Quantitative Methods

Lecture 4

Determinants and
Systems of linear equations

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BAKVM

Outline of the lecture

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- Determinants
- Systems of linear equations
- Linear mappings

The determinant of a square matrix

Let $n \in \mathbb{N}$ be a natural number and let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

The **determinant** of the matrix A is denoted by

$$\det A \qquad \text{or} \qquad |A|$$

We define the determinant inductively for $n = 1, 2, 3, 4, 5, \ldots$

The determinant of a square matrix

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The determinant of a 1×1 matrix is

$$\det(a_{11}) = |a_{11}| = a_{11}$$

i.e. the determinant is equal to the single entry a_{11} of the matrix.

The determinant of a square matrix

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The determinant of a 2×2 matrix is calculated by the formula

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Notice the scheme of the formula:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$+$

$-$

The determinant of a square matrix

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The determinant of a 3×3 matrix is calculated by the formula – **Sarrus' Rule**

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{32}a_{23}a_{11}$$

Notice the scheme of the formula:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$+$

$-$

Laplace expansion of the determinant

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To calculate the determinant of a general $n \times n$ matrix A, we use the Laplace expansion.

The Laplace expansion is done in three steps:

— choose either a row $i_0 \in \{1, 2, \dots, n\}$ or choose a column $j_0 \in \{1, 2, \dots, n\}$ of the matrix, and keep the chosen row or column fixed;
it is wise to choose a row or column with a maximum number of zeros (0)

— assign the sign "$+$" or "$-$" to each entry
$a_{i_0 j}$ of the row by the formula $(-1)^{i_0+j}$ for $j = 1, 2, \dots, n$ or
$a_{i j_0}$ of the column by the formula $(-1)^{i+j_0}$ for $i = 1, 2, \dots, n$, respectively

— sum up the terms to calculate the determinant

Laplace expansion of the determinant

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For a natural number $n \in \mathbb{N}$,
let a matrix $A \in \mathbb{R}^{n \times n}$ be given.

Choose a row $i_0 \in \{1, 2, \dots, n\}$ or
choose a column $j_0 \in \{1, 2, \dots, n\}$.

Here, we choose the
row $i_0 = 2$, say.

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\dots$	a_{1n}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	$\dots$	a_{2n}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	$\dots$	a_{3n}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	$\dots$	a_{4n}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	$\dots$	a_{5n}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	$\dots$	a_{6n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{n1}	a_{n2}	a_{n3}	a_{n4}	a_{n5}	a_{n6}	$\dots$	a_{nn}

Laplace expansion of the determinant

Assign the sign “+” or “–” to each entry a_{ij} by using the formula $(-1)^{i+j}$ for $i = 1, 2, \dots, n$ and for $j = 1, 2, \dots, n$.

$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$\pm$
$-$	$+$	$-$	$+$	$-$	$+$	$\cdots$	$\mp$
$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$\pm$
$-$	$+$	$-$	$+$	$-$	$+$	$\cdots$	$\mp$
$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$\pm$
$-$	$+$	$-$	$+$	$-$	$+$	$\cdots$	$\mp$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\pm$	$\mp$	$\pm$	$\mp$	$\pm$	$\mp$	$\cdots$	$\pm$

Laplace expansion of the determinant

For a row $i = 1, 2, \ldots, n$ and for a column $j = 1, 2, \ldots, n$, we denote by

$$M_{ij}$$

the (i, j)-**minor** of the matrix A,

which is the determinant of the

matrix obtained by deleting the

row i and the column j.

Here $i = 2$ and $j = 3$ are chosen, say.

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	…	a_{1n}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	…	a_{2n}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	…	a_{3n}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	…	a_{4n}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	…	a_{5n}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	…	a_{6n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{n1}	a_{n2}	a_{n3}	a_{n4}	a_{n5}	a_{n6}	…	a_{nn}

Laplace expansion of the determinant

Chosen the row $i_0 = 2$, say, the determinant is

$$\det A =$$
$$+ (-1)^{i_0+1} a_{i_0 1} M_{i_0 1} +$$
$$+ (-1)^{i_0+2} a_{i_0 2} M_{i_0 2} +$$
$$+ \cdots +$$
$$+ (-1)^{i_0+n} a_{i_0 n} M_{i_0 n}$$

where $M_{i_0 j}$ is the (i_0, j)-**minor** of the matrix A.

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	…	a_{1n}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	…	a_{2n}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	…	a_{3n}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	…	a_{4n}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	…	a_{5n}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	…	a_{6n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{n1}	a_{n2}	a_{n3}	a_{n4}	a_{n5}	a_{n6}	…	a_{nn}

Laplace expansion of the determinant

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Chosen the row i_0, calculate the minor $M_{i_0 j}$ for $j = 1, 2, \dots, n$.

Here, the row $i_0 = 2$ and the column $j = 3$ are chosen, say. (They are to be deleted.)

To calculate the minor M_{ij}, i.e. the determinant of the $(n-1) \times (n-1)$ matrix, use the Laplace expansion recursively.

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$	a_{1n}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	$\cdots$	a_{2n}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	$\cdots$	a_{3n}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	$\cdots$	a_{4n}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	$\cdots$	a_{5n}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	$\cdots$	a_{6n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{n1}	a_{n2}	a_{n3}	a_{n4}	a_{n5}	a_{n6}	$\cdots$	a_{nn}

Three theorems on the determinant

Theorem. The matrix $A \in \mathbb{R}^{n\times n}$ is **non-singular** if and only if the determinant

$$\det A \neq 0$$

Theorem. The matrix $A \in \mathbb{R}^{n\times n}$ is **singular** if and only if the determinant

$$\det A = 0$$

Theorem. Given two matrices $A, B \in \mathbb{R}^{n\times n}$, it holds

$$\det(AB) = (\det A)(\det B)$$

The geometrical meaning of the determinant

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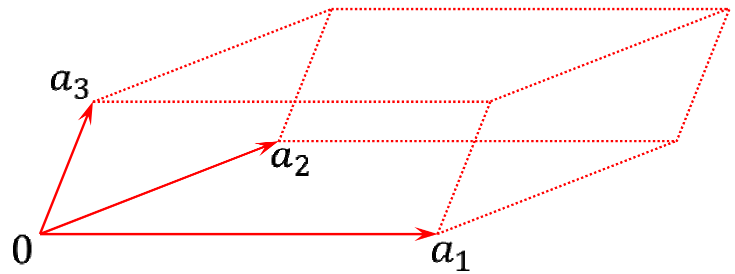
Let $n \in \mathbb{N}$ and let a matrix $A \in \mathbb{R}^{n \times n}$ be given.

The geometrical meaning of the determinant:

- The (absolute) value of the determinant is the volume of the parallelepiped spanned by the vectors $a_1, a_2, \dots, a_n$.
- The sign ("+" or "−") depends on the order of the vectors, i.e. the orientation of the basis.

$$A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$$



The transposed matrix

Let $m, n \in \mathbb{N}$ be natural numbers and let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix.

The transpose of the $m \times n$ matrix A is the $n \times m$ matrix B with the entries

$$b_{ji} = a_{ij} \qquad \text{for} \quad i = 1, 2, \ldots, m \quad \text{and} \;\; \text{for} \quad j = 1, 2, \ldots, n$$

$$A^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{mn} \end{pmatrix}$$

Some properties of the determinant

Let $n \in \mathbb{N}$ be natural numbers and let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

By the Laplace expansion of the determinant, it follows that

$$\det A = \det A^{\mathrm{T}}$$

By the above and by the geometrical meaning of the determinant, it follows that, if we multiply a single row of the matrix A by a constant $\lambda \in \mathbb{R}$, then the determinant of the new matrix is

$$\lambda \det A$$

Some properties of the determinant

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Let $n \in \mathbb{N}$ be natural numbers and let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

By the above and by the Laplace expansion of the determinant, if we change the order of two consecutive rows of the matrix A, then the determinant of the new matrix is

$$-\det A$$

By the above and by the geometrical meaning of the determinant, for any constant $\lambda \in \mathbb{R}$, if we add the λ-multiple of a row to another row of the matrix A, then the determinant of the new matrix is

$$\det A$$

To compute the determinant practically…

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The last two rules yield the Gaussian elimination method. The goal is to transform the matrix into a **triangular matrix**, i.e. the purpose is to turn all the entries below (or above, respectively) the main diagonal to zero (0).

When the matrix is triangular, then the determinant is computed easily:

$$\det A = a_{11}a_{22}\dots a_{nn}$$

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$	a_{1n}
0	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	$\cdots$	a_{2n}
0	0	a_{33}	a_{34}	a_{35}	a_{36}	$\cdots$	a_{3n}
0	0	0	a_{44}	a_{45}	a_{46}	$\cdots$	a_{4n}
0	0	0	0	a_{55}	a_{56}	$\cdots$	a_{5n}
0	0	0	0	0	a_{66}	$\cdots$	a_{6n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	0	0	0	0	$\cdots$	a_{nn}

To compute the inverse matrix practically…

Let a non-singular square matrix $A \in \mathbb{R}^{n \times n}$ be given.

Write the identity matrix I next to the matrix A:

$$(A|I)$$

Apply the Gaussian elimination method. The purpose is to turn the matrix A into the identity matrix; in the end, the identity matrix turns into the inverse matrix A^{-1}.

Systems of linear equations

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- System of linear equations
- Gaussian elimination
- Cramer’s rule

A system of linear equations

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Let $n \in \mathbb{N}$ be a natural number, let $A \in \mathbb{R}^{n \times n}$ be a (non-singular) square matrix, and let $b \in \mathbb{R}^n$ be a vector of the right-hand sides.

Now, our purpose is to find a solution $x \in \mathbb{R}^n$ to the system of the linear equations

$$Ax = b$$

or

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

A system of linear equations

Notice that the system of the linear equations

$$Ax = b$$

can also be written as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Frobenius' Theorem

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Let $m, n \in \mathbb{N}$ be natural numbers, let $A \in \mathbb{R}^{m \times n}$ be a matrix, and let $b \in \mathbb{R}^m$ be a vector of the right-hand sides.

The system of linear equations

$$Ax = b$$

has a solution if and only if

$$\operatorname{rank} A = \operatorname{rank}(A|b)$$

Moreover, the solution $x \in \mathbb{R}^n$ to the system $Ax = b$ is unique if and only if

$$\operatorname{rank} A = \operatorname{rank}(A|b) = n$$

A system of linear equations

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If the matrix $A \in \mathbb{R}^{n \times n}$ is non-singular, then there is the inverse matrix A^{-1} such that

$$AA^{-1} = I = A^{-1}A$$

where I is the identity matrix. Now, given the system of the linear equations

$$Ax = b$$

we calculate:

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

A system of linear equations

We have thus shown that the solution $x \in \mathbb{R}^n$ to the system of the linear equations

$$Ax = b$$

is

$$x = A^{-1}b$$

Thus, in theory, it is enough to compute the inverse matrix A^{-1}.

To solve a system of linear equations practically…

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Our purpose is to solve the system $Ax = b$.

Some rules, which hold:

- when an equation is multiplied by a constant, then the solution does not change
- when a multiple of an equation is added to another equation, then the solution does not change

These rules yield the Gaussian elimination method. The goal is to transform the matrix of the system into a triangular matrix. The system is solved easily then.

(Recall that the Gaussian elimination method can also be used to compute the determinant of a matrix, or to find the inverse of a matrix.)

To solve a system of linear equations practically…

Our purpose is to solve the system $Ax = b$.

Write the vector b of the right-hand sides next to the matrix A:

$$(A|b)$$

Apply the Gaussian elimination method. The purpose is to turn the matrix A into the triangular matrix (or into the identity matrix). The system is solved easily then.

Cramer’s Rule

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Let $n \in \mathbb{N}$ be a natural number, let $A \in \mathbb{R}^{n \times n}$ be a non-singular square matrix, and let $b \in \mathbb{R}^n$ be a vector of the right-hand sides. Our purpose is to solve the system

$$Ax = b$$

Let

$$\Delta = \det A$$

be the determinant of the matrix.

Assume that $\Delta \neq 0$.

Cramer's Rule

Let

$$\Delta_i = \det A_{i|b}$$

be the determinant of the matrix A whose i-th column is replaced by the column b of the right-hand sides for $i = 1, 2, \dots, n$.

Then

$$x_i = \frac{\Delta_i}{\Delta} \qquad \text{for} \quad i = 1, 2, \dots, n$$

is the solution to the system of linear equations $Ax = b$.