A NOTE ON HOMOGENEOUS LINEAR PROGRAMMING

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Abstract
We consider the homogeneous variant of linear programming problems in the setting of a module over a linearly ordered commutative associative ring. We formulate the optimality condition for the primal and for the dual problem; by combining the aforementioned results, we obtain the Strong Duality Theorem for the homogeneous linear programming problems. Albeit the results are based on a general discrete variant of Farkas' Lemma, which has been published recently, further restrictive assumptions are necessary to prove the results. We also propose a simple application of the results – an extension of the FMEA (Failure Mode and Effects Analysis) method – in business decision making of small and medium-sized enterprises.

Keywords: homogeneous linear programming, strong duality theorem, failure mode and effects analysis
JEL codes: C60, C61

1. Introduction

Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, a column vector $\boldsymbol{b} \in \mathbb{R}^{m}$, and a row vector $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times n}$, recall the classical pair of the primal and dual problem of linear programming:

$$
\begin{array}{llll}
(\mathrm{P_c}) & \text{maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\
 & \text{subject to} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}
\end{array}
\qquad
\begin{array}{llll}
(\mathrm{D_c}) & \text{minimize} & \boldsymbol{u}^{\mathrm{T}}\boldsymbol{b} \\
 & \text{subject to} & \boldsymbol{u}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{c}^{\mathrm{T}} \\
 & & \boldsymbol{u}^{\mathrm{T}} \ge \boldsymbol{0}^{\mathrm{T}}
\end{array}
$$

where $\boldsymbol{x} \in \mathbb{R}^{n}$ and $\boldsymbol{u}^{\mathrm{T}} \in \mathbb{R}^{1\times m}$ are variables. The *Weak Duality Theorem* holds true: if $\boldsymbol{x} \in \mathbb{R}^{n}$ and $\boldsymbol{u}^{\mathrm{T}} \in \mathbb{R}^{1\times m}$ is a feasible solution to the primal and dual problem, respectively, then $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \le \boldsymbol{u}^{\mathrm{T}}\boldsymbol{b}$. It is well known (see, e.g., Franklin, 2002) that the *Strong Duality Theorem* also holds true: problem ($\mathrm{P_c}$) has an optimal solution if and only if problem ($\mathrm{D_c}$) has an optimal solution; if $\boldsymbol{x}^* \in \mathbb{R}^{n}$ and $\boldsymbol{u}^{*\mathrm{T}} \in \mathbb{R}^{1\times m}$ is an optimal solution to the primal and dual problem, respectively, then $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* = \boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}$.

Let $\boldsymbol{x}^* \in \mathbb{R}^{n}$ and $\boldsymbol{u}^{*\mathrm{T}} \in \mathbb{R}^{1\times m}$ be any feasible solution to the primal and dual problem, respectively. It is easy to see that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* = \boldsymbol{u}^{*\mathrm{T}}\boldsymbol{A}\boldsymbol{x}^* \le \boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}$, hence $\boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) \le 0$. Notice that it holds $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* = \boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}$ if and only if $\boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) = 0$. The latter equation is the *complementarity condition*, whose meaning is as follows. Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}^{1\times n}$ be the rows of the matrix $\boldsymbol{A}$, let $b_1, \dots, b_m \in \mathbb{R}$ be the components of the vector $\boldsymbol{b}$, let $c_1, \dots, c_n \in \mathbb{R}$ be the components of the row vector $\boldsymbol{c}^{\mathrm{T}}$, and let $u_1^*, \dots, u_m^* \in \mathbb{R}$ be the components of the solution $\boldsymbol{u}^{*\mathrm{T}}$. Then the complementarity condition $\boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) = 0$ holds true if and only if the following pairs of equivalent implications hold true: $u_i^* > 0 \Longrightarrow \boldsymbol{a}_i\boldsymbol{x}^* = b_i$ and $\boldsymbol{a}_i\boldsymbol{x}^* < b_i \Longrightarrow u_i^* = 0$ for each $i = 1, \dots, m$. Consequently, the following optimality conditions can be obtained:

Let $\boldsymbol{x}^* \in \mathbb{R}^{n}$ be any feasible solution to the primal problem and let $I = \{\, i \in \{1, \dots, m\} : a_i x^* = b_i \,\}$ be the set of the indices of the active primal constraints. Then $\boldsymbol{x}^*$ is an optimal solution to the primal problem if and only if there exist non-negative $u_i^* \in \mathbb{R}$, for $i \in I$, such that $\boldsymbol{c}^{\mathrm{T}} = \sum_{i\in I} u_i^* \boldsymbol{a}_i$.

Let $\boldsymbol{u}^{*\mathrm{T}} \in \mathbb{R}^{1\times m}$ be any feasible solution to the dual problem and let $I = \{\, i \in \{1, \dots, m\} : u_i^* > 0 \,\}$ be the set of the indices of the active dual variables. Then $\boldsymbol{u}^{*\mathrm{T}}$ is an optimal solution to the dual problem if and only if there exists a solution $\boldsymbol{x}^* \in \mathbb{R}^{n}$ such that $\boldsymbol{A}\boldsymbol{x}^* \le \boldsymbol{b}$ and $\boldsymbol{a}_i\boldsymbol{x}^* = b_i$ for $i \in I$.

Our purpose is to introduce a pair of the primal and dual problem of homogeneous linear programming (in the setting of a module over a linearly ordered commutative associative ring) and to study possible generalizations of the above results for the pair of the problems.

2. Concepts and notation

Let R be a non-trivial linearly ordered commutative associative ring. (The ring R need not be unital; that is, it need not possess the unit element, neutral with respect to multiplication.) Additionally, let V be a linearly ordered module over the linearly ordered ring R. The relation of the linear ordering of the ring R and module V will be denoted by the symbol $\leq$ and $\preccurlyeq$, respectively. Finally, let W be a module over the ring R.

For a non-negative natural number m, let $\alpha_1, \dots, \alpha_m: W \to R$ be linear forms, which make up the linear mapping $A: W \to R^m$. (Notice that the mapping $A: W \to R^m$ generalizes the concept of the matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and the linear forms $\alpha_1, \dots, \alpha_m$, which the mapping A consists of, correspond to the rows $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ of the matrix $\boldsymbol{A}$, which we could see in the Introduction.) For any vectors $\boldsymbol{\lambda}, \boldsymbol{b} \in R^m$, we always stipulate that they consist of the components $\lambda_1, \dots, \lambda_m$ and $b_1, \dots, b_m$, respectively, and we define their scalar product by $\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} = \sum_{i=1}^{m} \lambda_i b_i$. Accordingly, the linear form $\boldsymbol{\lambda}^{\mathrm{T}}A: W \to R$ is defined by $\boldsymbol{\lambda}^{\mathrm{T}}Ax = \sum_{i=1}^{m} \lambda_i(\alpha_i x)$ for every $x \in W$, where $\alpha_i x$ is the value of α_i at x. For a $\boldsymbol{u} \in V^m$, we always stipulate that it consists of the components $u_1, \dots, u_m$ and we define the linear mapping $\iota\boldsymbol{u}^{\mathrm{T}}: R^m \to V$ by $\iota\boldsymbol{u}^{\mathrm{T}}: \boldsymbol{\lambda} \mapsto \sum_{i=1}^{m} \lambda_i u_i$ for every $\boldsymbol{\lambda} \in R^m$. We then have $\iota\boldsymbol{u}^{\mathrm{T}}\boldsymbol{b} = \sum_{i=1}^{m} b_i u_i$ and $\iota\boldsymbol{u}^{\mathrm{T}}A: W \to V$ is the composition of both mappings $A: W \to R^m$ and $\iota\boldsymbol{u}^{\mathrm{T}}: R^m \to V$, so that $\iota\boldsymbol{u}^{\mathrm{T}}Ax = \sum_{i=1}^{m} (\alpha_i x) u_i$ for every $x \in W$.

The symbol $\boldsymbol{0}$ denotes the column vector consisting of m zeros of the ring R and the inequalities $Ax \leq \boldsymbol{0}$ and $\boldsymbol{\lambda} \geq \boldsymbol{0}$ are understood componentwise, that is $\alpha_i x \leq 0$ and $\lambda_i \geq 0$, respectively, for every $i = 1, \dots, m$, for every $x \in W$, and for every $\boldsymbol{\lambda} \in R^m$. Given a column vector $\boldsymbol{b} \in R^m$and a positive scalar $t \in R$, that is $t > 0$, then $\boldsymbol{b}t$ is the column vector consisting of the components $b_1 t, \dots, b_m t$, where $b_1, \dots, b_m$ are the components of the vector $\boldsymbol{b}$. Subsequently, the inequality $Ax \leq \boldsymbol{b}t$ is also understood componentwise, that is $\alpha_i x \leq b_i t$ for every $i = 1, \dots, m$ and for every $x \in W$. The symbol $\boldsymbol{0}^{\mathrm{T}}$ denotes the row vector consisting of m zeros of the module V and the inequality $\boldsymbol{u}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ is understood componentwise, that is $u_i \succcurlyeq 0$ for every $i = 1, \dots, m$.

The symbol o denotes the zero linear form $o: W \to R$. The symbol 0 denotes either the zero of the ring R or the zero of the module V. The meaning of the symbol 0 will be clear from the context.

Finally, let $\gamma: W \to V$ be another linear mapping. Given a constant $r \in R$, then $r\gamma$ denotes the r-multiple of γ. That is, we have $r\gamma x = r(\gamma x)$, where γx is the value of γ at x, for every $x \in W$.

3. Basic results

The following key result – a discrete variant of Farkas' Lemma – has been published recently (Bartl, 2020):

Lemma 1 (Farkas' Lemma). *Let R be a non-trivial linearly ordered commutative ring (which need not be associative), let W be a module over the ring R such that $(\lambda\mu)x = \lambda(\mu x)$ for all $\lambda, \mu \in R$ and for all $x \in W$, let V be a linearly ordered module over the linearly ordered ring R, and let $A: W \to R^m$ with $\gamma: W \to V$ be linear mappings. It then holds: if*

$$\forall x \in W\colon\ Ax \leq \boldsymbol{0} \Longrightarrow \gamma x \preccurlyeq 0\,,$$

then

$$\exists r \in R,\, r > 0,\, \exists \boldsymbol{u} \in V^m,\, \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}\colon\ \iota\boldsymbol{u}^{\mathrm{T}}A = r\gamma\,.$$

By using Farkas' Lemma 1, another key result – a discrete variant of Gale's Theorem of the alternative (Fan, 1956; Gale, 1960; Bartl, 2007) – can be established instantly:

Theorem 2 (Gale's Theorem of the alternative). *Let R be a non-trivial linearly ordered commutative associative ring, let W be a module over the ring R, let $A: W \to R^m$ be a linear mapping, and let $\boldsymbol{b} \in R^m$ be a vector. It then holds: if*

$$\nexists x \in W \ \nexists t \in R,\ t > 0\colon\ \ Ax \leq \boldsymbol{b}t\,,$$

then

$$\exists \boldsymbol{\lambda} \in R^m, \boldsymbol{\lambda} \geq \boldsymbol{0}\colon\ \ \boldsymbol{\lambda}^{\mathrm{T}}A = o \ \ \wedge\ \ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} < 0\,.$$

Proof. There is no $x \in W$ and no positive $t \in R$ to solve $Ax \leq \boldsymbol{b}t$ if and only if $Ax \leq \boldsymbol{b}t$ implies $t \leq 0$, that is

$$\forall \binom{x}{t} \in W \times R\colon\ \ (A \quad -\boldsymbol{b})\binom{x}{t} \leq 0 \implies (o \quad 1)\binom{x}{t} \leq 0\,.$$

By considering the ring R as the module V, that is $V = R$, and by using Farkas' Lemma 1, there exist a non-negative $\boldsymbol{\lambda} \in R^m$ and a positive $r \in R$ such that $\boldsymbol{\lambda}^{\mathrm{T}}A = o$ and $-\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} = r1$. By treating the latter equality, we obtain $\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} = -r < 0$, which concludes the proof. ■

Remark. Due to the arbitrary choice of the vector $\boldsymbol{b} \in R^m$, we need to assume additionally that the ring R is associative in the proof of Theorem 2 in order that the mapping $R \ni t \mapsto \boldsymbol{b}t \in R^m$ is linear.

4. Homogeneous linear programming

Let R be a non-trivial linearly ordered commutative associative ring, let W be a module over the ring R, and let V be a linearly ordered module over the linearly ordered ring R. Consider the set $\mathcal{V} = \{\, u/t : u \in V,\ t \in R,\ t > 0 \,\}$ of "fractions" with positive denominator. We define a quasi-ordering of the set $\mathcal{V}$ as follows. For vectors $u_1, u_2 \in V$ and for scalars $t_1, t_2 \in R$ such that $t_1, t_2 > 0$, we define that $u_1/t_1 \succcurlyeq u_2/t_2$ if and only if $t_2u_1 \succcurlyeq t_1u_2$. Moreover, we define that $u_1/t_1 \approx u_2/t_2$ if and only if $t_2u_1 = t_1u_2$.

Let a linear mapping $A: W \to R^m$, a vector $\boldsymbol{b} \in R^m$, and a linear mapping $\gamma: W \to V$ be given. We consider the following pair of the primal and dual problem of *homogeneous linear programming*:

(P)	maximize	$\gamma x/t$	(D)	minimize	$\iota\boldsymbol{u}^{\mathrm{T}}\boldsymbol{b}/r$
	subject to	$Ax \leq \boldsymbol{b}t$		subject to	$\iota\boldsymbol{u}^{\mathrm{T}}A = r\gamma$
		$t > 0$			$\boldsymbol{u}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$
					$r > 0$

where the pairs $(x,t) \in W \times R$ and $(\boldsymbol{u}, r) \in V^m \times R$ are variables. The values of the objective functions lie in the set $\mathcal{V}$, whose quasi-ordering has been defined above. We are now going to formulate and prove the respective generalizations of the main results presented in the Introduction.

Theorem 3 (Weak Duality Theorem). *Let $(x,t) \in W \times R$ and $(\boldsymbol{u}, r) \in V^m \times R$ be a feasible solution to problem* (P) *and* (D)*, respectively. Then*

$$\gamma x/t \preccurlyeq \iota\boldsymbol{u}^{\mathrm{T}}\boldsymbol{b}/r\,.$$

Proof. Since the solutions are feasible, we have $r\gamma x = \iota\boldsymbol{u}^{\mathrm{T}}Ax \preccurlyeq \iota\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{b}t) = t\iota\boldsymbol{u}^{\mathrm{T}}\boldsymbol{b}$. By the definition of the quasi-ordering of the set $\mathcal{V}$, we obtain $\gamma x/t \preccurlyeq \iota\boldsymbol{u}^{\mathrm{T}}\boldsymbol{b}/r$, which concludes the proof. ■

Corollary 4. *Let $(x^*, t^*) \in W \times R$ and $(\boldsymbol{u}^*, r^*) \in V^m \times R$ be a feasible solution to* (P) *and* (D)*, respectively. If $\gamma x^*/t^* \approx \iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}/r^*$, then (x^*, t^*) and $(\boldsymbol{u}^*, r^*)$ is an optimal solution to* (P) *and* (D)*, respectively.*

Remark. Let $(x^*, t^*) \in W \times R$ and $(\boldsymbol{u}^*, r^*) \in V^m \times R$ be a feasible solution to (P) and (D), respectively. Observe that $\gamma x^*/t^* \approx \iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}/r^*$, that is $r^*\gamma x^* = t^*\iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}$, if and only if $\iota\boldsymbol{u}^{*\mathrm{T}}(Ax^* - \boldsymbol{b}t^*) = 0$, which is the *complementarity condition* now.

Our main goal is to study whether the Strong Duality Theorem holds for problems (P) and (D). We shall follow the outline used in Bartl (2007).

Recall that an element $a \in R$ is a *zero divisor* if and only if $a \neq 0$ and there exists a non-zero $b \in R$, that is $b \neq 0$, such that $ab = 0$. The linearly ordered ring R is *weakly Archimedean* if and only if, for every $a, b \in R$ such that $0 < a < b$, there exists a $\lambda \in R$ such that $b \leq \lambda a$. The module V is R*-torsion free* if and only if, for every $r \in R$ and for every $u \in V$, we have $ru \neq 0$ if both $r \neq 0$ and $u \neq 0$ and r is not a zero divisor.

We shall need the following additional hypotheses about the ring R and module V.

Hypothesis. The non-trivial linearly ordered commutative associative ring R and the linearly ordered module V over the linearly ordered ring R are such that:

(H1) There exists at least one element $K \in R$ such that $K > 0$ and K is not a zero divisor.

(H2) The ring R is weakly Archimedean.

(H3) The module V is R-torsion free.

We also introduce further notation as follows. Let $I \subseteq \{1, \dots, m\}$ be a set of indices and choose a $\boldsymbol{u} \in V^m$. We then put $\iota\boldsymbol{u}_I^{\mathrm{T}} A_I = \sum_{i \in I} \iota u_i \alpha_i$ and also $\iota\boldsymbol{u}_I^{\mathrm{T}} A_I x = \sum_{i \in I} (\alpha_i x) u_i$ for $x \in W$. For $x \in W$ and for $t \in R$, the inequality $A_I x \leq \boldsymbol{b}_I t$ means that $\alpha_i x \leq b_i t$ for every $i \in I$. Likewise, the inequality $\boldsymbol{u}_I^{\mathrm{T}} \succcurlyeq \boldsymbol{0}_I^{\mathrm{T}}$ and $\boldsymbol{b}_I t - A_I x > \boldsymbol{0}_I$ means that $u_i \succcurlyeq 0$ and $b_i t - \alpha_i x > 0$, respectively, for every $i \in I$.

Lemma 5 (Optimality condition for the primal problem). *Let R be a non-trivial linearly ordered commutative associative ring, let W be a module over the ring R, and let V be a linearly ordered module over the linearly ordered ring R. Additionally, assume that hypotheses* (H1)–(H3) *hold true.*

Let $(x^*, t^*) \in W \times R$ be any feasible solution to problem (P). *Let $I = \{ i \in \{1, \dots, m\} : a_i x^* = b_i t^* \}$ be the set of the indices of the active primal constraints. Then (x^*, t^*) is an optimal solution to problem* (P) *if and only if*

$$\exists r^* \in R,\ r^* > 0,\ \exists \boldsymbol{u}_I^* \in V^I,\ \boldsymbol{u}_I^{*\mathrm{T}} \succcurlyeq \boldsymbol{0}_I^{\mathrm{T}}\colon\ \iota\boldsymbol{u}_I^{*\mathrm{T}} A_I = r^* \gamma\,.$$

Proof. Let $J = \{1, \dots, m\} \setminus I$ denote the complement of the index set I.

The "if" part is easy. Put $u_i^* \coloneqq 0$ for every $i \in J$. We then have a $\boldsymbol{u}^* \in V^m$ such that $\boldsymbol{u}^{*\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ and $r^*\gamma = \iota\boldsymbol{u}_I^{*\mathrm{T}} A_I = \iota\boldsymbol{u}_I^{*\mathrm{T}} A_I + \iota\boldsymbol{u}_J^{*\mathrm{T}} A_J = \iota\boldsymbol{u}^{*\mathrm{T}} A$. It follows that $(\boldsymbol{u}^*, r^*)$ is a feasible solution to problem (D). Moreover, we have $r^*\gamma x^* = \iota\boldsymbol{u}_I^{*\mathrm{T}} A_I x^* = \iota\boldsymbol{u}_I^{*\mathrm{T}}(\boldsymbol{b}_I t^*) = \iota\boldsymbol{u}_I^{*\mathrm{T}}(\boldsymbol{b}_I t^*) + \iota\boldsymbol{u}_J^{*\mathrm{T}}\big(\boldsymbol{b}_J t^*\big) = \iota\boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{b}t^*) = t^*\iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}$, which means that $\gamma x^*/t^* \approx \iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}/r^*$. By Corollary 4, it follows that (x^*, t^*) is an optimal solution to problem (P).

We prove the "only if" part now. It is our purpose to show that $A_I x \leq \boldsymbol{b}_I t$ implies $t^*\gamma x \preccurlyeq t\gamma x^*$ for every $x \in W$ and for every $t \in R$. By using Farkas' Lemma 1 (with the module W replaced by the module $W \times R$), we then obtain that $\iota\boldsymbol{u}_I^{*\mathrm{T}} A_I = r t^* \gamma$ and $\iota\boldsymbol{u}_I^{*\mathrm{T}} \boldsymbol{b}_I = r\gamma x^*$ for some non-negative $\boldsymbol{u}_I^{*\mathrm{T}} \in V^I$ and for some positive $r \in R$. By putting $r^* \coloneqq rt^*$, we shall be done.

Let $x \in W$ and $t \in R$ be such that $A_I x \leq \boldsymbol{b}_I t$. (Notice that $t \in R$ is not restricted in sign.) Distinguish two cases: either (A) it holds $t^* + t > 0$, or (B) it holds $t^* + t \leq 0$. In case (A), put $(x^{**}, t^{**}) \coloneqq (x^*, t^*)$. In case (B), there exists a $\lambda^* \in R$ such that $0 < -t < -t - t \leq \lambda^* t^*$ because the ring R is weakly Archimedean by (H2). It is obvious that $\lambda^* > 0$, and we may assume wlog that λ^* is not a zero divisor. (By (H1), there exists a $K > 0$ which is not a zero divisor. It is easy to see that if $\lambda^* > 0$ is a zero divisor, then $0 < \lambda^* < K$. Since $t^* > 0$, it holds $\lambda^* t^* \leq K t^*$, so it is enough to put $\lambda^* \coloneqq K$.) In case (B), put $(x^{**}, t^{**}) \coloneqq (\lambda^* x^*, \lambda^* t^*)$ and notice that the solution (x^{**}, t^{**}) is also feasible to (P) and $\gamma x^{**}/t^{**} = \gamma(\lambda^* x^*)/(\lambda^* t^*) \approx \gamma x^*/t^*$, that is $t^*\lambda^*\gamma x^* = \lambda^* t^* \gamma x^*$, which means that the solution is also optimal. We conclude in either of the cases that (x^{**}, t^{**}) is an optimal solution to (P) and it holds $t^{**} + t > 0$.

Consider now the solution $(x^{**} + x,\ t^{**} + t)$. Let us split the constraints $A(x^{**} + x) \leq \boldsymbol{b}(t^{**} + t)$ into two systems $A_I(x^{**} + x) \leq \boldsymbol{b}_I(t^{**} + t)$ and $A_J(x^{**} + x) \leq \boldsymbol{b}_J(t^{**} + t)$. Note that $A_I x^{**} = \boldsymbol{b}_I t^{**}$ in either of the cases (A) or (B) by the definition of the set I. Since $x \in W$ and $t \in R$ are such that $A_I x \leq \boldsymbol{b}_I t$, it follows that the first system $A_I(x^{**}+x) \leq \boldsymbol{b}_I(t^{**}+t)$ is satisfied. We consider two cases: either $A_J(x^{**}+x) \leq \boldsymbol{b}_J(t^{**}+t)$, or $A_J(x^{**}+x) \nleq \boldsymbol{b}_J(t^{**}+t)$.

Assume that $A_J(x^{**}+x) \leq \boldsymbol{b}_J(t^{**}+t)$. Then the solution $(x^{**}+x,\ t^{**}+t)$ is feasible to (P) and, since (x^{**},t^{**}) is optimal to (P), it follows that $\gamma(x^{**}+x)/(t^{**}+t) \preccurlyeq \gamma x^{**}/t^{**}$, that is $t^{**}\gamma(x^{**}+x) \preccurlyeq (t^{**}+t)\gamma x^{**}$, hence $t^{**}\gamma x \preccurlyeq t\gamma x^{**}$.

Assume now that $A_J(x^{**}+x) \nleq \boldsymbol{b}_J(t^{**}+t)$, which means that $\alpha_i x - b_i t > b_i t^{**} - \alpha_i x^{**}$ for some $i \in J$. Let $J^{>} = \{\, i \in J : \alpha_i x - b_i t > b_i t^{**} - \alpha_i x^{**} \,\}$. Since $b_i t^{**} - \alpha_i x^{**} > 0$ and since the ring R is weakly Archimedean by (H2), there exists a $\lambda_i \in R$ such that $\alpha_i x - b_i t \leq \lambda_i (b_i t^{**} - \alpha_i x^{**})$ for every $i \in J^{>}$. Let $\lambda \coloneqq \max_{i \in J^{>}} \lambda_i$. It is obvious that $\lambda > 0$, and we may assume wlog that λ is not a zero divisor. (By (H1), there exists a $K > 0$ which is not a zero divisor. It is easy to see that if $\lambda > 0$ is a zero divisor, then $0 < \lambda < K$. Since $(b_i t^{**} - \alpha_i x^{**}) > 0$, it then holds $\lambda(b_i t^{**} - \alpha_i x^{**}) \leq$ $\leq K(b_i t^{**} - \alpha_i x^{**})$ for every $i \in J^{>}$. So we let $\lambda \coloneqq K$.)

Considering this $\lambda > 0$, which is not a zero divisor, we have $A_J x - \boldsymbol{b}_J t \leq \left(\boldsymbol{b}_J t^{**} - A_J x^{**}\right)\lambda$. Since $A_I x \leq \boldsymbol{b}_I t$, it follows that $A_I x - \boldsymbol{b}_I t \leq \boldsymbol{0}_I = (\boldsymbol{b}_I t^{**} - A_I x^{**})\lambda$. Rewriting the inequalities, we obtain $A_I(\lambda x^{**}+x) \leq \boldsymbol{b}_I(\lambda t^{**}+t)$ with $A_J(\lambda x^{**}+x) \leq \boldsymbol{b}_J(\lambda t^{**}+t)$, and we conclude that the solution $(\lambda x^{**}+x,\ \lambda t^{**}+t)$ is feasible to (P). Since the solution (x^{**},t^{**}) is optimal, it is easy to see that the solution $(\lambda x^{**},\lambda t^{**})$ is optimal too. It follows hence $\gamma(\lambda x^{**}+x)/(\lambda t^{**}+t) \preccurlyeq$ $\preccurlyeq \gamma(\lambda x^{**})/(\lambda t^{**})$, or $\lambda t^{**}\gamma(\lambda x^{**}+x) \preccurlyeq (\lambda t^{**}+t)\gamma(\lambda x^{**})$, therefore $\lambda t^{**}\gamma x \preccurlyeq \lambda t \gamma x^{**}$. Since $\lambda > 0$ is not a zero divisor and the module V is R-torsion free by (H3), it follows that $t^{**}\gamma x \preccurlyeq t\gamma x^{**}$.

In both cases ($A_J(x^{**}+x) \leq \boldsymbol{b}_J(t^{**}+t)$ or $A_J(x^{**}+x) \nleq \boldsymbol{b}_J(t^{**}+t)$), we have concluded that $t^{**}\gamma x \preccurlyeq t\gamma x^{**}$. In case (A), we directly have that $t^{*}\gamma x \preccurlyeq t\gamma x^{*}$. In case (B), we have that $\lambda^{*} t^{*}\gamma x \preccurlyeq$ $\preccurlyeq \lambda^{*} t\gamma x^{*}$. Since $\lambda^{*} > 0$ is not a zero divisor and the module V is R-torsion free by (H3), it follows that $t^{*}\gamma x \preccurlyeq t\gamma x^{*}$. Having shown that $A_I x \leq \boldsymbol{b}_I t$ implies $t^{*}\gamma x \preccurlyeq t\gamma x^{*}$ for every $x \in W$ and for every $t \in R$, it follows by Farkas' Lemma 1 that $\iota \boldsymbol{u}_I^{*\mathrm{T}} A_I = r t^{*}\gamma$ and $\iota \boldsymbol{u}_I^{*\mathrm{T}} \boldsymbol{b}_I = r\gamma x^{*}$ for some non-negative $\boldsymbol{u}_I^{*\mathrm{T}} \in V^I$ and for some positive $r \in R$. By putting $r^{*} \coloneqq r t^{*}$, we are done. ■

We shall also need the following additional hypothesis about the ring R and module V.

Hypothesis. The non-trivial linearly ordered commutative associative ring R and the linearly ordered module V over the linearly ordered ring R are such that:

(H4) For every positive $u \in V$, that is $u \succ 0$, and for every positive $\lambda \in R$, that is $\lambda > 0$, there exists a positive $\varepsilon \in V$, that is $\varepsilon \succ 0$, such that $\lambda\varepsilon \preccurlyeq u$.

Lemma 6 (Optimality condition for the dual problem). *Let R be a non-trivial linearly ordered commutative associative ring, let W be a module over the ring R, and let V be a linearly ordered module over the linearly ordered ring R. Additionally, assume that the module V is non-trivial and that hypotheses* (H1)–(H4) *hold true.*

Let $(\boldsymbol{u}^{*}, r^{*}) \in V^m \times R$ be any feasible solution to problem (D). *Let $I = \{\, i \in \{1,\dots,m\} : u_i^{*} \succ 0 \,\}$ be the set of the indices of the active dual variables. Then $(\boldsymbol{u}^{*}, r^{*})$ is an optimal solution to problem* (D) *if and only if*

$$\exists t^{*} \in R,\ t^{*} > 0,\ \exists x^{*} \in W,\ Ax^{*} \leq \boldsymbol{b}t^{*}\colon\ A_I x^{*} = \boldsymbol{b}_I t^{*}.$$

Proof. Let $J = \{1,\dots,m\} \setminus I$ denote the complement of the index set I.

We prove the "if" part. Notice that (x^{*},t^{*}) is a feasible solution to problem (P). By using that $\boldsymbol{u}_J^{*\mathrm{T}} = \boldsymbol{0}_J^{\mathrm{T}}$, we obtain that $r^{*}\gamma x^{*} = \iota\boldsymbol{u}^{*\mathrm{T}} A x^{*} = \iota\boldsymbol{u}_I^{*\mathrm{T}} A_I x^{*} + \iota\boldsymbol{u}_J^{*\mathrm{T}} A_J x^{*} = \iota\boldsymbol{u}_I^{*\mathrm{T}}(\boldsymbol{b}_I t^{*}) + \iota\boldsymbol{u}_J^{*\mathrm{T}}\left(\boldsymbol{b}_J t^{*}\right) =$ $= \iota\boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{b}t^{*}) = t^{*}\iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}$, which means that $\gamma x^{*}/t^{*} \approx \iota\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{b}/r^{*}$. By Corollary 4, it follows that $(\boldsymbol{u}^{*}, r^{*})$ is an optimal solution to problem (D).

It remains to prove the "only if" part. We prove it indirectly. Assume that there is no positive $t^{*} \in R$ and no $x^{*} \in W$ such that $Ax^{*} \leq \boldsymbol{b}t^{*}$ and $A_I x^{*} = \boldsymbol{b}_I t^{*}$. Equivalently, there is no $x^{*} \in W$ and no positive $t^{*} \in R$ to solve $A_I x^{*} \leq \boldsymbol{b}_I t^{*}$ and $-A_I x^{*} \leq -\boldsymbol{b}_I t^{*}$ and also $A_J x^{*} \leq \boldsymbol{b}_J t^{*}$. By Gale's Theorem 2, there exist non-negative $\boldsymbol{\lambda}_I^{+}, \boldsymbol{\lambda}_I^{-} \in V^I$ and a non-negative $\boldsymbol{\lambda}_J \in V^J$ such that it holds

$\boldsymbol{\lambda}_I^{+\mathrm{T}} A_I - \boldsymbol{\lambda}_I^{-\mathrm{T}} A_I + \boldsymbol{\lambda}_J^{\mathrm{T}} A_J = o$ and $\boldsymbol{\lambda}_I^{+\mathrm{T}} \boldsymbol{b}_I - \boldsymbol{\lambda}_I^{-\mathrm{T}} \boldsymbol{b}_I + \boldsymbol{\lambda}_J^{\mathrm{T}} \boldsymbol{b}_J < 0$. Put $\boldsymbol{\lambda}_I := \boldsymbol{\lambda}_I^{+} - \boldsymbol{\lambda}_I^{-}$. We then have $\boldsymbol{\lambda}_I^{\mathrm{T}} A_I + \boldsymbol{\lambda}_J^{\mathrm{T}} A_J = o$ and $\boldsymbol{\lambda}_I^{\mathrm{T}} \boldsymbol{b}_I + \boldsymbol{\lambda}_J^{\mathrm{T}} \boldsymbol{b}_J < 0$.

We may assume wlog that $\boldsymbol{\lambda}_I^{\mathrm{T}} \boldsymbol{b}_I + \boldsymbol{\lambda}_J^{\mathrm{T}} \boldsymbol{b}_J = \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b}$ is not a zero divisor. (By (H1), there exists a $K > 0$ which is not a zero divisor. If $\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b}$ is a zero divisor, then we have $0 < -\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} < K$. Since the ring is weakly Archimedean by (H2), there exists a $t \in R$ such that $0 < K \leq -t\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b}$, whence $t\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b}$ is not a zero divisor. Put $\boldsymbol{\lambda} := \boldsymbol{\lambda} t$.)

Considering the given solution $(\boldsymbol{u}^*, r^*)$, distinguish two cases: either (A) it holds that r^* is not a zero divisor, or (B) it holds that r^* is a zero divisor. In case (A), put $(\boldsymbol{u}^{**}, r^{**}) := (\boldsymbol{u}^*, r^*)$. In case (B), there exists a $K > 0$ which is not a zero divisor by (H1), therefore it holds $0 < r^* < K$. Since the ring R is weakly Archimedean by (H2), there exists a $\lambda^* \in R$ such that $0 < K \leq \lambda^* r^*$, whence $\lambda^* r^*$ is not a zero divisor. We may assume wlog that λ^* is not a zero divisor. (We have $0 < \lambda^* < K$ otherwise, hence $\lambda^* r^* \leq K r^*$, so it is enough to let $\lambda^* := K$.) Since the module V is R-torsion free by (H3), we have $u_i^* \succ 0$ if and only if $\lambda^* u_i^* \succ 0$ for $i = 1, \dots, m$. In case (B), put $(\boldsymbol{u}^{**}, r^{**}) :=$ $:= ((\lambda^* \boldsymbol{u}^{*\mathrm{T}})^{\mathrm{T}}, \lambda^* r^*)$ and observe that the solution $(\boldsymbol{u}^{**}, r^{**})$ is feasible to (D) and also $\iota \boldsymbol{u}^{**\mathrm{T}} \boldsymbol{b} / r^{**} =$ $= \iota(\lambda^* \boldsymbol{u}^{*\mathrm{T}}) \boldsymbol{b} / (\lambda^* r^*) \approx \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b} / r^*$, that is $r^* \lambda^* \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b} = \lambda^* r^* \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b}$. We conclude in either of the cases that the solution $(\boldsymbol{u}^*, r^*)$ is optimal if and only if the solution $(\boldsymbol{u}^{**}, r^{**})$ is optimal, moreover it holds $I = \{\, i \in \{1, \dots, m\} : u^{**} \succ 0 \,\}$ and r^{**} is not a zero divisor.

Now, it is our purpose to find a positive $\varepsilon \in V$, that is $\varepsilon \succ 0$, such that $u_i^{**} + \lambda_i \varepsilon \succcurlyeq 0$ for every $i = 1, \dots, m$. Let $I^{<} = \{\, i \in I : \lambda_i < 0 \,\}$. We distinguish two cases: either $I^{<} = \emptyset$, or $I^{<} \neq \emptyset$.

Assume first that $I^{<} = \emptyset$. As the module V is non-trivial by assumption, there exists a positive $\varepsilon \in V$. Since $\boldsymbol{u}^{**\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ and $\boldsymbol{\lambda} \geq \boldsymbol{0}$, it is easy to see that $u_i^{**} + \lambda_i \varepsilon \succcurlyeq 0$ for every $i = 1, \dots, m$.

Assume now that $I^{<} \neq \emptyset$. By (H4), there exists a positive $\varepsilon_i \in V$ such that $-\lambda_i \varepsilon_i \preccurlyeq u_i^{**}$ for $i \in I^{<}$. Let $\varepsilon := \min_{i \in I^{<}} \varepsilon_i$. Observe that $0 \preccurlyeq -\lambda_i \varepsilon \preccurlyeq u_i^{**}$, hence $0 \preccurlyeq u_i^{**} + \lambda_i \varepsilon$ for $i \in I^{<}$. Since we have $\lambda_i \geq 0$ for $i \in \{1, \dots, m\} \setminus I^{<}$, put together, we obtain that $0 \preccurlyeq u_i^{**} + \lambda_i \varepsilon$ for every $i = 1, \dots, m$.

We thus have $\boldsymbol{u}^{**\mathrm{T}} + (\boldsymbol{\lambda}\varepsilon)^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$. Recall that $\boldsymbol{\lambda}^{\mathrm{T}} A = o$ and $\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} < 0$. Since $\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b}$ is not a zero divisor and the module V is R-torsion free by (H3), we have $\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} \varepsilon \prec 0$. Consider $\boldsymbol{u} := \boldsymbol{u}^{**} + \boldsymbol{\lambda}\varepsilon$. Since $\iota \boldsymbol{u}^{\mathrm{T}} A = \iota \boldsymbol{u}^{**\mathrm{T}} A + \iota(\boldsymbol{\lambda}\varepsilon)^{\mathrm{T}} A = \iota \boldsymbol{u}^{**\mathrm{T}} A + \iota \varepsilon \boldsymbol{\lambda}^{\mathrm{T}} A = r^{**}\gamma + o = r^{**}\gamma$ and $\boldsymbol{u}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$, it follows that $(\boldsymbol{u}, r^{**})$ is a feasible solution to problem (D). It also holds $\iota \boldsymbol{u}^{\mathrm{T}} \boldsymbol{b} = \iota \boldsymbol{u}^{**\mathrm{T}} \boldsymbol{b} + \iota(\boldsymbol{\lambda}\varepsilon)^{\mathrm{T}} \boldsymbol{b} = \iota \boldsymbol{u}^{**\mathrm{T}} \boldsymbol{b} + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} \varepsilon \prec$ $\prec \iota \boldsymbol{u}^{**\mathrm{T}} \boldsymbol{b}$. Since r^{**} is not a zero divisor and the module V is R-torsion free by (H3), it follows that $r^{**} \iota \boldsymbol{u}^{\mathrm{T}} \boldsymbol{b} \prec r^{**} \iota \boldsymbol{u}^{**\mathrm{T}} \boldsymbol{b}$, that is $\iota \boldsymbol{u}^{\mathrm{T}} \boldsymbol{b} / r^{**} \prec \iota \boldsymbol{u}^{**\mathrm{T}} \boldsymbol{b} / r^{**}$, which means that $(\boldsymbol{u}^{**}, r^{**})$ is not an optimal solution problem (D), equivalently $(\boldsymbol{u}^*, r^*)$ not an optimal solution to problem (D) either. The proof is finished thus. ∎

By combining Lemma 5 and Lemma 6, our main result, that is the following Strong Duality Theorem for problems (P) and (D), is obtained easily.

Theorem 7 (Strong Duality Theorem). *Let R be a non-trivial linearly ordered commutative associative ring, let W be a module over the ring R, let V be a linearly ordered module over the linearly ordered ring R, and let hypotheses* (H1)–(H3) *hold true. It then holds:*

(I) *If $(x^*, t^*) \in W \times R$ is an optimal solution to problem* (P)*, then there exists an optimal solution $(\boldsymbol{u}^*, r^*) \in V^m \times R$ to problem* (D) *and it holds $\gamma x^*/t^* \approx \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b} / r^*$.*

(II) *If $(\boldsymbol{u}^*, r^*) \in V^m \times R$ is an optimal solution to problem* (D)*, the module V is non-trivial, and hypothesis* (H4) *also holds true, then there exists an optimal solution $(x^*, t^*) \in W \times R$ to problem* (P) *and it holds $\gamma x^*/t^* \approx \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b} / r^*$.*

Proof. (I) Let $(x^*, t^*) \in W \times R$ be an optimal solution to problem (P) and let $I = \{\, i \in \{1, \dots, m\} : a_i x^* = b_i t^* \,\}$. Let $J = \{1, \dots, m\} \setminus I$ be the complement of the index set I. By Lemma 5, there exist a positive $r^* \in R$ and a non-negative $\boldsymbol{u}_I^* \in V^m$ such that $r^*\gamma = \iota \boldsymbol{u}_I^{*\mathrm{T}} A_I$. Put $u_i^* := 0$ for every $i \in J$. We have a $\boldsymbol{u}^* \in V^m$ such that $\boldsymbol{u}^{*\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ and $r^*\gamma = \iota \boldsymbol{u}_I^{*\mathrm{T}} A_I = \iota \boldsymbol{u}_I^{*\mathrm{T}} A_I + \iota \boldsymbol{u}_J^{*\mathrm{T}} A_J = \iota \boldsymbol{u}^{*\mathrm{T}} A$, which means that $(\boldsymbol{u}^*, r^*)$ is a feasible solution to problem (D). Moreover, we have $r^*\gamma x^* = \iota \boldsymbol{u}_I^{*\mathrm{T}} A_I x^* = \iota \boldsymbol{u}_I^{*\mathrm{T}} (\boldsymbol{b}_I t^*) =$

$= \iota \boldsymbol{u}_I^{*\mathrm{T}}(\boldsymbol{b}_I t^*) + \iota \boldsymbol{u}_J^{*\mathrm{T}}(\boldsymbol{b}_J t^*) = \iota \boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{b} t^*) = t^* \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b}$, which means that $\gamma x^*/t^* \approx \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b}/r^*$. By Corollary 4, the solution $(\boldsymbol{u}^*, r^*)$ is optimal.

(II) Let $(\boldsymbol{u}^*, r^*) \in V^m \times R$ be an optimal solution to problem (D) and let $I = \{ i \in \{1, \dots, m\} : u_i^* \succ 0 \}$. Let $J = \{1, \dots, m\} \setminus I$ be the complement of the index set I. By Lemma 6, there exist a positive $t^* \in R$ and an $x^* \in W$ such that $A_I x^* = \boldsymbol{b}_I t^*$ and $A_J x^* \le \boldsymbol{b}_J t^*$. It follows that (x^*, t^*) is a feasible solution to problem (P). Additionally, it holds $r^* \gamma x^* = \iota \boldsymbol{u}^{*\mathrm{T}} A x^* = \iota \boldsymbol{u}_I^{*\mathrm{T}} A_I x^* + \iota \boldsymbol{u}_J^{*\mathrm{T}} A_J x^* =$ $= \iota \boldsymbol{u}_I^{*\mathrm{T}}(\boldsymbol{b}_I t^*) + \iota \boldsymbol{u}_J^{*\mathrm{T}}(\boldsymbol{b}_J t^*) = \iota \boldsymbol{u}^{*\mathrm{T}}(\boldsymbol{b} t^*) = t^* \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b}$, which means that $\gamma x^*/t^* \approx \iota \boldsymbol{u}^{*\mathrm{T}} \boldsymbol{b}/r^*$. By Corollary 4, the solution (x^*, t^*) is optimal. ■

5. An application in business decision making

Following Bartl (2019), we propose a simple application of homogeneous linear programming in the context of SMEs. In Subsection 5.1, we introduce a special linearly ordered commutative associative ring R, which we shall use. In Subsection 5.2, we describe a simple decision making problem (an extension of the FMEA method) and a mathematical model of the decision making problem in terms of homogeneous linear programming. In Subsection 5.3, we briefly discuss other special linearly ordered commutative rings.

5.1. A special linearly ordered commutative ring

We construct the special linearly ordered commutative associative ring in two steps. First, consider the ring S consisting of all rational numbers of the form $m \times 10^n$ for all integer m and n. In words, the auxiliary ring S consists of all numbers that can be written by using a finite number of (decimal) digits, such as 5, 12.345, -3.14, but not $1/3 = 0.333 \dots$ This ring S is a subring of the field of the real numbers $\mathbb{R}$, therefore the arithmetical operations (addition, subtraction, and multiplication) as well as its linear ordering are defined in the usual way. Second, let the special linearly ordered commutative associative ring R consist of all the formal power series of the form $\sum_{n=-\infty}^{+\infty} a_n x^n$, that is

$$\cdots + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 x^0 + a_{-1} x^{-1} + a_{-2} x^{-2} + a_{-3} x^{-3} + \cdots$$

where "x" is a formal variable and the coefficients $a_n \in S$ for all $n \in \mathbb{Z}$ where only finitely many of a_n's are non-zero. (Here S is the above constructed ring. Alternatively, we can consider $S = \mathbb{R}$ or $S = \mathbb{Q}$, or any other suitable ring.)

The addition and subtraction are defined in the usual way. That is, for $\sum_{n=-\infty}^{+\infty} a_n x^n \in R$ and for $\sum_{n=-\infty}^{+\infty} b_n x^n \in R$, we have $\sum_{n=-\infty}^{+\infty} a_n x^n + \sum_{n=-\infty}^{+\infty} b_n x^n = \sum_{n=-\infty}^{+\infty} (a_n + b_n) x^n$ and also $\sum_{n=-\infty}^{+\infty} a_n x^n - \sum_{n=-\infty}^{+\infty} b_n x^n = \sum_{n=-\infty}^{+\infty} (a_n - b_n) x^n$. We note that the element $0 = \sum_{n=-\infty}^{+\infty} a_n x^n$, where $a_n = 0$ for all $n \in \mathbb{Z}$, is neutral with respect to addition.

The multiplication is defined by using the rule that $x^m \times x^n = x^{m+n}$. That is, for $\sum_{n=-\infty}^{+\infty} a_n x^n \in R$ and for $\sum_{n=-\infty}^{+\infty} b_n x^n \in R$, we have that $\sum_{n=-\infty}^{+\infty} a_n x^n \times \sum_{n=-\infty}^{+\infty} b_n x^n =$ $= \sum_{n=-\infty}^{+\infty} (\sum_{m=-\infty}^{+\infty} a_{n-m} \times b_m) x^n$. We note that the element $1 = x^0 = \sum_{n=-\infty}^{+\infty} a_n x^n$, where $a_0 = 1$ and $a_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, is neutral with respect to multiplication.

The ring is ordered lexicographically by using the rule that $x^m \ll x^n$ if and only if $m < n$. That is, for $\sum_{n=-\infty}^{+\infty} a_n x^n \in R$ and for $\sum_{n=-\infty}^{+\infty} b_n x^n \in R$, we have $\sum_{n=-\infty}^{+\infty} a_n x^n < \sum_{n=-\infty}^{+\infty} b_n x^n$ if and only if there exists an $n_0 \in \mathbb{Z}$ such that $a_{n_0} < b_{n_0}$ and $a_n = b_n$ for all $n \in \mathbb{Z}$ such that $n > n_0$.

Both rings S and R are commutative, associative, unital, and do not contain zero divisors. The motivation behind the use of the ring S is that, when an expert (decision maker) writes down some number with a practical meaning, such as some score or probability of an event, then the number will consist of a finite number of digits. In other words, the expert cannot write down more than finitely many digits in practice. The motivation behind the use of the ring R is that it provides a finer resolution than the usual numerical scale. To illustrate this idea, consider the sample space $\Omega = \{1, 2\}$ with the probability mass function defined as $p_1 = 0.5 + 10x^{-1}$ and $p_2 = 0.5 - 10x^{-1}$. Then both elementary events $\{1\}$ and $\{2\}$ are (about) equally probable, but the event $\{1\}$ is "slightly more" probable than $\{2\}$.

5.2 A simple decision making problem: an extension of the FMEA method

The FMEA (Failure Mode and Effects Analysis) method (Stamatis, 2003) is a tool to identify serious risks; it can also be used in Six Sigma. Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ be the set of the risks under consideration; we assume for simplicity that the number of the risks is finite. Each risk ω receives three scores: probability P_ω is the likelihood of the occurrence of the risk, severity S_ω is the score of the worst impact of the risk, and detection D_ω is the likelihood that the risk will *not* be detected until its severe impact shows up. It is usual to take the scores from the scale $\{1, 2, \dots, 10\}$, where 1 and 10 represent the mildest and the most serious, respectively, value. The RPN (Risk Priority Number) of the risk ω is a number ranging from 1 (risk of little account) to 1000 (serious hazard); it is the product of the three scores, that is $RPN_\omega = P_\omega S_\omega D_\omega$.

In the FMEA method, it also makes sense to use the ring R introduced in the previous subsection. We then choose the scores $P_\omega, S_\omega, D_\omega$ from the scale $\mathcal{S} = \{ r \in R : 0 < r \leq 10 \}$. Recall that 0 is the element $0 = \sum_{n=-\infty}^{+\infty} a_n x^n$ with $a_n = 0$ for all $n \in \mathbb{Z}$, and 10 is the element $10x^0 = \sum_{n=-\infty}^{+\infty} b_n x^n$ with $b_0 = 10$ and $b_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Assume that, for each risk $\omega \in \Omega$, its probability score $P_\omega \in \mathcal{S}$ and its severity score $S_\omega \in \mathcal{S}$ are given and fixed, but its detection score $D_\omega \in \mathcal{S}$ can be decreased to $D'_\omega \in \mathcal{S}$ if full attention is paid to the risk ω. Full attention means attention of unit intensity. We assume, however, that attention of intensity of no more than $I_\omega \in R$ can be paid to the risk ω. We assume that $\sum_{\omega \in \Omega} I_\omega > 1$ and $0 < I_\omega < 1$ for $\omega \in \Omega$, where 1 is the element $1x^0 = \sum_{n=-\infty}^{+\infty} a_n x^n$ with $a_0 = 1$ and $a_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. If attention of intensity $x_\omega \in R$, such that $0 \leq x_\omega \leq I_\omega$, is paid to the risk ω, then its detection score is decreased proportionally to $D_\omega - (D_\omega - D'_\omega)x_\omega$, so that its RPN is mitigated to $RPN'_\omega = P_\omega S_\omega D_\omega - P_\omega S_\omega (D_\omega - D'_\omega) x_\omega$. The task is to divide the available unit attention among the risks so that the maximum of the mitigated RPN's is minimized.

This simple decision making problem can be expressed in terms of homogeneous linear programming as follows:

$$
\begin{array}{lll}
\text{minimize} & y/t & (1)\\
\text{subject to} & P_\omega S_\omega D_\omega t - P_\omega S_\omega (D_\omega - D'_\omega) x_\omega \leq y \quad \text{for} \quad \omega \in \Omega & \\
& \sum_{\omega \in \Omega} x_\omega \leq 1t & \\
& 0 \leq x_\omega \leq I_\omega t \quad \text{for} \quad \omega \in \Omega & \\
& t > 0 &
\end{array}
$$

where $y \in R$ and $x_\omega \in R$ for $\omega \in \Omega$ and also $t \in R$ are variables. The value of the variable x_ω is the intensity of the attention paid to the risk ω, and $t > 0$ is the homogenizing variable; its value can be seen as the total intensity of the attention. The variable y is auxiliary and it is used to find the maximum of the mitigated RPN's. We can rewrite problem (1) into the form of primal problem (P) as follows:

$$
\begin{array}{lll}
\text{maximize} & -y/t & (2)\\
\text{subject to} & -y - P_\omega S_\omega (D_\omega - D'_\omega) x_\omega \leq -P_\omega S_\omega D_\omega t \quad \text{for} \quad \omega \in \Omega & \\
& \sum_{\omega \in \Omega} x_\omega \leq 1t & \\
& x_\omega \leq I_\omega t \quad \text{for} \quad \omega \in \Omega & \\
& -x_\omega \leq 0t \quad \text{for} \quad \omega \in \Omega & \\
& t > 0 &
\end{array}
$$

Notice that, in this primal problem (2), the primal variable module is $W = R \times R^\Omega$ and the module V of the objective values is identified with the ring itself, that is $V = R$. The dual problem then takes the form:

$$
\begin{array}{lll}
\text{minimize} & (\sum_{\omega \in \Omega} -P_\omega S_\omega D_\omega u_\omega + 1w + \sum_{\omega \in \Omega} I_\omega v_\omega + \sum_{\omega \in \Omega} 0 z_\omega)/r & (3)\\
\text{subject to} & \sum_{\omega \in \Omega} -u_\omega = -1r & \\
& -P_\omega S_\omega (D_\omega - D'_\omega) u_\omega + 1w + 1v_\omega - 1z_\omega = 0r \quad \text{for} \quad \omega \in \Omega & \\
& w, u_\omega, v_\omega, z_\omega \geq 0 \quad \text{for} \quad \omega \in \Omega & \\
& r > 0 &
\end{array}
$$

where $z \in R$ and $u_\omega, v_\omega, w_\omega \in R$ for $\omega \in \Omega$ and also $r \in R$ are variables. Here $r > 0$ is the homogenizing variable. We can simplify problem (3) to:

$$\begin{array}{lll} \text{maximize} & (\sum_{\omega\in\Omega} P_\omega S_\omega D_\omega u_\omega - \sum_{\omega\in\Omega} I_\omega v_\omega - w)/r & \\ \text{subject to} & \sum_{\omega\in\Omega} u_\omega = 1r & \\ & P_\omega S_\omega (D_\omega - D'_\omega) u_\omega - v_\omega - w \leq 0r \qquad \text{for} \quad \omega \in \Omega & \\ & u_\omega, v_\omega, w \geq 0 \qquad \text{for} \quad \omega \in \Omega & \\ & r > 0 & \end{array} \tag{4}$$

Recall that R is a non-trivial linearly ordered commutative associative ring and notice that the additional hypotheses (H1)–(H4) are also satisfied. It follows hence that Strong Duality Theorem 7 holds for problems (2) and (3). Since problem (2) and (3) is equivalent with problem (1) and (4), respectively, it follows that the Strong Duality Theorem holds for problems (1) and (4) as well.

5.3 Other special linearly ordered commutative rings

The motivation behind the use of the rings S and R was explained at the end of Subsection 5.1. Notice, however, that we used another special linearly ordered commutative ring in the applications described in Bartl (2019), see Bartl (2017, Example 1); this ring is associative, contains zero divisors, but does not satisfy (H2), which is the reason why we have not used it here. More examples of linearly ordered commutative rings can be found in Bartl (2017, Examples 2–4), see also Bartl (2020, Examples 4.1, 5.1, 5.2, and 7.1). We do not go into the details due to the lack of space.

6. Remarks

Let R be a non-trivial linearly ordered commutative associative ring, let W be a module over the ring R, and let V be a linearly ordered module over the linearly ordered ring R.

Let $a, K \in R$ be positive elements such that a is a zero divisor and K is not a zero divisor. Observe that $0 < a < K$.

It is then easy to see that, if the ring R satisfies (H1) and (H2), then there are no zero divisors in the ring. (Assume for the sake of a contradiction that $a \in R$ is a positive zero divisor; that is, there exists a positive $b \in R$ such that $ab = 0$. Let $K \in R$ be a positive element, provided by (H1), which is not a zero divisor. Since $0 < a < K$ and the ring R is weakly Archimedean by (H2), there exists a $\lambda \in R$ such that $0 < K \leq \lambda a$. Since $b > 0$ and the ring is linearly ordered, it holds $0 \leq Kb \leq (\lambda a)b$. By the associativity, we have $(\lambda a)b = \lambda(ab) = \lambda 0 = 0$, whence $Kb = 0$, which is a contradiction because K is not a zero divisor. We conclude that there are no zero divisors in the ring.)

7. Conclusion

The classical variant of Farkas' Lemma (Farkas, 1902) says in symbols that ($\forall \boldsymbol{x} \in \mathbb{R}^n$: $\boldsymbol{Ax} \leq \boldsymbol{0} \Rightarrow \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq 0$) $\Leftrightarrow$ ($\exists \boldsymbol{u}^{\mathrm{T}} \in \mathbb{R}^{1\times m}$, $\boldsymbol{u}^{\mathrm{T}} \geq \boldsymbol{0}^{\mathrm{T}}$: $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{c}^{\mathrm{T}}$), where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is a matrix and $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times n}$ is a row vector. Like latter statement of Farkas' Lemma with the equation ($\boldsymbol{u}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{c}^{\mathrm{T}}$) yields the constraints of the dual problem ($\mathrm{D_c}$), the consequent of the discrete variant of Farkas' Lemma (Lemma 1) with the equation ($\boldsymbol{u}^{\mathrm{T}}A = r\gamma$) led us naturally to formulate the constraints of the dual problem (D), cf. Lemma 5. The constraints of the primal problem ($\mathrm{P_c}$) are given by the system of linear inequalities ($\boldsymbol{Ax} \leq \boldsymbol{b}$) considered in the classical variant of Gale's Theorem of the alternative (Fan, 1956; Gale, 1960). Likewise, the system of linear inequalities ($Ax \leq \boldsymbol{b}t$) considered in Gale's Theorem 2 yields the constraints of the primal problem (P), cf. Lemma 6. The objective functions of both problems (P) and (D) are made up easily then.

Given the new variant of Farkas' Lemma (Lemma 1), it was our purpose to investigate whether the Strong Duality Theorem can also be proved for problems (P) and (D). Albeit the hypothesis of the associativity of the ring R is unnecessary in the new variant of Farkas' Lemma (it is relaxed to the hypothesis that $(\lambda\mu)x = \lambda(\mu x)$ for all $\lambda, \mu \in R$ and for all $x \in W$ in Lemma 1), we used it several times when proving our results (Lemma 5 and Lemma 6 in particular). Moreover, we also used additional hypotheses (H1)–(H3) and (H4) to prove our results. We showed in Section 6, however, that there are no zero divisors in the ring R under these hypotheses. Then, the ring R being commutative, associative and without any zero divisors, it can naturally be extended into the corresponding field F of fractions. Consequently, Strong Duality Theorem 7 also follows from the already known results for the continuous case (Bartl, 2007), see Bartl and Dubey (2017, Remark 8), which is a disappointing finding. In other words, Strong Duality Theorem 7 cannot be seen as a new result due to the strong additional hypotheses, which we used to prove Lemma 5 and Lemma 6.

We thus ask whether the Strong Duality Theorem for problems (P) and (D) holds even if the hypothesis of the associativity of the ring R and/or additional hypotheses (H1)–(H3) and (H4) are relaxed, or if it holds in the special case of some of the rings discussed in Subsection 5.3.

Acknowledgements

This work was supported by the Czech Science Foundation under grant number GAČR 21-03085S, and by the Ministry of Education, Youth and Sports of the Czech Republic within the Institutional Support for Long-Term Development of a Research Organization in 2021.

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