SILESIAN UNIVERSITY
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Mathematics in economics

Lecture 5

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Extremes of a function of two real variables

Local vs global extremes.

Bounded vs unbounded extremes.

Necessary condition for the extreme:

$$\partial$$

$$\partial \qquad \partial$$

A point satisfying equalities above is called a stationary (critical) point. However, this condition is not sufficient

Extremes of a function of two real variables

In a critical point can be maximum, minimum or an inflection point. To decide which situation occurrs, we use the second derivatives and a matrix called *hessian*:

$$H_f(x,y) =$$

Then we use Sylvester s theorem.

Extremes of a function of two real variables

We denote: $D_1 = \frac{\partial}{\partial}$ and $D_2 = H_f(C)$. Then:

If D2>0, then we have an extreme. Moreover,
If D1>0, we have a minimum, if D1<0, we have a maximum.

IF D2<0, we have an inflection point.

If D2 = 0, we cannot decide.

Extremes of a function of two real variables - Problem 1

Find extremes of the function $f(x,y) =$.

Solution:

We start with the first derivatives:

$$\frac{\partial}{\partial} \quad - \quad \frac{\partial}{\partial}$$

Both derivatives must be 0, which yields the critical point C [0,0].

Extremes of a function of two real variables - Problem 1 – cont.

Now we compute all second derivatives and hessian:

∂ ∂ ∂ $\partial\partial$ $\partial\partial$

$H_f(x,y) =$

We substitute point C into hessian: $Hf(0,0) =$.

Because $D_2<0$, the point C is an inflection point.

Extremes of a function of two real variables - Problem 2

Find extremes of the function $f(x,y) = \quad - \quad + \quad$.

Solution:
We start with the first derivatives:

$$\frac{\partial f}{\partial x} = \; x - \; y = \qquad\qquad \frac{\partial f}{\partial y} = \; y + $$

Both derivatives must be 0, which yields the critical point *C* [1/2,1/2].

Extremes of a function of two real variables - Problem 2 – cont.

Now we compute all second derivatives and hessian:

$$\frac{\partial^2 f}{\partial x^2} = \qquad \frac{\partial^2 f}{\partial y^2} = \qquad \frac{\partial^2 f}{\partial x \partial y} =$$

$$H_f(x,y) = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}$$

Substituting point C into hessian yields the same result.

Because $D_2 < 0$, the point C is an inflection point.

Extremes of a function of two real variables - Problem 3

Find extremes of the function $f(x,y) = \;{}^{4} + \;$.

Solution:
We start with the first derivatives:

$$\frac{\partial f}{\partial x} = x^3 = \qquad\qquad \frac{\partial f}{\partial y} = y^3 =$$

Both derivatives must be 0, which yields the critical point *C* [0,0].

Extremes of a function of two real variables - Problem 3 – cont.

Now we compute all second derivatives and hessian:

$$\frac{\partial}{\partial} = 12x^2 \qquad \frac{\partial}{\partial} \qquad \frac{\partial f}{\partial x \partial y} =$$

$$H_f(x,y) = \begin{bmatrix} \; \end{bmatrix} \qquad H_f(C) = \begin{bmatrix} \; \end{bmatrix}$$

Because $D_2 = 0$, We cannot decide the nature of C.
But how do we know it is certainly a minimum?

Problem 4

Find the maximum of the revenue function:

$$TR(Q_1,Q_2) = \quad + \quad - \quad -$$

Solution:
We start with the first derivatives:

$$\partial \quad \partial \quad 2) = -$$

$$\partial \quad \partial \quad 2) = -$$

Both derivatives must be 0, which yields the critical point
C [12.5,2].

Problem 4 – cont.

Now we compute all second derivatives and hessian:

$$H_f(x,y) = \begin{bmatrix} & \\ & \end{bmatrix} \qquad H_f(C) = \begin{bmatrix} & \\ & \end{bmatrix}$$

Because $D_2 > 0$, we have an extreme.
Because $D_1 < 0$, we have a maximum.

Problems to solve – 1 (Assignment 7)

Find extremes of the following functions:

$f\,x,y = \quad + \quad - \quad +$

$f(x,y) = \quad - \quad +$

$f(x,y) = 2xy - 3x^2 - 2y^2 + 10$

$f(x,y) = y - \frac{x^3}{3} + \ln(x - y)$

$f(x,y) =$

Indefinite integral

Integration is a reverse procedure to differentiation.

Notation: $\int \quad + $

Legend:

$\int$ …. Integration sign

f(x) …. Integrated function

C …. Integration constant

Indefinite integral - cont

Indefinite integral is a linear operator:

$$\int \qquad \int$$

$$\int \qquad \int \qquad \int$$

We compute integrals with the use of formulas above, and with the use of the table of elementary integrals, see the next slide.

Indefinite integral – elementary integrals

$f(x)$	$\int f(x)dx$
0	C
1	$x + C$
x^n	$\frac{x^{n+1}}{n+1} + C$
e^x	$e^x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$
a^x	$\frac{a^x}{\ln a} + C$

Indefinite integral – elementary integrals

$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\frac{1}{\cos^2 x}$	$\mathrm{tg}x + C$
$-\frac{1}{\sin^2 x}$	$\mathrm{cotg}x + C$
$\frac{1}{1+x^2}$	$\mathrm{arctg}x + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} arctg \frac{x}{a} + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x + C$
$\frac{1}{\sqrt{1\pm x^2}}$	$\ln\left\lvert x+\sqrt{1\pm x^2}\right\rvert + C$

Indefinite integral - examples

$$\int \quad = \frac{x^3}{3} +$$

$$\int \quad \int \quad \int \quad \int \quad \int \quad \frac{x^4}{4} + \quad x^3 + \quad x^2 + +$$

$$\int \quad \int^{1} \quad \overset{4}{x^3} \quad 2\sqrt[4]{x^3} +$$

$$\int^{1} \quad \int \quad x \quad 1 +$$

Indefinite integral - examples

Indefinite integral – integration methods

For more complicated integration we use sutitable integration methods:

- Method per partes
- Partial fractions
- Substitutions

All these methods will be demonstrated on examples.

Indefinite integral – rational functions

By a rational function we mean the function of the form:

$$\frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomials.

In the first step we find the roots x_i of the denominator in order to rearrange the denominator into a product.

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\quad - \quad \cdot \quad - \quad \cdot \quad - \quad}$$

Indefinite integral – rational functions – cont.

Then, the situation splits into three possible cases:

1. All roots of a denominator are single. Then we obtain the following partial fractions:

$$\frac{P(x)}{Q(x)} =$$

2. Some root, for example x_1, is of order higher than 1:

$$\frac{P(x)}{Q(x)} =$$

Indefinite integral – rational functions – cont.

3. A denominator or its part given as a quadratic polynomial has no roots:

$$\frac{P(x)}{Q(x)} = \quad + \quad + \cdot \quad - \qquad + \quad + \qquad -$$

Coeffcients in numerators are unknown and must be computed by clearing a denomiantor and solving a subsequent equation.

Integration of rational functions – Problem 1

Solve: $\int \frac{3x}{\ }$.

Solution: The rational function is of case 1, with roots -2 and 1. Therefore, we obtain the following division into partial fractions:

$$\frac{3x+}{x+\quad -} \qquad \frac{A}{\ +} \qquad \frac{B}{\ -}$$

Now we clear the denominator:

$$3x + = \quad - \quad + \quad +$$

And we get two equations: $\begin{matrix} 3 = & + \\ 3 - & - \end{matrix}$

Integration of rational functions – Problem 1-cont.

Solving the equation yields: $A = \quad = \quad .$

Hence: $\frac{3x+}{x+} \quad - \quad + \quad -$

Now, we can integrate:

$$\int \quad \int \quad \int \quad - \quad - +$$

We will continue in Lecture 6.

Thank you for your attention!