Quantitative Methods

Lecture 6

Differential calculus



BAKVM

Outline of the lecture

- The derivative of a function
- Derivatives of elementary functions





Motivation: instantaneous velocity and the line tangent to the graph of a function

Rules to calculate the derivative

Chain rule



Example – instantaneous velocity of a mass point:

Let a mass point move along a line (horizontally, say).

Let f(t) denote the distance of the point from the origin at time t.

Let t_1 and t_2 , such that $t_1 < t_2$, be two times.

Then the average velocity of the point between the times t_1 and t_2 is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Now, what happens if t_1 is fixed and t_2 tends to t_1 ?

The average velocity of the point between the times t_1 and t_2 is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

What happens if t_1 is fixed and t_2 tends to t_1 ?

Considering the shorter and shorter time intervals, we get

$$\lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

which is the instantaneous velocity of the mass point at the time t_1

(if the limit exists)





Example – the line tangent to the graph of a function:

Let f be a "smooth" function.

Choose a point x_0 . Our purpose is to

find the line tangent to the graph of the function f at the point $[x_0, f(x_0)]$.

That is, we are seeking for a line

y = ax + b

that is tangent to the graph of the function f at the point $[x_0, f(x_0)]$.

First, our task is to find the slope a of the tangent line.

Let $h \neq 0$ be a small non-zero real number.

Then the ratio

$$\frac{f(x_0+h)-f(x_0)}{h}$$

is the slope of the secant line

passing through the points $[x_0, f(x_0)]$ and $[x_0 + h, f(x_0 + h)]$.

Now, let $h \rightarrow 0$.





Then

$$a = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is the slope of the tangent line at the point $[x_0, f(x_0)]$ (if the limit exists).

It then also holds:

$$a = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$



To find the constant b in the equation

$$y = ax + b$$

of the line tangent to the graph of the function f at the point $[x_0, f(x_0)]$, once the coefficient a is known, it suffices to put

$$f(x_0) = ax_0 + b$$
$$b = f(x_0) - ax_0$$



Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.

Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is **the first derivative** of the function f at the point x_0 (if the limit exists). If the limit does not exist — the function is not differentiable at x_0 , i.e. the function has no derivative at x_0 .

If the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is finite — the derivative is finite.

If the limit is infinite — then the derivative is infinite.

Theorem. If the derivative $f'(x_0)$ exists and is finite, then the function f is continuous at the point x_0 .



Rules to calculate the derivative I



Let *f* be a function such that its derivatives at a point $x \in \mathbb{R}$ exists and is finite. That is, f'(x) exists and is finite. Moreover, let $c \in \mathbb{R}$ be any constant.

It then holds:

 $(c \times f)'(x) = c \times f'(x)$

Rules to calculate the derivative II

Let f and g be two functions such that their derivatives at a point $x \in \mathbb{R}$ exist and are finite. That is, f'(x) and g'(x) exist and are finite.

It then holds:

(f+g)'(x) = f'(x) + g'(x)(f-g)'(x) = f'(x) - g'(x)



Rules to calculate the derivative III

Let f and g be two functions such that their derivatives at a point $x \in \mathbb{R}$ exist and are finite. That is, f'(x) and g'(x) exist and are finite.

It then holds:

$$(f \times g)'(x) = (f'(x) \times g(x)) + (f(x) \times g'(x))$$

If $g(x) \neq 0$, then $\left(\frac{f}{g}\right)'(x) = \frac{\left(f'(x) \times g(x)\right) - \left(f(x) \times g'(x)\right)}{g^2(x)}$



Chain rule



Let *g* be a function such that its derivatives at a point $x \in \mathbb{R}$ exists and is finite. Let *f* be a function such that its derivatives at the point g(x) exists and is finite. Then:

 $(f \circ g)'(x) = f'(g(x)) \times g'(x)$

Examples: Derivatives of elementary functions I



Let f(x) = const.

Then f'(x) = 0

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\text{const.} - \text{const.}}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Examples: Derivatives of elementary functions II



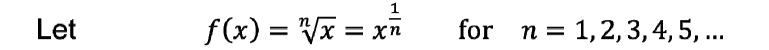
Let
$$f(x) = x^n$$
 for $n = 1, 2, 3, 4, 5, ...$

Then f'(x) =

$$(x) = n \times x^{n-1}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} =$$
$$= \lim_{h \to 0} \frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{k} x^{n-2} h^2 + \dots + h^n - x^n}{h} = nx^{n-1}$$

Examples: Derivatives of elementary functions III



Then
$$f'(x) = \frac{1}{n \times \sqrt[n]{x^{n-1}}} = \frac{1}{n} \times x^{\frac{1}{n}-1}$$

— calculate the derivative of the function inverse $(\sqrt[n]{x})$ to the function $f(x) = x^n$



Examples: Derivatives of elementary functions IV



Let
$$f(x) = x^{\frac{p}{q}} = \sqrt[q]{x^p}$$
 for $p, q = 1, 2, 3, 4, 5, ...$

Then
$$f'(x) = \frac{p}{q} \times \sqrt[q]{x^{p-q}} = \frac{p}{q} \times x^{\frac{p-q}{q}}$$

-

— calculate the derivative of the composite function $(\sqrt[q]{x^p})$ – chain rule

Examples: Derivatives of elementary functions IV



Let $f(x) = x^{\lambda}$ for $\lambda \in \mathbb{R}$

Then $f'(x) = \lambda \times x^{\lambda-1}$

— consider the limit of
$$x^{\lambda_n}$$
 for $\lambda_n = \frac{p_n}{q_n} \rightarrow \lambda$

Examples: Derivatives of elementary functions V



- $f(x) = e^x$ $f'(x) = e^x$ Let
- Then

Examples: Derivatives of elementary functions VI



- Let $f(x) = \sin x$
- Then $f'(x) = \cos x$

Let $f(x) = \cos x$

Then $f'(x) = -\sin x$

Examples: Derivatives of elementary functions VII

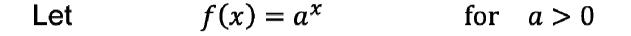


Let
$$f(x) = \tan x$$

Then
$$f'(x) = \frac{1}{\cos^2 x} = \frac{1}{(\cos x)^2}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Examples: Derivatives of elementary functions VIII



Then $f'(x) = a^x \times \ln a$

$$f(x) = a^{x} = e^{\ln(a^{x})} = e^{x \ln a}$$
$$f'(x) = e^{x \ln a} \times (1 \times \ln a) = a^{x} \times \ln a$$



The rules of differentiation



Let f(x) and g(x) be functions with the derivative in the interval $J \subseteq R$ hen:

i)
$$[c \cdot f(x)]' = c \cdot f'(x)$$

ii) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
iii) $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
iv) $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}, g(x) \neq 0$
v) $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Derivatives of elementary functions



		sinx	cosx
f(x)	f(x)	cosx	$-\sin x$
		4-	1
konstanta	0	tgx	$\overline{\cos^2 x}$
<i>x</i>	1	cotgx	$-\frac{1}{\sin^2 x}$
x^n	nx^{n-1}		
e ^x	e^{x}	arcsinx	$\frac{1}{\sqrt{1-x^2}}$
$\ln x$	1		$\frac{\sqrt{1-x}}{1}$
	x	arccosx	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \cdot \ln a$		1
$\log_a x$	1	arctgx	$\frac{1}{1+x^2}$
	$x \ln a$	arccotgx	1
		arceotgi	$1 + x^2$

Examples



 $y = x^{2} \Rightarrow y' = 2 \cdot 1 = 2$ $y = 6x^{3} - 5x + 4 \Rightarrow y' = 18x^{2} - 5$ $y = \frac{1}{x^{2}} \Rightarrow y' = -\frac{2}{x^{3}}$ $y = 5^{x} \Rightarrow y' = 5^{x} \cdot \ln 5$ $y = \ln(x^{2} - 4) \Rightarrow y' = \frac{1}{x^{2} - 4} \cdot (2x)$ $y = x \cdot e^{x} \Rightarrow y' = 1 \cdot e^{x} + x \cdot e^{x}$