Tautologies

Example 1: What do you notice about each sentence below?

1.	A number is even or a number is not even.
2.	Cheryl passes math or Cheryl does not pass math.
3.	It is raining or it is not raining.
4.	A triangle is isosceles or a triangle is not isosceles.

Each sentence in Example 1 is the disjunction of a statement and its negation. Each of these sentences can be written in symbolic form as $p \vee \sim p$. Recall that a disjunction is false if and only if both statements are false; otherwise it is true. By this definition, $p \vee \sim p$ is always true, even when statement p is false or statement ~p is false! This is illustrated in the truth table below:

p	~p	$p \vee \sim p$
T	F	T
F	T	T

The compound statement $p \vee \sim p$ consists of the individual statements p and ~p. In the truth table above, $p \vee \sim p$ is always true, regardless of the truth value of the individual statements. Therefore, we conclude that $p \vee \sim p$ is a tautology.

Definition: A compound statement, that is always true regardless of the truth value of the individual statements, is defined to be a **tautology**.

Let's look at another example of a tautology.

Example 2: Is $(p \wedge q) \rightarrow p$ a tautology?

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Solution: The compound statement $(p \wedge q) \rightarrow p$ consists of the individual statements p, q, and $p \wedge q$. The truth table above shows that $(p \wedge q) \rightarrow p$ is true regardless of the truth value of the individual statements. Therefore, $(p \wedge q) \rightarrow p$ is a tautology.

In the examples below, we will determine whether the given statement is a tautology by creating a truth table.

Example 3: Is $x \rightarrow (x \vee y)$ a tautology?

x	y	$x \vee y$	$x \rightarrow (x \vee y)$

T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Solution: Yes; the truth values of $x \rightarrow (x \vee y)$ are {T, T, T, T}.

Example 4: Is $\sim b \rightarrow b$ a tautology?

b	~b	$\sim b \rightarrow b$
T	F	T
F	T	F

Solution: No; the truth values of $\sim b \rightarrow b$ are {T, F}.

Example 5: Is $(p \vee q) \rightarrow (p \wedge q)$ a tautology?

p	q	$(p \vee q)$	$(p \wedge q)$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Solution: No; the truth values of $(p \vee q) \rightarrow (p \wedge q)$ are {T, F, F, T}.

Example 6: Is $[(p \rightarrow q) \wedge p] \rightarrow p$ a tautology?

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Solution: Yes; the truth values of $[(p \rightarrow q) \wedge p] \rightarrow p$ are {T, T, T, T}.

Example 7: Is $(r \rightarrow s) \leftrightarrow (s \rightarrow r)$ a tautology?

r	s	$r \rightarrow s$	$s \rightarrow r$	$(r \rightarrow s) \leftrightarrow (s \rightarrow r)$

T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Solution: No; the truth values of (r → s) ↔ (s → r) are {T, F, F, T}.

Summary: A compound statement that is always true, regardless of the truth value of the individual statements, is defined to be a tautology. We can construct a truth table to determine if a compound statement is a tautology.

Exercises

1. What is the truth value of r ∨ ~r?

- True
- False
- Not enough information was given.
- None of the above.

RESULTS BOX:

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2. Is the following statement a tautology? s → ~s

- Yes
- No
- Not enough information was given.
- None of the above.

RESULTS BOX:

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3. Is the following statement a tautology? [(p ∨ q) ∧ ~p] → q

- Yes
- No

☐ Not enough information was given.
☐ None of the above.

RESULTS BOX:

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4. Is the following statement a tautology? $\sim(x \vee y) \leftrightarrow (\sim x \wedge \sim y)$

☐ Yes
☐ No
☐ Not enough information was given.
☐ None of the above.

RESULTS BOX:

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5. Is the following statement a tautology? $a \wedge \sim a$

☐ Yes
☐ No
☐ Not enough information was given
☐ None of the above

RESULTS BOX:
