SILESIAN UNIVERSITY
SCHOOL OF BUSINESS ADMINISTRATION IN KARVINA

Mathematics in Economics

Lecture 3

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Mathematics in Economics/PMAT

Function properties

In this lecture we will examine the basic function properties.

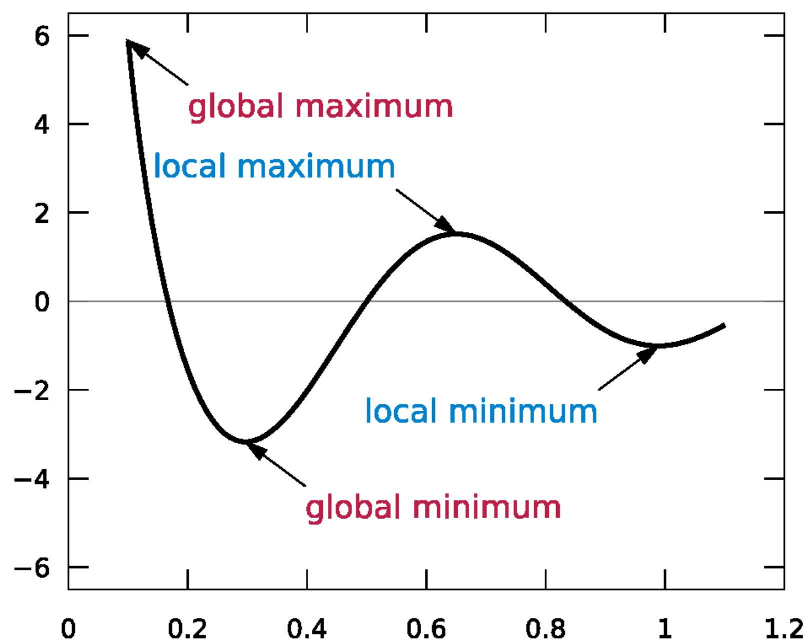
Monotonicity: A function y = f(x) is called **increasing** on the interval J = (a,b) if: $\forall \quad \in \quad < \quad <$

Also, a function y = f(x) is increasing if f (x) is positive on J.

Decreasing function is defined analogically.

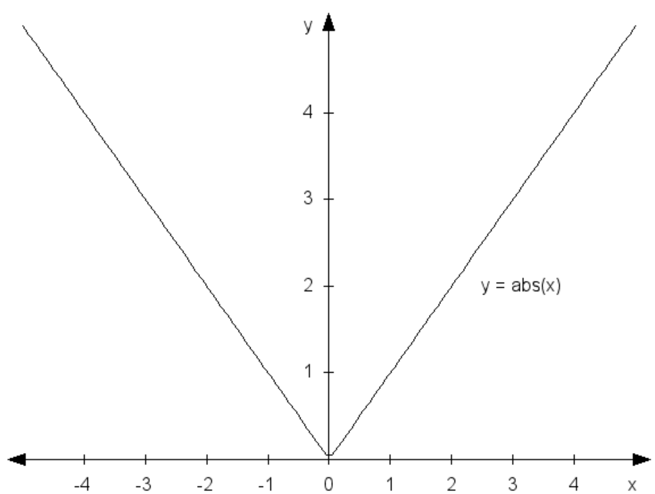
Extrema: A function y = f(x) has a local **maximum** at the point *a* if on some neighborhood of *a* the value f(a) is the highest.

A **minimum** is defined analogicall



Function properties

To **find extremes** of a function y = f(x), a **first derivative test** can be used: if a function has an extreme at a point *a*, and the derivative f (*a*) exists, then **f (*a*) = 0**. However, contrary is not true. Also, extremes might be at points where f (*x*) does not exist, see the picture below:



Function properties

The function $y = f(x)$ is said to be *even*, if $f(x) = f(-x)$ for all x. Geometrically speaking, a function is even when its graph is symmetrical with regard to the axis y.

The function $y = f(x)$ is said to be *odd*, if $f(x) = -f(-x)$ for all x. Geometrically speaking, a function is odd when its graph is symmetrical with regard to the point 0.

The function $y = f(x)$ is said to be *bounded from above*, if all values $f(x)$ are lower than some real number H.
The function $y = f(x)$ is said to be *bounded from below*, if all values $f(x)$ are higher than some real number L.
If a function is bounded from above and from below, it is called *bounded*.

Function properties

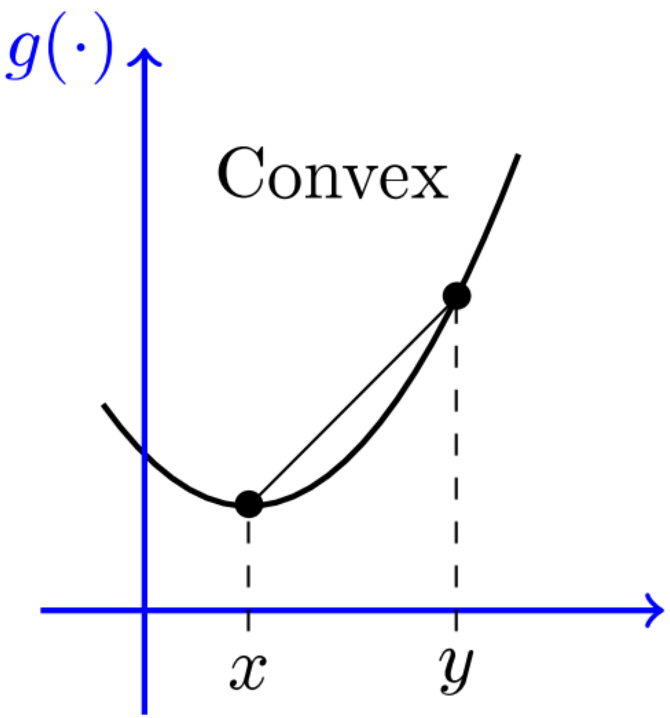
A function y = f(x) is called *periodical* if there exists real p such that f(x) = f(x+np) for all n. The *p* is called a period. Typical periodical functions are goniometric functions.

A function y = f(x) is called *convex* on the interval J, if f (x) > 0 for all x from J. (It has a shape of a valley)

A function y = f(x) is called *concave* on the interval J, if f (x) < 0 for all x from J. (It has a shape of a hill).

See the next slide.
Points at which the second derivative changes its sign are called *inflection points*.

Convex and concave functions



A usual procedure

When examining function properties, we usually follow the following structure:

1. Domain, odd/even, periodicity.
2. Limits at discontinuities and infinity.
3. Intersections with x and y axes.
4. The first derivative, its roots.
5. Extremes and monotonicity.
6. The second derivative and its roots.
7. Concave/convex intervals, inflection points
8. Asymptoties
9. Range.
10. A graph.

Solved problem 1

Find the properties of the function $y = \quad - \quad + \quad$.

We will follow the steps from the previous slide.

1. The domain: because the function is polynomial, D(f) = R.

Checking odd/even function requirements: the function is nor even, nor odd. Also, it is not periodical.

2. Limits in infinity: $\lim\limits_{x\to-\infty}(x^3 - \quad + \quad = -\infty$

$$\lim_{x\to\infty}(x^3 - \quad + \quad = \infty$$

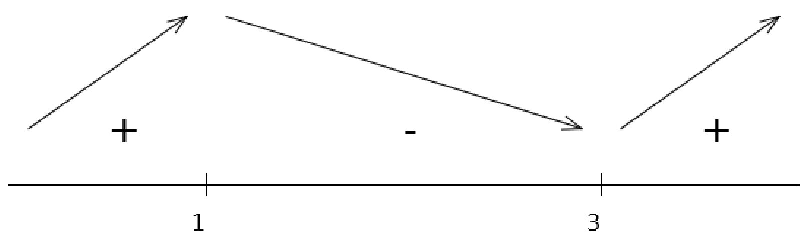
Solved problem 1 - cont.

3. Intersections with axes:
Let x = 0, then y = 0. We have the first intercept [0,0].
Let y = 0, then $0 = \quad - \quad + \quad = \quad -$
From the last equality we obtain two roots: x = 0 and x = 3.
Therefore, we attain two intercepts with the axis x: [0,0] and [3,0]. The first intercept is the same as for axis y.

4. The first derivative: $f'(x) = \quad - \quad + \quad$ = 0.
The roots of this quadratic equality are x = 1 and x = 3.
These points are called stationary points and might be Maximum, minimum or inflection point.

Solved problem 1 - cont.

5. Extremes: by checking the signs of the first derivative (see the picture below)
We find that for x = 1 the function has its local maximum, and x = 3 is a local minimum. (Also, we could applied the second derivative test).

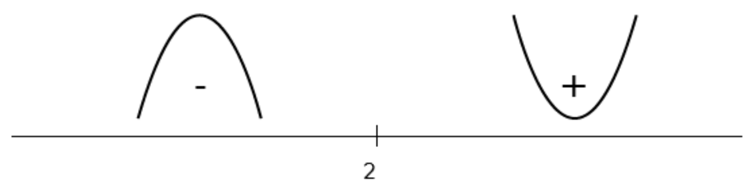


Monotonicity: from the picture above we see that the function is decreasing in (1,3) interval, and increasing elsewhere.

Solved problem 1 - cont.

6. The second derivative: $f''(x) = \quad - \quad$ = 0.
The root: x = 2. In this point an inflection point could be.

7. We check whether the sign of the second derivative changes in 2: yes, so we have the inflection point (see the picture below).



From the picture, intervals of convexity and concavity are clear.

Solved problem 1 - cont.

8. Asymptotes: asymptote is a line such that the distance between the curve and the line approaches zero as they tend to infinity. They can be vertical, horizontal or oblique. In our case there is no horizontal or vertical asymptote.

However, we can check an oblique asymptote y = ax + b:

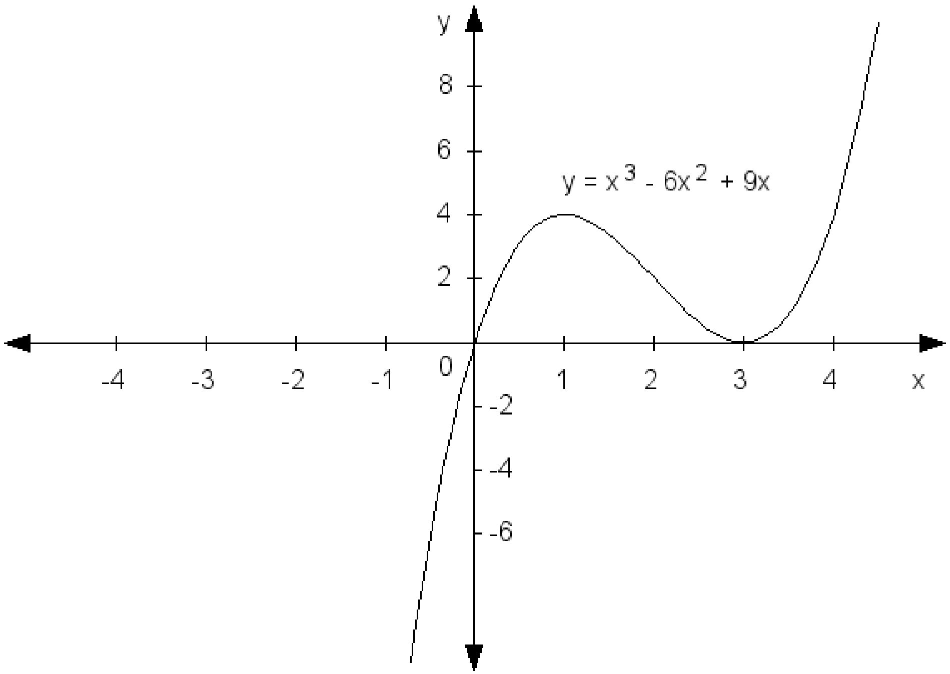
$$a = {}_{x\to\infty} \frac{f(x)}{x} = {}_{\to\infty} \quad - \quad + \quad = \infty$$

If the result is not a real number, the asymptote does not exist.

9. Range of the function: D(f) = R.

Solved problem 1 - end

10. The graph:



Solved problem 2

Find extremes of the function $y = \quad .$

Solution: we compute the first derivative and its roots:

$$y' = \quad + \quad = \quad +$$

The root is x = -1, it is the only stationary point. The nature of this point can be checked by sign of the derivative at both intervals on the left and right to -1. Because the derivative changes its sign from minus to plus, the point x = -1 is a local minimum.

Solved problem 3

Find asymptotes of the function $y = \frac{x^2}{x-}$.

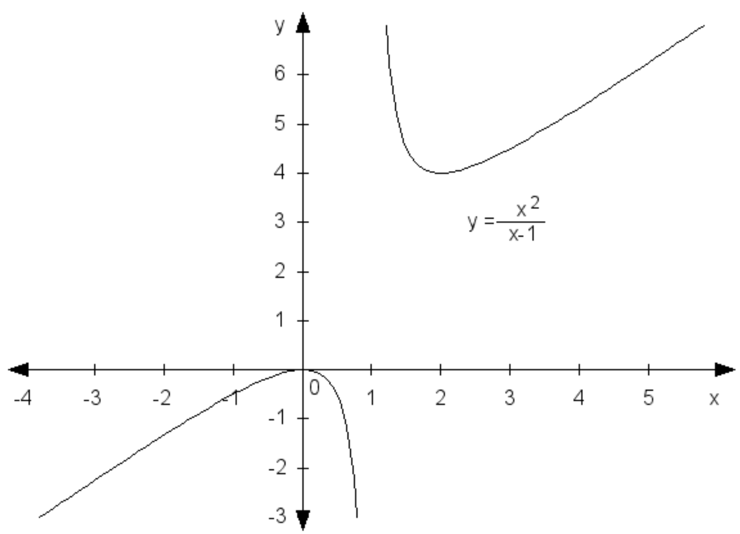
Solution: a vertical asymptote is determined from the domain of a function. In our case, x cannot be 1, therefore there exists the vertical asymptote x = 1.
A horizontal asymptote can be considered a special case of an oblique asymptote, so we can skip it for a while.
The oblique asymptote y = ax + b:

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{\to\infty} \frac{\frac{x^2}{\;}}{x} = \lim_{\to\infty} \frac{x^2}{x(x-} -$$

$$b = \lim_{x\to+\infty} \quad - \quad = \lim_{\to+\infty} \left[\quad \right] \left[\quad \right]$$

Solved problem 3 – cont.

The graph of the function $y = \frac{x^2}{x-}$.



Solved problem 4

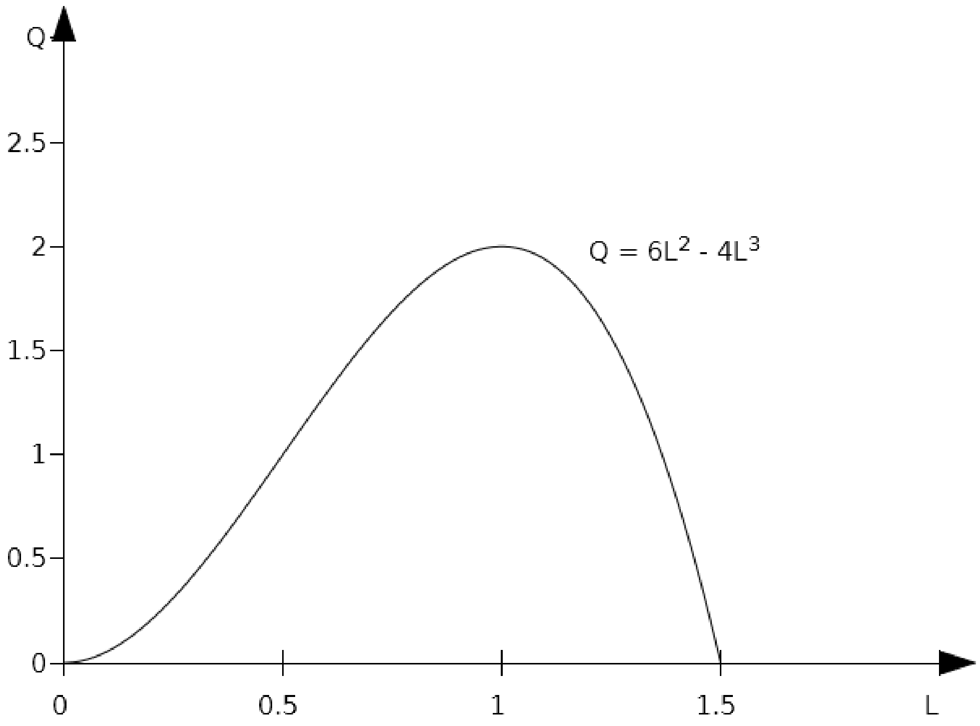
Find extremes of the production function $Q = \quad - \quad$. Draw its graph.

Solution:

The first derivative is $Q' = \quad - \quad$, we find roots of the first derivative: L = 0 and L = 1. By the use of the second derivative, or by checking signs of the first derivative, we obtain that L = 0 is a local minimum and L = 1 is a local maximum. Therefore, the highest (optimal) production is achieved when L = 1.

The graph is provided on the next slide.

Solved problem 4 – cont.



Solved problem 5

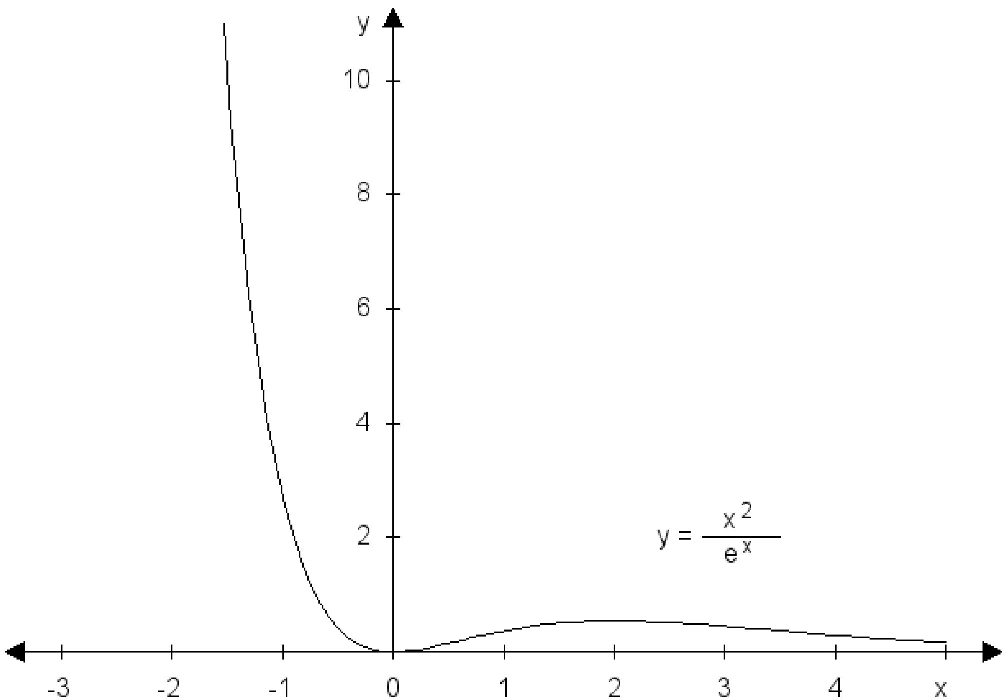
Find extremes and draw the graph of the function: $y = \frac{x^2}{e^x}$

Solution:
First, we compute the first derivative:

$$y' = \frac{2x}{e^x} = \frac{}{e^x}$$

As can be seen, the roots of the equation y = 0 are
x = 0 and x = 2. By checking the signs we obtain:
x = 0 is a local minimum and x = 2 is a local maximum.

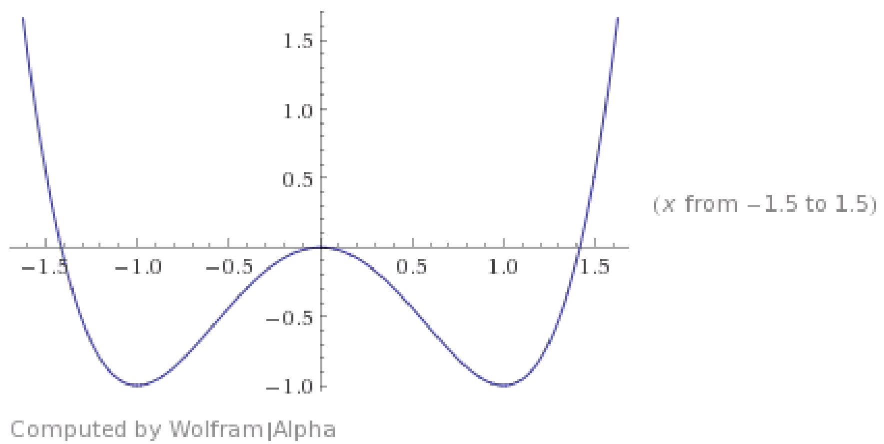
Solved problem 5 – cont.



Problems to solve 1 (Assignment 4)

Find all properties of the function $y = \quad -$

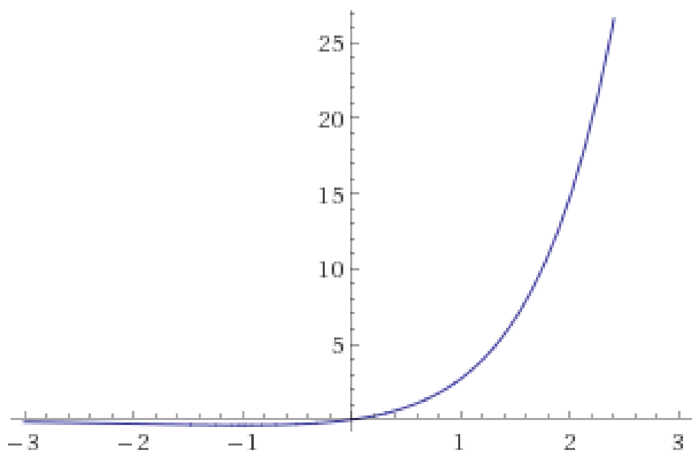
Hint:



Problems to solve 2 (Assignment 4)

Find all properties of the function $y = \;\cdot\;$.

Hint:



(x from −3 to 3)

Computed by Wolfram|Alpha

Problems to solve 3 (Assignment 5)

Find extremes of the following functions:

$f(x) = \quad - \quad +$

$f(x) = - \quad +$

$f(x) = \quad \cdot$

$f(x) = \dfrac{\quad}{x +}$

$f(x) = \quad + \quad +$

$f(x) = \dfrac{\quad}{\ln x}$

Problems to solve 3 (Assignment 5)

Find the maximum of total revenue function $TR(Q) = - \quad + \quad -$

Find the minimum of total cost function: $TC(Q) = \quad - \quad +$

Find the maximum of the profit function: $PR(Q) = \quad + \quad - \quad .$

Find the maximum of total revenue function: $TR(Q) = - \quad + \quad + \quad .$

Thank you for your attention!