SILESIAN
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# Mathematics in Economics

## Lecture 2

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# Introduction to differential calculus of one real variable
# The derivative of a function

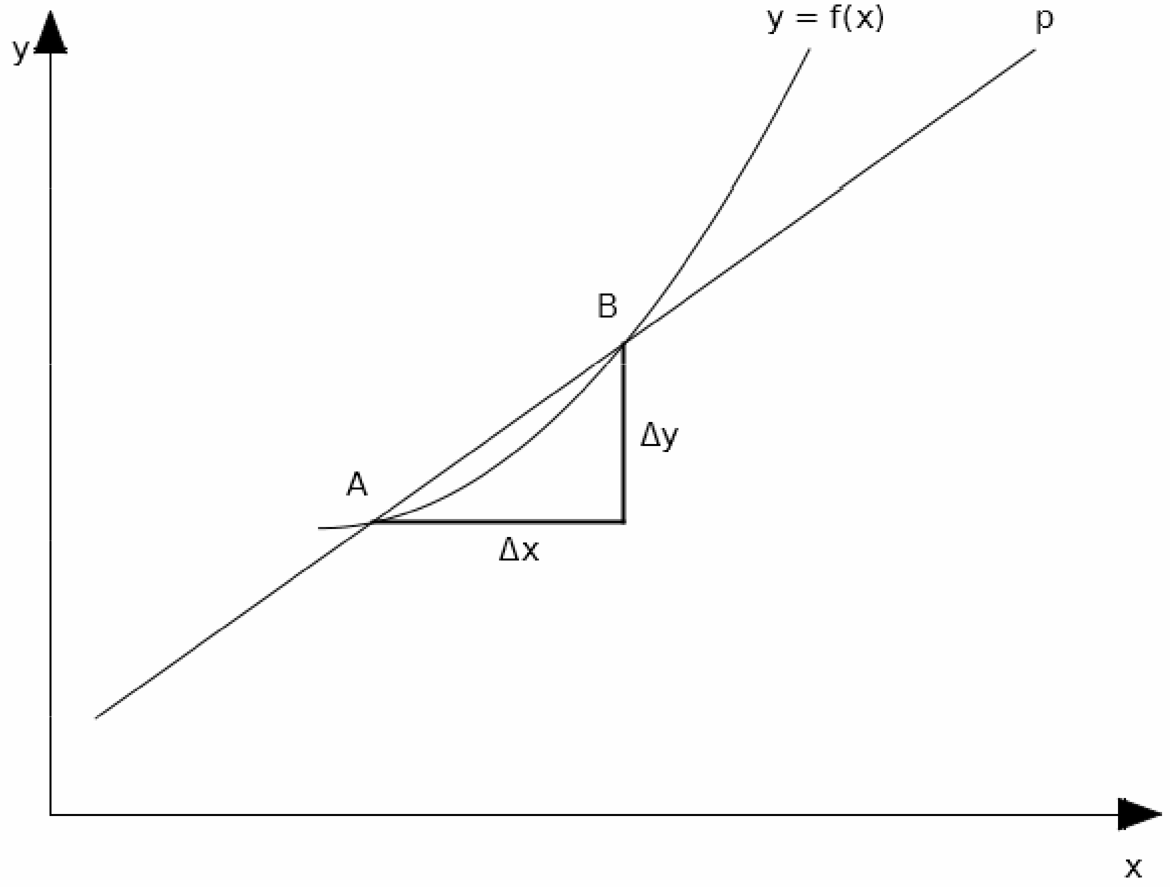
Let y = f (x) be a function of one real variable. Then the derivative of the function f is defined as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative is usually denoted f´(x) or y´. The process of finding a derivative is called differentiation.

Geometric interpretation: the derivative of a function f at a point x is equal to the slope of a tangent line to the curve at the point x.

# Geometric interpretation of a derivative



# The rules of differentiation

Let f(x) and g(x) be functions with the derivative in the interval $J \subseteq R$ Then:

i) $[c \cdot f(x)]' = c \cdot f'(x)$

ii) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

iii) $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

iv) $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$, $g(x) \neq 0$

v) $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

# Derivatives of elementary functions

| $f(x)$ | $f'(x)$ |
|---|---|
| konstanta | $0$ |
| $x$ | $1$ |
| $x^n$ | $nx^{n-1}$ |
| $e^x$ | $e^x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $a^x$ | $a^x \cdot \ln a$ |
| $\log_a x$ | $\frac{1}{x \ln a}$ |

| | |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\mathrm{tg}x$ | $\frac{1}{\cos^2 x}$ |
| $\mathrm{cotg}x$ | $-\frac{1}{\sin^2 x}$ |
| $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\mathrm{arctg}x$ | $\frac{1}{1+x^2}$ |
| $\mathrm{arccotg}x$ | $-\frac{1}{1+x^2}$ |

# Examples

$$y = x^2 \Rightarrow y' = 2 \cdot 1 = 2$$

$$y = 6x^3 - 5x + 4 \Rightarrow y' = 18x^2 - 5$$

$$y = \frac{1}{x^2} \Rightarrow y' = -\frac{2}{x^3}$$

$$y = 5^x \Rightarrow y' = 5^x \cdot \ln 5$$

$$y = \ln\left(x^2 - 4\right) \Rightarrow y' = \frac{1}{x^2 - 4} \cdot (2x)$$

$$y = x \cdot e^x \Rightarrow y' = 1 \cdot e^x + x \cdot e^x$$

# Derivatives of higher orders

If a function f´(x) can be differentiated, we obtain the second derivative of f(x), denoted as f´´(x), and so on.

First derivatives are used to find monotonicity and extremas of functions. The second derivative is useful in finding concavity and concavity or inflexion points. The use of derivatives of the order 3 and higher are rather rare.

$y = \ln x \Rightarrow y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$

Example: $y = \ln x \Rightarrow y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$

# Differential of a function

The differential of a function y = f(x) is denoted as dy, and is defined as follows: $dy = f'(x)dx$ .

The differential expresses an increment of the dependent variable dy in respect to the increment of independent variable dx. Also, the differential isused to linearization of more complex functions.

Example: Find the the differential of the function $y = x^2$ at a point x = 4.

Solution: dy=2xdx, and substituting x = 4 we obtain: dy = 8dx.

# The logarithmic differentiation

For functions of the type $y = f(x)^{g(x)}$ we use the so called logarithmic differentiation:

$$y = f(x)^{g(x)}$$

$$\ln y = \ln f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x) \, /'$$

$$\frac{1}{y} y' = g'(x) \cdot \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x)$$

$$y' = y \left[ g'(x) \cdot \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x) \right]$$

$$y' = f(x)^{g(x)} \left[ g'(x) \cdot \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x) \right]$$

# Taylor and Maclaurin series

Let a function y = f(x) be differentiable of the order n at a point a, then it can be approximated by the Taylor series of the form:

$$T_n(f,a,x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \quad \dots \quad + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

- If a = 0, we obtain a special case of the Taylor series, called Maclaurin series:

$$T_n(f,0,x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \quad \dots \quad + \frac{f^{(n)}(0)}{n!}x^n + R_{n+1}(x)$$

# Maclaurin series of selected functions

| Function | Maclaurin series | Range of convergence |
|---|---|---|
| sinx | $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$ | $(-\infty,\infty)$ |
| cosx | $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$ | $(-\infty,\infty)$ |
| exp(x) | $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$ | $(-\infty,\infty)$ |
| $\ln(x+1)$ | $x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\dots$ | $(-1,1\rangle$ |

# Taylor and Maclaurin series

Example: Find the Maclaurin series of the function $y = e^x$ .

Solution:

$$f(0) = e^0 = 1,$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1,$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1,$$

$$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1,$$

........

Therefore, we obtain:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

# Economic applications of derivatives

The elasticity of a function y = f(x):

$$E(x) = \lim_{\Delta x \to 0} \frac{x}{y} \frac{\Delta y}{\Delta x} = \frac{x}{y} \frac{dy}{dx} = \frac{x}{y} y'$$

The price elasticity of demand:

$$E(P) = -\frac{P}{Q} \frac{dQ}{dP}$$

The price elasticity of supply:

$$E(P) = \frac{P}{Q} \frac{dQ}{dP}$$

# Economic applications of derivatives – cont.

Marginal product of labour:

$$MP_L = \frac{dQ}{dL} = Q'(L)$$

Marginal revenue:

$$MR = \frac{dTR(Q)}{dQ}$$

Marginal cost:

$$MC = \frac{dTC}{dQ}$$

# Solved problems 1

Find the derivative of the function $y = x^2 \cdot \ln x$

Solution: The given function is a product of two elementary functions, so we must use the product rule.

We obtain:

$$y' = 2x \cdot \ln x + x^2 \cdot \frac{1}{x},$$

$$y' = 2x \cdot \ln x + x.$$

# Solved problems 2

Find the differential of the function $y = x^3 + 1$. Also, find the increment dy for x = 2 and dx = 0.1

Solution: $y' = 3x^2$ and the differential is: $dy = 3x^2 dx$

The increment dy: $dy = 3 \cdot 2^2 \cdot 0.1 = 1.2$

# Solved problems 3

Find the Taylor series of the function $y = \sqrt{x}$ at the point a = 1.

Solution:

$$f(1) = \sqrt{1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4\sqrt{x^3}} \Rightarrow f'(1) = -\frac{1}{4\sqrt{1^3}} = -\frac{1}{4}$$

.....

Therefore, the Taylor series is given as follows:

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \ldots$$

# Solved problems 4

Find the Maclaurin series of the function $y = \sqrt{x+1}$ .

Solution:

$$f(0) = \sqrt{0+1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \Rightarrow f'(1) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4\sqrt{(x+1)^3}} \Rightarrow f'(1) = -\frac{1}{4\sqrt{(0+1)^3}} = -\frac{1}{4}$$

.....

Therefore, the Taylor series is given as follows:

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \ldots$$

# Solved problems 5

Find the derivative of the function $y = x^x$ .

Solution:

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \cdot \ln x \;/'$$

$$\frac{1}{y} y' = \ln x + x \cdot \frac{1}{x}$$

$$\frac{1}{y} y' = \ln x + 1$$

$$y' = y\left[\ln x + 1\right]$$

$$y' = x^x \left[\ln x + 1\right]$$

# Solved problems 6

Find the derivative of the function $y = x^{\ln x}$.

Solution:

$$y = x^{\ln x}$$

$$\ln y = \ln x^{\ln x}$$

$$\ln y = \ln x \cdot \ln x \,/'$$

$$\frac{1}{y} y' = \frac{1}{x} \ln x + \ln x \cdot \frac{1}{x}$$

$$\frac{1}{y} y' = \frac{2}{x} \ln x$$

$$y' = y\left[\frac{2}{x} \ln x\right]$$

$$y' = x^{\ln x} \frac{2}{x} \ln x$$

# Solved problems 7

Find marginal revenue MR (x) of the total revenue $TR(x) = x^3 - 2x^2 + 5x + 5$ and marginal costs of the total costs $TC(x) = 120x^4 - \ln x$ .

Solution: $$MR(x) = \frac{dTR(x)}{dx} = 3x^2 - 4x + 5$$

$$MC(x) = \frac{dTC(x)}{dx} = 480x^3 - \frac{1}{x}$$

# Problems to solve 1

Differentiate:

$$y = 1 + x + x^2 + x^3 + x^4$$

$$y = 24x^5 - 3x^2 + 8x - 4$$

$$y = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$$

$$y = \frac{3}{x^4} - 2\sqrt[3]{x} + \frac{4}{\sqrt[4]{x^3}}$$

$$y = 3^x - 2\log x + \sqrt{\sqrt[3]{x^2}}$$

$$y = 4tgx - \cot gx$$

# Problems to solve 2

Differentiate:

$$y = \left(x^2 + 1\right) \cdot e^x$$

$$y = \left(x^2 + 4\right) \cdot \sin x$$

$$y = \frac{2x^2 - 3x + 1}{x}$$

$$y = \frac{x}{\ln x}$$

$$y = \frac{x^2 - 1}{x^2 + 1}$$

$$y = \sqrt{x^2 + 4x}$$

$$y = \ln\left(4x + 1\right)$$

# Problems to solve 3

Find the Maclaurin series of the following functions:

$$y = \sin x$$

$$y = \cos x$$

$$y = e^{2x}$$

$$y = e^{3x}$$

$$y = \sqrt{x+4}$$

$$y = \ln(x+3)$$

**Thank you for your attention!**