TWO APPLICATIONS OF A LINEARLY ORDERED RING IN BUSINESS DECISION MAKING

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Abstract
We propose two applications of a special linearly ordered commutative ring with zero divisors in business decision making. First, we consider an enterprise facing several future scenarios (events) with the likelihood (expectation or probability) and the worst impact score of each event being given. We propose that the likelihoods and scores attain values in the special linearly ordered ring. The undesired impact of an event can be mitigated if the enterprise makes an investment into preventive measures. The goal is to find an optimal allocation of a limited budget so as to minimize the overall expected impact score. Second, we briefly note that it also makes sense to use the special linearly ordered ring with zero divisors in the FMEA (Failure Mode and Effects Analysis) method, and, in line with the first application, we propose an extension the method. We illustrate the applications by simple examples.

Keywords: decision making under risk, failure mode and effects analysis, linear programming, linearly ordered rings, optimal allocation
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1. Introduction

Consider a manager (decision maker) who evaluates some threats to an enterprise or risks in the production process; alternatively, the manager can evaluate new opportunities for the enterprise, etc.

Each threat or opportunity is understood as an undesirable or desirable, respectively, event. We assume for simplicity that the number of the events under consideration is finite. Furthermore, an impact score of each event is given. The impact score can be either a number from a certain scale (such as from 0 to 100, say), or the score can mean the estimated costs of the damage caused by the event. Alternatively, the score can mean the estimated profit brought by an opportunity if it happens. Let us, however, introduce the convention that the event is a risk and the score of its undesirable impact is positive. The score of desirable impacts of opportunities will be negative then.

Thus, let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ be the set of the future scenarios or events under consideration. Moreover, for each $\omega \in \Omega$, let P_ω be the likelihood (or expectation or probability) that the event ω occurs and let S_ω be the score of the worst impact of the event ω. We assume that the likelihoods P_ω and the scores S_ω are positive and non-zero, respectively, for $\omega \in \Omega$. (If the outcome of the event ω is desirable or undesirable, then $S_\omega < 0$ or $S_\omega > 0$, respectively. Events that are impossible ($P_\omega = 0$) or have no impact ($S_\omega = 0$) need not be considered.)

One of the decision making methods suggests that the manager divides the events into four categories: events of (a) high likelihood and significant impact, (b) low likelihood but significant impact, (c) high likelihood but negligible impact, (d) low likelihood and negligible impact. Schematically, the events can be grouped into the four parts of a square as follows:

(a)	(c)
(b)	(d)

Then, the manager should tackle the events in the upper left corner, i.e. of category (a), first; then, once this is done, the manager should deal with the events on the diagonal, i.e. of categories (b) and (c). And the manager should not deal with the events in the lower right corner, i.e. of category (d), at all. Although the last rule may seem questionable, notice the events of category (a), (b) and (c) should be treated first. If there are no such events, then the events of category (d) can be "refined", i.e. split into new categories (a), (b), (c), (d), and the analysis can be repeated.

The purpose of this paper is to note that the effect of the above described decision making method, i.e., neglecting the events of category (d), can be achieved by a suitable application of a special linearly ordered commutative ring. We also notice that it makes sense to use the special linearly ring in the FMEA (Failure Mode and Effects Analysis) method.

2. A special linearly ordered commutative ring

The ring is an algebraic structure where the algebraic operations of addition, subtraction, and multiplication are defined; the addition is commutative and associative, and the multiplication is distributive with respect to the addition. If the multiplication is also commutative and the ring is endowed with a relation of linear ordering compatible with the operations, we say the ring is commutative and linearly ordered. In this paper, we shall not go into details, but we refer the interested reader to a textbook on algebra, such as Procházka (1990). We have established a discrete variant of Farkas' Lemma in the setting of a linearly ordered commutative ring (Bartl and Dubey, 2017, Bartl, 2017, Bartl, submitted). It turns out that one of the examples of the linearly ordered commutative rings given in Bartl (2017) finds applications in business decision making. We reproduce the example (Bartl, 2017, Example 1, Bartl, submitted, Example 7.2) of the linearly ordered commutative ring as follows.

Consider the set of the real numbers $\mathbb{R}$. Introduce a new positive infinitesimal element δ. That is, the element δ is positive ($\delta > 0$), but it is not among the standard real numbers ($\delta \notin \mathbb{R}$), yet it is smaller than any positive real number ($0 < \delta < a$ for all positive $a \in \mathbb{R}$).

Now, the special linearly ordered commutative ring R shall consist of all sums of the form $a + b\delta$ with $a, b \in \mathbb{R}$. Formally, we have $R = \{a + b\delta : a, b \in \mathbb{R}\}$. The addition and subtraction are defined in the usual way. That is, we have $(a + b\delta) + (c + d\delta) = (a + c) + (b + d)\delta$ and $(a + b\delta) - (c + d\delta) = (a - c) + (b - d)\delta$. Using the rule that $\delta^2 = \delta\delta = 0$ (annihilates), the multiplication is also defined in the usual way, so that we have $(a + b\delta)(c + d\delta) =$ $= (ac) + (ad + bc)\delta + (bd)\delta^2 = (ac) + (ad + bc)\delta$. For example, we have $1 + 2\delta + 3 + 4\delta =$ $= 4 + 6\delta$ and $(5 + 6\delta)(7 + 8\delta) = 35 + 82\delta$.

Finally, we endow the ring with a linear ordering. We order the elements of the ring by using the above rule that the element δ is infinitely less than any positive real number. That is, we have $(a + b\delta) \leq (c + d\delta)$ if and only if either $a < c$, or $a = c$ and $b \leq d$. For example, the next relations hold true: $-1 + 100\delta < 0 - 1000\delta < 1 + 0\delta < 2 - 1\delta < 2 + 1\delta < 3 - 2\delta$.

3. A mathematical model

We notice that the rules of the decision making method, which was described in the Introduction, can be modelled mathematically by assigning the positive and non-zero elements of the ring R to the likelihoods P_ω and the impact scores S_ω, respectively, of the events $\omega \in \Omega$. (Recall that $S_\omega > 0$ or $S_\omega < 0$ iff the impact is undesirable or desirable, i.e., the event ω is a risk or an opportunity, respectively.) High likelihoods receive values of the form $a + b\delta$ with $a > 0$, while low likelihoods are evaluated by $a + b\delta$ with $a = 0$ and $b > 0$. Similarly, significant impacts receive values of the form $a + b\delta$ with $a \neq 0$, while negligible impacts are evaluated by $a + b\delta$ with $a = 0$ and $b \neq 0$.

For an event $\omega \in \Omega$, the product $P_\omega S_\omega = a_\omega + b_\omega\delta$, with $a_\omega, b_\omega \in \mathbb{R}$, is the event's weighted impact score. Now, observe that the event ω is of high likelihood and significant impact, i.e. of category (a), iff $a_\omega \neq 0$, it is of low likelihood but significant impact or high likelihood but negligible impact, i.e. of category (b) or (c), iff $a_\omega = 0$ and $b_\omega \neq 0$, and it is of low likelihood and negligible impact, i.e. of category (d), iff the result is zero. The rules of the decision making method described in the Introduction have been modelled mathematically thus. Moreover, the sum $\sum_{\omega\in\Omega} P_\omega S_\omega$ can be understood as the overall vulnerability index of the enterprise.

Now, assume that the likelihood P_ω of the occurrence of the event $\omega \in \Omega$ is given, but its undesired impact can be mitigated (or can even be turned into a desirable one) if the enterprise makes an investment into preventive measures against the event. In particular, if the amount I_ω of money is invested into the preventive measures against the event ω, then the event's impact score will decrease to S'_ω. We assume that the amount $I_\omega \in \mathbb{R}$ is a positive real number, but the new impact score $S'_\omega \in R$ is again a value in the special ring R. We also assume that $S'_\omega < S_\omega$, i.e., the undesired impact is decreased.

The enterprise can choose the amount of money which it invests into the preventive measures against the event $\omega \in \Omega$. Assuming $I_\omega > 0$, the invested amount $i_\omega \in \mathbb{R}$ must be such that $0 \le i_\omega \le I_\omega$, meaning that it makes no sense to invest more than I_ω into the preventive measures. Then, we assume the resulting impact score is decreased proportionally, i.e., the resulting impact score is $s_\omega = S_\omega - (S_\omega - S'_\omega) i_\omega / I_\omega$.

Finally, we assume that the total budget for the investments is limited by the amount $I \in \mathbb{R}$, which is a positive real number. Then, the goal of the enterprise is to invest into the preventive measures so that the overall vulnerability index $\sum_{\omega\in\Omega} P_\omega s_\omega = $ $= \sum_{\omega\in\Omega} P_\omega S_\omega - P_\omega (S_\omega - S'_\omega) i_\omega / I_\omega$ is minimized and the available budget I is not exceeded. Removing the constant term and inverting the sign of the second term in the objective function, this problem can be formulated mathematically as follows:

$$\text{maximize} \qquad \sum_{\omega\in\Omega} P_\omega \frac{S_\omega - S'_\omega}{I_\omega} i_\omega \tag{1}$$

$$\text{subject to} \qquad \sum_{\omega\in\Omega} i_\omega \le I\,,$$

$$0 \le i_\omega \le I_\omega \qquad \text{for} \quad \omega \in \Omega$$

where $i_\omega \in \mathbb{R}$ are variables for $\omega \in \Omega$. Since the coefficients $P_\omega (S_\omega - S'_\omega)/I_\omega = a_\omega + b_\omega \delta$, with $a_\omega, b_\omega \in \mathbb{R}$, are fixed and the variables i_ω are real numbers, the problem is a simple problem of lexicographic linear programming if the set $\Omega = \{\omega_1, \dots, \omega_n\}$ is finite. (More generally, we could assume that the set $\Omega = \{\omega_1, \omega_2, \omega_3, \dots\}$ is countable. We should assume then that the coefficients a_ω and b_ω are bounded so that the sum $\sum_{\omega\in\Omega} (a_\omega + b_\omega \delta) i_\omega$ converges.)

Thus if the set Ω of the events is finite, the above lexicographic linear programming problem can be solved easily, which can help the enterprise to allocate the budget optimally so that the overall vulnerability index of the enterprise is minimized.

4. An example

An enterprise estimates that one of the four risks or scenarios $\mathrm{A, B, C, D}$ will arise in near future. Thus, we consider the set $\Omega = \{\mathrm{A, B, C, D}\}$. Based on the analysis, the enterprise estimates the likelihoods of the scenarios as follows: $P_\mathrm{A} = 0.6 - 300\delta$, $P_\mathrm{B} = 0 + 800\delta$, $P_\mathrm{C} = 0.4 - 700\delta$, $P_\mathrm{D} = 0 + 200\delta$. Recall that δ is a positive infinitesimal element less than any positive real number ($0 < \delta < a$ for all positive $a \in \mathbb{R}$), and it annihilates when multiplied by itself ($\delta\delta = 0$); see Section 2 for the details. It holds $\sum_{\omega\in\Omega} P_\omega = 1$ in this particular example, i.e., the positive likelihoods P_ω can be understood as (generalized) probabilities. Moreover, the enterprise estimates the (undesired) impact scores of the scenarios as follows: $S_\mathrm{A} = 10 + 0\delta$, $S_\mathrm{B} = 60 + 80\delta$, $S_\mathrm{C} = 0 + 60\delta$, $S_\mathrm{D} = 0 + 90\delta$.

The enterprise analysed each scenario further, and concluded that the impact of each of the scenarios can be mitigated if preventive measures are adopted. It is possible to decrease the impact scores to the levels as follows: $S'_\mathrm{A} = 0 + 80\delta$, $S'_\mathrm{B} = -3 - 60\delta$, $S'_\mathrm{C} = -20 + 0\delta$, $S'_\mathrm{D} = -1 + 0\delta$. The costs of the respective preventive measures, however, will be: $I_\mathrm{A} = 600$, $I_\mathrm{B} = 700$, $I_\mathrm{C} = 800$, $I_\mathrm{D} = 100$. If a partial investment into the preventive measures against a scenario $\omega \in \Omega$ is made, then the (undesired) impact is mitigated proportionally. Notice that scenario A possesses a minor risk since it cannot be mitigated completely ($S'_\mathrm{A} > 0$, but $0 < S'_\mathrm{A} < a$ for all positive $a \in \mathbb{R}$), while scenarios $\mathrm{B, C, D}$ stand for opportunities actually because their impacts can be made negative ($S'_\mathrm{B} < -3$, $S'_\mathrm{C} = -20$, $S'_\mathrm{D} = -1$), i.e. desirable, if some investment is made.

The total budget which the enterprise can invest into the preventive measures is $I = 1000$. Denote by $i_\mathrm{A}, i_\mathrm{B}, i_\mathrm{C}, i_\mathrm{D}$ the amounts of money invested into the preventive measures against the scenarios $\mathrm{A, B, C, D}$, respectively. For a scenario $\omega \in \Omega$, if $i_\omega = I_\omega$, then the impact score is mitigated to $s_\omega = S'_\omega$, and it is mitigated to $s_\omega = S_\omega - (S_\omega - S'_\omega) i_\omega / I_\omega$ if $0 \le i_\omega \le I_\omega$. The goal is to minimize the enterprise's overall vulnerability index $\sum_{\omega \in \Omega} P_\omega s_\omega$ so that the available budget I is not exceeded, i.e., subject to $\sum_{\omega \in \Omega} i_\omega \le I$ and $0 \le i_\omega \le I_\omega$ for all $\omega \in \Omega$.

It is easy to see that, instead of minimizing the sum $\sum_{\omega \in \Omega} P_\omega s_\omega$, we can equivalently maximize the sum $\sum_{\omega \in \Omega} P_\omega (S_\omega - S'_\omega) i_\omega / I_\omega$. Denoting $P_\omega (S_\omega - S'_\omega)/I_\omega = a_\omega + b_\omega \delta$ for $\omega \in \Omega$, a few simple calculations (see Section 2 for the rules) yield

$$a_\mathrm{A} + b_\mathrm{A} \delta = \tfrac{1}{100} - \tfrac{508}{100} \delta, \qquad a_\mathrm{C} + b_\mathrm{C} \delta = \tfrac{1}{100} - \tfrac{1747}{100} \delta,$$

$$a_\mathrm{B} + b_\mathrm{B} \delta = 0 + 72\delta, \qquad a_\mathrm{D} + b_\mathrm{D} \delta = 0 + 2\delta.$$

Put together, we obtain the next linear programming problem:

$$\text{maximize} \quad \left(\tfrac{1}{100} i_\mathrm{A} + 0 i_\mathrm{B} + \tfrac{1}{100} i_\mathrm{C} + 0 i_\mathrm{D}\right) + \left(-\tfrac{508}{100} i_\mathrm{A} + 72 i_\mathrm{B} - \tfrac{1747}{100} i_\mathrm{C} + 2 i_\mathrm{D}\right)\delta \tag{2}$$

$$\begin{aligned} \text{subject to} \quad & i_\mathrm{A} + i_\mathrm{B} + i_\mathrm{C} + i_\mathrm{D} \le 1000, \\ & 0 \le i_\mathrm{A} \le 600, \\ & 0 \le i_\mathrm{B} \le 700, \\ & 0 \le i_\mathrm{C} \le 800, \\ & 0 \le i_\mathrm{D} \le 100. \end{aligned}$$

We solve this linear programming problem in two steps.

In the first step, we maximize

$$\tfrac{1}{100} i_\mathrm{A} + 0 i_\mathrm{B} + \tfrac{1}{100} i_\mathrm{C} + 0 i_\mathrm{D}$$

subject to the above constraints. It is easy to see that the optimal value is equal 10 and that a solution $[i_\mathrm{A}, i_\mathrm{B}, i_\mathrm{C}, i_\mathrm{D}]$ is optimal if and only if $i_\mathrm{A} + i_\mathrm{C} = 1000$, $0 \le i_\mathrm{A} \le 600$, $0 \le i_\mathrm{C} \le 800$, and $i_\mathrm{B} = i_\mathrm{D} = 0$.

In the second step, we add the above constraints that describe the optimal solutions of the first step ($i_\mathrm{A} + i_\mathrm{C} = 1000$, $0 \le i_\mathrm{A} \le 600$, $0 \le i_\mathrm{C} \le 800$, $i_\mathrm{B} = i_\mathrm{D} = 0$) to the constraints of the original problem, and maximize

$$-\tfrac{508}{100} i_\mathrm{A} + 72 i_\mathrm{B} - \tfrac{1747}{100} i_\mathrm{C} + 2 i_\mathrm{D}$$

subject to the larger collection of the constraints. It is easy to see that the optimal solution is $i_\mathrm{A} = 600$, $i_\mathrm{C} = 400$, and $i_\mathrm{B} = i_\mathrm{D} = 0$, and the optimal value is -10036.

Put together, the enterprise minimizes the (undesired) impacts if it invests the amounts $i_\mathrm{A} = 600$, $i_\mathrm{B} = 0$, $i_\mathrm{C} = 400$, $i_\mathrm{D} = 0$ into the preventive measures against the scenarios $\mathrm{A, B, C, D}$, respectively, yielding the overall vulnerability index of the enterprise $\sum_{\omega \in \Omega} P_\omega s_\omega = (6 + 45024\delta) - (10 - 10036\delta) = -4 + 55060\delta$, i.e., the enterprise may expect a desired impact of new opportunities.

5. A note on the FMEA method

The FMEA (Failure Mode and Effects Analysis) method (Stamatis, 2003) is a tool to identify serious risks; it can also be used in Six Sigma. An event ω under consideration receives three scores: probability P_ω is the likelihood of the occurrence of the event, severity S_ω is the score of the worst impact of the event, and detection D_ω is the likelihood that the event will *not* be detected until its severe impact shows up. It is usual to take the scores from the scale $\{1, 2, \ldots, 10\}$, where 1 and 10 represent the mildest and the most serious, respectively, value. The RPN (Risk Priority Number) of the event ω

is a number ranging from 1 (risk of little account) to 1000 (serious hazard); it is the product of the three scores, i.e. $RPN_\omega = P_\omega S_\omega D_\omega$.

In the FMEA method, it also makes sense to use the ring R presented in Section 2. We then choose the scores $P_\omega, S_\omega, D_\omega$ from the scale $\mathcal{S} = \{a + b\delta : a, b \in \mathbb{R}$ are such that $0 \le a \le 10$, and $b > 0$ if $a = 0$, and $b \le 0$ if $a = 10\}$, say. If any of the three quantities is considered negligible, it receives a score of the form $a + b\delta$ with $a = 0$ and $b > 0$; it receives a score with $a > 0$ otherwise. If exactly one of the three quantities is negligible, then the resulting RPN of the event is also negligible. If at least two of the three quantities are negligible, then the RPN of the event is zero. If none of the three scores is negligible, then the RPN of the event is of the form $a + b\delta$ with $a > 0$, i.e., the event receives due attention.

6. An extension of the FMEA method and an example

Considering the mathematical model presented in Section 3, it is straightforward to extend the FMEA method in a similar way.

Let Ω be a (finite) set of events that the enterprise is considering. For each event $\omega \in \Omega$, let its probability $P_\omega \in \mathcal{S}$, severity $S_\omega \in \mathcal{S}$, and detection $D_\omega \in \mathcal{S}$ scores be given, where $\mathcal{S}$ is the scale introduced in Section 5.

Assume that the three scores of an event $\omega \in \Omega$ can be decreased to $P'_\omega \in \mathcal{S}$, $S'_\omega \in \mathcal{S}$, $D'_\omega \in \mathcal{S}$ if the enterprise invests the (positive) amounts $I^P_\omega \in \mathbb{R}$, $I^S_\omega \in \mathbb{R}$, $I^D_\omega \in \mathbb{R}$ into the respective preventive measures. If a smaller amount is invested, then the respective score is decreased proportionally.

Thus, if $i^P_\omega, i^S_\omega, i^D_\omega \in \mathbb{R}$ such that $0 \le i^P_\omega \le I^P_\omega$, $0 \le i^S_\omega \le I^S_\omega$, and $0 \le i^D_\omega \le I^D_\omega$ are the amounts invested into the preventive measures to decrease the probability, severity, and detection scores, respectively, then the RPN of the event $\omega \in \Omega$ will be

$$RPN_\omega = \left(P_\omega - \frac{P_\omega - P'_\omega}{I^P_\omega} i^P_\omega\right)\left(S_\omega - \frac{S_\omega - S'_\omega}{I^S_\omega} i^S_\omega\right)\left(D_\omega - \frac{D_\omega - D'_\omega}{I^D_\omega} i^D_\omega\right).$$

Now, the goal is to find an optimal allocation of the investments into the preventive measures so that the maximum RPN ($\max_{\omega\in\Omega} RPN_\omega$) is minimized and a given positive budget $I \in \mathbb{R}$ is not exceeded. The problem can then be formulated mathematically as follows:

$$\begin{aligned}
&\text{minimize} && w && (3)\\
&\text{subject to} && RPN_\omega \le w \quad \text{for} \quad \omega \in \Omega\,, \\
& && \textstyle\sum_{\omega\in\Omega} i^P_\omega + i^S_\omega + i^D_\omega \le I\,, \\
& && 0 \le i^P_\omega \le I^P_\omega \quad \text{for} \quad \omega \in \Omega\,, \\
& && 0 \le i^S_\omega \le I^S_\omega \quad \text{for} \quad \omega \in \Omega\,, \\
& && 0 \le i^D_\omega \le I^D_\omega \quad \text{for} \quad \omega \in \Omega\,,
\end{aligned}$$

which is a problem with non-linear constraints. We propose an efficient solution method at the end of this section below.

As an illustration, consider only one event for simplicity, i.e., we consider $\Omega = \{\mathrm{A}\}$. The probability score $P_\mathrm{A} = 5 + 100\delta$ can be decreased to $P'_\mathrm{A} = 1 - 100\delta$ if the amount $I^P_\mathrm{A} = 300$ is invested. The severity score $S_\mathrm{A} = 6 + 200\delta$ can be decreased to $S'_\mathrm{A} = 2 - 200\delta$ if the amount $I^S_\mathrm{A} = 400$ is invested. And the detection score $D_\mathrm{A} = 7 + 300\delta$ can be decreased to $D'_\mathrm{A} = 3 - 300\delta$ if the amount $I^D_\mathrm{A} = 500$ is invested. The investments into the preventive measures are limited by the available budget $I = 500$. Then, a couple of simple calculations show that the

$$RPN_\mathrm{A} = \left(5 + 100\delta - \frac{4+200\delta}{300} i^P_\mathrm{A}\right)\left(6 + 200\delta - \frac{4+400\delta}{400} i^S_\mathrm{A}\right)\left(7 + 300\delta - \frac{4+600\delta}{500} i^D_\mathrm{A}\right)$$

and the problem is to

$$\begin{aligned} &\text{minimize} && RPN_{\mathrm{A}} && (4) \\ &\text{subject to} && i_{\mathrm{A}}^P + i_{\mathrm{A}}^S + i_{\mathrm{A}}^D \le 500\,, \\ &&& 0 \le i_{\mathrm{A}}^P \le 300\,, \\ &&& 0 \le i_{\mathrm{A}}^S \le 400\,, \\ &&& 0 \le i_{\mathrm{A}}^D \le 500\,. \end{aligned}$$

Following Section 4, we solve the problem in two stages. In the first stage, we solve the problem

$$\begin{aligned} &\text{minimize} && \left(5 - \tfrac{1}{75} i_{\mathrm{A}}^P\right)\left(6 - \tfrac{1}{100} i_{\mathrm{A}}^S\right)\left(7 - \tfrac{1}{125} i_{\mathrm{A}}^D\right) && (5) \\ &\text{subject to} && i_{\mathrm{A}}^P + i_{\mathrm{A}}^S + i_{\mathrm{A}}^D \le 500\,, \\ &&& 0 \le i_{\mathrm{A}}^P \le 300\,, \\ &&& 0 \le i_{\mathrm{A}}^S \le 400\,, \\ &&& 0 \le i_{\mathrm{A}}^D \le 500\,. \end{aligned}$$

To solve this problem, we approximate the objective function around the point $[i_{\mathrm{A}}^P, i_{\mathrm{A}}^S, i_{\mathrm{A}}^D] = [0, 0, 0]$ linearly:

$$\mathrm{z} \approx 5 \times 6 \times 7 - 6 \times 7 \times \tfrac{1}{75} i_{\mathrm{A}}^P - 5 \times 7 \times \tfrac{1}{100} i_{\mathrm{A}}^S - 5 \times 6 \times \tfrac{1}{125} i_{\mathrm{A}}^D =$$

$$= 210 - \tfrac{840}{1500} i_{\mathrm{A}}^P - \tfrac{525}{1500} i_{\mathrm{A}}^S - \tfrac{360}{1500} i_{\mathrm{A}}^D\,.$$

Next, we increase the variable with the largest (most negative) coefficient, i.e. the variable i_{A}^P, as much as possible, until either the limit I_{A}^P is reached or the budget I is exhausted. Thus, we fix its value at $i_{\mathrm{A}}^P \coloneqq 300$, and we approximate the objective function around the point $[i_{\mathrm{A}}^P, i_{\mathrm{A}}^S, i_{\mathrm{A}}^D] = [300, 0, 0]$, with the variable i_{A}^P fixed, linearly:

$$z \approx \left(5 - \tfrac{1}{75} \times 300\right) \times \left(6 \times 7 - 7 \times \tfrac{1}{100} i_{\mathrm{A}}^S - 6 \times \tfrac{1}{125} i_{\mathrm{A}}^D\right) = 42 - \tfrac{35}{500} i_{\mathrm{A}}^S - \tfrac{24}{500} i_{\mathrm{A}}^D\,.$$

Next, we again increase the variable with the largest coefficient, i.e. the variable i_{A}^S, as much as possible, until either the limit $I_{\mathrm{A}}^S = 400$ is reached or the remaining budget $I - i_{\mathrm{A}}^P = 200$ is exhausted. Thus, we fix its value at $i_{\mathrm{A}}^S \coloneqq 200$. Since the budget is exhausted now, we have solved the problem thus; the optimal solution is $[i_{\mathrm{A}}^P, i_{\mathrm{A}}^S, i_{\mathrm{A}}^D] = [300, 200, 0]$, and there is no need for the second stage.

The above simple example also suggests how we can solve the general problem (3) to minimize $\max_{\omega \in \Omega} RPN_\omega$ subject to $\sum_{\omega \in \Omega} i_\omega^P + i_\omega^S + i_\omega^D \le I$ and $0 \le i_\omega^P \le I_\omega^P$, $0 \le i_\omega^S \le I_\omega^S$, $0 \le i_\omega^D \le I_\omega^D$ for all $\omega \in \Omega$ efficiently. We solve the problem in two stages. In the first stage, we consider the real parts (without the infinitesimal element δ) of the RPN_ω in the objective function. Starting with zero investments, as above, we choose the event $\omega \in \Omega$ such that its RPN_ω is maximal; if there are two or more events with the same maximal value, we consider all such events, or rather, we consider a group of those events. Next, we approximate the real parts of the RPN_ω of the events linearly, as above. Then, we increase the respective variables with the largest coefficients (one variable from each RPN_ω) so that the equality of the (decreasing) maximal values of RPN_ω in the group is preserved until either the upper limit of a variable is reached, or the budget is exhausted, or the common maximal value of the decreasing RPN_ω decreases to a value of a $RPN_{\omega'}$ which was originally less than the common maximal value; the event ω' must be added to the group of the events whose common maximal value is decreased.

The process stops when all three variables i_ω^P, i_ω^S, i_ω^D of an event $\omega \in \Omega$ are fixed at their upper bounds (since we consider $\max_{\omega \in \Omega} RPN_\omega$, the maximum will not decrease any more), or the budget I is exhausted, or the coefficients of the real parts by the non-fixed variables i are zero, in which case we proceed with the second stage.

In the second stage, we consider the infinitesimal parts (with the element δ) of the RPN_{ω}. We then continue analogously as in the first stage.

Notice that we could apply this two-stage procedure to solve problem (1) or example (2) given in Section 3 or 4, respectively, too.

7. Conclusion

We have proposed two applications of a special linearly ordered commutative ring with zero divisors (Section 2) in business decision making. The first application consists in an optimal allocation of a limited budget in order to minimize the overall vulnerability index of an enterprise (Section 3); we have illustrated the application by a simple example (Section 4). The second application is within the FMEA (Failure Mode and Effects Analysis) method (Section 5). As an extension of the FMEA method, we have considered the problem of an optimal allocation of a limited budget in order to minimize the maximum RPN (Risk Priority Number) of the events; moreover, we have also proposed an efficient procedure to solve the respective mathematical models (Section 6).

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