Inicializace duální simplexové metody (fáze I)

- řešíme úlohy

$$(P') \quad \min c^T x \qquad Ax = b \qquad x \geqq 0$$

$$(D') \quad \max y^T b \qquad y^T A \leqq c^T$$

- úlohy chceme řešit duální simplexovou metodou
- neznáme výchozí duálně přípustnou bázi.

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Postup fáze I:

- předpokládejme, že známe nějakou (libovolnou) bázi $B_0 \subseteq \{1, \dots, n\}$.
- například, jestliže známe hodnost matice A, tak známe i max. lineárně nezávislou množinu sloupců A_{B_0}, která generuje ostatní sloupce matice A.
- uvažujme např. hodnost $A = m_1 \leqq m < n$

počet prvků v bázi

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- zvolme $B_0 = \{1, 2, \dots, m_1\} \subseteq \{1, \dots, n\}$ ..... je báze
- řešme soustavu

$$y^T A_{B_0} = c^T_{B_0}$$

..... má aspoň jedno řešení y_0^T

- položme $N_0 = \{1, \dots, n\} \setminus B_0$

a indexy z N_0 rozdělme do dvou skupin:

$$N_0^> = \{ j \in N_0 \, ; \, y_0^T a_j > c_j \}$$

$$N_0^\leq = \{ j \in N_0 \, ; \, y_0^T a_j \leq c_j \}$$

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- jestliže $N_0^> = \emptyset$, pak máme duálně přípustnou bázi → nyní pokračujeme ve výpočtu fází II.
- jinak $(N_0^> \neq \emptyset)$ pro každý index $j \in N_0^>$ zavedeme novou umělou proměnnou s_j a uvažujeme novou duální úlohu
- pro jednoduchost uvažujme dále, že

$$N_0^> = \{ m_1 + 1, m_1 + 2, \dots, m_2 \},$$

kde $m_2 \leq n$

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(pro D')

$$\max \quad s_{m_1+1} + \dots + s_{m_2}$$

z. p.

$$
\begin{array}{lllll}
y^T a_1 & & & & \leq c_1 \\
\vdots & & & & \vdots \\
y^T a_{m_1} & & & & \leq c_{m_1} \\
y^T a_{m_1+1} & + \; s_{m_1+1} & \ddots & & \leq c_{m_1+1} \\
\vdots & & & & \vdots \\
y^T a_{m_2} & & & + \; s_{m_2} & \leq c_{m_2} \\
y^T a_{m_2+1} & & & & \leq c_{m_2+1} \\
\vdots & & & & \vdots \\
y^T a_m & & & & \leq c_m \\
 & - \; s_{m_1+1} & \ddots & & \leq y_0^T a_{m_1+1} - c_{m_1+1} + 1 \\
 & & & - \; s_{m_2} & \leq y_0^T a_{m_2} - c_{m_2} + 1 \\
 & s_{m_1+1} & \ddots & & \leq 0 \\
 & & & s_{m_2} & \leq 0
\end{array}
$$

výchozí duálně příp. báze

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Úlohu vyřešíme.

Jestliže v optimu je $s = 0$ (tj. $s_{m_1+1} = \dots = s_{m_2} = 0$) a nedošlo k degeneraci, potom všechny proměnné/podmínky "$s_j \leq 0$" jsou v bázi, přidané podmínky vymažeme, zůstane duálně přípustná báze. $\to$ pokračujeme fází II.

Jestliže v optimu je $s \neq 0$ ... některé $s_j < 0$, potom konec výpočtu: původní úloha (D') není přípustná.

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Wolfeho metoda pro úlohy kvadratického programování

Uvažujeme úlohu

$$\min \ \frac{1}{2} x^T Q x + p^T x$$

s.t. $\quad Ax = b$

$\qquad x \geq 0$

kde $Q \in \mathbb{R}^{n \times n}$ .... symetrická.

Q ... pozitivně definitní, jestliže $p \neq 0$

Q ... pozitivně semidefinitní, -"- $p = 0$

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Pozn.: Symetrie matice Q není na újmu obecnosti:

když A je obecná, tak

$$x^T A x = x^T (Ax) = (Ax)^T x = (x^T A^T) x = x^T A^T x$$

Tedy

$$x^T A x = \frac{1}{2} x^T A x + \frac{1}{2} x^T A^T x = x^T \left(\frac{A + A^T}{2} \right) x$$

symetrická matice

4 18-17:17

Wolfe's method for problems of quadratic programming

Consider the problem of quadratic programming in the following form:

$$\min \ \frac{1}{2} x^T Q x + p^T x$$
$$\text{s.t.} \quad Ax = b, \quad x \geq 0$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $p^T \in \mathbb{R}^{1 \times n}$

and $Q \in \mathbb{R}^{n \times n}$ ... symmetric positive definite if $p^T \neq 0^T$

– " – semidefinite if $p^T = 0^T$

$x \in \mathbb{R}^n$ .... variables

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Let us consider:

Let $\underline{x}^*$ be an optimal solution to the problem.

- choose any $y \in \mathbb{R}^n$ such that:

$$Ay = 0 \quad \text{and} \quad x^*_j = 0 \Rightarrow y_j \geq 0$$

- then consider the point

$$x^* + y \cdot \varepsilon \quad \text{with } \varepsilon \searrow 0$$

- then observe that

$$A(x^* + y \cdot \varepsilon) = b$$
$$(x^* + y \cdot \varepsilon) \geq 0$$

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hence (x^* ... optimum ... min)

$$\frac{1}{2}(x^{*T}+\varepsilon\cdot y^T)\cdot Q\cdot(x^*+y\cdot\varepsilon)+\lambda^T(x^*+y\cdot\varepsilon)\geqq\frac{1}{2}x^{*T}Qx^*+\lambda^Tx^*$$

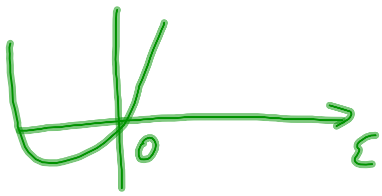
so

$$\frac{1}{2}\varepsilon\cdot y^TQx^*+\frac{1}{2}x^{*T}Qy\cdot\varepsilon+\frac{1}{2}\varepsilon\cdot y^TQy\cdot\varepsilon+\lambda^Ty\cdot\varepsilon\geqq 0$$

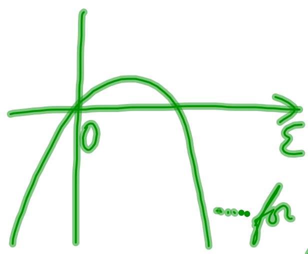
so (Q ... symmetric)

$$\frac{1}{2}y^TQy^T\cdot\varepsilon^2+(x^{*T}Q+\lambda^T)\cdot y\cdot\varepsilon\geqq 0$$

for all $\varepsilon\geqq 0$ ($\varepsilon\searrow 0$)



or



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It follows $\underbrace{(x^{*T}Q+\lambda^T)\cdot y}_{\text{the coefficient of the linear term must be }\geqq 0}\geqq 0$

So we have (if x^* ... is optimal):

$$\left.\begin{array}{l}\forall y\in\mathbb{R}^n:\ Ay=0\\ \quad\&\ \underline{y_j\geqq 0}\ \text{... for } j\in\{1,\dots,n\} \text{ such that } x^*_j=0\end{array}\right\}\Rightarrow(x^{*T}Q+\lambda^T)\cdot y\geqq 0$$

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Recall the Farkas lemma:

$A \in \mathbb{R}^{m \times n}$, $c^T \in \mathbb{R}^{1 \times n}$

$\forall x \in \mathbb{R}^n: \; Ax \leq 0 \implies c^T x \leq 0$

iff

$\exists \mu^T \in \mathbb{R}^{1 \times m}, \mu^T \geq 0^T: \; c^T = \mu^T A$

Equivalently:

$\forall x \in \mathbb{R}^n: \; Ax \geq 0 \implies c^T x \geq 0$

iff

$\exists \mu^T \in \mathbb{R}^{1 \times m}, \mu^T \geq 0^T: \; c^T = \mu^T A$

We have:

$$\forall y \in \mathbb{R}^n: \left. \begin{array}{l} Ay \geq 0 \\ -Ay \geq 0 \\ e_j^T y \geq 0 \quad \text{for } x^*_j = 0 \end{array} \right\} \implies (x^{*T} Q + \lambda^T) y \geq 0$$

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Therefore, by the Farkas lemma:

$\exists \mu^{+T}, \mu^{-T} \in \mathbb{R}^{1 \times m}, \mu^{+T}, \mu^{-T} \geq 0^T, \; \exists \nu_j \geq 0$ for j such that $x^*_j = 0$:

$$(x^{*T} Q + \lambda^T) = \mu^{+T} A - \mu^{-T} A + \sum_{\substack{j=1 \\ x^*_j = 0}}^{n} \nu_j e_j^T$$

Equivalently:

$\exists \mu^T \in \mathbb{R}^{1 \times m} \; \exists \nu_j \geq 0$ for j $x^*_j = 0$:

$$x^{*T} Q + \lambda^T + \mu^T A = \sum_{\substack{j=1 \\ x^*_j = 0}}^{n} \nu_j e_j^T$$

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Introduce a vector $v \in \mathbb{R}^n$ so that:

$$v_j = \begin{cases} \text{that } v_j \cdots\cdots & \text{if } x_j^* = 0 \\ 0 \quad \cdots\cdots & \text{if } x_j^* \neq 0 \end{cases}$$

That is: $v^T x^* = 0$
(equivalently)

So we proved:

Theorem: If x^* is an optimal (min) solution to the quadratic programming problem $\min \frac{1}{2} x^T Q x + p^T x$
$Ax = b,\ x \geq 0$,
then
$\exists\, u^T \in \mathbb{R}^{1\times m}\ \exists\, v^T \in \mathbb{R}^{1\times n},\ v^T \geq 0 : x^T Q + p^T + u^T A = v^T$
$\&\ v^T x^* = 0$

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Example: $\min\ x^2 - 5x + 7y$
s.t. $3x + 4y = 6$
$x, y \geq 0$

$Q = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$
$p^T = (-5\ \ 7)$
$A = (3\ \ 4)$
$b = (6)$

Solve the system:

"$x^T Q + p^T + u^T A = v^T$" $\quad x \geq 0,\ v^T \geq 0$ $\ \&\ v^T x = 0$
$\&\quad Ax = b$

So
$$2x - 5 + 3u = v_1 \qquad x \cdot v_1 = 0$$
$$7 + 4u = v_2 \qquad y \cdot v_2 = 0$$
$$3x + 4y = 6$$

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Simpler approach:

$$y = \frac{6-3x}{4} \geq 0 \qquad \min x^2 - 5x + 7\,\frac{6-3x}{4}$$

$$3x \leq 6 \qquad \min x^2 - \frac{41}{4}x + \frac{21}{2}$$

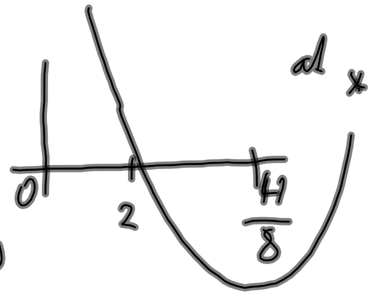
$x \geq 0$

$y \geq 0$

$x \leq 2$

global min is at $-\frac{b}{2a}$

at $x = \frac{41}{8} > 5$



So the optimal solution

is $\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \nu_1 = 0$

$$2 \cdot 2 - 5 + 3\mu = 0 \Rightarrow \mu = \frac{1}{3}$$

$$7 + 4\mu = \nu_2 \Rightarrow \nu_2 = \frac{25}{3}$$

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Another example:

$$\min -x^2 - y^2 \qquad Q = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

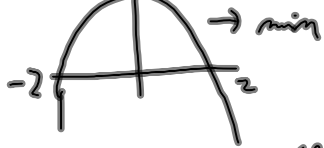
$$\text{s.t.} \quad x + y = 2 \qquad c^T = (0 \;\; 0)$$

$$x, y \geq 0 \qquad A = (1 \;\; 1)$$

$$b = (2)$$

Two optimal points:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \;\&\; \begin{pmatrix} x^{**} \\ y^{**} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



Solve: " $Ax = b$ $\qquad x \geq 0 \quad \nu^T x = 0$

$x^T Q + c^T + \mu^T A = \nu^T$ $\quad \nu^T \geq 0$

$$-2x + \mu = \nu_1 \qquad x \cdot \nu_1 = 0 \qquad \text{Solutions: } \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mu = 4, \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$-2y + \mu = \nu_2 \qquad y \cdot \nu_2 = 0 \qquad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mu = 4, \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$x + y = 2 \qquad \text{Third solution: } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mu = 2, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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The last example shows that the condition is necessary only.

The next result gives a sufficient condition.

<u>Theorem:</u> Let Q .... positively semidefinite

x^* .... a feasible solution .... satisfies $Ax^* = b$, $x^* \geq 0$

μ^*, $\nu^* \geq 0$ solve the system $x^T Q + p^T + \mu^T A = \nu^T$, $\nu^T \cdot x = 0$

Then $\underline{x^*}$ is an optimal solution to

$$\min \tfrac{1}{2} x^T Q x + p^T x$$
$$\text{s.t.} \quad Ax = b,\ x \geq 0$$

Moreover,
if Q is positively definite,
then $\underline{x^*}$ is the only optimal solution

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<u>Proof:</u> Let $x \in \mathbb{R}^n$, $Ax = b, x \geq 0$, be any feasible solution.

Let $y = x - x^*$. Then $Ay = 0$

& $y_j \geq 0$ .... for j $x_j^* = 0$

Then

$$\tfrac{1}{2} x^T Q x + p^T x = \tfrac{1}{2}(x^* + y)^T Q (x^* + y) + p^T (x^* + y)$$
$$= \tfrac{1}{2} x^{*T} Q x^* + p^T x^* + (p^T + x^{*T} Q) y + \tfrac{1}{2} y^T Q y$$

We also have

$$x^{*T} Q + p^T + \mu^{*T} A = \nu^{*T} \quad \Big|\, () \cdot y$$

$$(x^{*T} Q + p^T) \cdot y + \mu^{*T} \underbrace{Ay}_{=0} = \nu^{*T} y$$

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Notice:

$$r^{*T} y = \sum_{j=1}^{n} r_j^* \cdot y_j \geqq 0$$

- if $x_j^* = 0$, then $r_j^* \geqq 0$ & $y_j \geqq 0$
- if $x_j^* > 0$, then $r_j^* = 0$ $\Rightarrow r^{*T} y \geqq 0$

So we have:

$$(x^{*T} Q + p^T) \cdot y \geqq 0$$

Then:

$$\frac{1}{2} x^T Q x + p^T x = \frac{1}{2} x^{*T} Q x^* + p^T x^* + \underbrace{(x^{*T} Q + p^T) y}_{\geqq 0} + \underbrace{\frac{1}{2} y^T Q y}_{\substack{\geqq 0 \\ > 0 \dots \text{if}}}$$

q.e.d.

if Q is positively definit

and x ≠ x*

4 25-17:27

A remark on the existence of an optimal solution.

Theorem: If Q is positively definite and the problem is feasible ... $\begin{matrix} Ax = b \\ x \geqq 0 \end{matrix}$ ... has a solution x_0

then the problem has an optimal solution.

Proof: Since $x \in \mathbb{R}^n$ ... we are in a space of finite dimension

then $\exists \alpha > 0 \ \forall x \in \mathbb{R}^n : x^T Q x \geqq \alpha \cdot \|x\|^2$

4 25-18:01

Let $\|x\|$ .... Euclidian norm .... any norm
e.g.

Then the unit sphere

$$S = \{x \in \mathbb{R}^n;\ \|x\| = 1\}$$ .... is a compact set

$f(x) := x^T Q x$ .... is continuous

$\Rightarrow f$ ... attains its global minimum on S at $\bar{x} \in S$, say

$$\forall x \in S: \quad x^T Q x \geq \underbrace{\bar{x}^T Q \bar{x}}_{\alpha}$$

$x \in \mathbb{R}^n$ ... general:

$x = 0$ .... obvious

$x \neq 0$ .... then $\frac{x}{\|x\|} \in S \Rightarrow \frac{x^T}{\|x\|} Q \frac{x}{\|x\|} \geq \bar{x}^T Q \bar{x}$

$\left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1$

$$x^T Q x \geq \bar{x}^T Q \bar{x} \cdot \|x\|^2$$

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Then

$$\frac{1}{2} x^T Q x + \lambda^T x \geq \frac{1}{2} \alpha \|x\|^2 - \|\lambda\| \cdot \|x\| \overset{*)}{>}$$

$$> \frac{1}{2} x_0^T Q x_0 + \lambda^T x_0 \quad \text{if } \|x\| > R$$

where $R > 0$ is sufficiently large

*) $\pm \lambda^T x \leq |\lambda^T x| \leq \|\lambda\| \cdot \|x\|$

Schwarz inequality

So the minimum value must be attained in the set $C = \{x \in \mathbb{R}^n;\ Ax = b, x \geq 0, \|x\| \leq R\}$, which is compact, the objective function is continuous $\Rightarrow$ attains its global minimum in C

q.e.d.

4 25-18:10

Wolfe's algorithm

We are solving

$$\min \tfrac{1}{2} x^T Q x + p^T x$$
$$\text{s.t.} \qquad Ax = b$$
$$x \geqq 0$$

We assume:

Q ⋯ symmetric positively definite

or Q ⋯ symmetric positively semidefinite & $p^T = 0$.

4 25-18:18

Then to solve the problem is equivalent

to solving the system

$$Ax = b \quad \& \quad x \geqq 0$$

$$x^T Q + p^T + u^T A = v^T \quad \& \quad v^T x = 0$$
$$v^T \geqq 0$$

or rather

$$Qx + A^T u - I v = -p$$
$$Ax \qquad\qquad = b \quad \& \quad v^T x = 0$$
$$x, v \geqq 0$$

write $u = u^+ - u^-$, $u^+, u^- \geqq 0$

Apply primal simplex method

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≈ Idea: use the primal simplex method

- if x_j is basic, then $\underline{v_j}$ $\underline{\text{must}}$ be non-basic (must not enter the basis)
- if v_j is basic, then $\underline{x_j}$ $\underline{\text{must}}$ be non-basic
- if x_j and v_j are non-basic, then any of them can enter the basis

4 25-18:27

Wolfe's algorithm:

I. solve the system $Ax = b$, $x \geq 0$

$\Downarrow$

apply the (primal / OR dual) simplex method to the problem

$$\min\ \sigma^T x \quad \text{s.t.}\quad Ax = b,\ x \geq 0$$

[if no solution, we are done

Let $\underline{x}'$ be a solution found.
Actually, the $\underline{x}'$ is a basic solution (because we applied the simplex method).

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So $\underline{m}$ variables are basic:

$B' \subseteq \{1, \dots, m\}$ ... the final basis found

x'_i for $i \in B'$ .... basic variables

$|B'| = m$ ... if rank $A = m$

$|B'| \leqq m$ ... in general

If – by chance – it holds

$Qx' = -p$ ... then we are done

x' ... is an optimal solution to the q.p. problem

4 25-18:36

II. Assume that $Qx' \neq -p$

Then, introduce new variables

$$R_1, R_2, \dots, R_m \geqq 0$$

Determine constant numbers

$$d_1, d_2, \dots, d_m \in \{-1, +1\}$$

so that

$$q_j x' + d_j \cdot \underbrace{R_j}_{\geqq 0} = -p_j \quad \text{for } j = 1, \dots, m$$

the j-th row of matrix Q

4 25-18:40

- if $q_{j\cdot}x' < p_j \rightsquigarrow$ then $d_j := +1$
  $R_j = p_j - q_{j\cdot}x'$
- if $q_{j\cdot}x' > p_j \rightsquigarrow$ then $d_j := -1$
  $R_j = q_{j\cdot}x' + p_j$
- if $q_{j\cdot}x' = p_j \rightsquigarrow$ then chose $d_j \in \{+1,-1\}$
  $R_j = 0$

Form the diagonal matrix

$$D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix}$$

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III. Solve the problem

$$\min \; \sigma^T x + \sigma^T u^+ - \sigma^T u^- + \sigma^T v + e^T R \;\to \min$$

$$\text{s.t.} \quad \begin{aligned} Qx + A^T u^+ - A^T u^- - I v + D R &= -p \\ Ax &= b \end{aligned}$$

$$x, v, R \geq 0, \quad u^+, u^- \geq 0$$

columns $B' \subseteq \{1,\dots,m\}$
(from step I.)

Apply the primal simplex method.

The initial feasible basis is in red.

4 25-18:48

III. Solve the problem (*)

$$\min \ \sigma^T x + \sigma^T u^+ - \sigma^T u^- + \sigma^T v + e^T R$$

$$\text{s.t.} \quad Qx + A^T u^+ - A^T u^- - I v + D R = -p \ \vert s$$

$$A x = b \ \vert r$$

$$x, v, R \geq 0, \quad u^+, u^- \geq 0$$

- Apply the primal simplex algorithm to solve the problem (*).
(We know the initial basis.)

5 2-16:06

- In step 4. of the primal simplex method ("$\exists l \in N: y^T a_l > c_l$" ...... choice of the pivot column), we apply the exclusion rule:
  - if some $\underline{x_j}$ is in the present basis, then corresponding $\underline{v_j}$ must not enter the basis now – choose another pivot column

5 2-16:31

- if some $\underline{v}_j$ is in the present basis, then the corresponding $\underline{x}_j$ must not enter the basis now – choose another pivot column.

Questions:

1. Is the choice possible?
   → Wolfe's Theorems
2. Is the algorithm finite?
   → yes, if we avoid degeneracy
   ? Does the Bland least index rule work even here (with the exclusion rule)?

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Let $x, v, r \geqq 0$ and $u \in \mathbb{R}^m \dots u^+, u^- \geqq 0$

be some feasible solution to the problem (*).

Assume that $v^T x = 0$ holds. Assume that no further pivot step observing the exlucion rule is possible.

Divide the index set $\{1, \dots, n\}$ into three parts:

$$J_x^+ = \{ j \in \{1, \dots, n\} \ ; \ x_j > 0 \}$$

$$J_v^+ = \{ j \in \{1, \dots, n\} \ ; \ v_j > 0 \}$$

$$J_0 = \{ j \in \{1, \dots, n\} \ ; \ x_j = v_j = 0 \}$$

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Denote:

$a_1, \dots, a_n$ .... columns of matrix A

$a^1, \dots, a^m$ .... rows of matrix A

$q_1, \dots, q_n$ .... columns/rows of matrix Q (symmetric)

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Consider the problem (P)

$$\min \quad \sum_{j=1}^{m} 1 \cdot z_j \to \min$$

$$\text{s.t.} \quad \sum_{j \in J_x^+ \cup J_0} q_j \cdot x_j + \sum_{i=1}^{m} a^i u_i - \sum_{j \in J_v^+ \cup J_0} e_j \cdot v_j + \sum_{j=1}^{m} d_j \cdot z_j = -p \quad \mathbb{R}^n$$

$$\sum_{j \in J_x^+ \cup J_0} a_j \cdot x_j = b \quad \mathbb{R}^m$$

We are assuming that no further pivot step observing the exclusion rule in problem (*) is possible. Equivalently: the solution x,v,z ≥ 0, u is optimal to this problem (P).

$$x_j \geq 0, \quad u_i \in \mathbb{R}, \quad v_j \geq 0, \quad z_j \geq 0$$

Let μ be the optimal value of this problem (P).

5 2-16:47

Theorem: (Wolfe)

If Q is positively definite, then the optimal value of (P) is $\mu = 0$.

Proof:

Let $[r^*; s^*]$ be an optimal solution of the problem dual to (P).

Then $\mu = r^{*T} b - s^{*T} p$

5 2-16:52

and it holds:

(α) $r^{*T} a_j + s^{*T} q_j = 0$ for $j \in J_x^+$

(β) $r^{*T} a_j + s^{*T} q_j \leqq 0$ for $j \in J_0$

(γ) $s^{*T} a^i = 0$ for $i = 1, \dots, m$

(δ) $-s^{*T} e_j = 0$ for $j \in J_w^+$

(ε) $-s^{*T} e_j \leqq 0$ for $j \in J_0$

(ξ) $s^{*T} d_j \leqq 1$ for $j = 1, \dots, n$

5 2-17:07

Now, we show that $s^{*T} Q s^* \le 0$

It holds:

$$(r^{*T} a_j + s^{*T} q_j) \cdot s_j^* \begin{cases} \le 0 & \text{for } j \in J_0 \text{ by } (\beta)+(\varepsilon) \\ = 0 & \text{for } j \in J_x^+ \text{ by } (\alpha) \\ = 0 & \text{for } j \in J_r^+ \text{ by } (\delta) \end{cases}$$

sum up:

$$\underbrace{r^{*T} A s^*}_{=0 \text{ by } (\gamma)} + s^{*T} Q s^* \le 0$$

So it follows $s^{*T} Q s^* \le 0$

Q .... positively definite

Hence: $s^* = 0$

So the optimal value is:

$$\mu = r^{*T} b - s^{*T} h = r^{*T} b = r^{*T} A x =$$

$$= \sum_{j=1}^{n} r^{*T} a_j \cdot x_j = \sum_{j \in J_x^+} \underbrace{r^{*T} a_j}_{=0 \text{ by } (\alpha)} \cdot x_j + \sum_{j \in J_r^+ \cup J_0} r^{*T} a_j \cdot \underbrace{x_j}_{=0 \text{ because } j \notin J_x^+}$$

$= 0$ q.e.d.

__Theorem:__ (Wolfe)

If Q is positively semi-definite and $p^T = \sigma^T$, then the optimal value $\mu = 0$.

__Proof:__ Let $[r^*, s^*]$ be an optimal solution to the problem dual to (P).

We already know that
$$s^{*T} Q s^* \leqq 0$$

5 9-16:14

Now, for any $y \in \mathbb{R}^n$ and for any $\varepsilon \in \mathbb{R}$, it holds

$$(s^* + y\varepsilon)^T \cdot Q \cdot (s^* + y\varepsilon) \geqq 0$$ ...... because Q is positively semi-definite

hence:
$$s^{*T} Q s^* + \varepsilon \cdot (y^T Q s^* + s^{*T} Q y) + \varepsilon^2 \cdot (y^T Q y) \geqq 0$$

hence:
$$\varepsilon^2 \cdot (y^T Q y) + \varepsilon \cdot (2 \cdot y^T Q s^*) \geqq -s^{*T} Q s^* \geqq 0$$

for all $\varepsilon \in \mathbb{R}$, consider $\varepsilon \to 0$

It follows hence:
$$y^T Q s^* = 0$$ ...... for all $y \in \mathbb{R}^n$

5 9-16:20

So we have: $Q s^* = 0$

And the optimal value of the objective function:

$$\mu = r^{*T} b - s^{*T} \underbrace{p}_{=0} = r^{*T} b = r^{*T} A x = \sum_{j=1}^{m} r^{*T} a_j x_j =$$

$$= \sum_{j \in J_x^+} \underbrace{r^{*T} a_j x_j}_{= -s^{*T} q_j} + \sum_{j \in J_r^+ \cup J_0} r^{*T} a_j \underbrace{x_j}_{=0} =$$

by (α) because $j \notin J_x^+$

5 9-16:28

$$= \sum_{j \in J_x^+} \underbrace{-s^{*T} q_j}_{=0} x_j = 0$$

because $Q s^* = 0$

or $s^{*T} Q = 0^T$

q.e.d.

5 9-16:36

Remark: In the theorems we assumed:

Q ... positively definite and p^T any

or Q ... positively semi-definite and $p^T = 0^T$.

A trick due to E. M. L. Beale:

If Q is positively semi-definite and $p^T \neq 0^T$, then consider the matrix $(Q + \delta I)$ with $\delta > 0$, $\delta \searrow 0$.

Then $Q + \delta I$ is positively definite.

5 9-16:43

Example:

$$\min \; x_1^2 + x_1 x_2 + x_2^2 - 5x_2$$

$$\text{s.t.} \quad x_1 + x_2 = 1$$

$$x_1, x_2 \geqq 0$$

... the optimal solution is $x_1 = 0$, $x_2 = 1$

Find the solution by Wolfe's method:

Solution: The above is a problem of the form

$$\min \tfrac{1}{2} x^T Q x + p^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geqq 0$$

with

$Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $p = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$

$A = (1 \;\; 1)$ $b = (1)$

5 9-16:49

<u>Step I:</u> solve $\min \ \sigma^T x$ s.t. $Ax = b$, $x \geqq 0$ by the simplex method to find a feasible basic solution.

$$\min \ 0x_1 + 0x_2$$
$$\text{s.t.} \ \boxed{x_1} + x_2 = 1$$
$$x_1, x_2 \geqq 0$$

For example: $B' = \{x_1\}$ is a primal feasible basis

and $x_1' = 1$, $x_2' = 0$ is the corresponding basic solution

5 9-16:54

? $Qx' = -p$?

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad \ldots\ldots \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

<u>Step II:</u> Introduce new variables $R_1, R_2 \geqq 0$ and numbers $d_1, d_2 \in \{+1, -1\}$

$$2 - 1 \cdot R_1 = 0 \qquad \rightsquigarrow \qquad d_1 = -1 \qquad R_1 = 2$$
$$1 + 1 \cdot R_2 = 5 \qquad\qquad d_2 = +1 \qquad R_2 = 4$$

5 9-16:59

Step III: Solve the problem:

$$\min \; 0^T x + 0^T \mu + 0^T v + e^T R$$
$$\text{s.t.} \quad Qx + A^T\mu - Iv + DR = -p$$
$$Ax = b$$
$$x, v, R \geqq 0, \quad \mu \text{ ... unrestricted}$$

$$\min \; 0x_1 + 0x_2 + 0\mu + 0v_1 + 0v_2 + 1R_1 + 1R_2$$

$2x_1 + 1x_2 + 1\mu - 1v_1 - 1R_1$	$= 0$	$\mid s_1$
$1x_1 + 2x_2 + 1\mu - 1v_2 + 1R_2$	$= 5$	$\mid s_2$
$1x_1 + 1x_2$	$= 1$	$\mid r$

$$x_1, x_2, v_1, v_2, R_1, R_2 \geqq 0, \quad \mu \in \mathbb{R}$$

Initial basis: $B = \{x_1, R_1, R_2\}$

5 9-17:02

Iteration 1: step 1: basis: $B = \{x_1, R_1, R_2\}$

basic solution $x_1 = 1$, $R_1 = 2$, $R_2 = 4$

? is it dual feasible ?

step 2: solve $y^T A_B = c_B^T$

$2s_1 + 1s_2 + 1r = 0$		$s_1 = -1$
$-1s_1 = 1$		$s_2 = +1$
$+1s_2 = 1$		$r = 1$

? does it hold $y^T A_N \leqq c_N^T$

- columns x_2 and v_1 ... are violated

$(2 > 0$ and $1 > 0)$

5 9-17:10

step 4: l ... can be $\underline{x_2}$ or $\underline{s_1}$

because of the exclusion rule: l .. cannot be $\underline{s_1}$

so $l = x_2$ ... the only possibility

- solve: $a_l = A_B \, \rho_B$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \rho_{x_1} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \cdot \rho_{s_1} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \rho_{s_2}$$

$\rho_{x_1} = 1$

$\rho_{s_1} = 1$

$\rho_{s_2} = 1$

5 9-17:17

step 6: $\lambda = \min \left\{ \frac{x_i}{\rho_i} \; ; \; i \in B \, , \, \rho_i > 0 \right\}$

$$\lambda = \min \left\{ \underset{x_1}{\frac{1}{1}} \, , \, \underset{s_1}{\frac{2}{1}} \, , \, \underset{s_2}{\frac{4}{1}} \right\}$$

$k = x_1$ ... leaves the basis

$l = x_2$ ... enters the basis

The new basis is:

$B = \{ x_2 \, , \, s_1 \, , \, s_2 \}$

5 9-17:21

Iteration 2: step 1: solve $A_B x_B = b$

basis: $B = \{x_2, R_1, R_2\}$

$$\begin{array}{lllll} 1x_2 & -1R_1 & & = 0 \\ 2x_2 & & +1R_2 & = 5 \\ 1x_2 & & & = 1 \end{array}$$

$x_2 = 1$

$R_1 = 1$

$R_2 = 3$

step 2: solve $y^T A_B = c_B^T$

$$\begin{array}{lllll} 1s_1 & +2s_2 & +1r & = 0 \\ -1s_1 & & & = 1 \\ & +1s_2 & & = 1 \end{array}$$

$s_1 = -1$

$s_2 = 1$

$r = -1$

N_1 ... the only violated column

5 9-17:26

step 4: $l = N_1$ ... enters the basis

solve $a_l = A_B R_B$

$$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} R_{x_2} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} R_{R_1} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} R_{R_2}$$

$R_{x_2} = 0$

$R_{R_1} = 1$ ...... in step 6: $k = R_1$ leaves the basis

$R_{R_2} = 0$

The new basis is: $B = \{x_2, N_1, R_2\}$

5 9-17:47

Iteration 3: step 1: solve $A_B x_B = b$

$$\begin{array}{llll} 1x_2 & -1 s_1 & & = 0 \\ 2x_2 & & +1 r_2 & = 5 \\ 1x_2 & & & = 1 \end{array} \qquad \begin{array}{l} x_2 = 1 \\ s_1 = 1 \\ r_2 = 3 \end{array}$$

step 2: solve $y^T A_B = c_B^T$

$$\begin{array}{llll} 1 s_1 & +2 s_2 & +1 r & = 0 \\ -1 s_1 & & & = 0 \\ & 1 s_2 & & = 1 \end{array} \qquad \begin{array}{l} s_1 = 0 \\ s_2 = 1 \\ r = -2 \end{array}$$

Columns u is violated $(1 \neq 0)$

5 9-17:55

Step 4: $l = u$ ... enters the basis

$(l = u^+)$

Solve: $a_l = A_B r_B$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot r_{x_2} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \cdot r_{s_1} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_{r_2}$$

$r_{x_2} = 0$

$r_{s_1} = -1$

$r_{r_2} = 1$ ..... step 6: $k = r_2$ ... leaves the basis

5 9-18:00

Iteration 4: The new basis is: $B = \{x_2, u^+, v_1\}$

- solve $A_B x_B = b$

$$\begin{array}{llll} 1x_2 & +1u^+ & -1v_1 & =0 \\ 2x_2 & +1u^+ & & =5 \\ 1x_2 & & & =1 \end{array} \qquad \begin{array}{l} x_2 = 1 \\ u^+ = 3 \\ v_1 = 4 \end{array}$$

step 2: - solve $y^T A_B = c_B^T$

$$\begin{array}{llll} 1s_1 & +2s_2 & +1r & =0 \\ 1s_1 & +1s_2 & & =0 \\ -1s_1 & & & =0 \end{array} \qquad \begin{array}{l} s_1 = 0 \\ s_2 = 0 \\ r = 0 \end{array}$$

This basis is optimal ! $\Longrightarrow$ $x_2 = 1$ ... optimal ($x_1 = 0$)

5 9-18:06

5 9-18:10

Wolfeho metoda pro úlohu kvadratického programování

Uvažujme úlohu

$$\min \ \frac{1}{2} x^T Q x + p^T x$$
$$\text{s.t.} \quad Ax = b, \quad x \geqq 0$$

Q – symetrická
Q – positivně definitní, jestliže $p \neq 0$
Q – positivně semidefinitní, jestliže $p = 0$

Úvaha: Nechť x^* – je optimální řešení

– zvol $y \in \mathbb{R}^n$ tak, že $Ay = 0$

$$x_j^* = 0 \Rightarrow y_j \geqq 0$$

– uvažuj bod $x^* + y \cdot \varepsilon$ pro $\varepsilon \searrow 0$

$\rightsquigarrow$ jistě platí $A(x^* + y\varepsilon) = b$

$$x^* + y\varepsilon \geqq 0$$

a $\frac{1}{2}(x^* + y\varepsilon)^T Q (x^* + y\varepsilon) + p^T(x^* + y\varepsilon) \geqq \frac{1}{2} x^{*T} Q x^* + p^T x^*$

$$\varepsilon^2 \cdot \frac{1}{2} y^T Q y + \varepsilon \cdot x^{*T} Q y + \varepsilon p^T y \geqq 0 \quad \text{pro } \varepsilon \searrow 0$$

$$\Downarrow$$

$$\left(x^{*T} Q + p^T\right) y \geqq 0$$

Tedy:

$$\left.\begin{array}{l} Ay = 0 \\ y_j \geqq 0 \text{ -- pro } j \text{ -- } x_j^* = 0 \end{array}\right\} \Rightarrow \left(x^{*T} Q + p^T\right) y \geqq 0$$

$\overset{\text{Farkas}}{\Longleftrightarrow}$

$$\exists u \in \mathbb{R}^{1 \times m} \ \exists v_j \geqq 0 \text{ -- pro } x_j^* = 0 : \quad x^{*T} Q + p^T = u^T A + \sum_{\substack{j=1 \\ x_j = 0}}^{n} v_j \cdot e_j^T$$

neboli:

$$\exists u^T \in \mathbb{R}^{1 \times m} \quad \exists v_j \geqq 0 \text{ -- pro } x_j^* = 0 : \quad x^{*T} Q + p^T + u^T A = \underbrace{\sum_{\substack{j=1 \\ x_j = 0}}^{n} v_j e_j^T}_{= v^T \geqq 0}$$

$\textcircled{!}$: $v^T x^* = 0$

Tedy:

Věta: Jestliže bod $\underline{x^*}$ je opt. řeš. úlohy (QP), potom

$$\exists u^T \in \mathbb{R}^{1 \times m} \ \exists v^T \in \mathbb{R}^{1 \times n}, \ v^T \geqq 0^T : \quad x^{*T} Q + p^T + u^T A = v^T$$
$$v^T x^* = 0.$$

Příklad

$$\min\ x^2 - 5x + 7y$$
$$3x + 4y = 6\ ,\quad x, y \geq 0$$

$Q = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$A = (3\ \ 4)\qquad p = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

$b = (6)$

Řešme soustavu $\quad Ax = b$

$$x^TQ + p^T + u^TA = v^T \qquad v^T \geq 0^T$$
$$x \geq 0 \qquad v^Tx = 0$$

bod optima: $\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \rightsquigarrow (2\ \ 0) \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + (-5\ \ 7) + \frac{1}{3} \cdot (3\ \ 4)$

$$\swarrow$$

$$u = \frac{1}{3} \qquad v = \begin{pmatrix} 0 \\ \frac{25}{3} \end{pmatrix}$$

Příklad:

$$\min\ -x^2 - y^2$$
$$x + y = 2\ ,\quad x, y \geq 0$$

Body optima: $\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ a $\begin{pmatrix} x^{**} \\ y^{**} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Řešme soustavu, $\quad Ax = b$

$$x^TQ + p^T + u^TA = v^T \rightsquigarrow \qquad x + y \geq 0$$
$$-2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_1 - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u^* = (4) \qquad u^{**} = (4)$$
$$v^* = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \qquad v^{**} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Třetí řešení:

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (\hat{u}) = (2) \qquad (\hat{v}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

...... není optimální $\Rightarrow$ je to pouze podmínka nutná.

..... zde totiž: $Q = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ .... není pos. semidef.!

Postačující podmínka:

Věta: Nechť Q ... pozitivně semidefinitní

x^* ... přípustné řešení ... $Ax^* = b$, $x^* \geqq 0$

u^*, $v^* \geqq 0$ ..... řeší podmínky $x^{*T} Q + p^T + u^{*T} A = v^{*T}$

$v^{*T} x^* = 0$

Potom x^* je optimální.

Jestliže Q – pos. def., potom x^* je jediné opt. řeš.

Důkaz:

Nechť x je jakékoliv přípustné řešení $Ax = b$, $x \geqq 0$

Polož $y = x - x^*$ ...... $x = x^* + y$

Pak $Ay = 0$

$y_j \geqq 0$ ... jestliže $x_j^* = 0$

$$\tfrac{1}{2} x^T Q x + p^T x = \tfrac{1}{2}(x^* + y)^T Q (x^* + y) + p^T (x^* + y)$$

$$= \tfrac{1}{2} x^{*T} Q x^* + p^T x^* + (p^T + x^{*T} Q) \cdot y + \tfrac{1}{2} y^T Q y$$

Máme $x^{*T} Q + p^T + u^{*T} A = v^{*T}$ $\quad| \cdot y$

$$x^{*T} Q y + p^T y + u^{*T} \underbrace{Ay}_{=0} = v^{*T} y$$

neboli $(x^{*T} Q + p^T) y = \underbrace{v^{*T} y}_{\geqq 0} \geqq 0$ ... protože $v_j > 0 \Rightarrow x_j^* = 0 \Rightarrow y_j \geqq 0$

Tudíž máme

$$\tfrac{1}{2} x^T Q x + p^T x = \tfrac{1}{2} x^{*T} Q x^* + p^T x^* + \underbrace{v^{*T} y}_{\geqq 0} + \underbrace{\tfrac{1}{2} y^T Q y}_{\geqq 0} \geqq \tfrac{1}{2} x^{*T} Q x^*$$

Navíc: nerovnost je ostrá, jestliže Q je pos. def. $\Rightarrow$

$\Rightarrow x^*$ – je jediné opt. řeš.

c. b. d.

Existence řešení

<u>Věta:</u> Jestliže Q je positivně definitní

a $\begin{array}{c} Ax = b \\ x \geqq 0 \end{array}$ ... má řešení ~ úloha je přípustná,

$x_0 \in \mathbb{R}^n$ ... řešení příp.

potom úloha má opt. řeš.

<u>Důkaz:</u>

$x \in \mathbb{R}^n$ .... v konečné dimenzi $\Rightarrow \exists \alpha > 0 : \forall x \in \mathbb{R}^n : \; x^T Q x \geqq \alpha \cdot \|x\|^2$

potom $\frac{1}{2} x^T Q x + p^T x \geqq \frac{1}{2} \alpha \cdot \|x\|^2 - \|p\| \cdot \|x\| > \frac{1}{2} x_0^T Q x_0 + p^T x_0$

jestliže $\|x\| > R$, kde $R > 0$ je dost. velké

$R = \frac{\|p\| + \sqrt{\|p\|^2 + 2\alpha\left(\frac{1}{2} x_0^T Q x_0 + p^T x_0\right)}}{\alpha}$ $\quad R \geqq \|x_0\|$

Tudíž se stačí omezit na množinu

$$\{x \in \mathbb{R}^n ; \; Ax = b, \; x \geqq 0, \; \|x\| \leqq R\}$$

která je kompaktní, tudíž $\frac{1}{2} x^T Q x + p^T x$ na ní nabývá minima <u>c. b. d.</u>

<u>Wolfeho algoritmus</u>

Předpokládejme Q ... pos. def.

nebo Q ... pos. semidef. a $p^T = 0^T$

Potom vyřešit

$$\begin{array}{ll} \min & \frac{1}{2} x^T Q x + p^T x \\ \text{z.p.} & Ax = b \\ & x \geqq 0 \end{array}$$

$\longleftrightarrow$ ekvivalentní

vyřešit

$$\begin{array}{l} Ax = b \\ x \geqq 0 \\ x^T Q + p^T + u^T A = v^T \\ v^T \geqq 0 \quad v^T x = 0 \\ u^T \in \mathbb{R}^{1 \times m} \end{array}$$

- řeš $\begin{array}{c} Ax = b \\ x \geqq 0 \end{array}$ ..... $\begin{array}{l} \min \; 0^T x \\ Ax = b \\ x \geqq 0 \end{array}$ $\rightsquigarrow x'$ ... řešení .... je bazické ..... $\underline{m}$ proměnných je > 0

$\Downarrow$

polož $u = 0$ a $v = 0$ $\rightsquigarrow$ platí $v^T x' = 0$ $\quad x'$ ... řešení

jestliže náhodou platí $x^T Q = -p$ ... jsme hotovi

jestliže $x^T Q \neq -p$, potom zavedeme nové proměnné

$z_1, \dots, z_n \geqq 0$

tedy $Qx' \neq -p$ $\qquad$ q_i … i-tý řádek

Nechť $\quad q_i x' + d_i z_i = -p_i$ $\quad$ … kde $d_i = \pm 1$ … zvoleny pevně tak, aby $z_i \geq 0$

$q_i x' < -p_i \rightsquigarrow d_i = 1$, $\quad z_i = -q_i x' - p_i$

$q_i x' > -p_i \rightsquigarrow d_i = -1$, $\quad z_i = q_i x' + p_i$

Máme tedy diagonální matici

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

<u>a řešíme úlohu:</u>

$$\min \quad o^T x + o^T u^+ + o^T u^- + o^T v + e^T z$$

$$\text{s.t.} \quad \underbrace{Qx}_{} + A^T u^+ - A^T u^- - I v + \underbrace{Dz}_{} = -p \quad | \; s \qquad (*)$$

$$\underbrace{Ax}_{} = b \quad | \; r$$

ve výchozí bázi jsou ty, které byly bazické při řešení úlohy $\min o^T x$, $Ax = b$, $x \geq 0$

$x, v, z \geq 0$, $\quad u^+, u^- \geq 0$ $\quad$ ve výchozí bázi

Úlohu řešíme simplexovou metodou.

Ale musíme dodržet podmínku $\quad v^T x = 0$

vylučovací pravidlo: $\quad v_j$ … v bázi $\Rightarrow$ x_j … nesmí být v bázi

$\qquad\qquad\qquad\qquad\;\; x_j$ … v bázi $\Rightarrow$ v_j … nesmí být v bázi

Test optimality: – řeš $u^T A_B = c^T_B$ … řeš: $\quad s^T Q_B + r^T A_B = o^T_B \longrightarrow$ např. $r^T = o^T$, $s^T = o^T$

$? \; s^T D = e^T$ $\qquad ? \; s^T I \leq o^T \, ?$

$s = e^T D \leq o$, $\; d \leq o$

$s^T A^T \overset{!}{=} o$

Mějme nějaké přípustné řešení $x, v, z \geq 0, u$ úlohy $(*)$ takové, že $v^T x = 0$

Množinu indexů $\{1, \dots, n\}$ rozdělme na tři části

jsou navzájem disjunktní

$J_x^+ \cup J_v^+ \cup J_0 = \{1, \dots, n\}$

$$J_x^+ = \{ i \in \{1, \dots, n\};\; x_i > 0 \}$$
$$J_v^+ = \{ i \in \{1, \dots, n\};\; v_i > 0 \}$$
$$J_0 = \{ i \in \{1, \dots, n\};\; x_i = v_i = 0 \}$$

$a_1, \dots, a_n$ – sloupce matice A

$a^1, \dots, a^m$ – sloupce matice A^T

Uvažujme úlohu

$$\min \quad \sum_{j=1}^{n} 1 \cdot R_j$$

$$\text{s.p.} \quad \sum_{j\in J_x^+ \cup J_0} x_j \cdot q_j + \sum_{i=1}^{m} u_i \cdot a^i - \sum_{j\in J_v^+ \cup J_0} v_j \cdot e_j + \sum_{j=1}^{n} R_j \cdot d_j = -p \qquad 1s \qquad (P)$$

$$\sum_{j\in J_x^+ \cup J_0} x_j \cdot a_j \qquad\qquad = b \qquad 1r$$

$$x_j \geqq 0, \quad u_i \in R \qquad v_j \geqq 0 \qquad R_j \geqq 0$$

Optimální hodnotu této úlohy označme μ.

Věta: (Wolfeho)

Jestliže Q je positivně definitní, potom optimální hodnota $\mu = 0$.

Důkaz:

Nechť r a s je optimální řešení úlohy duální k (P).

Potom platí:

$$\mu = r^T b - s^T p$$

a dále

$$(\alpha) \qquad r^T a_j + s^T q_j = 0 \qquad \text{pro } j \in J_x^+$$

$$(\beta) \qquad r^T a_j + s^T q_j \leqq 0 \qquad \text{pro } j \in J_0$$

$$(\gamma) \qquad s^T a^i = 0 \qquad \text{pro } i = 1, \dots, m$$

$$(\delta) \qquad -s^T e_j = 0 \qquad \text{pro } j \in J_v^+$$

$$(\varepsilon) \qquad -s^T e_j \leqq 0 \qquad \text{pro } j \in J_0$$

$$(\zeta) \qquad s\, d_j \leqq 1 \qquad \text{pro } j = 1, \dots, n$$

Nejprve ukážeme, že $s^T Q s \leqq 0$.

Ukazujeme, že $s^T Q s \leq 0$

Tvrdíme, že

$$(r^T a_{j\cdot} + s^T q_{j})\cdot s_j \;\begin{cases} \leq 0 & \text{pro } j \in J_0 & \text{... plyne z } (\beta)+(\varepsilon) \\ = 0 & \text{pro } j \in J_x^+ & \text{... z } (\alpha) \\ = 0 & \text{pro } j \in J_v^+ & \text{... z } (\delta) \end{cases}$$

sečíst:

$$\underbrace{r^T A s}_{=0 \text{ – podle } (\gamma)} + s^T Q s \leq 0$$

$$\Downarrow$$

$$s^T Q s \leq 0$$

Nyní: Q ... positivně definitní $\Rightarrow$ $s = 0$

optimální hodnota účelové funkce $\mu = r^T b - s^T p = r^T b = r^T A x = \sum_{j=1}^{n} r^T a_{j\cdot} x_j$

$$= \mu = \sum_{j \in J_x^+} \underbrace{r^T a_{j\cdot} x_j}_{=0 \text{ ... podle } (\alpha)} + \sum_{j \in J_v^+ \cup J_0} r^T a_{j\cdot} \underbrace{x_j}_{=0 \text{ ... protože } j \notin J_x^+} = 0$$

c.b.d.

Tudíž: Jestliže v pivotačním kroku Wolfeho metody nemáme optimální řešení ($\sum_{j=1}^{m} z_j = 0$), potom lze provést pivotační krok a přitom zachovat podmínku $v^T x = 0$.

Věta: (Wolfeho)

Jestliže Q je positivně semidefinitní a $p^T = o^T$, potom optimální hodnota úlohy (P) je $\mu = 0$.

Důkaz:

Nechť r^T a s^T je optimální řešení úlohy duální k (P).

Již víme, že $s^T Q s \leqq 0$

Nyní $\forall y \in \mathbb{R}^n \ \forall \varepsilon \in \mathbb{R}$:

$$(s + \varepsilon y)^T Q (s + \varepsilon y) \geqq 0 \quad \text{--- protože } Q \text{ je positivně semidef.}$$

$$s^T Q s + \varepsilon \cdot (y^T Q s + s^T Q y) + \varepsilon^2 (y^T Q y) \geqq 0$$

$$2 \varepsilon \, y^T Q s + \varepsilon^2 y^T Q y \geqq - s^T Q s \geqq 0 \quad \text{pro } \varepsilon \to 0$$

$$\Downarrow$$

$$y^T Q s = 0 \quad \text{--- pro každé } y \in \mathbb{R}^n$$

$$\Downarrow$$

$$Q s = o \qquad \text{neboli } s^T Q = o^T$$

Optimální hodnota cílové funkce:

$$\mu = r^T b - s^T \underbrace{p}_{=o} = r^T b = r^T A x = \sum_{j=1}^{n} r^T a_j \cdot x_j =$$

$$= \mu = \sum_{j \in J_x^+} \underbrace{r^T a_j \cdot x_j}_{\substack{= - r^T q_j \\ \text{podle } (\alpha)}} + \sum_{j \in J_r^+ \cup J_0} \underbrace{r^T a_j \cdot x_j}_{\substack{=0 \\ j \notin J_x^+}} = \sum_{j \in J_x^+} - \underbrace{s^T q_j}_{=0} \cdot x_j = 0$$

protože $s^T q_j$ ... je j-tá složka vektoru

$$s^T q_j = s^T Q e_j = e_j^T Q^T s = e_j^T \underbrace{Q s}_{= o \text{ ... víme}} = 0$$

Tudíž $\mu = 0$

c.b.d.

Příklad:

$$\min \quad x_1^2 + x_1 x_2 + x_2^2 - 5x_2$$

$$\text{s.t.} \quad x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

..... opt. řeš. je $x_1 = 0$, $x_2 = 1$

Je to úloha

$$\min \quad x^T Q x + p^T x$$

$$Ax = b$$

$$x \geq 0$$

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{..... je positivně definitní} \qquad p = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \end{pmatrix}$$