## Mathematics in Economics – lecture 9

## Indefinite integral

Integration is a reverse procedure to differentiation.

Notation:

$$\int f(x)dx = F(x) + C$$

Legend: $\int$ …. Integration sign – indefinite integral; $f(x)$ …. Integrated function;

$F(x)$ … antiderivative of $f(x)$; …$C$ …. Integration constant

Indefinite integral is a linear operator:

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

We compute integrals with the use of formulas above, and with the use of the table of elementary integrals:

**Indefinite integral – elementary integrals**

| $f(x)$ | $\int f(x)dx$ |
|---|---|
| $0$ | $C$ |
| $1$ | $x + C$ |
| $x^n$ | $\frac{x^{n+1}}{n+1} + C$ |
| $e^x$ | $e^x + C$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + C$ |
| $\frac{1}{ax+b}$ | $\frac{1}{a}\ln\lvert ax+b\rvert + C$ |
| $a^x$ | $\frac{a^x}{\ln a} + C$ |

| | |
|---|---|
| $\sin x$ | $-\cos x + C$ |
| $\cos x$ | $\sin x + C$ |
| $\frac{1}{\cos^2 x}$ | $\mathrm{tg} x + C$ |
| $-\frac{1}{\sin^2 x}$ | $\mathrm{cotg} x + C$ |
| $\frac{1}{1+x^2}$ | $\mathrm{arctg} x + C$ |
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a} arctg \frac{x}{a} + C$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin x + C$ |
| $-\frac{1}{\sqrt{1-x^2}}$ | $\arccos x + C$ |
| $\frac{1}{\sqrt{1\pm x^2}}$ | $\ln\left\|x+\sqrt{1\pm x^2}\right\| + C$ |

**Indefinite integral - examples**

1) $\int \left(6x^2 + \sqrt{x}\right) dx =$

2) $\int (2x^2 + x - 15) dx =$

3) $\int (6x^3 + 7x^2 - 6x + 3) dx =$

4) $\int (3x + 1)(4x - 1) dx =$

5) $\int (2x + 3)^2 dx =$

6) $\int \frac{4x^3 + 3x^2 - 5x}{x^2} dx =$

**7)**

$$\int\left(x^3+2x^2+6x+1\right)dx=\int x^3dx+2\int x^2dx+6\int xdx+\int 1dx=$$

8)

$$\int\sqrt[3]{x}dx=\int x^{\frac{1}{3}}dx=\frac{x^{\frac{4}{3}}}{\frac{4}{3}}=\frac{3\sqrt[4]{x^3}}{4}+C$$

9)

$$\int\left(2x+\frac{5}{x}\right)dx=x^2+5\ln|x|+C$$

10)

$$\int\left(5\sin x-2\cos x+3^x\right)dx=-5\cos x-2\sin x+\frac{3^x}{\ln 3}+C$$

## Indefinite integral – integration methods

For more complicated integration we use suitable integration methods:

- Substitutions
- Method per partes

All these methods will be demonstrated on examples.

## I) Integration by a substitution

We use a substitution typically in the following cases:

- When an integrand contains an internal function.
- When an integrand contains lnx or exp(x).
- When an integrand contains goniometric functions.
- When an integrand contains square roots.

Problem 1

$$\int (2x+1)^4 \, dx$$

$$\int (2x+1)^4 \, dx = \begin{vmatrix} 2x+1=t \\ 2dx=dt \end{vmatrix} = \int t^4 \frac{dt}{2} = \frac{1}{2}\int t^4 dt = \frac{1}{2}\frac{t^5}{5} + C = \frac{t^5}{10} + C = \frac{(2x+1)^5}{10} + C$$

A note: We substitute not only an integrand, but also dx!

Problem 2

$$\int e^{2x+3} dx$$

$$\int e^{2x+3} dx = \begin{vmatrix} 2x+3=t \\ 2dx=dt \end{vmatrix} = \int e^t \cdot \frac{dt}{2} = \frac{1}{2}\int e^t dt = \frac{1}{2}e^t + C = \frac{1}{2}e^{2x+3} + C$$

Problem 3

$$\int \cos(5x-4) \, dx$$

$$\int \cos(5x-4) \, dx = \begin{vmatrix} 5x-4=t \\ 5dx=dt \end{vmatrix} = \int \cos t \frac{dt}{5} = \frac{1}{5}\int \cos t dt = \frac{\sin t}{5} + C = \frac{\sin(5x-4)}{5} + C$$

Problem 4

$$\int \frac{\ln x}{x} dx$$

$$\int \frac{\ln x}{x} dx = \begin{vmatrix} \ln x = t \\ \frac{1}{x} dx = dt \end{vmatrix} = \int t dt = \frac{t^2}{2} + C = \frac{\ln^2 x}{2} + C$$

Problem 5

$$\int \frac{5}{x\ln^2 x}dx$$

$$\int \frac{5}{x\ln^2 x}dx = \left| \begin{matrix} \ln x = t \\ \frac{1}{x}dx = dt \end{matrix} \right| = \int \frac{5}{t^2}dt = -\frac{5}{t} + C = \frac{-5}{\ln|x|} + C$$

Problem 6 Usually, we substitute (square roots).

$$\int \sqrt{4x+1}dx$$

$$\int \sqrt{4x+1}dx = \left| \begin{matrix} \sqrt{4x+1} = t \\ 4x+1 = t^2 \\ 4dx = 2tdt \end{matrix} \right| = \int t \cdot \frac{tdt}{2} = \frac{1}{2}\int t^2 dt = \frac{1}{2}\frac{t^3}{3} + C = \frac{t^3}{6} + C = \frac{\left(\sqrt{4x+1}\right)^3}{6} + C$$

Problem 7

$$\int x\sqrt{x^2-1}dx$$

$$\int x\sqrt{x^2-1}dx = \left| \begin{matrix} \sqrt{x^2-1} = t \\ x^2-1 = t^2 \\ 2xdx = 2tdt \\ xdx = tdt \end{matrix} \right| = \int t \cdot tdt = \int t^2 dt = \frac{t^3}{3} + C = \frac{\left(\sqrt{x^2-1}\right)^3}{3} + C$$

**HOMEWORK**

A] $\int (3x-2)^4 \, dx$

B] $\int \sin(1-5x)dx$

C] $\int \frac{\ln^2 x}{x} dx$

D] $\int \cos^4 x \sin x dx$

E] $\int \sqrt{2x+5}dx$

F] $\int x\sqrt{x^2+2}dx$

## II) Integration by parts (per partes method)

Per partes method (integration by parts) is used for integration of a product of two functions.

Let $u(x)$ and $v(x)$ be two functions. Then, we obtain:

$$(u \cdot v)' = u'v + uv'$$

$$uv' = (uv)' - u'v$$

$$\int uv'dx = \int (uv)'dx - \int u'vdx$$

$$\int uv'dx = uv - \int u'vdx$$

The last formula is “per partes“ formula.

Problem 1

$$\int x \cdot e^x dx$$

$$\int x \cdot e^x dx = \begin{vmatrix} u = x, v' = e^x \\ u' = 1, v = e^x \end{vmatrix} = xe^x - \int 1 \cdot e^x dx = xe^x - e^x + C = (x-1)e^x + C$$

Note: a choice of $u$ and $v'$ is important. An incorrect choice leads to a growing difficulty of a problem.

Problem 2

$$\int x \cdot \ln x dx$$

$$\int x \ln x dx = \begin{vmatrix} u = \ln x, v' = x \\ u' = \frac{1}{x}, v = \frac{x^2}{2} \end{vmatrix} = \frac{x^2}{2}\ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Problem 3

$$\int x \sin x dx$$

$$\int x \sin x dx = \begin{vmatrix} u = x, v' = \sin x \\ u' = 1, v = -\cos x \end{vmatrix} = -x\cos x - \int (-\cos x) dx = -x\cos x + \sin x + C$$

**HOMEWORK**

A] $\int (2x+1).e^{x}dx$

B] $\int x^{2}.lnxdx$

C] $\int 3x.cosxdx$