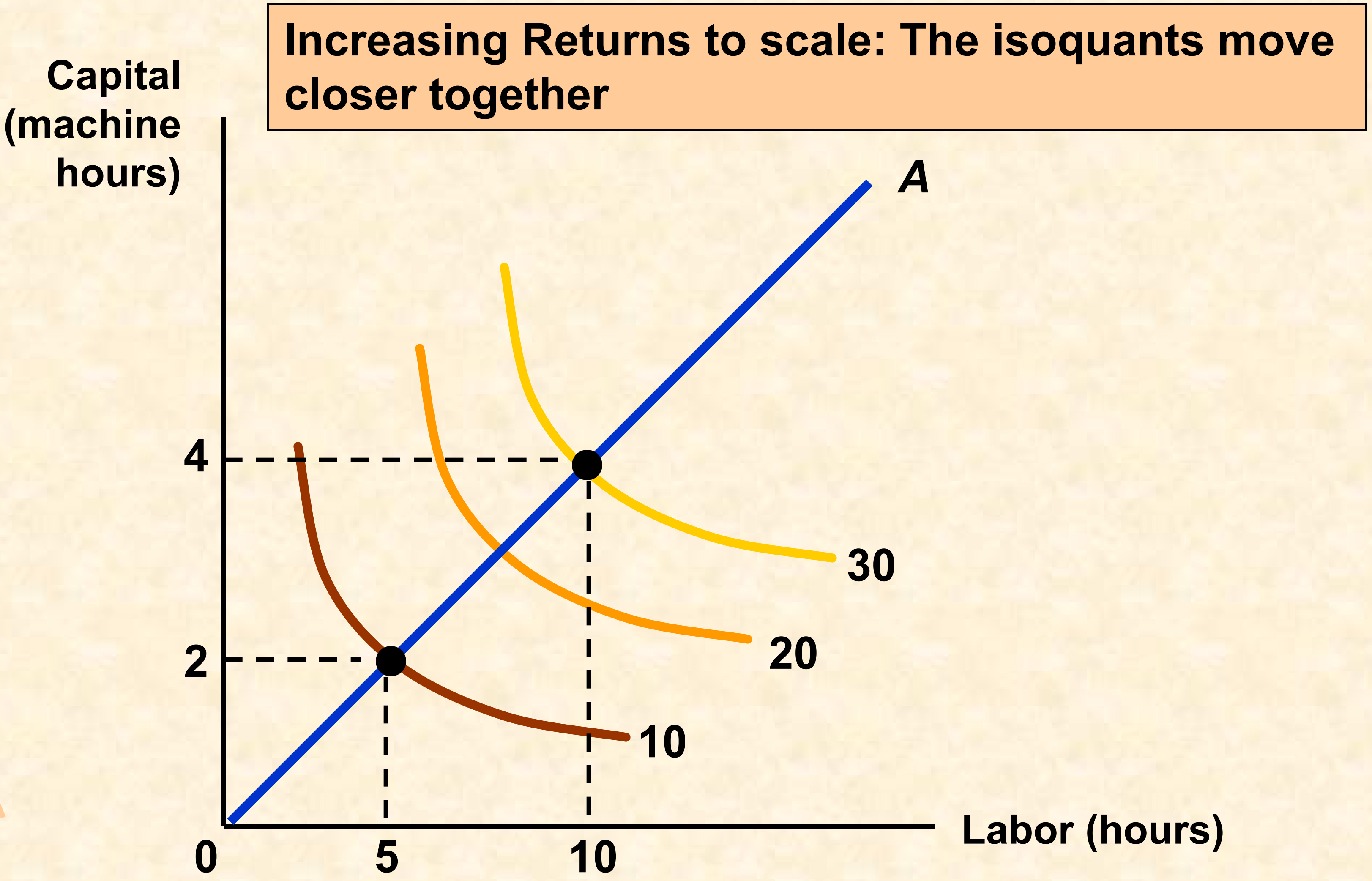
- Theme - how increases in all input quantities affect the quantity of output the firm can produce.
- When inputs have positive marginal products, a firm's total output must increase when the quantities of all inputs are increased simultaneously

$$\text{Returns to scale} = \frac{\%\Delta\,(\text{quantity of output})}{\%\Delta\,(\text{quantity of } \textit{all} \text{ inputs})}$$

Returns to scale

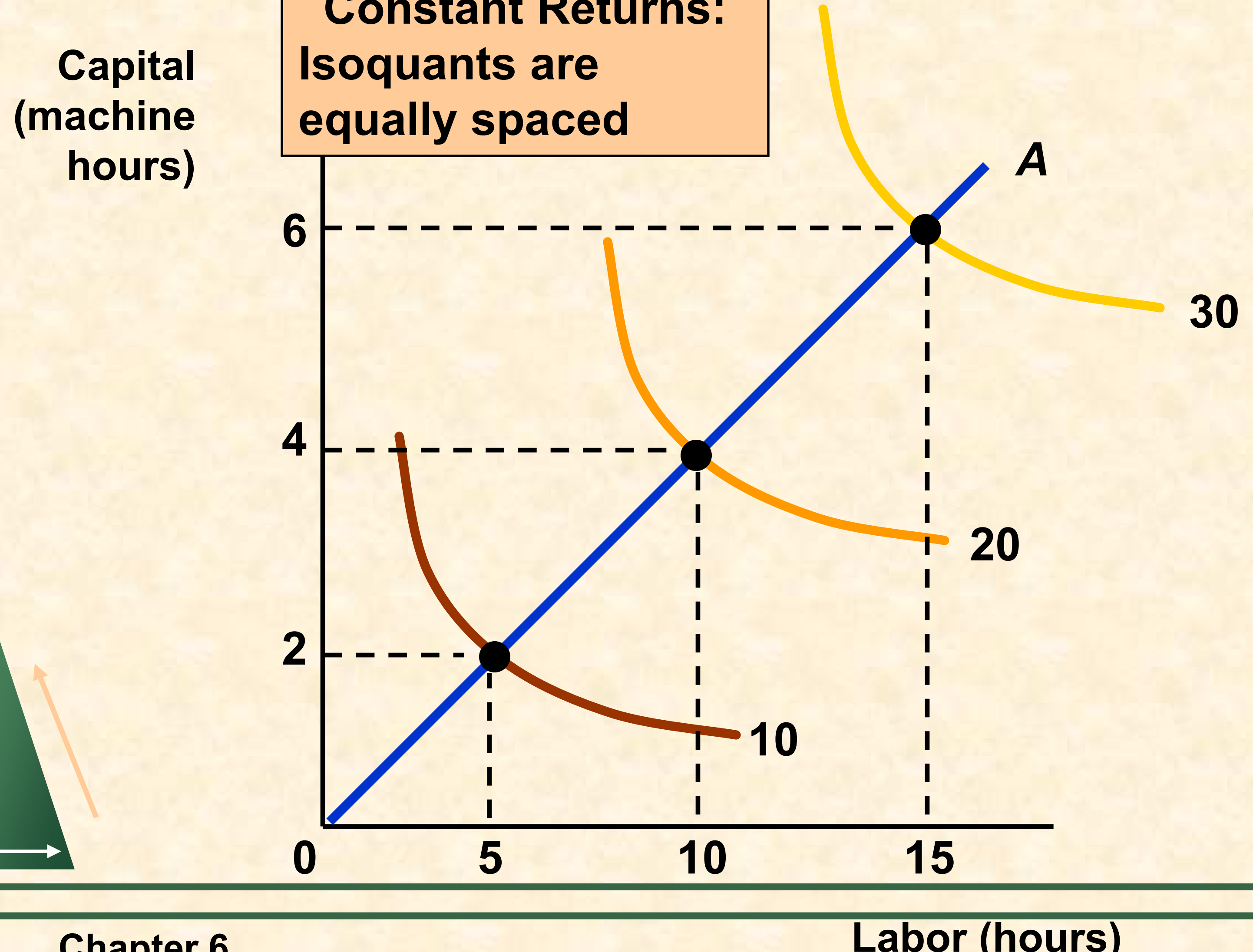
- **increasing returns to scale.** In this case, a proportionate increase in all input quantities results in a greater than proportionate increase in output.
- **constant returns to scale.** In this case, a proportionate increase in all input quantities results in the same proportionate increase in output.
- **decreasing returns to scale**. In this case, a proportionate increase in all input quantities results in a less than proportionate increase in output.

Firm Size and Output: Increasing Returns to Scale



Firm Size and Output: Constant Returns to Scale

Constant Returns: Isoquants are equally spaced



Firm Size and Output: Decreasing Returns to Scale