Why and How to overcome General Equilibrium Theory

Glötzl, Erhard

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Why and How to overcome General Equilibrium Theory*

Newtonian Constrained Dynamic Models as a New Approach to Describe Economic Dynamics in Analogy to Physics. A Unified Look at Neoclassical Models, Keynesian Models and Game Theory.

Erhard Glötzl

Linz, Austria

erhard.gloetzl@gmail.com

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Abstract

For more than 100 years economists have tried to describe economics in analogy to physics, more precisely to classical Newtonian mechanics. The development of the Neoclassical General Equilibrium Theory has to be understood as the result of these efforts. But there are many reasons why General Equilibrium Theory is inadequate: 1. No true dynamics. 2. The assumption of the existence of utility functions and the possibility to aggregate them to one "Master" utility function. 3. The impossibility to describe situations as in "Prisoners Dilemma", where individual optimization does not lead to a collective optimum. This paper aims at overcoming these problems. It illustrates how not only equilibria of economic systems, but also the general dynamics of these systems can be described in close analogy to classical mechanics.

To this end, this paper makes the case for an approach based on the concept of constrained dynamics, analyzing the economy from the perspective of "economic forces" and "economic power" based on the concept of physical forces and the reciprocal value of mass. Realizing that accounting identities constitute constraints in the economy, the concept of constrained dynamics, which is part of the standard models of classical mechanics, can be applied to economics. Therefore it is reasonable to denote such models as Newtonian Constraint Dynamic Models (NCD-Models)

Such a framework allows understanding both Keynesian and neoclassical models as special cases of NCD-Models in which the power relationships with respect to certain variables are one-sided. As mixed power relationships occur more frequently in reality than purely one-sided power constellations, NCD-models are better suited to describe the economy than standard Keynesian or Neoclassic models.

A NCD-model can be understood as "Continuous Time", "Stock Flow Consistent", "Agent Based Model", where the behavior of the agents is described with a general differential equation for every agent. In the special case where the differential equations can be described with utility functions, the behavior of every agent can be understood as an individual optimization strategy. He thus seeks to maximize his utility. However, while the core assumption of neoclassical models is that due to the "invisible hand" such egoistic individual behavior leads to an optimal result for all agents, reality is often defined by "Prisoners Dilemma" situations, in which individual optimization leads to the worst outcome for all. One advantage of NCD-models over standard models is that they are able to describe also such situations, where an individual optimization strategy does not lead to an optimum result for all agents. This will be illustrated in a simple example.

In conclusion, the big merit and effort of Newton was, to formalize the right terms (physical force, inertial mass, change of velocity) and to set them into the right relation. Analogously the appropriate terms of economics are force, economic power and change of flow variables. NCD-Models allow formalizing them and setting them into the right relation to each other.

Content

1. Introduction

For more than 100 years economists have tried to describe the economy in analogy to physics, more precisely to classical mechanics. The neoclassical General Equilibrium Theory has to be understood as the result of these efforts. But the orientation of economics towards physics has been implemented only partially, especially the dynamics of mechanical systems have been omitted completely. This paper therefore seeks to analyze economic models in perfect analogy to Newtonian mechanics, illustrating that not only equilibria but also the general dynamics of economic system with all their disequilibria can be described using the framework provided by classical mechanics.

The formalization of the physical concepts of force and mass by Isaac Newton revolutionized physics and was the basis for the entire following development of the discipline. Similarly, this contribution aims at developing a formalization of the concepts of economic force and power in order to establish a single consistent structure for the description of economic systems.

Chapter 2 provides a short overview over the historic attempts to find similarities between economics and physics.

In chapter 3 we then set out to explain the principal ideas of this approach with easy examples.

Chapter 4 describes the formal structure of such “Newtonian Constrained Dynamic Models” (NCD-Models), based on the concepts of economic force and economic power. The label “Newtonian” stems from the fact that the basic equations describe the change of flow variables just like in Newtonian mechanics. The label “constrained” refers to the fact that the economy is often subject to constraints. Accounting identities constitute the most important class of such constraints, which provoke constraining economic forces, in perfect analogy to classical mechanics.

Similar to potential forces in physics, we look especially at those cases where economic forces can be expressed as gradients of a utility function. Economic models in which the equations can be expressed with a single master utility function are a special case. This is important with respect to the fact that neoclassic always assumes such a master utility function to exist and that economic systems are determined by its maximum. In no way is it however the case that the maximization of such a master utility function, if it exists, also leads to the optimal total utility for all agents.

NCD-models of the economy have the same mathematical structure as the classical Newtonian mechanics with constraints. Both in physics and in the economy there are two types of variables. The stock variables $x(t)=\left(x_1(t),\ldots,x_n(t)\right)$ and the flow variables $y(t)=\left(y_1(t),\ldots,y_n(t)\right)$ which are defined by the condition:

$$\dot{x}_i(t)=y_i(t)$$

In physics, stock variables refer to position and flow variables to velocity. In economics typical stock variables are capital, debt, etc. i.e. the quantities found in the balance sheet. Typical flow variables are consumption, investment, work, etc. i.e. those quantities which lead to a change in the balance sheet. In mathematical formal terms however also other variables such as prices can be regarded as flow-variables.

The Newtonian behavioral equations for a masse point with mass *M* can either be expressed as second degree differential equations of the position variables or as equivalently as first degree system of differential equations of the position and velocity variables. In the following we always chose the latter form of expression. The change of the velocity coordinates, i.e. the change of flow variables, is described by the physical forces f_i. For reasons of simplicity we only investigate autonomous forces – forces, which are not explicitly dependent on velocity.

$$\begin{aligned} x_i'(t) &= y_i(t) \\ y_i'(t) &= \frac{1}{M}.f_i(x(t), y(t)) \end{aligned} \qquad \text{<1.1>}$$

The simplified form (for the full form see chapter 4) of the basic equations of NCD-models can be stated analogously to physics as the following:

$$\begin{aligned} x_i'(t) &= y_i(t) \\ y_i'(t) &= \mu_i . f_i(x(t), y(t)) \end{aligned} \qquad \text{<1.2>}$$

The functions f_i denote the economic forces. The parameters μ_i can be interpreted as economic power. Economic power therefore is formally equivalent to the reciprocal value of mass. In contrast to mass it is however fully dependent on the coordinates. This concept of economic power allows to interpret the common Keynesian and neoclassical algebraic models as economic NCD-models with one-sided power structures, i.e. models in which certain power factors $\mu_i \to \infty$ and/or $\mu_k = 0$. Standard equilibrium models can be understood as states of NCD-models in which it holds that the economic forces $f_i = 0$.

The factors μ_i are often interpreted as the adjustment speed y_i. This interpretation is only partly correct for two reasons:

- A variable does not adjust on its own. It can only be adjusted by the actions of an agent. The factors μ_i are therefore rather characteristics of the agents than characteristics of the variables
- Even clearer this can be shown by the general equations of NCD-models (see chapter 4). In the general form equation <1.2> reads:

$$\begin{aligned} x_i' &= y_i \\ y_i' &= \sum_j \mu_i^j f_i^j \left(x(t), y(t)\right) \end{aligned}$$

or in the case of there being an additional constraint

$$\begin{aligned} x_i' &= y_i \\ y_i' &= \sum_j \mu_i^j f_i^j \left(x(t), y(t)\right) + \lambda \frac{\partial ZB}{\partial y_i} \\ ZB\left(x, y\right) &= 0 \end{aligned}$$

These equations do not allow an interpretation of the factors μ_i^j as adjustment speeds. They can however be interpreted as the power of an agent j to change the variable y_i by exercising the economic force f_i^j.

In practice there are two common differences between Newtonian models in physics and NCD-models in economics: (1) the physical forces are often only dependent on the position coordinates (i.e. stock variables *x*) while economic forces are most commonly only dependent on flow variables *y*. (2) A common but not general difference is that in physics constraints are predominantly holonomic, while constraints in economics are nearly always non-holonomic.

In chapter 5 we will discuss different closures of economic models and their implicit statement about economic power relationships. Especially we discuss the case that for certain power factors it holds that $\mu_i^j = 0$, which means that an agent j has no power to influence a flow variable y_i, or that he does not wish to do so. This case corresponds to a closure by dropping some of the equations of an over-determined system of equations. We also show that the implementation of Lagrangian multipliers could be interpreted as a special closure of an over-determined equation system.

In chapter 6 we demonstrate that standard economic models can be seen as special cases of NCD-models, in particular as NCD-models with one-sided power relations, that means with power factors which are just zero or infinite. That means that in contrarst to standard economic models, NCD-Models allow also describing situations with mixed power relations.

In chapter 7 we then illustrate the methodology with an NCD-model for the two institutional sectors households and businesses.

An NCD-model describes the behavior of a system in which every agent follows an individual optimization strategy, in order to increase his individual utility. The assumption that this egoistic behavior leads to an overall optimal result for all agents via the 'invisible hand' is at the core of standard economic theory. In many real situations this assumption is however incorrect, as reality is often determined by prisoners dilemma situations in which individual optimization leads to the worst outcome for all agents. In chapter 8 we will present a NCD-model for a continuous state, continuous-time prisoners dilemma, which can be reduced to the standard prisoners dilemma if described with discrete time and two states (cooperation, defection). This method to describe problems of game theory with continuous time and differential equations can be used also for more general problems in game theory. Because of the characterization with differential equations the continuous-time approach is usually easier to solve than the discrete time models.

It needs to be stated that in general not only in the economy, but also in society as a whole, prisoners dilemma situations (and other game theoretical situations) are frequent. Government regulations and laws have to be understood as attempts to overcome the dilemma. Such laws can be modeled as constraints within economic NCD models, which create a situation in which individual optimization might indeed lead to a general optimum. A more detailed discussion under which conditions this is possible is offered elsewhere. [1]

Chapter 9 concludes with an overview over the conceptual and methodological advantages of NCD-models for the understanding of the economy and the dynamics of general economic systems.

[1] Glötzl, E. `The prisoners dilemma as NCD-model. The conditions under which individual optimization leads to a general optimum.´

2. Literature Review

2.1. Economics and Physics

Since the beginnings of modern economics the endeavor to construct the discipline along the principles of physics has been omnipresent. Already Adam Smith showed his fascination of Newton in 'History of Astronomy' (A. Smith, 1795), a fascination that also reveals itself in the methodology of his economic theory as numerous studies show (for an overview over the literature see (Redman, 1993)). For instance Smith's theory of value, developed in 'The Wealth of Nations' (A. Smith, 1776), is to be regarded as the counterpart to the concept of energy in physics. In its essence the Smithian theory of value was adopted by all following classical economists. In this point of view value is conserved just like energy within the circular flow (Mirowski 1989).

As a result of the impressive scientific advances in the field of physics and chemistry during the 18th and 19th century, the social sciences increasingly tried to imitate the methodology of the natural sciences. Due to the complex and interdependent structure of social phenomena these attempts were of limited success. Only in the field of economics the orientation towards the methodology of physics seemed promising by focusing exclusively on competitive markets, prices and quantities and limiting investigation to rational human behavior (Rothschild, 2002).

The decisive step in this development was brought by Léon Walras' General Equilibrium Theory (Walras, 1874),, and the simultaneously published contributions by Stanley Jevons and the introduction of the 'calculus of pleasure and pain'. This work marked the end of the era of classical economics and was the birth of neoclassical economics. The assumption that the behavior of all economic agents could be described by utility functions was at the core of this new school of thought. All economic questions involving psychological and social factors were deliberately ignored. Until today these central principles are the foundation of standard economics. The Arrow-Debreu General Equilibrium Model, is seen as the first complete model describing a general equilibrium based on the Walrasian theory (Arrow & Debreu, 1954).

The endeavor to identify further similarities between physics and economics, as well as the goal to still increase the orientation of the methodology of economics towards economics was continued by Paul Samuelson. It was his work which was decisive for mathematics to become the standard method in economics. Moreover, Samuelson identified several similarities between physics and economics, arguing that classical thermodynamics and neoclassical economics are related in their common search of a basis for the optimization of observed behavior. In physics this is achieved by maximizing free energy, in economics by maximizing utility (James B. Cooper, 2010; J. B. Cooper & Russell, 2011). In a similar vein Smith und Foley (2008) attempt to adopt the model structure of thermodynamics as well as the principle of entropy in economics and show under which circumstances and conditions this is possible (E. Smith & Foley, 2008).

In contrast to that, other authors such as Kümmel (2011) have tried to investigate the consequences of the existence of the first and second law of thermodynamics within the economy, rather than trying to find suitable analogies for economics.

2.2. Economics and Power

> *„Economics as a separate science is unrealistic and misleading if taken as a guide in practice. It is one element – a very important element, it is true - in a wider study, the science of power.“* (Russell, 1938, p. 108)

The goal to imitate physics led to the fact that questions of power were ignored for two distinct reasons. On the one hand there was the idea that while power relations might play a role in the short term, in the long run are irrelevant due to inevitable economic laws. This argument is most prominently made in 'Macht oder ökonomisches Gesetz' by Eugen von Bahm-Böwerk (Böhm-Bawerk, 1914). To some extent the idea can also be found in later discussions, for example in the Lucas-critique. On the other hand as a result of the self-imposed restriction to follow a strictly mathematical methodology questions of power were left to the disciplines of psychology and the social sciences.

Those economic theories which explicitly deal with questions of power, such as Marxian theory where class struggle and distribution put power relations center stage (Foley, 1986) or parts of institutional economics, have been marginalized and are a small minority in modern economics. In contrast, neoclassical orthodoxy limits itself to monopoly power of companies and negotiating power of workers on the labor market in its understanding of power, as the AS-AD model which can be found in every standard economics textbook (see for instance Blanchard & Illing, 2009). This view of power fully neglects the fact that in reality all agents have a more or less pronounced power to assert their interest, be it in the market process or by influencing the political and social framework. Finally, power can not only be a means to economic actions but an end in itself (Rothschild, 2002).

2.3. Closure of economic models

An important body of literature has dealt with the problem of closure of economic models. Closure is the task of making an under- or over-determined equation system, usually including macroeconomic accounting identities, solvable. Therefore, "[…] prescribing closures boils down to stating which variables are endogenous or exogenous[…]"(Taylor, 1991, p. 41), as some behavioral equations need to be omitted to yield a determined system. Already in 1956, Kaldor set out to investigate the model structures of different schools of economic thought and thereby implicitly also discussed diverse closures of Ricardian, Marxist, Keynesian and Neoclassical models (Kaldor, 1955). In a similar vein Sen (1963) further showed that in fact Neo-classical and Neo-keynesian models of distribution can be derived from the same equation system and differ in their essence the choice of which equations are dropped i.e. in the assumptions about causality. Marglin (1987) on the other hand approaches the problem from the other direction and argues that Neo-classical, Neo-keynesian and Neo-marxist models have a common underdetermined core equational system which is closed using different behavioral rules. More recently, BarBosa-Filho (2001, 2004) investigated three alternative closures of Keynesian models with investment, net exports or autonomous consumption as driving force of aggregate demand.

2.4. The invisible hand does not always lead to the optimum

Adam Smith's analysis of the economy and his theory that egoistic behavior of all agents will lead to the optimal result in the end, often summarized under the metaphor of the 'invisible hand', is a central thought in economics until today. This is the case even though many authors have shown that individual optimization does not necessarily lead to an overall optimum. For instance John Nash, the founder of game theory, showed that individually optimal behavior can lead to stable equilibria which constitute the worst scenario for all players (Nash, 1951). Throughout the second half of the 20th century there has been significant work, not least with experiments, trying to understand to what extent such prisoners dilemmas play a role in reality as Giza (2013) illustrates.

This method to describe problems of game theory with continuous time and differential equations can be used also for more general problems in game theory (Cvitanic 20011). Because of the characterization with differential equations the continuous-time approach is usually easier to solve than the discrete time models (Sannikov 2012).

3. The basic principles illustrated in easy examples

In chapter 6 we will postulate the general case how common economic models can be described as special cases of NCD models. In order to illustrate the basic ideas several simple examples will be discussed in the following.

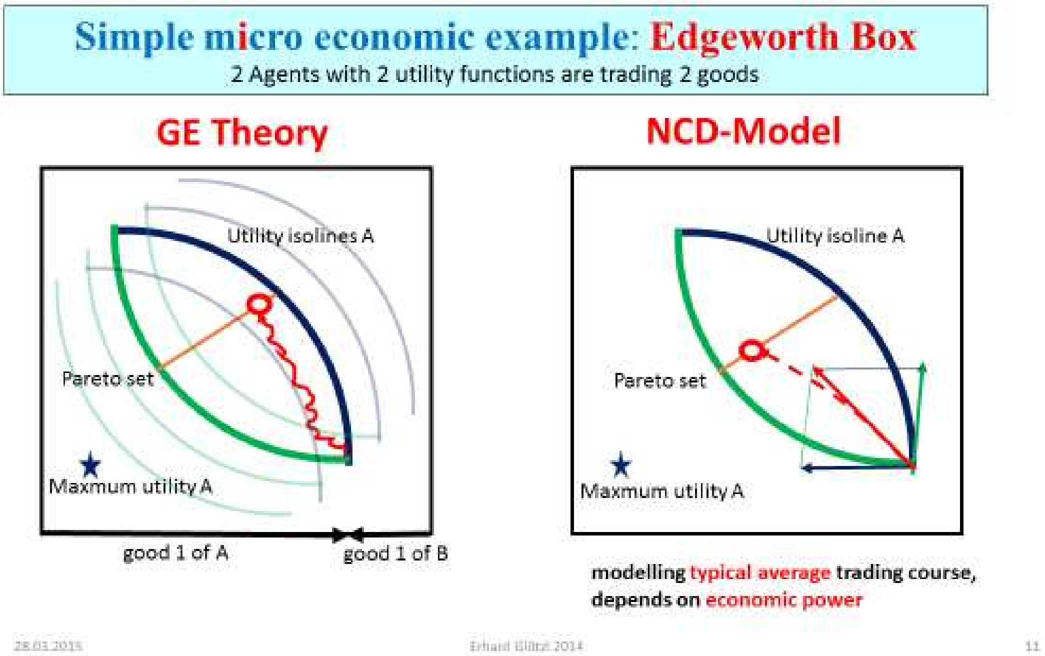
3.1. Microeconomic example: Edgeworth-Box

An Edgeworth-Box is a graphic tool in microeconomics designed to describe the equilibrium in a pure barter economy with only two agents A, B and two goods. Starting from an allocation of the goods between the agents, they reach a Pareto-optimum by trading along a contract curve. In this optimum the utility of no agent can be increased without simultaneously decreasing that of another agent. Equilibrium Theory makes no assertions about the contract curve, i.e. the way how the optimum is reached, nor which of the possible Pareto-equilibria is reached.

The nature of an NCD model lies exactly in describing the dynamics of the contract curve. Modeling the contract curve also yields the position of the equilibrium.

Evidently, it cannot be predicted in the individual case on which contract curve the agents reach a result which is beneficial for both (Pareto-optimum). It makes sense however, to understand the typical negotiation path as mean of the negotiation paths in similar situations and to model the typical negotiation path of two agents in terms of an NCD model in the following way:

The negotiation strategy of both agents is based on optimizing their individual utility function. Each agent will therefore employ an 'economic force' in the direction which corresponds to the highest increase of his utility function. The more his gain in utility, the higher will be the force he employs. The direction and magnitude can be described exactly by the gradient of the utility function, which is perpendicular to the lines of constant utility. The extent to which an agent can achieve his goal does not only depend on the force he and the other agent employed, but also on their respective 'economic power'. The actual change in the allocation of goods will therefore be directed towards the resulting force of the economic forces employed by the agents, weighted by their respective power factors. Evidently, the negotiation result also depends on the agents' power factors.



Denoting:

$x_1^A, x_2^A, x_1^B, x_2^B$ the amounts of goods 1, 2 of the agents A, B

$x_1^{A\prime}, x_2^{A\prime}, x_1^{B\prime}, x_2^{B\prime}$ the change over time of goods 1, 2 of the agents A, B

$U^A(x_1^A, x_2^A)$ and $U^B(x_1^B, x_2^B)$ the utility functions of A, B

μ^A, μ^B the respective economic power factors of A, B

m_1, m_2 the total amounts of goods 1, 2

then the above can be formalized in the following way:

$$x_1^{A\prime} = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} + \mu^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_1^A} = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} + \mu^B \frac{\partial U^B((m_1 - x_1^A),(m_2 - x_2^A))}{\partial x_1^A}$$
$$x_2^{A\prime} = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} + \mu^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_2^A} = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} + \mu^B \frac{\partial U^B((m_1 - x_1^A),(m_2 - x_2^A))}{\partial x_2^A}$$

The equilibrium (Pareto-optimum) which is dependent on the respective economic power factors is then given as

$$0 = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} + \mu^B \frac{\partial U^B((m_1 - x_1^A),(m_2 - x_2^A))}{\partial x_1^A}$$
$$0 = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} + \mu^B \frac{\partial U^B((m_1 - x_1^A),(m_2 - x_2^A))}{\partial x_2^A}$$

3.2. Macroeconomic example: 'Saving vs. Investment'

3.2.1. Problem

Two aspects will be illustrated with this simple example:

1. How can the analogy between economic models and physics be understood? The answer is that NCD models describe the dynamics of economics in analogy to movement under constraints in classical mechanics.
2. Is saving the precondition for investment or is the opposite the case and saving follows from investment? Or put differently: What is the relationship between the neoclassical assumption that saving leads to investment to the Keynesian assumption that saving follows from investment? The answer is that in the end it depends on the distribution of power between savers and investors.

3.2.2. Physics: Movement on an inclined plane:

Denoting:

x_1, x_2 the spatial coordinates,

v_1, v_2 the velocity coordinates and

v_1', v_2' their derivatives with respect to time

M the inertial mass

f_1, f_2 the coordinates of the forces exerted on the mass M

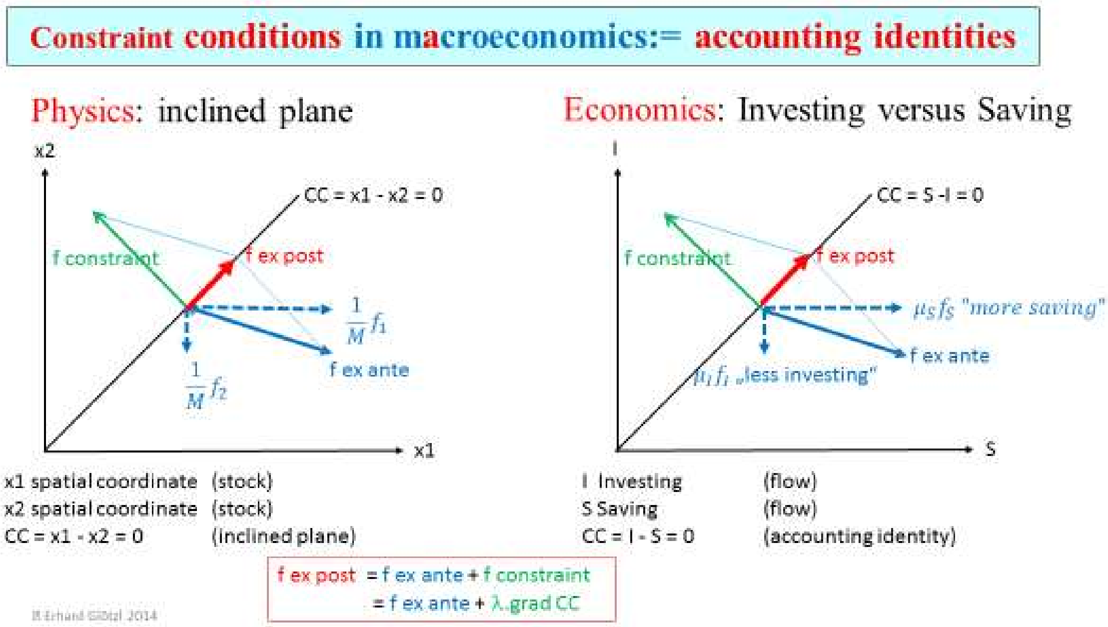
$ZB(x_1, x_2) = x_1 - x_2 = 0$ the constraint describing the inclined plane with 45°

λ the Lagrange-multiplier

The movement of the mass point on the inclined plane is the described by the following Newton-Lagrange equations:

$$
\begin{aligned}
v_1' &= \frac{1}{M} f_1 + \lambda \frac{\partial ZB}{\partial x_1} = \frac{1}{M} f_1 + \lambda \\
v_2' &= \frac{1}{M} f_2 + \lambda \frac{\partial ZB}{\partial x_2} = \frac{1}{M} f_2 - \lambda \\
ZB(x_1, x_2) &= x_1 - x_2 = 0
\end{aligned}
\qquad \text{<3.1>}
$$

The respective first terms $\frac{1}{M} f_1$ und $\frac{1}{M} f_2$ describe the coordinates of the 'ex ante' force while the respective second terms $\lambda \frac{\partial ZB}{\partial x_1}$ and $\lambda \frac{\partial ZB}{\partial x_2}$ describe the coordinates of the 'constraint force'. The sum of both terms is denoted as 'ex post' force, as it describes the factual resulting movement under the constraint.



3.2.3. The NCD model and 'Saving vs. Investment' in analogy to physics

Denoting:

I Investment and S Saving and I', S' their derivatives with respect to time

IF any investment function, e.g. $I = IF := i_0 + i_1 Y$

SF any saving function, e.g.

$S = SF := s_0 + s_1 Y$ or $\quad (s_0, s_1$ constant, Y $BIP)$

$S = SF := s_L\, p_L\, L - s_P\, P \quad$ (s_L saving rate of work income, p_L hourly wage, L work, s_P saving rate of profit, P profit)

μ_S, μ_I the economic power of the savors and investors respectively

$ZB(I,S) = I - S = 0$ the accounting identity of investment and saving as a constraint

λ Lagrange multiplier

Assuming that the investor will try to invest harder the more he is currently behind his investment plan (investment function) and vice versa, his behaviour can be expressed formally defining the economic force f_I he employs to change his investment in the following way:

$$f_I = (IF - I)$$

Defining in turn

$$f_S = (SF - S)$$

a NCD model can be set up easily in the following way:

$$\begin{aligned} I' &= \mu_I f_I + \lambda \frac{\partial ZB}{\partial I} = \mu_I (IF - I) + \lambda \frac{\partial ZB}{\partial I} = \mu_I (IF - I) + \lambda \\ S' &= \mu_S f_S + \lambda \frac{\partial ZB}{\partial S} = \mu_S (SF - S) + \lambda \frac{\partial ZB}{\partial S} = \mu_S (SF - S) + \lambda \\ ZB(I,S) &= I - S = 0 \end{aligned} \qquad \text{<3.2>}$$

It is visible immediately that the movement on an inclined plane and the development of saving and investment can be described by the analogous equations <3.1> and <3.2>. The only substantial difference lies in the fact that the mass M is independent of the coordinates while the power factors μ_I, μ_S are dependent on them. It is also typical that forces in physics only depend on the spatial coordinates (here x_1, x_2), while in economics they often only depend on the flow variables (here I, S).

3.2.4. Necolassical and Keynsian conceptions of investment and as special cases of a NCD model with one-sided power relations

Dividing equation (2) of the NCD model <3.2>

$$\begin{aligned} &(1) \qquad I' = \mu_I (IF - I) + \lambda \\ &(2) \qquad S' = \mu_S (SF - S) + \lambda \\ &(3) \qquad ZB(I,S) = I - S = 0 \end{aligned}$$

by μ_S and letting $\mu_S \to \infty$ yields

$$\begin{aligned} &(1) \qquad I' = \mu_I (IF - I) + \lambda \\ &(2) \qquad S = SF \\ &(3) \qquad ZB(I,S) = I - S = 0 \end{aligned}$$

Setting $\mu_I = 0$ in (1) and taking the derivative of the constraint (3) equation (1) reads:

$$(1) \qquad I' = S'$$

This equation can however be derived from (3) by differentiation and can therefore be omitted. This can be interpreted as the change of I only depending on the change of S and the constraint.

The power relations $\mu_S = \infty$ and $\mu_I = 0$ therefore describe the neoclassical assumption that investment is entirely determined by saving. Exercising the same transformations with the power factors $\mu_I = \infty$ $\mu_S = 0$ yields the contrary Keynesian perspective that saving is determined entirely by the investment behavior.

A major conclusion is that in reality power relations will neither correspond to the neoclassical nor the Keynsian perception. In reality mixed power relations are to be assumed. This in turn means that the reality can be described more adequately using NCD models.

3.2.5. General equilibrium model under constraint as a stationary solution of a NCD model

Starting point for the general equilibrium theory are utility functions. The utility functions corresponding to the economic force $f = (f_1, f_2)$:

$$f_I = (IF - I)$$
$$f_S = (SF - S)$$

are given as

$$U_I = \frac{1}{2}(IF - I)^2$$
$$U_S = \frac{1}{2}(SF - S)^2$$

In this case the individual utility functions can be aggregated to one 'master utility function' MU

$$MU = \frac{1}{2}(IF - I)^2 + \frac{1}{2}(SF - S)^2$$

so that the gradient of MU yields the economic force:

$$f = grad\, MU$$

In terms of coordinates this means:

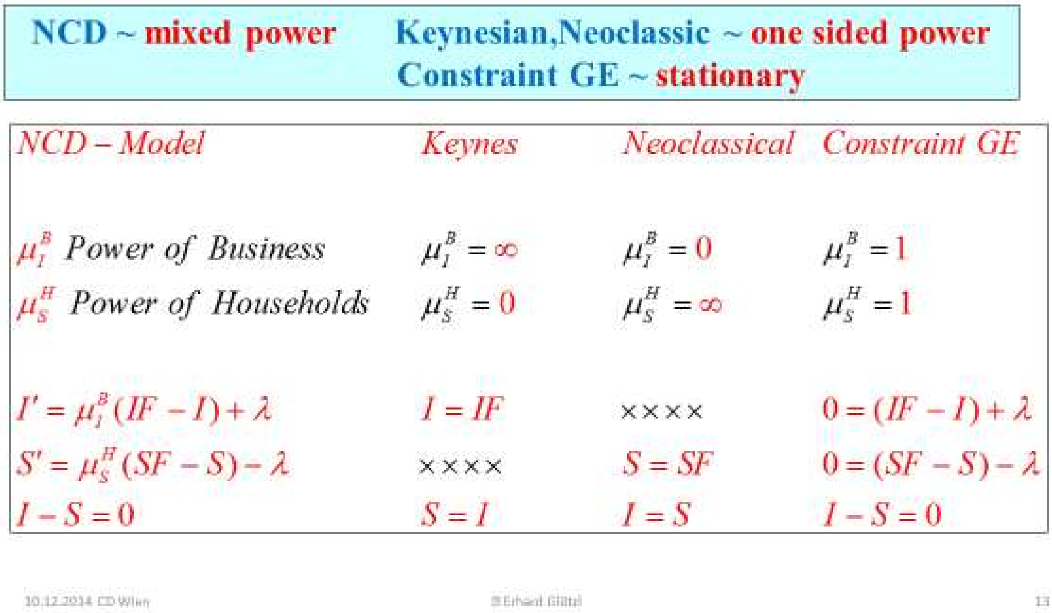
$$f_I = \frac{\partial MU}{\partial I} = (IF - I)$$
$$f_S = \frac{\partial MU}{\partial S} = (SF - S)$$

The model for general equilibrium theory is given by the maximization of the master utility function under the constraints $ZB(I,S) = I - S = 0$, i.e. by setting the gradient of MU under consideration of the Lagrange multipliers zero. This yields the equation system:

$$
\begin{aligned}
0 &= \frac{\partial MU}{\partial I} + \lambda \frac{\partial ZB}{\partial I} = (IF - I) + \lambda \\
0 &= \frac{\partial MU}{\partial S} + \lambda \frac{\partial ZB}{\partial S} = (SF - S) - \lambda \qquad \text{<3.3>} \\
ZB(I,S) &= I - S = 0
\end{aligned}
$$

Starting on the other hand from NCD-Modell <3.2>

it becomes obvious that the General Equilibrium model <3.3> corresponds to the stationary solutions of the NCD-model with power factors $\mu_I = 1$ and $\mu_S = 1$.



NCD ~ mixed power	Keynesian, Neoclassic ~ one sided power Constraint GE ~ stationary		
NCD – Model	*Keynes*	*Neoclassical*	*Constraint GE*
μ_I^B *Power of Business*	$\mu_I^B = \infty$	$\mu_I^B = 0$	$\mu_I^B = 1$
μ_S^H *Power of Households*	$\mu_S^H = 0$	$\mu_S^H = \infty$	$\mu_S^H = 1$
$I' = \mu_I^B(IF - I) + \lambda$	$I = IF$	$\times\times\times\times$	$0 = (IF - I) + \lambda$
$S' = \mu_S^H(SF - S) - \lambda$	$\times\times\times\times$	$S = SF$	$0 = (SF - S) - \lambda$
$I - S = 0$	$S = I$	$I = S$	$I - S = 0$

3.3. ‘Creditor vs. debtor’ as analogous model to ‘saving vs. investment’

In a closed economy the sum of all receivables R is always equal to the sum of all debts D, i.e. the accounting identity (constraint) $R = D$ always holds. The development of these quantities with respect to time is on the one hand dependent on the sum of creditors’ and debtor’s respective interest and on the other hand on their power to assert their interest[2]. Therefore the ‘Creditor vs. debtor’ model can be understood in full analogy to the ‘saving vs. investment’ model.

[2] In Glötzl (1999) the ‘fundamental paradox of money economies’ is postulated, describing that in an economy in which credits are measured in monetary quantities, the power of the sum of creditors to increase their credit will always be greater than the power of the sum of debtors to decrease their debt. In other words there is an ‘powerlessness of the debtors’ and a „power of the creditors’. These power relations are the reason for debt traps and the constant growth of credit and debt.

Similar Models

• Model	variables		constraint condition
• Inclined plane	x1	x2	x1=x2
• Investment versus Saving	I	S	I=S
• Creditors versus Debitors	R	D	R=D

3.4. Subsistence economy

3.4.1. Overview

A particularly simple economic case for our model is a subsistence economy with only one agent. More precisely that means there is only one agent, who produces the consumption goods C for his own consumption (and therefore produces no investment goods), who produces these goods with his own work L and without capital and who consumes all produced goods immediately without storing anything. We first set up this model with the General Equilibrium and then with the Keynesian approach. The aim is not to ascertain which model is ‘correct’, but to demonstrate the mathematical structure of the model equations associated with these model types. Then we will describe this subsistence economy with two NCD models with different utility functions and show:

1. the fundamental analogy of these NCD models to the movement of a masse point on an inclined plane

2. that the General Equilibrium model can also be interpreted as a state of the first NCD model where all economic forces are equal to zero.

3. That the Keynesian model can be interpreted as a special case of the second NCD model with one-sided power relations.

3.4.2. The General Equilibrium model

The first pillar for the neoclassical equilibrium model is the agent’s utility function U , which is the sum of the utility functions U_C for the consumption C and U_L for the work L . As for our purposes the form of the specific utility function is not relevant for reasons of simplicity we chose

$$U(C,L) = U_C(C) + U_L(L) = 2\sqrt{C} - \frac{1}{2}L^2$$

The second pillar is a Cobb-Douglas production function. As capital is not used as a factor of production it takes the form:

$$Y = L$$

The third pillar is the accounting identity for the use of Y:

$$Y = C$$

This gives a General Equilibrium model which consists of the maximization of U under the constraint $ZB(C,L) = C - L = 0$. This yields the model equations:

$$\begin{array}{lll} \frac{\partial U}{\partial C} + \lambda \frac{\partial ZB}{\partial C} = 0 & bzw, & \frac{1}{\sqrt{C}} + \lambda = 0 \\ \frac{\partial U}{\partial L} + \lambda \frac{\partial ZB}{\partial L} = 0 & bzw. & -L - \lambda = 0 \\ ZB(C,L) = 0 & bzw. & C - L = 0 \end{array} \qquad \text{<3.4>}$$

The immediate solution is given by: $C = L = 1$.

The associated stock variable for consumption $SC := \int C\,dt$ of the flow variable C and the associated stock variable for work $SL := \int L\,dt$ of the flow variable L are irrelevant in this model. Overall the model structure is therefore as described in <1.1>, when the constraints are taken into account additionally.

3.4.3. The Keynesian Model

The first pillar for the Keynesian model is a consumption function which we will assume to be of the following form for reasons of simplicity:

$$C = i_0 + i_1 Y$$

The second pillar is again the accounting identity for the use of Y:

$$Y = C$$

This yields the model equations:

$$\begin{array}{l} C = i_0 + i_1 Y \\ Y = C \end{array} \qquad \text{<3.5>}$$

The result is given immediately by: $C = \dfrac{i_0}{1 - i_1}$

Again it holds that the associated stock variable for consumption $SC := \int C\,dt$ of the flow variable C and the associated stock variable for work $SL := \int L\,dt$ of the flow variable L are irrelevant in this model. Overall the model structure is therefore also as lid out in <1.1>, when the constraints are taken into account additionally.

3.4.4. The first NCD-Modell

The first NCD model starts from the same utility function as the neoclassical model does:

$$U(C,L) = 2\sqrt{C} - \frac{1}{2}L^2$$

The more an agent's utility increases through consumption and the bigger his economic power μ_C , the stronger is the economic force the agent employs to increase his consumption. Similarly, the agent will employ a greater economic force to reduce his work, the more his utility thereby increases and the bigger his economic power μ_L is. This yields the two behavioral equations:

$$\begin{aligned} C' &= \mu_C \frac{\partial U}{\partial C} = \mu_C \frac{1}{\sqrt{C}} \\ L' &= \mu_L \frac{\partial U}{\partial L} = -\mu_L L \end{aligned}$$

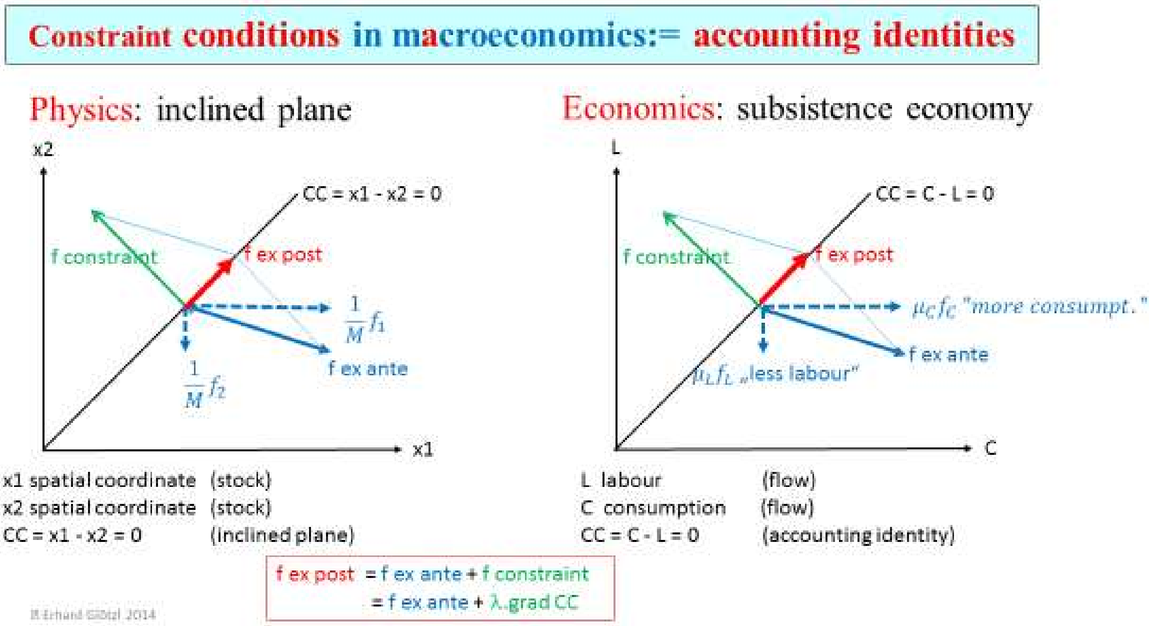
These equations describe the dynamics without taking into account the constraint. This constraint is defined by the fact that the agent can only consume what he produced. The dynamics without the constraint will be called 'ex ante' dynamics in the following. Choosing the units adequately the constraint reads:

$$ZB(C,L) = C - L = 0$$

Due to this constraint there are additional constraint forces, which are perpendicular to the constraint. This means that the constraint forces are a multiple λ (Lagrange multiplier) of the constraint's gradient. The basic system of equations is then given as:

$$\begin{aligned} SC' &= C \\ SL' &= L \\ C' &= \mu_C \frac{\partial U}{\partial C} + \lambda \frac{\partial ZB}{\partial C} = \mu_C \frac{1}{\sqrt{C}} + \lambda \\ L' &= \mu_L \frac{\partial U}{\partial L} + \lambda \frac{\partial ZB}{\partial L} = -\mu_L L - \lambda \\ ZB(C,L) &= C - L = 0 \end{aligned} \qquad \text{<3.6>}$$

This system of equations describes the true 'ex post' dynamics which result from the constraint. The structure is identical to <1.2> . The 'ex post' dynamics of the NCD model is therefore given by the interaction between the 'ex ante' forces and the constraint force (see figure 1).



3.4.5. The second NCD model

The second NCD model starts from a utility function U_C for the consumption:

$$U_C(C,Y) = -\frac{1}{2}(i_0 + i_1 Y - C)^2$$

This utility function can be interpreted in the following way: the agent (for whatever reason) desires a consumption in the amount $(i_0 + i_1 Y)$. His utility increases the closer his consumption approaches his desired consumption $(i_0 + i_1 Y)$. The utility function U_L for labor L is assumed to be identical to the previous case. The agent's utility function U therefore reads:

$$U(C,L,Y) = U_C(C,Y) + U_L(L) = -\frac{1}{2}(i_0 + i_1 Y - C)^2 - \frac{1}{2}L^2$$

and due to the accounting identities for the creation and use of Y (choosing adequate units) it holds that:

$$\begin{aligned} ZB_1(L,Y) &= L - Y = 0 \\ ZB_2(C,Y) &= C - Y = 0 \end{aligned} \qquad \text{<3.7>}$$

From ZB_1 results that $Y = L$. When this is inserted into the other model equations ZB_1 can be omitted and which yields:

$$
\begin{aligned}
&SC' = C \\
&SL' = L \\
&C' = \mu_C \frac{\partial U}{\partial C} + \lambda \frac{\partial ZB}{\partial C} = \mu_C (i_0 + i_1 L - C) + \lambda \\
&L' = \mu_L \frac{\partial U}{\partial L} + \lambda \frac{\partial ZB}{\partial L} = -\mu_L L - \lambda \\
&ZB(C, L) = C - L = 0
\end{aligned}
\qquad \text{<3.8>}
$$

3.4.6. The analogy between NCD models and movement on an inclined plane

Looking at the constrained movement of a mass point M on an inclined plane with an inclination of 45 degrees, which is subject to two forces f_1, f_2 in the direction of the two coordinate axis (see figure 1), reveals a close analogy to the previously shown models of a subsistence economy. The model equations are:

$$
\begin{aligned}
&x_1' = y_1 \\
&x_2' = y_2 \\
&y_1' = \frac{1}{M} f_1 + \lambda \\
&y_2' = \frac{1}{M} f_2 - \lambda \\
&ZB(x_1, x_2) = x_1 - x_2 = 0
\end{aligned}
\qquad \text{<3.9>}
$$

The general structure of NCD model equations of the two NCD models shown in <3.6> is nearly identical to those of movement on an inclined plane <3.9> . The only differences are:

1. In classical mechanics the inertial mass is not dependent on the direction in which the mass point is accelerated. In contrast the economic power can assume different values for different variables.

2. The economic forces in this example depend solely on flow variables, as is mostly the case in economics. In contrast the physical forces very often depend solely on stock variables (spatial coordinates), as is the case in this example.

3. The constraints are non-holonomic in the subsistence economy model as is nearly always the case in economics. This means they also depend on flow variables. The constraints present in the movement on an inclined plane however only depend on stock variables (spatial coordinates). The constraints are therefore holonomic which is very often the case in mechanics.

3.4.7. The General Equilibrium Model as a state without power of the first NCD model

It is apparent that the general equilibrium model <3.4> identical with the results from the first NCD model <3.6> , in which ‘ex post’ forces disappear with $\mu_C = 1$ and $\mu_L = 1$.

3.4.8. The Keynsian model as NCD model with one-sided power relations

In the following we demonstrate that the Keynesian model <3.5> is identical with the second NCD model with the special power factors $\mu_C = \infty$ and $\mu_L = 0$. If in equation <3.8> $\mu_L = 0$ is chosen the system of equations reads:

$$
\begin{array}{ll}
(1) & SC' = C \\
(2) & SL' = L \\
(3) & C' = \mu_C(i_0 + i_1 L - C) + \lambda \\
(4) & L' = -\lambda \\
(5) & ZB(C,L) = C - L = 0
\end{array}
$$

This system of differential and algebraic equations can be simplified by taking the derivative of the algebraic equation (5), which is the constraint.

$$C' - L' = 0$$

Equation 4 then reads $\lambda = -C'$. Inserting this then yields:

$$
\begin{array}{ll}
(1) & SC' = C \\
(2) & SL' = L \\
(3) & C' = \frac{1}{2}\mu_C(i_0 + i_1 L - C) \\
(4) & L' = C' \\
(5) & ZB(C,L) = C - L = 0
\end{array}
$$

Equation (4) can be omitted as it directly follows from equation (5). Dividing (3) by μ_C and letting $\mu_C \to \infty$ the system of equations is:

$$
\begin{array}{ll}
(1) & SC' = C \\
(2) & SL' = L \\
(3) & C = i_0 + i_1 L \\
(5) & ZB(C,L) = C - L = 0
\end{array}
$$

Due to <3.7> it holds that $L = Y$. Therefore the model equations are identical with those in <3.5> and therefore identical to the Keynesian model.

4. Model Equations of general Newtonian Constrained Dynamic Models (NCD models)

4.1. The general structure of NCD models

For any number of agents (independent from the fact whether these agents are individual economic agents of a representative agent for a certain group or sector) the general concept of NCD models can be described verbally in the following way:

- Starting from an economic state at time t, which is described by n stocks x_i and n flows y_i $(i = 1, \dots, n)$, every one of m agents $(j = 1, \dots, m)$ is interested in changing this state and has an economic power μ_i^j to assert his interest.
- Therefore, he employs an economic force f_i^j to change the flows in the direction which is beneficial for him. The effective force is directly proportional to the economic force f_i^j he employed and his economic power μ_i^j. The interaction between all forces and power factors determine the 'ex ante' dynamics.
- l constraints ZB_k $(k = 1, .., l)$, such as accounting identities evoke l additional constraint forces. The 'ex post' dynamics is determined by n interest-led forces (times the power factors μ_i^j) plus l constraint forces. The l constraint forces are given analogously to classical mechanics as the l Lagrange multipliers λ_k times the gradient of ZB_k.

As the models can be formulated substantially more easily using continuous time and differential equations rather than difference equations in addition to better revealing analogies to classical mechanics, this approach will be chosen. Generally an equivalent formulation in discrete time would however always be possible. Similarly, adding stochastic terms to the model would not pose any problem. For reasons of simplicity this will not be done in the following.

The general structure of NCD models can be illustrated with the following equational model: (number of variables $(i = 1, \dots, n)$, number of agents $(j = 1, \dots, m)$, number of constraints $(k = 1, \dots, l)$

$$\boxed{\begin{aligned} x_i' &= y_i \\ y_i' &= \sum_{j=1}^{m} \mu_i^j f_i^j(x, y) + \sum_{k=1}^{l} \lambda_k \frac{\partial ZB_k(x, y)}{\partial y_i} \\ ZB_k(x, y) &= 0 \end{aligned}} \qquad \text{<4.1>}$$

Put simply, NCD models can be regarded as SFC (Stock-Flow-Consistent) and potentially also AB (Agent-Based) models with continuous time and constraints.

4.1.1. Comment 1:

When it holds that for an i_0 and a certain j_0 that $\mu_{i_0}^{j_0} \to \infty$, the differential equation reads

$$y_{i_0}' = \sum_{j=1}^{m} \mu_{i_0}^{j} f_{i_0}^{j_0}(x,y) + \sum_{k=1}^{l} \lambda_k \frac{\partial ZB_k(x,y)}{\partial y_{i_0}} \qquad \text{<4.2>}$$

by dividing by $\mu_{i_0}^{j_0}$ this yields the algebraic equation:

$$0 = f_{i_0}^{j_0}(x,y)$$

This means that also algebraic behavioral equations can be interpreted ad NCD behavioral equations with infinite power factors.

4.1.2.Comment 2:

A special case of Comment 1 is to look at the models with one or more parameters p_m :

$$\begin{aligned}
&x_i' = y_i \\
&y_i' = \sum_{j=1}^{m} \mu_i^j f_i^j(x,y,p) + \sum_{k=1}^{l} \lambda_k \frac{\partial ZB_k(x,y,p)}{\partial y_i} \\
&p_m = f_m^p(x,y,p) \\
&ZB_k(x,y,p) = 0
\end{aligned}$$

4.1.3.Comment 3:

If it holds for a certain i_1 and a certain j_1 that for all $j \neq j_1$ it is true that $\mu_{i_1}^j = 0$, the differential equation will read

$$y_{i_1}' = \mu_{i_1}^{j_1} f_{i_1}^{j_1}(x,y) + \sum_{k=1}^{l} \lambda_k \frac{\partial ZB_k(x,y)}{\partial y_{i_1}}$$

In this case the power factor $\mu_{i_1}^{j_1}$ can also be interpreted as adjustment speed. This interpretation is however only partially adequate due to two reasons:

- A variable does not adjust on its own, it can only be adjusted by an agent's actions. The factors μ are therefore rather characteristics of the agents than of the variables.
- Most importantly the interpretation of the factors as adjustment speeds is not tenable anymore for the general case of the behavioral equation of NCD models in <4.1>. They can however very well be interpreted as the power of agent j to change variable y_i when applying a force f_i^j.

4.1.4.Comment 4:

In terms of comment 2 and 3 the 'parameters' can either be seen as a variable with infinite adjustment speed or as a variable with an associated agent who possesses infinite power to change it.

4.2. NCD-models with individual utility functions

For economic models the case in which the economic forces can be described as gradients of an individual utility function U^j of an agent j is of special importance. It is only dependent on the flow variables, i.e. if it holds:

$$f_i^j(x,y) = \frac{\partial U^j(y)}{\partial y_i}$$

The path-independent economic force $grad_y U^j(y)$ associated to the utility function $U^j(y)$ describes the 'rational' preferences of agent j. For these cases the basic system of equations reads:

$$\begin{aligned} x_i' &= y_i \\ y_i' &= \sum_{j=1}^{m} \mu_i^j \frac{\partial U^j(y)}{\partial y_i} + \sum_{k=1}^{l} \lambda_k \frac{\partial ZB_k(x,y)}{\partial y_i} \\ ZB_k(x,y) &= 0 \end{aligned}$$

This system of equations can be interpreted in the following way: the more an agent's individual utility will increase, the higher will be the 'rational' preference respectively the economic interest and thereby the economic force an agent will employ in order to change a variable. The factual change arises as an interplay of all these forces and constraint forces. It is thus the resultant force of the agents' individual optimization strategies.

A core assumption of standard economics is that in a market economy the 'invisible hand' will lead to an optimal result for all market participants, or put more widely, that total utility will be maximal when all market participants seek to maximize their own utility. That this in no way is always the case will be illustrated with the example of a continuous prisoners dilemma in chapter 8. NCD models allow investigating the question under which circumstances this core assumption of market economics is fulfilled or which constraints are necessary so that individual optimization leads to a general optimum. These problems will be discussed more in depth elsewhere[3].

4.3. NCD models with a master utility function

Modern neoclassical models, especially DSGE models, in general do not assume that every agent tries to maximize his own utility, rather they assume that the entire economic system is determined by the maximization of one single function. For the sake of a clear distinction we will call this utility function 'master utility function' MU. In case such a master utility function exists the basic system of equations can be written as:

[3] E. Glötzl, The prisoners dilemma as an NCD model. The conditions under which individual optimization leads to a general optimum. *(work in process)*

$$
\begin{aligned}
&x_i' = y_i \\
&y_i' = \frac{\partial MU(y)}{\partial y_i} + \sum_{k=1}^{l} \lambda_k \frac{\partial ZB_k(x,y)}{\partial y_i} \\
&ZB_k(x,y) = 0
\end{aligned}
$$

With respect to the master utility function to major questions arise:

1. Under which conditions does such a master utility function exist such that its maximization determines the entire system? This question is often called the problem of aggregability of utility functions. As to content this question entails under which conditions it is justified to describe an economic system as neoclassical model.
2. Under which conditions does the maximization of the master utility function MU also lead to the maximization of total utility GU, if defined as sum of the utility of all agents?

In order to answer the first question whether a master utility function exists three sufficient conditions can be defined, which we always formulate only for two individual utility functions with two flow variables for the sake of simplicity. It therefore needs to be illustrated under which conditions for 2 individual utility functions $U^A(y_1,y_2), U^B(y_1,y_2)$ and individual power factors $\mu_1^A, \mu_1^B, \mu_2^A, \mu_2^B$ a master utility function MU exists so that:

$$
\begin{aligned}
\mu_1^A \frac{\partial U^A(y_1,y_2)}{\partial y_1} + \mu_1^B \frac{\partial U^B(y_1,y_2)}{\partial y_1} = \frac{\partial MU(y_1,y_2)}{\partial y_1} \\
\mu_2^A \frac{\partial U^A(y_1,y_2)}{\partial y_2} + \mu_2^B \frac{\partial U^B(y_1,y_2)}{\partial y_2} = \frac{\partial MU(y_1,y_2)}{\partial y_2}
\end{aligned}
$$

This holds in the following 3 cases:

1. „quasi-linear“:

$$
\begin{aligned}
&\textit{If}\ \ U^A(y_1,y_2) = a_0 + p_1^A y_1 + p_2^A y_2, \qquad U^B(y_1,y_2) = b_0 + p_1^B y_1 + p_2^B y_2 \\
&\Rightarrow MU(y_1,y_2) := a_0 + (\mu_1^A p_1^A + \mu_1^B p_1^B) y_1 + b_0 + (\mu_2^A p_2^A + \mu_2^B p_2^B) y_2 \ \textit{is a master utility function}
\end{aligned}
$$

2. „independent“:

$$
\begin{aligned}
&\textit{If}\ U^A(y_1,y_2) = U^A(y_1)\ \text{und}\ U^B(y_1,y_2) = U^B(y_2) \\
&\Rightarrow MU(y_1,y_2) := \mu_1^A U^A(y_1) + \mu_2^B U^B(y_2)\ \textit{is a master utility function}
\end{aligned}
$$

2. „uniform power“:

$$
\begin{aligned}
&\textit{If}\ \ \mu^A := \mu_1^A = \mu_2^A\ \text{und}\ \mu^B := \mu_1^B = \mu_2^B \\
&\Rightarrow MU(y_1,y_2) := \mu^A U^A(y_1,y_2) + \mu^B U^B(y_1,y_2)\ \textit{is a master utility function}
\end{aligned}
$$

For the continuous prisoners dilemma in chapter 8 condition 1 ‘quasi-linear’ is fulfilled and for the example in chapter 7 the condition 2 ‘independent’ is fulfilled.

Defining total utility as $GU := U^A + U^B$ it becomes clear from the above examples that in general $GU \neq MU$ and that the maximization of the master utility function MU does not necessarily lead to a maximization of the total utility function GU. An example for that is the continuous prisoners dilemma in chapter 8. In answer of the second question from the above conditions it becomes obvious immediately that it holds that:

$$1.\ \mathit{If\ all}\ \mu_i^j = 1 \qquad \Rightarrow \qquad MU = GU$$

$$2.\ \mathit{If\ all}\ \mu_i^j = \mu \qquad \Rightarrow \qquad a.)\ MU = \mu\, GU$$

$$b.)\ MU\ \mathit{maximal} \Leftrightarrow GU\ \mathit{maximal}$$

5. Closure

5.1. Problem description

The aim of economic models is to be able to calculate the state of the system i.e. the endogenous variables with given exogenous variables. For a model with n endogenous variables to have a unique solution n linearly independent equations are necessary.

If there are less than n linearly independent equations the model is underdetermined. That means that there is a multitude of solutions. If there are more than n linearly independent equations, the model is over-determined. That means that it does not have a solution in general. Especially when an additional constraint such as an accounting identity or an equilibrium condition in the form of

$$ZB(x, y) = 0$$

is introduced to an otherwise solvable system with n behavioral equations for n variables, it will normally be over-determined. The procedure to adjust an over-determined system in order for it to be uniquely solvable is called 'closure'. That means to modify a system of equations so that the number of endogenous variables is equal to the number of linearly independent equations. In fact the choice of a closure determines which variables are considered endogenous and which exogenous.

In principle there are two possibilities for the closure:

1. Dropping the respective number of equations ('drop closure'). Economic theories are often characterized exactly by which equations are dropped (Sen, 1963).
2. Introducing the respective number of additional endogenous variables. The most important special case is the introduction of Lagrange multipliers ('Lagrange closure').

For reasons of simplicity we will show both possibilities each with an economic model with 3 variables and 1 constraint.

5.2. Drop-Closure

The over-determined algebraic economic model:

$$\begin{aligned} 0 &= f_1(x, y) \\ 0 &= f_2(x, y) \\ 0 &= f_3(x, y) \\ ZB(x, y) &= 0 \end{aligned} \qquad \langle 5.1 \rangle$$

can be made solvable by for instance dropping the third equation to read:

$$\begin{aligned} 0 &= f_1(x, y) \\ 0 &= f_2(x, y) \\ ZB(x, y) &= 0 \end{aligned}$$

The over-determined economic system of differential equations:

$$\begin{aligned} y_1' &= f_1(x,y) \\ y_2' &= f_2(x,y) \\ y_3' &= f_3(x,y) \\ ZB(x,y) &= 0 \end{aligned} \qquad \text{<5.2>}$$

can be made solvable by dropping for instance the third equation to read:

$$\begin{aligned} y_1' &= f_1(x,y) \\ y_2' &= f_2(x,y) \\ ZB(x,y) &= 0 \end{aligned} \qquad \text{<5.3>}$$

5.3. Lagrange Closure for an algebraic model

An over-determined algebraic model:

$$\begin{aligned} 0 &= f_1(x,y) \\ 0 &= f_2(x,y) \\ 0 &= f_3(x,y) \\ ZB(x,y) &= 0 \end{aligned}$$

can be translated into a solvable NCD system by introducing the arbitrary parameter $\mu = (\mu_1, \mu_2, \mu_3)$ and the additional variable λ (Lagrange multiplier):

$$\begin{aligned} y_1' &= \mu_1 f_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1} \\ y_2' &= \mu_2 f_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2} \\ y_3' &= \mu_3 f_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3} \\ ZB(x,y) &= 0 \end{aligned}$$

5.4. Lagrange Closure for a model of differential equations

The general form of an over-determined economic system of differential equations:

$$\begin{aligned} y_1' &= \mu_1 f_1(x,y) \\ y_2' &= \mu_2 f_2(x,y) \\ y_3' &= \mu_3 f_3(x,y) \\ ZB(x,y) &= 0 \end{aligned} \qquad \text{<5.4>}$$

can also be translated into a solvable NCD model by introducing the additional variables λ (Lagrange multipliers):

$$
\begin{aligned}
y_1' &= \mu_1 f_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1} \\
y_2' &= \mu_2 f_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2} \\
y_3' &= \mu_3 f_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3} \\
ZB(x,y) &= 0
\end{aligned}
\qquad \text{<5.5>}
$$

5.5. Drop Closure of algebraic models as manifestation of one-sided power relations

Under the respective regularity conditions the following proposition holds:

<u>Proposition 1</u>:

The algebraic system of equations

$$
\begin{aligned}
0 &= f_1(x,y) \\
0 &= f_2(x,y) \\
ZB(x,y) &= 0
\end{aligned}
$$

is equivalent to the NCD model

$$
\begin{aligned}
&(1) \qquad y_1' = \mu_1 f_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1} \\
&(2) \qquad y_2' = \mu_2 f_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2} \\
&(3) \qquad y_3' = \mu_3 f_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3} \\
&(4) \qquad ZB(x,y) = 0
\end{aligned}
$$

if the power factors are chosen such that:

$$
\begin{aligned}
&(5) \qquad \mu_1 = \mu_2 = \infty \\
&(6) \qquad \mu_3 = 0
\end{aligned}
$$

From proposition 1 follows the interpretation that a drop closure of algebraic models as in 5.2 is equivalent to a Lagrange closure of algebraic models as in 5.3, with the power factors defined as $\mu_1 = \mu_2 = \infty$ und $\mu_3 = 0$. This means that a drop closure of an over-determined algebraic model is equivalent to setting one-sided power relations.

Proof:

From (5) it follows that the economic forces $f_1(x,y)$ **und** $f_1(x,y)$ are effective immediately and unrestrictedly so that the system goes into a state in which these forces are 0 immediately. When (1) is divided by μ_1 and (2) by μ_2 respectively, $\mu_1 \to \infty$ and $\mu_2 \to \infty$, yields:

$$
\begin{array}{ll}
(1) & 0 = f_1(x,y) \\
(2) & 0 = f_2(x,y) \\
(3) & y_3' = \mu_3 f_3(x,y) + \lambda \dfrac{\partial ZB(x,y)}{\partial y_3} \\
(4) & ZB(x,y) = 0
\end{array}
$$

By (6) it is stated that nobody has powers to influence y_3 directly or that nobody wants to influence them. This means that y_3 is exclusively determined by 'market forces' and the constraint forces.

In concrete we show that under the condition (6) λ is given solely by (3), (4), (6) and that equation (3) takes the form $ZB' = 0$ As $ZB' = 0$ follows from (4) equation (3) can be omitted which yields:

$$
\begin{array}{ll}
(1) & 0 = f_1(x,y) \\
(2) & 0 = f_2(x,y) \\
(4) & ZB(x,y) = 0
\end{array}
$$

$$
\begin{array}{llll}
\textit{Because of } (4): & & (7) & ZB' = 0 \\
& \Rightarrow & (8) & \dfrac{\partial ZB}{\partial y_1} y_1' + \dfrac{\partial ZB}{\partial y_2} y_2' + \dfrac{\partial ZB}{\partial y_3} y_3' = 0 \\
\textit{Because of } (3),(6): & & (9) & y_3' = \lambda \dfrac{\partial ZB}{\partial y_3} \\
\textit{Because of } (7),(8): & & (10) & \dfrac{\partial ZB}{\partial y_1} y_1' + \dfrac{\partial ZB}{\partial y_2} y_2' + \dfrac{\partial ZB}{\partial y_3} \lambda \dfrac{\partial ZB}{\partial y_3} = 0 \\
& \Rightarrow & (11) & \lambda = -\dfrac{1}{(\dfrac{\partial ZB}{\partial y_3})^2} (\dfrac{\partial ZB}{\partial y_1} y_1' + \dfrac{\partial ZB}{\partial y_2} y_2')
\end{array}
$$

$\textit{Put } (6),(11) \textit{ into } (3), \textit{ so one gets for } (3):$

$$
y_3' = -\frac{1}{(\dfrac{\partial ZB}{\partial y_3})^2} (\frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2') \frac{\partial ZB}{\partial y_3}
$$

$$
\textit{respectively} \quad \frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2' + \frac{\partial ZB}{\partial y_3} y_3' = 0
$$

$$
\textit{respectively} \quad ZB' = 0
$$

$ZB' = 0 \textit{ follows from } (4), \textit{ therefor } (3) \textit{ can be omittet}.$

Obviously, by a respective choice of power factors in the same way one gets:

(1)	$0 = f_1(x,y)$		(2)	$0 = f_2(x,y)$
(3)	$0 = f_3(x,y)$	*bzw.*	(3)	$0 = f_3(x,y)$
(4)	$ZB(x,y) = 0$		(4)	$ZB(x,y) = 0$

From this follows the direct interpretation:

Which equations are omitted in a drop closure is an implicit assumption about power relations. A drop closure omitting the i[th] equation corresponds to the assumptions

1. y_i cannot be influenced by any agent and is therefore determined only by market forces and the constraint,
2. all other y_k with $k \neq j$ of the agents are fully determined so that the system immediately goes into the state in which the economic forces $f_k = 0 \; for \; k \neq j$.

5.6. Drop closure of systems of differential equations as manifestation of economic powerlessness

Under the respective regularity and differentiability conditions the following proposition holds:

Proposition 2:
For each system of differential equations such as <5.6>

$$\begin{aligned} y_1' &= f_1(x,y) \\ y_2' &= f_2(x,y) \\ ZB(x,y) &= 0 \end{aligned} \qquad \text{<5.6>}$$

there are functions $\tilde{f} = (\tilde{f}_1, \tilde{f}_{2,}, \tilde{f}_3)$ so that the system of differential equations is equivalent to an NCD model of the form <5.7>

$$\begin{aligned} &(1) \quad y_1' = \mu_1 \tilde{f}_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1} \\ &(2) \quad y_2' = \mu_2 \tilde{f}_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2} \\ &(3) \quad y_3' = \mu_3 \tilde{f}_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3} \\ &(4) \quad ZB(x,y) = 0 \end{aligned} \qquad \text{<5.7>}$$

with the power factor

$$(5) \qquad \mu_3 = 0$$

and reciprocally it holds that for every NCD model of the form <5.7> with $\mu_3 = 0$ there are functions $f = (f_1, f_2)$ so that this NCD model is equivalent to a system of differential equations of the form <5.6>.

Proposition 2 leads to the following interpretation: a drop closure of an over-determined system of differential equations in terms of 5.2 is equivalent to an NCD model of the form <5.7> with a power factor $\mu_3 = 0$. That means that the drop closure is synonymous to nobody having power to influence variable y_3 or that nobody wants to. This in the end means that y_3 is solely determined by market forces and the constraint.

Proof:

We first demonstrate the inverse:

Analogously to 5.5 we show that under the condition (5) λ is given solely by equations (3), (4), (5) and that equation (3) takes the form of $ZB' = 0$. As $ZB' = 0$ follows from (4), equation (3) can be omitted. Inserting λ yields

$$\begin{aligned} y_1' &= f_1(y_1, y_2, y_3) \\ y_2' &= f_2(y_1, y_2, y_3) \\ ZB(y_1, y_2, y_3) &= 0 \end{aligned}$$

Proof of the inverse:

$$Because\ of\ (4):\qquad (6)\qquad ZB' = 0$$

$$\Rightarrow (7)\qquad \frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2' + \frac{\partial ZB}{\partial y_3} y_3' = 0$$

$$Because\ of\ (3),(5):\qquad (8)\qquad y_3' = \lambda \frac{\partial ZB}{\partial y_3}$$

$$Because\ of\ (7),(8):\qquad (9)\qquad \frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2' + \frac{\partial ZB}{\partial y_3} \lambda \frac{\partial ZB}{\partial y_3} = 0$$

$$\Rightarrow (10)\qquad \lambda = -\frac{1}{(\frac{\partial ZB}{\partial y_3})^2} (\frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2')$$

Put $(5),(10)$ *into* (3), *then one gets for* (3):

$$y_3' = -\frac{1}{(\frac{\partial ZB}{\partial y_3})^2} (\frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2') \frac{\partial ZB}{\partial y_3}$$

$$respectively\ \frac{\partial ZB}{\partial y_1} y_1' + \frac{\partial ZB}{\partial y_2} y_2' + \frac{\partial ZB}{\partial y_3} y_3' = 0$$

$$respectively\ ZB' = 0$$

$ZB' = 0$ *follows from* (4) *therefore* (3) *can be omittet*.

Put (10) *into* $(1),(2)$ *one gets an system of eqations which can be solved with respect to* y_1', y_2'.

That means, that the equation sytem <5.7.> *with*

$(5)\qquad \mu_3 = 0$

becomes equivalent to: <5.8>

$$y_1' = f_1(y_1, y_2, y_3)$$
$$y_2' = f_2(y_1, y_2, y_3)$$
$$ZB(y_1, y_2, y_3) = 0$$

For the first case it holds that: starting from a system of differential equations

$$y_1' = f_1(x, y)$$
$$y_2' = f_2(x, y)$$
$$ZB(x, y) = 0$$

and setting

$$\tilde{f}_1 = f_1 - \frac{1}{\frac{\partial ZB}{\partial y_3}} y_3'$$

$$\tilde{f}_2 = f_2 - \frac{1}{\frac{\partial ZB}{\partial y_3}} y_3'$$

$$\tilde{f}_3 = arbitrary$$

yields in the same way that <5.7> with $\mu_3 = 0$ is equivalent to <5.6>.

6. Relationship between NCD models and standard types of economic models

6.1. NCD models and algebraic models

From proposition 1 in 5.5 follows: Each algebraic economic model is equivalent to a NCD model with one-sided power relations, i.e. a NCD model in which certain power factors are infinite and others equal to zero.

6.2. NCD models and systems of differential equations

From proposition 2 in 5.6 follows: Each economic system of differential equations is equivalent to a NCD model in which certain power factors are zero.

6.3. NCD models and general equilibrium models

A general economic equilibrium model with constraints with 3 variables and 1 constraint has the following general form (if necessary under consideration of comments 4.1.1 and 4.1.2 on the algebraic behavioral equations of parameters):

$$
\begin{aligned}
&x_1' = y_1 \\
&x_2' = y_2 \\
&x_3' = y_3 \\
&0 = f_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1} \\
&0 = f_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2} \\
&0 = f_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3} \\
&ZB(x,y) = 0
\end{aligned}
$$

This evidently describes exactly those solutions to the NCD model

$$
\begin{aligned}
&y_1' = f_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1} \\
&y_2' = f_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2} \\
&y_3' = f_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3} \\
&ZB(x,y) = 0
\end{aligned}
$$

for which the ‘ex post’ forces are 0. They can be described as the limit $v \to \infty$ of the NCD model

$$y_1' = v(f_1(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_1})$$
$$y_2' = v(f_2(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_2})$$
$$y_3' = v(f_3(x,y) + \lambda \frac{\partial ZB(x,y)}{\partial y_3})$$
$$ZB(x,y) = 0$$

<6.1>

when dividing by v. Here v can be interpreted as a factor for the adjustment speed. The assumption $v \to \infty$ therefore means that the system always jumps to the equilibrium immediately.

If a master utility function MU exists with

$$f_1(x,y) = \frac{\partial MU(x,y)}{\partial y_1}$$
$$f_2(x,y) = \frac{\partial MU(x,y)}{\partial y_2}$$
$$f_3(x,y) = \frac{\partial MU(x,y)}{\partial y_3}$$

and MU is concave, there is only one solution and for this solution MU is maximal. In the microeconomic standard general equilibrium models the economic forces $f = (f_{1,}, f_2, f_3)$ are also called excess demand.

Economic equilibrium models in general assume a concave master utility function to exist. This assumption and the equilibrium assumption, i.e. that the economic 'ex post' forces are always equal to 0 however constitute a strong restriction.

6.4. NCD models and DSGE models

With the exception of the stochastic element DSGE models are general equilibrium models with constraints and a master utility function MU, where not MU is maximized, but the total discounted future utility DU. DU is the integral over all future utility discounted with the interest rate β:

$$DU := \int_0^\infty e^{-\beta t} MU(x(t), y(t))\, dt$$

This constitutes a variational problem with the Lagrangian

$$\mathcal{L} = e^{-\beta t} MU(x(t), y(t))$$

The solutions to this variational problem are found within the solutions to the associated Euler equations.

Therefore, in case there is a master utility function which allows describing the economic forces of an NCD model, also the respective equilibrium model for the maximization of DU can be set up.

The inverse question whether the model equations of every general NCD model can be written as Euler equations of a variational problem leads to the so-called 'inverse Langrage problem'. The answer to this question is no, as not every ordinary system of differential equations can be expressed as Euler equations of a Lagrangian and in case there is a Lagrangian it is not uniquely determined (Santilli, 1978). This is relevant in so far as in mainstream economics only models which are derived from the variational problem of a discounted master utility function are investigated. This constitutes a major restriction and further shows the only partially completed adoption of Newtonian mechanics in economics.

Also for very simple NCD models there does not have to be a Lagrangian and if it exists it can be very complex. Two examples will makes this clear.

A simple example of a system of differential equation with a Lagrangian which is not uniquely determined is:

$$x' = y$$
$$y' = k.y$$

It describes a system with positive feedbacks for $k > 0$ (exponential growth) and with negative feedbacks for $k < 0$ (exponential decline). Already this extremely simple differential equation of the first order has two substantially different Lagrange functions:

$$\mathcal{L}_1(x, y) := \frac{1}{2} y^2 e^{-kt}$$
$$\mathcal{L}_2(x, y) := y \ln(\mathrm{y}) + kx$$

For the very simple system of differential equations

$$x_1' = y_1$$
$$x_2' = y_2$$
$$y_1' = x_1$$
$$y_2' = y_1$$

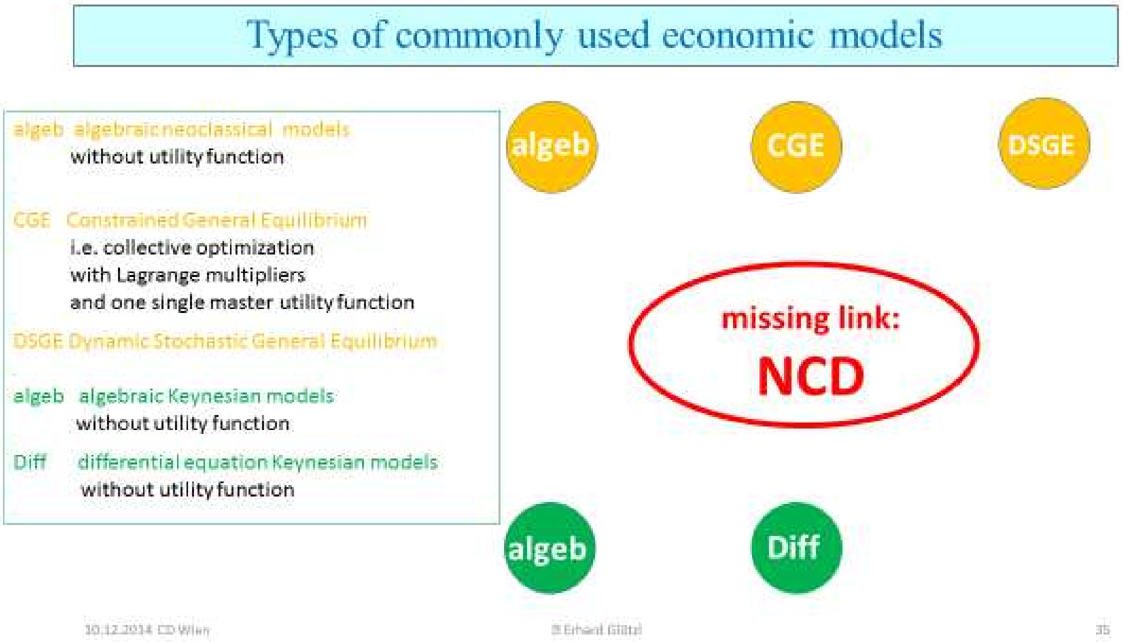
it can be shown that no Lagrangian exists (Prince & King, 2007).

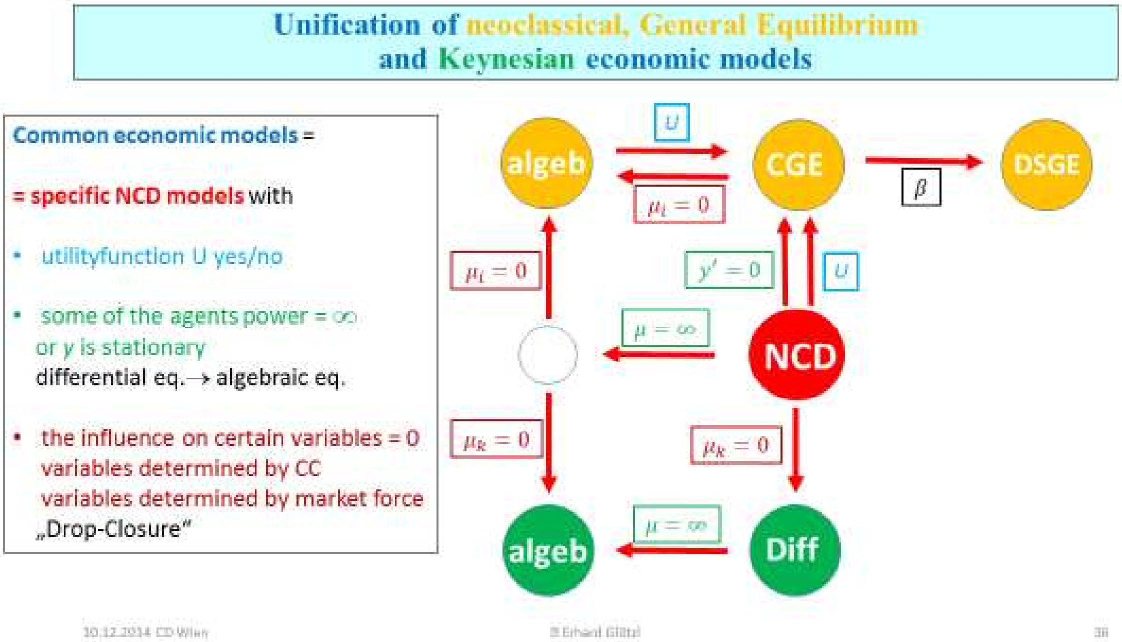
Differential equations and thereby NCD models therefore seem to be more adequate to describe economic systems in a general way.

6.5. NCD models and agent based models

In agent based models (ABM) a multitude of agents is assumed. The behavior of every agent is described. As the general structure of NCD models remains unchanged independent of the number of agents, NCD models with many agents can be regarded as , stock flow consistent, time continuous ABMs, with behavioral equations in the form of differential equations.

6.6. Schematic overview





7. The relationships illustrated with the example of a 2-sector NCD model

7.1. NCD model

We start from a NCD model with two agents H and B which represent the sectors households (H) and business (B). For each of the first six out of seven variables

K	*Capital*
I	*Investment*
p_L	*wagerate*
S	*saving*
L	*labour*
P	*profit*
Y	*GDP*

there is one differential equation. For K it holds by definition, for all others it is a behavioral equation expressing that the respective force employed by agent j (H or B) in order to change variable i is equal to the deviation of variable i from a desired state. μ_i^j denominates the economic power of agent j to influence variable i. In this model we assume that each agent j can only influences the one variable y_j respectively, i.e. $\mu_I^j = 0$ *für* $i \neq j$. This means that in this model the condition 2 'independent' discussed in 4.3 for the existence of a master utility function is fulfilled.

The 7th variable Y corresponds to a parameter with an algebraic behavioral equation, which in terms of 4.1.1 comment 1 and 4.1.2 comment 2 can be interpreted as border case of a respective differential equation with an infinite power factor.

In addition, the two accounting identities constitute constraints. There is one constraint for the production side of GDP and one for the use of saved funds for investment. From this the model equations for the NCD model can be derived.

(1) $K' = I$

(2) $I' = m_I^B(i_0 + i_1 Y - I) + l_2$ $\quad$ i_0, i_1 *constant (linear investment function)*

(3) $p_L' = m_{p_L}^B\left(\frac{\partial Y}{\partial L} - p_L\right) - l_1 L$ $\quad$ $\frac{\partial Y}{\partial L}$ *marginal productivity of labour*

(4) $S' = m_S^H(s_L p_L L + s_P P - S) - l_2$ $\quad$ s_L, s_P *share of savings out of labour income resp. profit*

(5) $L' = m_L^H(\bar{L} - L) - l_1 p_L$ $\quad$ $\bar{L}$ *intended labour of household*

(6) $P' = m_P^B(rY - P) - l_1$ $\quad$ *r intented profit rate*

(7) $Y = L^a K^{1-a}$ $\quad$ *Cobb - Douglas production function*

(8) $ZB_1 = Y - p_L L - P = 0$

(9) $ZB_2 = I - S = 0$

For the master utility function it holds that:

$$MU = -\frac{1}{2}[\mu_I^B(i_0 + i_1 Y - I)^2 + \mu_{p_L}^B(\frac{\partial Y}{\partial L} - p_L)^2 + \\ +\mu_S^H(s_L p_L L + s_P P - S)^2 + \mu_L^H(\bar{L} - L)^2 + \mu_P^B(rY - P)^2] \quad \text{<7.1>}$$

In the following we will show how Keynesian, neoclassical and general equilibrium models can be derived from this NCD model by assuming one-sided power relations ($\mu = 0$ *bzw.* $\mu = \infty$). Similarly the assumption of infinitely high adjustment speeds ($\nu = \infty$ in terms of <6.1>) yield equilibrium models with constraints.
In terms of content this approach corresponds to that of Sen (1963), considering that as per 5.5 the choice of one-sided power relations is equivalent to a drop closure.

This allows us to once again state the basic idea behind NCD models. NCD models are a way to overcome the allegedly insurmountable antagonism between different schools of economic models. These different model types can be expressed as versions and special cases of one single model, and only differ in the choice of different one-sided power relations or adjustment speeds. Moreover, NCD models make it possible to better depict the reality, as mixed power relations are more common than one-sided power relations.

7.2. Algebraic neoclassical model

Setting

$$\mu_I^B = \mu_P^B = 0$$
$$\mu_{p_L}^B = \mu_S^H = \mu_L^H = \infty$$

leads to dropping the equations (2) and (6) and yields

(1) $K' = I$

(3) $p_L = \frac{\partial Y}{\partial L}$

(4) $S = s_L p_L L + s_P P$

(5) $L = \bar{L}$

(7) $Y = L^{\alpha} K^{1-\alpha}$

(8) $ZB_1 = Y - p_L L - P = 0$

(9) $ZB_2 = I - S = 0$

This model corresponds to the neoclassical standard model with exogenously determined work $\bar{L}$.

7.3. Algebraic Keynesian model

Setting

$$\mu_L^H = \mu_P^B = 0$$
$$\mu_I^B = \mu_{p_L}^B = \mu_S^H = \infty$$

leads to dropping the equations (5) and (6) and yields

$$
\begin{array}{ll}
(1) & K' = I \\
(2) & I = i_0 + i_1 Y \\
(3) & p_L = \dfrac{\partial Y}{\partial L} \\
(4) & S = s_L p_L L + s_P P \\
(7) & Y = L^{\alpha} K^{1-\alpha} \\
(8) & ZB_1 = Y - p_L L - P = 0 \\
(9) & ZB_2 = I - S = 0
\end{array}
$$

This model corresponds to Keynes' approach in the General Theory where work and employment is not exogenously determined but determined by the market forces and constraints. This means nothing else than unemployment being possible.

7.4. General equilibrium model with constraints

Adding the adjustment speed of v in terms of <6.1> and letting $v \to \infty$, this yields the general equilibrium model

$$
\begin{array}{ll}
(1) & K' = I \\
(2) & 0 = \mu_I^B (i_0 + i_1 Y - I) + \lambda_2 \\
(3) & 0 = \mu_{p_L}^B (\dfrac{\partial Y}{\partial L} - p_L) - \lambda_1 L \\
(4) & 0 = \mu_S^H (s_L p_L L + s_P P - S) - \lambda_2 \\
(5) & 0 = \mu_L^H (\overline{L} - L) - \lambda_1 p_L \\
(6) & 0 = \mu_P^B (rY - P) - \lambda_1 \\
(7) & Y = L^{\alpha} K^{1-\alpha} \\
 & \\
(8) & ZB_1 = Y - p_L L - P = 0 \\
(9) & ZB_2 = I - S = 0
\end{array}
$$

As the condition for the existence of a master utility function: 2 'independent' discussed in chapter 4.3 is given the system of equations can be written with the master utility function from <7.1> as

(1) $K' = I$

(2) $0 = \dfrac{\partial MU}{\partial I} + \lambda_2$

(3) $0 = \dfrac{\partial MU}{\partial p_L} - \lambda_1 L$

(4) $0 = \dfrac{\partial MU}{\partial S} - \lambda_2$

(5) $0 = \dfrac{\partial MU}{\partial L} - \lambda_1 p_L$

(6) $0 = \dfrac{\partial MU}{\partial P} - \lambda_1$

(7) $Y = L^{\alpha} K^{1-\alpha}$

(8) $ZB_1 = Y - p_L L - P = 0$

(9) $ZB_2 = I - S = 0$

7.5. General equilibrium model with constraint and discounted utility function

With the master utility function <7.1> the discounted utility function DU with interest rate β can be set up:

$$DU = \int_0^{\infty} e^{-\beta t} MU(x(t), y(t))\, dt$$

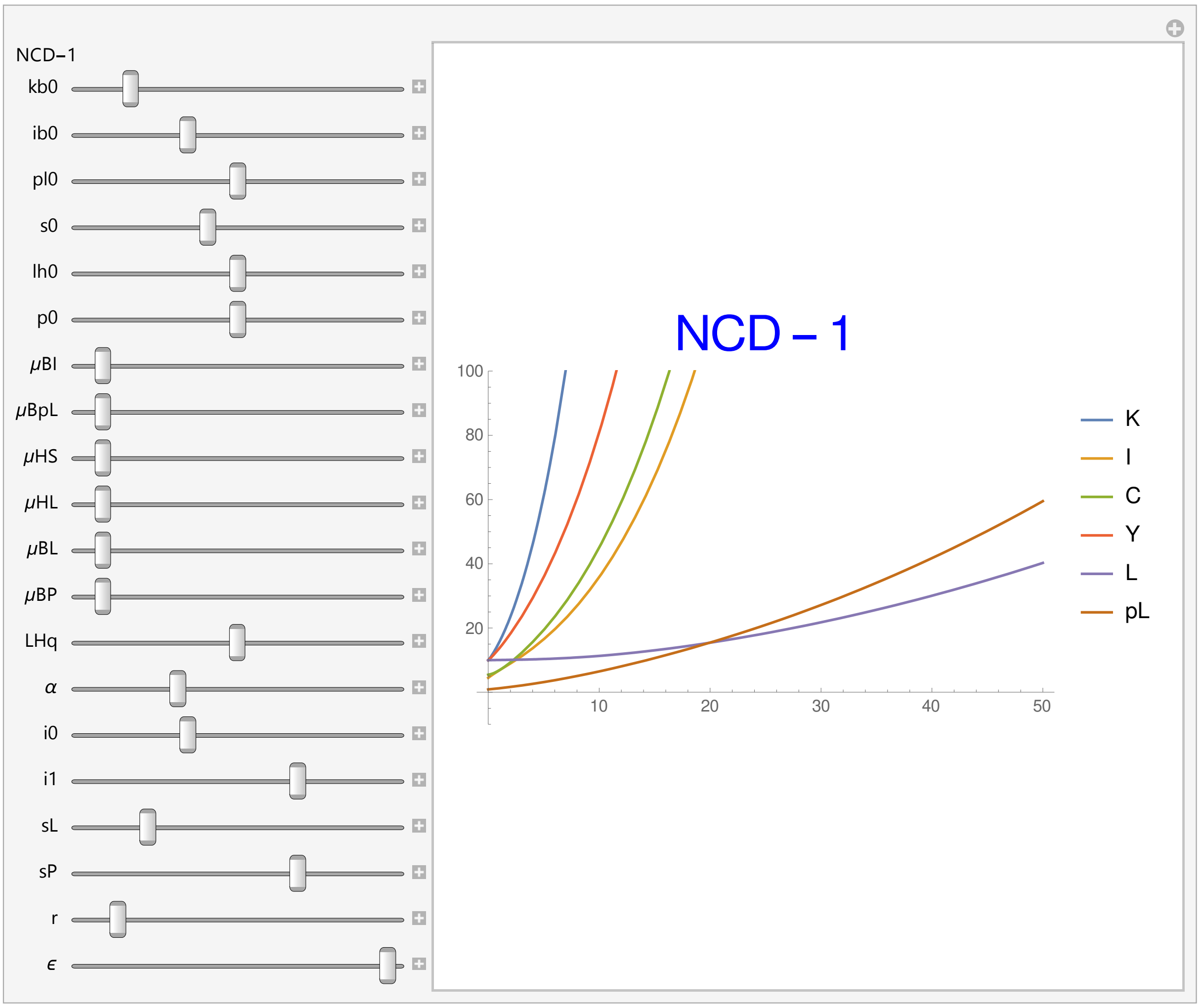
The maximization of DU leads to a variational problem with the respective Euler equations.

7.6. Computational results

The following results illustrate exemplarily the dependence of the system from the choice of power factors. Choosing one-sided power relations yields the above models.

The parameters $\bar{L} = 20,\ \alpha = 0.3,\ i_0 = 5,\ i_1 = 0.7,\ s_L = 0.2,\ s_P = 0.7,\ r = 0.1$ were not changed for any of the computations.

	μ_I^B	$\mu_{p_L}^B$	μ_S^H	μ_L^H	μ_P^B	v
NCD model 1	1	1	1	1	1	1
NCD model 2	1	10	10	1	1	1
Algebraic neoclassical	0	∞	∞	∞	0	1
Algebraic Keynesian	∞	∞	∞	0	0	1
General equilibrium with constraint	1	1	1	1	1	∞



NCD−1
kb0
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pl0
s0
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μBP
LHq
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i1
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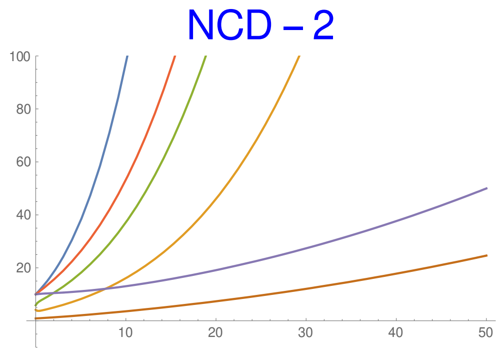
NCD – 2
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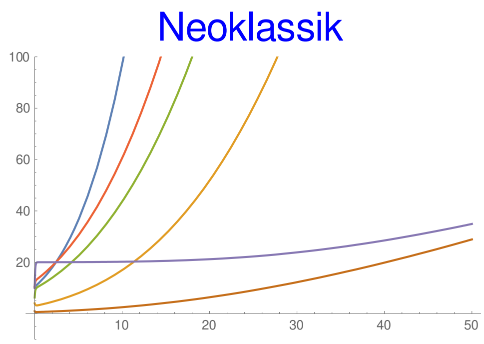
Neoklassik
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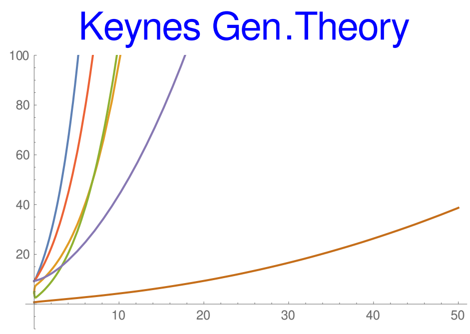
Keynes Gen.Theory
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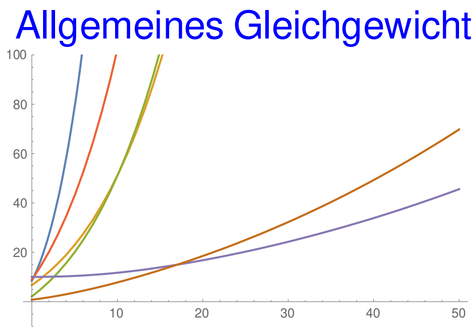
Allgemeines Gleichgewicht
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8. Continuous prisoners dilemma

An NCD model describes the behavior of a system in which every agent follows an individual optimization strategy. He tries to increase his individual utility. The core assumption of mainstream market economics is the assumption that these individual strategies lead to the general optimum through the 'invisible hand' of the market. In reality this is however not or at least not always the case.

In real situations prisoners dilemmas are very common. The individually best strategy for each agent leads to the overall worst solution. We show that especially such situations can be modeled very well using NCD models. As an example we build an NCD model of a continuous prisoners dilemma. Even though this NCD model can be described by a master utility function which even increases with time, the utility for both agents is decreasing continuously.

We start from two agents A, B and two variables y_1, y_2 and the special utility functions

$$U^A(y_1, y_2) = y_1 - 2y_2 + 2$$
$$U^B(y_1, y_2) = -2y_1 + y_2 + 2$$

First we show that in discrete time and with only one round this gives the payoff matrices of the classical prisoners dilemma, if the variables can only take the two discrete values, $y = 0$ for cooperation and $y = 1$ for defection:

$$\begin{pmatrix} U_A(0,1) & U_A(1,1) \\ U_A(0,0) & U_A(1,0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} U_B(0,1) & U_B(1,1) \\ U_B(0,0) & U_B(1,0) \end{pmatrix}_B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$$

In order to set up an NCD model for the continuous prisoners dilemma case we start from the following power factors.

$$\mu_1^A = 1, \ \mu_2^A = 0$$
$$\mu_1^B = 0, \ \mu_2^B = 1$$

These describe a situation in which each agent can only influence 'his' variable, i.e. he can only influence his own decisions. With these power factors and the utility functions U^A, U^B the behavioral equations for the NCD model of the continuous prisoners dilemma read:

$$y_1'(t) = 1.\frac{\partial U^A(y_1(\mathrm{t}), y_2(\mathrm{t}))}{\partial y_1} + 0.\frac{\partial U^B(y_1(\mathrm{t}), y_2(\mathrm{t}))}{\partial y_1} = \frac{\partial(\mathrm{y}_1 - 2\,\mathrm{y}_2 + 2)}{\partial y_1} = 1$$
$$y_2'(t) = 0.\frac{\partial U^A(y_1(\mathrm{t}), y_2(\mathrm{t}))}{\partial y_2} + 1.\frac{\partial U^B(y_1(\mathrm{t}), y_2(\mathrm{t}))}{\partial y_2} = \frac{\partial(-2\,\mathrm{y}_1 + \mathrm{y}_2 + 2)}{\partial y_2} = 1$$

which can be described with the master utility function:

$$MU(y_1(t), y_2(t)) = y_1(t) + y_2(t)$$
$$y_1' = \frac{\partial MU(y_1, y_2)}{\partial y_1} = 1$$
$$y_2' = \frac{\partial MU(y_1, y_2)}{\partial y_2} = 1$$

The solution follows as:

$$y_1(t) = t + y_1(0)$$
$$y_2(t) = t + y_2(0)$$

Therefore, the progression of the master utility function is given as:

$$MU(t) = y_1(t) + y_2(t) = 2t + y_1(0) + y_2(0)$$

This means that the master utility function continuously increases while the progression of the individual utility functions U^A, U^B of the agents. The total utility GU is given by

$$U^A(t) = y_1 - 2y_2 + 2 = t + y_1(0) - 2t - 2y_2(0) + 2 = -t + y_1(0) - 2y_2(0) + 2$$
$$U^B(t) = -2y_1 + y_2 + 2 = -2t - 2y_1(0) + t + y_2(0) + 2 = -t - 2y_1(0) + y_2(0) + 2$$
$$GU(t) = U^A(t) + U^B(t) = -2t - y_1(0) - y_2(0) + 4$$

which means that they are continuously decreasing.

9. Advantages of NCD models

In conclusion there are several advantages to the use of NCD models:

1. NCD models can be the bases for a new economic thinking in terms of:
   - Economic power
   - Economic force
   - Economic constraint force

2. NCD models allow a unified look onto many types of economic models.

3. NCD models give an understanding for the equivalence of different closures and the choice to assign different one-sided power relations.

4. In reality power is not purely one-sided. NCD models allow depicting real power relations better.

5. NCD models give a correct and precise understanding between 'ex ante' and 'ex post' dynamics.

6. NCD models allow describing real disequilibrium dynamics. Especially situations without equilibria (or in which utility functions are not concave) can be described well.

7. NCD models allow expressing real competition models, i.e. models in which individual optimization strategies do not lead to an overall optimal result. In reality such prisoners dilemma situations are very common.

8. NCD models can be the basis for a new theoretical understanding of e.g.:
   - Economic growth
   - Business cycles and economic crises
   - Analogies between physics and economics

9. NCD models can also be used for many practical tasks such as economic forecasting or modeling the impacts of fiscal or monetary policy.

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