# Quantitative Methods

# Lecture 7

L'Hospital's rule

SILESIAN UNIVERSITY
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BAKVM

# Outline of the lecture

- L'Hospital's rule
- Higher-order derivatives

## L'Hospital's rule

- Limits of the type $\frac{0}{0}$
- Higher-order derivatives
- Limits of the type $\frac{0}{0}$ revisited
- Limits of the type $\frac{?}{\infty}$
- Limits of the type $\infty - \infty$

# L'Hospital's Rule

Let $f$ and $g$ be two functions and let $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ not be an isolated point of the domains $\mathrm{D}_f$ and $\mathrm{D}_g$. Let us discuss the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

The case:

- $\lim_{x \to x_0} f(x)$ is finite & $\lim_{x \to x_0} g(x) \neq 0$ and is finite — easy,
  we already know the limit is the quotient of the limits
- $\lim_{x \to x_0} f(x)$ is finite & $\lim_{x \to x_0} |g(x)| = +\infty$ — easy, the limit is zero

# L'Hospital's Rule

Let $f$ and $g$ be two functions and let $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ not be an isolated point of the domains $\mathrm{D}_f$ and $\mathrm{D}_g$. Let us discuss the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

The cases:

- $\lim_{x \to x_0} |f(x)| = +\infty$ is finite & $g(x) \neq 0$ on a neighbourhood of the point $x_0$
- $\lim_{x \to x_0} f(x) \neq 0$ & $\lim_{x \to x_0} g(x) = 0$

— both are easy, the limit is either $+\infty$ or $-\infty$ or does not exist

# L'Hospital's Rule

Let $f$ and $g$ be two functions and let $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ not be an isolated point of the domains $D_f$ and $D_g$. Let us discuss the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

The cases:

- $\lim_{x \to x_0} |f(x)| = +\infty$ is finite & $g(x) \neq 0$ on a neighbourhood of the point $x_0$
- $\lim_{x \to x_0} f(x) \neq 0$ & $\lim_{x \to x_0} g(x) = 0$

— both are easy, the limit is either $+\infty$ or $-\infty$ or does not exist

## L’Hospital’s Rule

Let $f$ and $g$ be two functions and let $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ not be an isolated point of the domains $D_f$ and $D_g$.

Consider the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

in the case when

$$\lim_{x \to x_0} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to x_0} g(x) = 0$$

and

$$¡¡¡ \quad g(x) \neq 0 \quad \text{for every} \quad x \in (x_0 - \delta, x_0 + \delta) \quad \text{for some} \quad \delta > 0 \quad !!!$$

Then it is the limit of the type $\frac{0}{0}$

# Notice

A special case of the limit

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}$$

is for

$$f(x)=F(x)-F(x_0) \qquad \text{and} \qquad g(x)=x-x_0$$

If the function $F$ is continuous at the point $x_0$, then $\lim_{x\to x_0} f(x)=0$ and

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=F'(x_0)=f'(x_0)$$

if the limit exists.

# Theorem (L'Hospital's Rule)

Let $x_0 \in \overline{\mathbb{R}}$ not be an isolated point of either the domains $D_f$ and $D_g$ of functions $f$ and $g$, and let

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

Moreover, let there be some $\delta > 0$ such that

$$f'(x) \quad \text{and} \quad g'(x) \quad \text{exist} \quad \text{for every} \quad x \in (x_0 - \delta, x_0 + \delta)$$

and

$$g(x) \neq 0 \quad \text{and} \quad g'(x) \neq 0 \quad \text{for every} \quad x \in (x_0 - \delta, x_0 + \delta)$$

…

# Theorem (L'Hospital's Rule)

… Then: If the limit

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

exists (finite or infinite), then the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

also exists and it holds

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

# Higher-order derivatives

Let $I$ be an open interval and let a function $f$ be defined on the interval $I$.
It may happen that the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite for every point $x$ of the interval $I$.

We have a new function $f'$ defined on the interval $I$ thus.

Choose a point $x_0 \in I$. Then

$$f''(x_0) = \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0)}{h}$$

is the **second derivative** of the function $f$ at the point $x_0$ (if the limit exists).

# Higher-order derivatives

**Notation:**

$f'$ … the (first) derivative

$f''$ … the second derivative

$f'''$ … the third derivative

$f^{(4)}$ … the fourth derivative

$\vdots$

$f^{(n)}$ … the $n$-th derivative

We also put:

$f^{(0)} = f$ … the zeroth derivative of the function $f$, i.e. the original function $f$

# It also holds — Theorem

Let

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

and

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$
$$g'(x_0) = g''(x_0) = \cdots = g^{(n-1)}(x_0) = 0$$

the derivatives exist and are zero

and

$f^{(n)}$ exists and is finite

$g^{(n)}$ exists and is finite

and
$\exists \delta > 0 \ \forall x \in (x_0 - \delta, x_0 + \delta)$:
$g(x) \neq 0$

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f^{(n)}(x_0)}{g^{(n)}(x_0)}$$

# Theorem

Let $x_0 \in \overline{\mathbb{R}}$ not be an isolated point of either the domains $\mathrm{D}_f$ and $\mathrm{D}_g$ of functions $f$ and $g$, and let

$$\lim_{x \to x_0} |g(x)| = +\infty$$

Moreover, let there be some $\delta > 0$ such that

$$f'(x) \quad \text{and} \quad g'(x) \quad \text{exist} \quad \text{for every} \quad x \in (x_0 - \delta, x_0 + \delta)$$

and

$$g'(x) \neq 0 \quad \text{for every} \quad x \in (x_0 - \delta, x_0 + \delta)$$

…

# Theorem

… Then: If the limit

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

exists (finite or infinite), then the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

also exists and it holds

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

# Limit of the type $\infty-\infty$

Let

$$\lim f(x) = \lim g(x) = \pm\infty$$

Then

$$\lim f(x) - g(x) = \lim \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

i.e. the limit of the type $\infty - \infty$ is transformed into the type $\frac{0}{0}$

# Example

Find the limit

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Recall that $(\sin x)' = \cos x$ and that $(x)' = (x^1)' = 1x^0 = 1$.
Hence:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

## Example

Find the limit

$$\lim_{x\to 0}\frac{\arctan x}{x}$$

Recall that $(x)' = 1$, but we have to find $(\arctan x)'$ first.

Recall that $\tan x = \dfrac{\sin x}{\cos x}$ for $x \neq k\dfrac{\pi}{2}$ for $k \in \mathbb{Z}$, therefore

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} =$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

# Example (continued)

Next, it holds for every $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that

$$\arctan \tan x = x \qquad | \,()'$$

$$(\arctan \tan x)' = 1$$

$$\arctan' \tan x \times \frac{1}{\cos^2 x} = 1$$

$$\arctan' \tan x = \cos^2 x \qquad | \, x = \arctan y$$

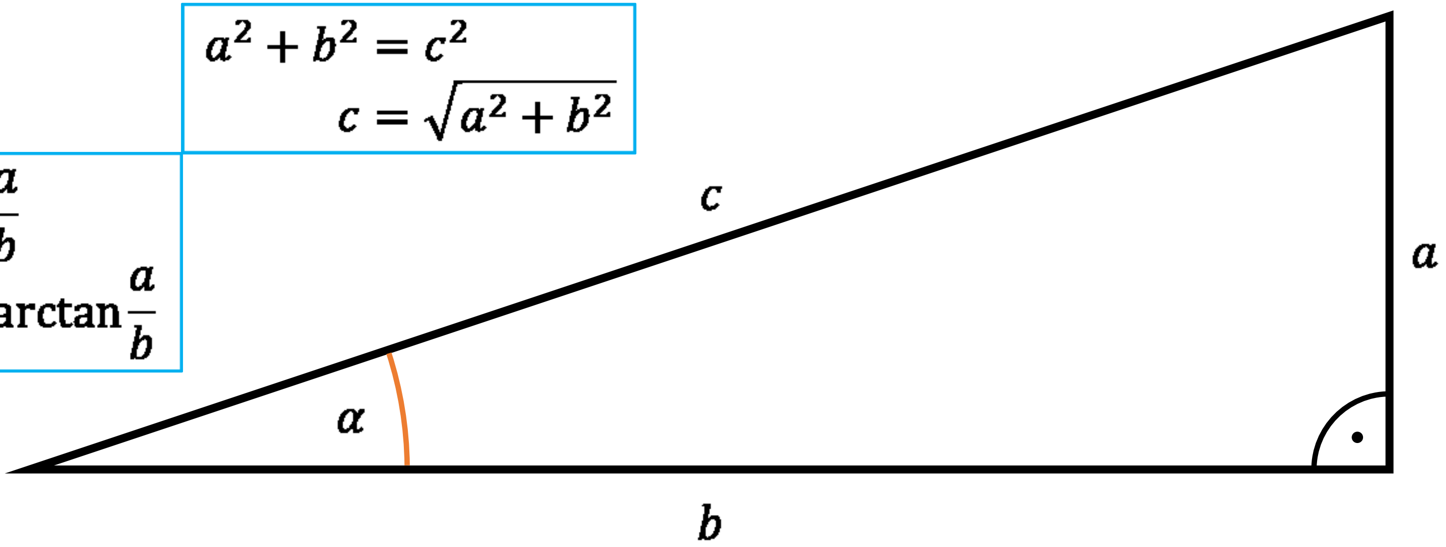
$$\arctan' y = \cos^2 \arctan y$$

# Example (continued)

Now, we know that $\arctan' y = \cos^2 \arctan y$ for all $y \in (-\infty, +\infty)$, but what is $\cos^2 \arctan y$ ?

$$a^2 + b^2 = c^2$$
$$c = \sqrt{a^2 + b^2}$$

$$\tan \alpha = \frac{a}{b}$$
$$\alpha = \arctan \frac{a}{b}$$




$$\cos \alpha = \frac{b}{c} = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\cos^2 \alpha = \frac{b^2}{a^2 + b^2}$$

$$\cos^2 \arctan \frac{a}{b} = \frac{b^2}{a^2 + b^2}$$

Assuming WLOG that $b = \pm 1$, we have obtained the result: $\arctan' a = \cos^2 \arctan a = \frac{1}{1+a^2}$

# Example (finished)

**Finally:**

$$\lim_{x\to 0}\frac{\arctan x}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{1} = \frac{\frac{1}{1+0}}{1} = \frac{1}{1} = 1$$

# Example

Find the limit

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{\arcsin x}\right)$$

We calculate:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{\arcsin x}\right)=\lim_{x\to 0}\frac{\arcsin x-\sin x}{\sin x\times\arcsin x}$$

Recall that $(\sin x)'=\cos x$, but we have to find $(\arcsin x)'$ first.

# Example (continued)

Next, it holds for every $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that

$$\arcsin \sin x = x \qquad |\,()'$$

$$(\arcsin \sin x)' = 1$$

$$\arcsin' \sin x \times \cos x = 1$$

$$\arcsin' \sin x = \frac{1}{\cos x} \qquad |\, x = \arcsin y$$

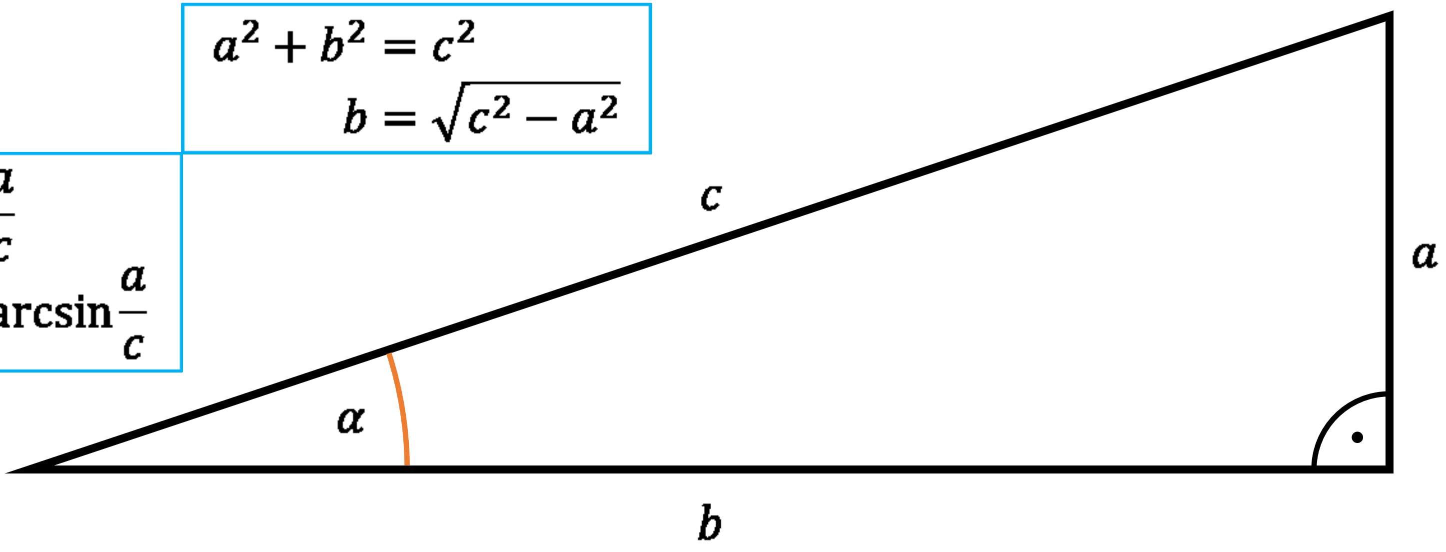
$$\arcsin' y = \frac{1}{\cos \arcsin y}$$

# Example (continued)

Now, we know that $\arcsin' y = \frac{1}{\cos \arcsin y}$ for all $y \in (-1, +1)$, but what is $\frac{1}{\cos \arcsin y}$?

$$a^2 + b^2 = c^2$$
$$b = \sqrt{c^2 - a^2}$$

$$\sin \alpha = \frac{a}{c}$$
$$\alpha = \arcsin \frac{a}{c}$$




$$\cos \alpha = \frac{b}{c} = \frac{\sqrt{c^2 - a^2}}{c}$$

$$\cos \arcsin \frac{a}{c} = \frac{\sqrt{c^2 - a^2}}{c}$$

Assuming WLOG that $c = 1$, we have obtained <u>the result</u>: $\arcsin' a = \frac{1}{\cos \arcsin a} = \frac{1}{\sqrt{1-a^2}}$

# Example (continued)

**Finally:**

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{\arcsin x}\right)=\lim_{x\to 0}\frac{\arcsin x-\sin x}{\sin x\times \arcsin x}=$$

$(fg)' = f'g + fg'$

$$=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}-\cos x}{\cos x\times \arcsin x+\sin x\times \frac{1}{\sqrt{1-x^2}}}=\cdots$$

# Example (finished)

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{\arcsin x}\right)=\cdots=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}-\cos x}{\cos x\times\arcsin x+\sin x\times\frac{1}{\sqrt{1-x^2}}}=$$

$(fg)'=f'g+fg'$

$$=\lim_{x\to 0}\frac{\frac{x}{\sqrt{1-x^2}^3}+\sin x}{-\sin x\times\arcsin x+\cos x\times\frac{1}{\sqrt{1-x^2}}+\cos x\times\frac{1}{\sqrt{1-x^2}}+\sin x\times\frac{x}{\sqrt{1-x^2}^3}}=$$

$$=\frac{0+0}{0+1+1+0}=0$$

$\left(f(g(x))\right)'=f'(g(x))\times g'(x)$

$\left(\frac{1}{\sqrt{1-x^2}}\right)'=\left((1-x^2)^{-\frac{1}{2}}\right)'=-\frac{1}{2}(1-x^2)^{-\frac{3}{2}}\times(-2x)=\frac{x}{\sqrt{1-x^2}^3}$