Quantitative Methods

Lecture 6

Differential calculus

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BAKVM

Outline of the lecture

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- The derivative of a function
- Derivatives of elementary functions

The derivative of a function

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Motivation: instantaneous velocity and the line tangent to the graph of a function

Rules to calculate the derivative

Chain rule

The derivative of a function

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Example – instantaneous velocity of a mass point:

Let a mass point move along a line (horizontally, say).

Let $f(t)$ denote the distance of the point from the origin at time t.

Let t_1 and t_2, such that $t_1 < t_2$, be two times.

Then the average velocity of the point between the times t_1 and t_2 is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Now, what happens if t_1 is fixed and t_2 tends to t_1?

The derivative of a function

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The average velocity of the point between the times t_1 and t_2 is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

What happens if t_1 is fixed and t_2 tends to t_1?

Considering the shorter and shorter time intervals, we get

$$\lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

which is **the instantaneous velocity of the mass point** at the time t_1

(if the limit exists)

The derivative of a function

Example – the line tangent to the graph of a function:

Let f be a “smooth” function.

Choose a point x_0. Our purpose is to

find the line tangent to the graph of the function f at the point $[x_0, f(x_0)]$.

That is, we are seeking for a line

$$y = ax + b$$

that is tangent to the graph of the function f at the point $[x_0, f(x_0)]$.

First, our task is to find the slope a of the tangent line.

The derivative of a function

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Let $h \neq 0$ be a small non-zero real number.

Then the ratio

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is the slope of the secant line

passing through the points $[x_0, f(x_0)]$ and $[x_0 + h, f(x_0 + h)]$.

Now, let $h \to 0$.

The derivative of a function

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Then

$$a = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is the slope of the tangent line at the point $[x_0, f(x_0)]$

(if the limit exists).

It then also holds:

$$a = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The derivative of a function

To find the constant b in the equation

$$y = ax + b$$

of the line tangent to the graph of the function f at the point $[x_0, f(x_0)]$,

once the coefficient a is known, it suffices to put

$$f(x_0) = ax_0 + b$$

$$b = f(x_0) - ax_0$$

The derivative of a function

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.

Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is **the first derivative** of the function f at the point x_0 (if the limit exists).

If the limit does not exist — the function is not differentiable at x_0, i.e. the function has no derivative at x_0.

The derivative of a function

If the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is finite — the derivative is finite.

If the limit is infinite — then the derivative is infinite.

Theorem. If the derivative $f'(x_0)$ exists and is finite, then the function f is continuous at the point x_0.

Rules to calculate the derivative I

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Let f be a function such that its derivatives at a point $x \in \mathbb{R}$ exists and is finite.

That is, $f'(x)$ exists and is finite.

Moreover, let $c \in \mathbb{R}$ be any constant.

It then holds:

$$(c \times f)'(x) = c \times f'(x)$$

Rules to calculate the derivative II

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Let f and g be two functions such that their derivatives at a point $x \in \mathbb{R}$ exist and are finite. That is, $f'(x)$ and $g'(x)$ exist and are finite.

It then holds:

$$(f+g)'(x) = f'(x) + g'(x)$$

$$(f-g)'(x) = f'(x) - g'(x)$$

Rules to calculate the derivative III

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Let f and g be two functions such that

their derivatives at a point $x \in \mathbb{R}$ exist and are finite.

That is, $f'(x)$ and $g'(x)$ exist and are finite.

It then holds:

$$(f \times g)'(x) = \big(f'(x) \times g(x)\big) + \big(f(x) \times g'(x)\big)$$

If $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{\big(f'(x) \times g(x)\big) - \big(f(x) \times g'(x)\big)}{g^2(x)}$$

Chain rule

Let g be a function such that its derivatives at a point $x \in \mathbb{R}$ exists and is finite.

Let f be a function such that its derivatives at the point $g(x)$ exists and is finite.

Then:

$$(f \circ g)'(x) = f'\big(g(x)\big) \times g'(x)$$

Examples: Derivatives of elementary functions I

Let $f(x) = \text{const.}$

Then $f'(x) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\text{const.} - \text{const.}}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Examples: Derivatives of elementary functions II

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Let $f(x) = x^n$ for $n = 1, 2, 3, 4, 5, \dots$

Then $f'(x) = n \times x^{n-1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} =$$

$$= \lim_{h \to 0} \frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{k} x^{n-2} h^2 + \cdots + h^n - x^n}{h} = n x^{n-1}$$

Examples: Derivatives of elementary functions III

Let $f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ for $n = 1, 2, 3, 4, 5, \ldots$

Then $f'(x) = \frac{1}{n \times \sqrt[n]{x^{n-1}}} = \frac{1}{n} \times x^{\frac{1}{n}-1}$

— calculate the derivative of the function inverse $(\sqrt[n]{x})$ to the function $f(x) = x^n$

Examples: Derivatives of elementary functions IV

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Let $f(x) = x^{\frac{p}{q}} = \sqrt[q]{x^p}$ for $p, q = 1, 2, 3, 4, 5, \ldots$

Then $f'(x) = \frac{p}{q} \times \sqrt[q]{x^{p-q}} = \frac{p}{q} \times x^{\frac{p-q}{q}}$

— calculate the derivative of the composite function $(\sqrt[q]{x^p})$ – chain rule

Examples: Derivatives of elementary functions IV

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Let $f(x) = x^{\lambda}$ for $\lambda \in \mathbb{R}$

Then $f'(x) = \lambda \times x^{\lambda - 1}$

— consider the limit of x^{λ_n} for $\lambda_n = \frac{p_n}{q_n} \to \lambda$

Examples: Derivatives of elementary functions V

Let $f(x) = \mathrm{e}^{x}$

Then $f'(x) = \mathrm{e}^{x}$

Examples: Derivatives of elementary functions VI

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Let $f(x) = \sin x$

Then $f'(x) = \cos x$

Let $f(x) = \cos x$

Then $f'(x) = -\sin x$

Examples: Derivatives of elementary functions VII

Let $f(x) = \tan x$

Then $f'(x) = \frac{1}{\cos^2 x} = \frac{1}{(\cos x)^2}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Examples: Derivatives of elementary functions VIII

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Let $f(x) = a^x$ for $a > 0$

Then $f'(x) = a^x \times \ln a$

$$f(x) = a^x = \mathrm{e}^{\ln(a^x)} = \mathrm{e}^{x \ln a}$$

$$f'(x) = \mathrm{e}^{x \ln a} \times (1 \times \ln a) = a^x \times \ln a$$

The rules of differentiation

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Let f(x) and g(x) be functions with the derivative in the interval $J \subseteq R$. Then:

i) $[c \cdot f(x)]' = c \cdot f'(x)$

ii) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

iii) $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

iv) $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}, \quad g(x) \neq 0$

v) $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Derivatives of elementary functions

$f(x)$	$f'(x)$
konstanta	0
x	1
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \cdot \ln a$
$\log_a x$	$\frac{1}{x \ln a}$

$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\mathrm{tg}\, x$	$\frac{1}{\cos^2 x}$
$\mathrm{cotg}\, x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\mathrm{arctg}\, x$	$\frac{1}{1+x^2}$
$\mathrm{arccotg}\, x$	$-\frac{1}{1+x^2}$

Examples

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$$y = x^2 \Rightarrow y' = 2 \cdot 1 = 2$$

$$y = 6x^3 - 5x + 4 \Rightarrow y' = 18x^2 - 5$$

$$y = \frac{1}{x^2} \Rightarrow y' = -\frac{2}{x^3}$$

$$y = 5^x \Rightarrow y' = 5^x \cdot \ln 5$$

$$y = \ln\left(x^2 - 4\right) \Rightarrow y' = \frac{1}{x^2 - 4} \cdot (2x)$$

$$y = x \cdot e^x \Rightarrow y' = 1 \cdot e^x + x \cdot e^x$$