Quantitative Methods

Lecture 8

Sketching the graph of a function

SILESIAN UNIVERSITY
SCHOOL OF BUSINESS ADMINISTRATION IN KARVINA

BAKVM

Outline of the lecture

- Sketching the graph of a function

  (functions increasing and decreasing, convex and concave,

  local minima and maxima, inflection points)

Sketching the graph of a function

To sketch the graph, determine:

- the domain D_f and the range R_f of the function
- the intersections with the coordinate axes x and y
- intervals of monotonicity ([non-]increasing/[non-]decreasing)
- intervals of convexity and concavity ([strictly] convex/[strictly] concave)
- local extrema and inflection points

Intervals of monotonicity and of convexity and concavity

- Function increasing, non-decreasing, non-increasing, and decreasing
- Convex combination of points
- Function strictly convex, convex, concave, and strictly concave
- Characterization by the derivatives of the first, second and higher order

Sketching the graph of a function

We already know: Let I be an interval and let f be a function defined on the I.

on the interval I, the function f is	…if and only if…	for every $x_1, x_2 \in I$ such that $x_1 < x_2$
increasing	…………	$f(x_1) < f(x_2)$
non-decreasing	…………	$f(x_1) \leq f(x_2)$
non-increasing	…………	$f(x_1) \geq f(x_2)$
decreasing	…………	$f(x_1) > f(x_2)$

Sketching the graph of a function

We already know: $f'(x)$ is the slope of the line tangent to the graph of the function f at the point x.

Given the line

$$y = ax + b$$

where a is its slope, we know:

— if $a > 0$, then the line goes up (increases)

— if $a = 0$, then the line is horizontal (constant)

— if $a < 0$, then the line goes down (decreases)

Sketching the graph of a function

Hence:

If $f'(x) > 0$ at every point x of an interval I,

then the function $f(x)$ is increasing on I.

Example (however): The function

$$y = x^3$$

is increasing, but

$$y' = 3x^2$$

is zero at $x = 0$.

Theorem

Let I be an open interval and let f be a function defined on I.

Assume that the first derivative $f'(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

I.	on the interval I, the function f is	…if and only if…	for every $x \in I$
	non-decreasing	………..	$f'(x) \geq 0$
	non-increasing	………..	$f'(x) \leq 0$
hence	**constant**	………..	$f'(x) = 0$

Theorem

Let I be an open interval and let f be a function defined on I.

Assume that the first derivative $f'(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

II.

If $f'(x) > 0$ for every $x \in I$, then the function f is **Increasing** on the interval I.

If $f'(x) < 0$ for every $x \in I$, then the function f is **decreasing** on the interval I.

Theorem

Let I be an open interval and let f be a function defined on I.

Assume that the first derivative $f'(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

III. The function f is **increasing** on the interval I if and only if

it is non-decreasing on it and

each open interval $J \subseteq I$ contains at least one $x \in J$ such that $f'(x) > 0$.

Theorem

Let I be an open interval and let f be a function defined on I.

Assume that the first derivative $f'(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

III. The function f is **decreasing** on the interval I <u>if and only if</u>

it is non-increasing on it and

each open interval $J \subseteq I$ contains at least one $x \in J$ such that $f'(x) < 0$.

Sketching the graph of a function

Analogously:

If $f''(x) > 0$ at every point x of an interval I,

then the first derivative $f'(x)$ is increasing on I.

What does that mean?

$\rightarrow$ The function $f(x)$ is strictly convex on I.

Sketching the graph of a function

Analogously:

If $f''(x) > 0$ at every point x of an interval I,

then the function $f(x)$ is strictly convex on I.

Example (however): The function

$$y = x^4$$

is strictly convex, but

$$y'' = 4 \times 3x^2$$

is zero at $x = 0$.

Theorem

Let I be an open interval and let f be a function defined on I.

Assume the second derivative $f''(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

I. on the interval I, the function f is	...if and only if...	for every $x \in I$
convex	....	$f''(x) \geq 0$
concave	....	$f''(x) \leq 0$
hence **linear (straight)**	....	$f''(x) = 0$

Theorem

Let I be an open interval and let f be a function defined on I.

Assume the second derivative $f''(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

II.

If $f''(x) > 0$ for every $x \in I$, then the function f is **convex** on the interval I.

If $f''(x) < 0$ for every $x \in I$, then the function f is **concave** on the interval I.

Theorem

Let I be an open interval and let f be a function defined on I.

Assume the second derivative $f''(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

III. The function f is **strictly convex** on the interval I if and only if

it is convex on it and

each open interval $J \subseteq I$ contains at least one $x \in J$ such that $f''(x) > 0$.

Theorem

Let I be an open interval and let f be a function defined on I.

Assume the second derivative $f''(x)$ (finite or infinite) exists at each point $x \in I$.

Then:

III. The function f is **strictly concave** on the interval I if and only if

it is concave on it and

each open interval $J \subseteq I$ contains at least one $x \in J$ such that $f''(x) < 0$.

Local extrema and inflection points

Function increasing / decreasing at a point

The point of a (strict) local maximum / minimum

The line tangent to the graph of a function at a point

Function strictly convex / concave at a point

The point of inflection

Function increasing and decreasing at a point

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0$.

The function f is **increasing at the point** x_0 if and only if

there exists a $\delta > 0$ such that

$$f(x) < f(x_0) \qquad \text{for all} \quad x \in (x_0 - \delta, x_0)$$

and

$$f(x_0) < f(x) \qquad \text{for all} \quad x \in (x_0, x_0 + \delta)$$

Function increasing and decreasing at a point

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0$.

The function f is **decreasing at the point** x_0 if and only if

there exists a $\delta > 0$ such that

$$f(x) > f(x_0) \qquad \text{for all} \quad x \in (x_0 - \delta, x_0)$$

and

$$f(x_0) > f(x) \qquad \text{for all} \quad x \in (x_0, x_0 + \delta)$$

Function increasing and decreasing at a point

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0,$ and assume that $f'(x_0)$ exists.

Theorem:

If $f'(x_0) > 0,$ then f is increasing at the point x_0.

If $f'(x_0) < 0,$ then f is decreasing at the point x_0.

Local extrema of a function

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0$.

There is a **local maximum** of the function f at the point x_0 if and only if there exists a $\delta > 0$ such that

$$f(x) \leq f(x_0) \qquad \text{for all} \quad x \in (x_0 - \delta, x_0 + \delta)$$

Local extrema of a function

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0$.

There is a **strict local maximum** of the function f at the point x_0 if and only if there exists a $\delta > 0$ such that

$$f(x) < f(x_0) \qquad \text{for all} \quad x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$$

increasing
↓
decreasing

Local extrema of a function

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0$.

There is a **strict local minimum** of the function f at the point x_0 if and only if

there exists a $\delta > 0$ such that

$$f(x) > f(x_0) \qquad \text{for all} \quad x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$$

decreasing
↓
increasing

Local extrema of a function

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0$.

There is a **local minimum** of the function f at the point x_0 if and only if there exists a $\delta > 0$ such that

$$f(x) \geq f(x_0) \qquad \text{for all} \quad x \in (x_0 - \delta, x_0 + \delta)$$

Local extrema of a function

The

- local maxima
- strict local maxima
- strict local minima
- local minima

are together called **local extrema** of the function.

<u>Theorem:</u> Let $f'(x_0)$ exist.

If there is a local extremum of the function f at the point x_0, then $f'(x_0) = 0$.

The line tangent to the graph of a function

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0$, and let $f'(x_0)$ exist.

Recall that the equation of the line tangent to the graph of the function f at the point $[x_0, f(x_0)]$ is

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

Function strictly convex / concave at a point

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0,$ and let $f'(x_0)$ exist.

The function f is **strictly convex at the point** x_0 if and only if

there exists a $\delta > 0$ such that

$$f(x) > f'(x_0)x + f(x_0) - f'(x_0)x_0 \qquad \text{for all} \quad x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$$

Function strictly convex / concave at a point

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0$, and let $f'(x_0)$ exist.

The function f is **strictly concave at the point** x_0 if and only if

there exists a $\delta > 0$ such that

$$f(x) < f'(x_0)x + f(x_0) - f'(x_0)x_0 \qquad \text{for all} \quad x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$$

Function strictly convex / concave at a point

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$

for some $\delta_0 > 0$, and let $f'(x_0)$ exist.

Theorem:

If $f''(x_0) > 0$, then f is strictly convex at the point x_0.

If $f''(x_0) < 0$, then f is strictly concave at the point x_0.

Points of inflexion

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0$, and let $f'(x_0)$ exist.

The point x_0 is a rising point of inflexion of the function f if and only if there exists a $\delta > 0$ such that

$$f(x) > f'(x_0)x + f(x_0) - f'(x_0)x_0 \qquad \text{for all} \quad x \in (x_0 - \delta, x_0)$$

and

$$f(x) < f'(x_0)x + f(x_0) - f'(x_0)x_0 \qquad \text{for all} \quad x \in (x_0, x_0 + \delta)$$

convex
↓
concave

Points of inflexion

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0$, and let $f'(x_0)$ exist.

The point x_0 is a falling point of inflexion of the function f if and only if there exists a $\delta > 0$ such that

$$f(x) < f'(x_0)x + f(x_0) - f'(x_0)x_0 \qquad \text{for all} \quad x \in (x_0 - \delta, x_0)$$

and

$$f(x) > f'(x_0)x + f(x_0) - f'(x_0)x_0 \qquad \text{for all} \quad x \in (x_0, x_0 + \delta)$$

concave
↓
convex

Points of inflexion

Let a function f and a point $x_0 \in \mathbb{R}$ be given.

Assume that the function f is defined on the whole interval $(x_0 - \delta_0, x_0 + \delta_0)$ for some $\delta_0 > 0,$ and let $f'(x_0)$ exist.

Theorem: Let $f''(x_0)$ exist.

If there is an inflection at the point $x_0,$ then $f''(x_0) = 0.$

Examples

<u>Example 1:</u>

$$y = x^3$$

and consider the point

$$x_0 = 0$$

We have

$$y' = 3x^2$$

hence

$$y'(x_0) = 0$$

but there is <u>no local extremum</u> at x_0 – the function is strictly increasing there!

Examples

Example 2:

$$y = x^4$$

and consider the point

$$x_0 = 0$$

We have

$$y' = 4x^3$$

hence

$$y'(x_0) = 0$$

and yes, there is a strict local minimum at x_0.

Theorem

Let

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

and let

$$f^{(n)}(x_0) \neq 0$$

Then:

If n is odd and $f^{(n)}(x_0) > 0$, then f is <u>increasing</u> at the point x_0.

If n is odd and $f^{(n)}(x_0) < 0$, then f is <u>decreasing</u> at the point x_0.

If n is even and $f^{(n)}(x_0) > 0$, then there is a <u>strict local minimum</u> of f at x_0.

If n is even and $f^{(n)}(x_0) < 0$, then there is a <u>strict local maximum</u> of f at x_0.

e.g. $n = 2$

Examples

Example 1:

$$y = x^3$$

and consider the point

$$x_0 = 0$$

We have

$$y' = 3x^2$$

hence

$$y'(x_0) = 0$$

but there is no local extremum at x_0 – the function is strictly increasing there!

Examples

Example 2:

$$y = x^4$$

and consider the point

$$x_0 = 0$$

We have

$$y' = 4x^3$$

hence

$$y'(x_0) = 0$$

and yes, there is a strict local minimum at x_0.