Mathematics in Economics – lecture 5

Differential calculus of two real variables

1) The domain of a function of two real variables

By the domain of a function $f(x,y)$ we mean all ordered pairs $[x,y] \in R^2$ for which the function f is defined.

The domain is usually a subspace of Cartesian coordinate system (CCS).

Find the domain of the function $f(x,y) = \sqrt{x+y-2}$.

Find the domain of the function $f(x,y) = \ln(25 - x^2 - y^2)$

Find the domain of the function $f(x,y) = \sqrt{x^2 + y^2 - 14}$

2) A derivative of a function of two real variables

Let $f(x,y)$ be a function of two real variables. Then, this function can be differentiated with regard to x and/or y. These derivatives are called *partial derivatives* and are denoted as follows:

$$\frac{\partial f(x,y)}{\partial y}, \quad f_y'(x,y), \quad f_y' \qquad \frac{\partial f(x,y)}{\partial x}, \quad f_x'(x,y), \quad f_x'$$

In mathematics, a partial derivative of a function of several variables is **its derivative with respect to one of those variables, with the others held constant** (as opposed to the total derivative, in which all variables are allowed to vary). Partial derivatives are used in vector calculus and differential geometry.

If first derivatives are differentiable again, we obtain second derivatives:

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 f}{\partial x \partial y} \quad \frac{\partial^2 f}{\partial y \partial x}$$

The latter two derivatives are called mixed derivatives. It can be proved they are equal: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Find the first and the second derivatives of a function:

a) $f(x,y) = 6x^2y^3 + 3x^3y^4 - 5x^2 + 3y^3$

b) $f(x,y) = 3x^5y^4 - 4x^2y^5 - 6e^{4x} + 3siny$

c) $f(x,y) = 5x^2y^5 - 7x^4y - 5cosx + 8y^2$

3) A total differential

Let $f(x,y)$ be differentiable at a point $C = [c_1, c_2, c_3]$. Then, the total differential is given as:

$$df(C) = \frac{\partial f}{\partial x}(C)dx + \frac{\partial f}{\partial y}(C)dy$$

The total differential expresses (approximately) an increment of a function, which depends on increments of *x* (*dx*) and *y* (*dy*).

For a function of two or more independent variables, the total differential of the function is **the sum over all of the independent variables of the partial derivative of the function with respect to a variable, times the total differential of that variable**.

1) Find the total differential of the function $f(x,y) = 3x^2 + 5xy + y$, at the point *C* [1,1,9] and for *dx* = 0.1, *dy* = 0.2

Solution:

We start with the first derivatives:

Substituting *C*:

Substituting into general form yields:

2) Find the total differential of the function $f(x,y) = 4y^2 - 2xy + x$, at the point *C* [2,1,…] and for *dx* = – 0.1, *dy* = 0.3

Problems to solve

1) Find the domain of the following functions:

$$f(x,y)=\sqrt{2x-y+3}$$

$$f(x,y)=\sqrt{x^2+y^2-4}+\frac{1}{x}$$

$$f(x,y)=\sqrt{x-y+1}+\sqrt{x+y+1}$$

$$f(x,y)=\log\left(y^2+x\right)$$

2) Find the partial derivatives of the first and second order:

$$f(x,y)=x^2+y^2$$

$$f(x,y)=x^2y^3+5x+y-1$$

$$f(x,y)=\ln\left(xy+y^4\right)$$

3) Find the total differential of the function

$$f(x,y)=x^3+5xy$$