A universal theorem of the alternative

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Abstract. We present a particular theorem of the alternative for finite systems of linear inequalities. The theorem is universal in the sense that other classical theorems of the alternative (Motzkin's Theorem and Tucker's Theorem) are implicit in it; the theorem itself is an extension of Farkas' Lemma. The presented result also generalizes and unifies both Dax's new theorem the alternative [Dax, A. (1993). *Annals of Operations Research*, 46, 11–60] and Rohn's residual existence theorem for linear equations [Rohn, J. (2010). *Optimization Letters*, 4, 287–292]. The universal theorem of the alternative is established by using Farkas' Lemma in the setting of a vector space over a linearly ordered (commutative or skew) field.

Keywords: Theorems of the alternative, systems of linear inequalities, Farkas' Lemma, linearly ordered vector spaces, linearly ordered fields.

JEL Classification: C65

1 Introduction

In this note, we present a universal theorem of the alternative which unifies both Dax's new theorem of the alternative (Dax, 1993) and Rohn's residual existence theorem for linear equations (Rohn, 2010). Moreover, the universal theorem contains Motzkin's Theorem of the alternative (Motzkin, 1934) and Tucker's Theorem of the alternative (Tucker, 1956) implicitly in it. The theorem is established by using Farkas' Lemma (Farkas, 1902), which has proved to be a result of importance in optimization and also in economics (Vohra, 2006). Short proofs of Farkas' Lemma include those given by Dax (1997), Broyden (1998; Roos and Terlaky, 1999), Komornik (1998), Scowcroft (2006, pp. 3535–3536, in the Introduction) and Bartl (2012a). See Fujimoto et al. (2018) for another proof; see also the references therein for further discussion of earlier proofs. Considering natural numbers $m, n, \nu \in \mathbb{N}$, recall the original results in the setting of the finite-dimensional vector space $\mathbb{R}^N$, where $N \in \mathbb{N}$ is a natural number.

Farkas' Lemma (Farkas, 1902)**.** *Let $\boldsymbol{A} \in \mathbb{R}^{m\times N}$ be a matrix and let $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$ be a row vector. It holds*

$$\forall \boldsymbol{x} \in \mathbb{R}^N\colon\ \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{0} \Rightarrow \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \le 0$$

if and only if

$$\exists \boldsymbol{u}^{\mathrm{T}} \in \mathbb{R}^{1\times m},\ \boldsymbol{u}^{\mathrm{T}} \ge \boldsymbol{0}^{\mathrm{T}}\colon\ \boldsymbol{c}^{\mathrm{T}} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{A}\,.$$

Dax's new theorem of the alternative (Dax, 1993, Sections 5.1 and 5.4)**.** *Let $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$ be a row vector, let $\boldsymbol{B}_1, \dots, \boldsymbol{B}_n \in \mathbb{R}^{1\times N}$ be one-row matrices, and let $w_1, \dots, w_n \in \mathbb{R}$ be positive weights. It holds*

$$\forall \boldsymbol{x} \in \mathbb{R}^N\colon\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \le w_1|\boldsymbol{B}_1\boldsymbol{x}| + \dots + w_n|\boldsymbol{B}_n\boldsymbol{x}|$$

if and only if

$$\exists v_1 \in \mathbb{R},\ -w_1 \le v_1 \le w_1,\ \dots,\ \exists v_n \in \mathbb{R},\ -w_n \le v_n \le w_n\colon\ \boldsymbol{c}^{\mathrm{T}} = v_1\boldsymbol{B}_1 + \dots + v_n\boldsymbol{B}_n\,.$$

Rohn's residual existence theorem for linear equations (Rohn, 2010, Theorem 2)**.** *Let $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$ be a row vector, let $\boldsymbol{Z} \in \mathbb{R}^{p\times N}$ be a matrix, and let $\{\boldsymbol{b}_1^{\mathrm{T}}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}}\} \subseteq \mathbb{R}^{1\times p}$ be any collection of row vectors. It holds*

$$\forall \boldsymbol{x} \in \mathbb{R}^N\colon\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \le \max\{\boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x}\}$$

if and only if

$$\exists v_1, \dots, v_\nu \in \mathbb{R},\ v_1, \dots, v_\nu \ge 0,\ v_1 + \dots + v_\nu = 1\colon\ \boldsymbol{c}^{\mathrm{T}} = (v_1\boldsymbol{b}_1^{\mathrm{T}} + \dots + v_\nu\boldsymbol{b}_\nu^{\mathrm{T}})\boldsymbol{Z}\,.$$

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Motzkin's Theorem (Motzkin, 1934; Motzkin, 1952, Theorem D6, p. 60)**.** *Let $\boldsymbol{A} \in \mathbb{R}^{m\times N}$ and $\boldsymbol{B} \in \mathbb{R}^{\nu\times N}$ be matrices. It holds*

$$\nexists \boldsymbol{x} \in \mathbb{R}^N\colon\ \boldsymbol{Ax} \le \boldsymbol{0} \wedge \boldsymbol{Bx} < \boldsymbol{0}$$

if and only if

$$\exists \boldsymbol{\lambda}^{\mathrm{T}} \in \mathbb{R}^{1\times m},\ \boldsymbol{\lambda}^{\mathrm{T}} \ge \boldsymbol{0}^{\mathrm{T}},\ \exists \boldsymbol{\mu}^{\mathrm{T}} \in \mathbb{R}^{1\times \nu},\ \boldsymbol{\mu}^{\mathrm{T}} \ge \boldsymbol{0}^{\mathrm{T}},\ \mu_1 + \cdots + \mu_\nu = 1\colon\ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{o}^{\mathrm{T}},$$

where $\boldsymbol{o}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$ is the row vector consisting of N zeros and $\mu_1, \dots, \mu_\nu \in \mathbb{R}$ are the components of $\boldsymbol{\mu}^{\mathrm{T}}$.

Tucker's Theorem (Tucker, 1956, Corollary 2A part (i))**.** *Let $\boldsymbol{A} \in \mathbb{R}^{m\times N}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times N}$ be matrices, where the matrix $\boldsymbol{B}$ consists of the rows $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n \in \mathbb{R}^{1\times N}$.* It holds

$$\nexists \boldsymbol{x} \in \mathbb{R}^N\colon\ \boldsymbol{Ax} \le \boldsymbol{0} \wedge \boldsymbol{Bx} \le \boldsymbol{0} \wedge \boldsymbol{b}_1\boldsymbol{x} + \cdots + \boldsymbol{b}_n\boldsymbol{x} = -1$$

if and only if

$$\exists \boldsymbol{\lambda}^{\mathrm{T}} \in \mathbb{R}^{1\times m},\ \boldsymbol{\lambda}^{\mathrm{T}} \ge \boldsymbol{0}^{\mathrm{T}},\ \exists \boldsymbol{\mu}^{\mathrm{T}} \in \mathbb{R}^{1\times n},\ \boldsymbol{\mu}^{\mathrm{T}} > \boldsymbol{0}^{\mathrm{T}}\colon\ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{o}^{\mathrm{T}},$$

where $\boldsymbol{o}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$ is the row vector consisting of N zeros.

2 Notation and Farkas' Lemma

Let F be a linearly ordered (commutative or skew) field. Additionally, let V be a linearly ordered vector space over the linearly ordered field F. The relation of the linear ordering of the field F and vector space V will be denoted by the symbol $\le$ and $\preccurlyeq$, respectively. Recall that an element $\lambda \in F$ is *positive* and *non-negative* if and only if $\lambda > 0$ and $\lambda \ge 0$, respectively. These two concepts are analogously defined for the elements of the vector space V. Finally, let W be a vector space over the field F.

For non-negative natural numbers m and n, and for positive natural numbers $\nu_1, \dots, \nu_n$, let $\alpha_1, \dots, \alpha_m\colon W \to F$ and $\beta_{j1}, \dots, \beta_{j\nu_j}\colon W \to F$ be linear forms, which make up the linear mappings $A\colon W \to F^m$ and $B_j\colon W \to F^{\nu_j}$, respectively, for $j = 1, \dots, n$. (The mappings $A\colon W \to F^m$ and $B_j\colon W \to F^{\nu_j}$ generalize the concept of the matrices $\boldsymbol{A} \in \mathbb{R}^{m\times N}$ and $\boldsymbol{B}_j \in \mathbb{R}^{\nu_j\times N}$, which we could see in the Introduction. Additionally, the linear forms $\alpha_1, \dots, \alpha_m\colon W \to F$ and $\beta_{j1}, \dots,$ $\beta_{j\nu_j}\colon W \to F$ correspond to the rows $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ and $\boldsymbol{b}_{j1}, \dots, \boldsymbol{b}_{j\nu_j}$ of the matrix $\boldsymbol{A}$ and $\boldsymbol{B}_j$, respectively, for $j = 1, \dots, n$.)

For any elements $\lambda, \mu \in F$ and for any vector $u \in V$, we put $\iota\lambda\mu = \mu\lambda$ and $\iota u\lambda = \lambda u$. In other words, the symbol 'ι' (Greek letter iota) means that the next two entities are to be transposed and multiplied in the new order.

Now, let m and ν be a non-negative and positive, respectively, natural number. As above, consider the linear mappings $A\colon W \to F^m$ and $B\colon W \to F^\nu$. For any $\boldsymbol{u} \in V^m$ and $\boldsymbol{v} \in V^\nu$, we stipulate that they consist of the components $u_1, \dots, u_m$ and $v_1, \dots, v_\nu$, respectively. Moreover, we define the linear mappings $\iota\boldsymbol{u}^{\mathrm{T}}A\colon W \to V$ and $\iota\boldsymbol{v}^{\mathrm{T}}B\colon W \to V$ by $\iota\boldsymbol{u}^{\mathrm{T}}Ax = \sum_{i=1}^{m}(\alpha_i x)u_i$ and $\iota\boldsymbol{v}^{\mathrm{T}}Bx = \sum_{k=1}^{\nu}(\beta_k x)v_k$, respectively, for every $x \in W$. The symbol $\boldsymbol{0}$ denotes the column vector consisting of m zeros of the field F. Additionally, the symbol $\boldsymbol{0}^{\mathrm{T}}$ denotes the row consisting of m or ν zeros of the vector space V. The inequalities $Ax \le \boldsymbol{0}$ and $\boldsymbol{u}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ as well as $\boldsymbol{v}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ are understood componentwise, that is $\alpha_i x \le 0$ and $u_i \succcurlyeq 0$ for every $i = 1, \dots, m$ and also $v_k \succcurlyeq 0$ for every $k = 1, \dots, \nu$, respectively. The symbol $\boldsymbol{e}$ denotes the column vector consisting of ν ones of the field F. We then have $\iota\boldsymbol{v}^{\mathrm{T}}\boldsymbol{e} = \iota v_1 1 + \cdots + \iota v_\nu 1 = v_1 + \cdots + v_\nu$. The symbol o denotes the zero linear mapping $o\colon W \to V$. If $m = 0$, then $\iota\boldsymbol{u}^{\mathrm{T}}A = o$ and the inequalities $Ax \le \boldsymbol{0}$ and also $\boldsymbol{u}^{\mathrm{T}} \succcurlyeq \boldsymbol{0}^{\mathrm{T}}$ are logically true by convention.

Finally, let $\gamma: W \to V$ be any linear mapping. (The mapping $\gamma: W \to V$ generalizes the concept of the row vector $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$, which we could see in the Introduction.)

Considering a non-negative natural number m, we now recall the following generalization of Farkas' Lemma formulated in the setting of a (possibly infinite-dimensional) vector space W over a (commutative or skew) linearly ordered field F, see Bartl (2007, Lemma 4.1).

Lemma 1 (Farkas' Lemma)**.** *Let F be a linearly ordered (commutative or skew) field, let W be a vector space over the field F, and let V be a linearly ordered vector space over the linearly ordered field F. Let $A: W \to F^m$ and $\gamma: W \to V$ be linear mappings. It then holds*

$$\forall x \in W:\ \ Ax \leq \mathbf{0} \ \Longrightarrow\ \gamma x \preccurlyeq 0$$

if and only if

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}}:\ \ \gamma = \iota \boldsymbol{u}^{\mathrm{T}} A\,.$$

See Bartl (2012a) for a very short proof. The original version of Farkas' Lemma presented in the Introduction is obtained by considering the field $F := \mathbb{R}$ of the real numbers, the finite-dimensional vector space $W := \mathbb{R}^N$, and the one-dimensional real line $V := \mathbb{R}^1$.

3 A universal theorem of the alternative

Considering non-negative natural numbers m and n and also positive natural numbers $\nu_1, \dots, \nu_n$, we present a new universal theorem of the alternative.

Theorem 2 (universal theorem of the alternative)**.** *Let F be a linearly ordered (commutative or skew) field, let W be a vector space over the field F, and let V be a linearly ordered vector space over the linearly ordered field F. Let $A: W \to F^m$ be a linear mapping, let $B_1: W \to F^{\nu_1}$, …, $B_n: W \to F^{\nu_n}$ be linear mappings consisting of the linear forms $\beta_{11}, \dots, \beta_{1\nu_1}: W \to F$, …, $\beta_{n1}, \dots, \beta_{n\nu_n}: W \to F$, respectively, let $w_1, \dots, w_n \in V$ be non-negative weights, and let $\gamma: W \to V$ be a linear mapping. It then holds*

$$\forall x \in W:\quad Ax \leq \mathbf{0} \ \Longrightarrow\ \gamma x \preccurlyeq \iota w_1 \max\{\beta_{11}x, \dots, \beta_{1\nu_1}x\} + \cdots + \iota w_n \max\{\beta_{n1}x, \dots, \beta_{n\nu_n}x\}$$

if and only if

$$\begin{aligned}&\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\\ &\exists \boldsymbol{v}_1 \in V^{\nu_1},\ \boldsymbol{v}_1^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\ \iota\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{e} = w_1,\ \ \dots,\ \ \exists \boldsymbol{v}_n \in V^{\nu_n},\ \boldsymbol{v}_n^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\ \iota\boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{e} = w_n:\quad \gamma = \iota\boldsymbol{u}^{\mathrm{T}}A + \iota\boldsymbol{v}_1^{\mathrm{T}}B_1 + \cdots + \iota\boldsymbol{v}_n^{\mathrm{T}}B_n\,.\end{aligned}$$

Proof. The implication

$$Ax \leq \mathbf{0} \ \Longrightarrow\ \gamma x \preccurlyeq \iota w_1 \max\{\beta_{11}x, \dots, \beta_{1\nu_1}x\} + \cdots + \iota w_n \max\{\beta_{n1}x, \dots, \beta_{n\nu_n}x\}$$

holds for every $x \in W$ if and only if the implication

$$Ax \leq \mathbf{0}\ \wedge\ B_1x \leq \boldsymbol{e}y_1\ \wedge \cdots \wedge\ B_nx \leq \boldsymbol{e}y_n \ \Longrightarrow\ \gamma x \preccurlyeq \iota w_1 y_1 + \cdots + \iota w_n y_n$$

holds for every $x \in W$ and for every $y_1, \dots, y_n \in F$, which equivalently means that

$$\forall \begin{pmatrix} x \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in W \dot{\times} F^n:\ \ \left\{\begin{matrix} Ax & & & \leq 0 \\ B_1x + \boldsymbol{e}y_1 & & & \leq 0 \\ \vdots & \ddots & & \vdots \\ B_nx & & + \boldsymbol{e}y_n & \leq 0 \end{matrix}\right\} \ \Longrightarrow\ \gamma x + \iota w_1 y_1 + \cdots + \iota w_n y_n \preccurlyeq 0\,.$$

By Farkas' Lemma 1, with the vector space W replaced by $W \dot{\times} F^n$, it equivalently holds that

$$\exists \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix} \in V^m \dot{\times} V^{\nu_1} \dot{\times} \cdots \dot{\times} V^{\nu_n},\ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix}^{\mathrm{T}} \succcurlyeq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}^{\mathrm{T}}:\quad \iota \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} A & & & \\ B_1 & \boldsymbol{e}\iota & & \\ \vdots & & \ddots & \\ B_n & & & \boldsymbol{e}\iota \end{pmatrix} = (\gamma \quad \iota w_1 \quad \dots \quad \iota w_n)\,.$$

It is thus equivalent to say that there exist a non-negative $\boldsymbol{u} \in V^m$ and a non-negative $\boldsymbol{v}_1 \in V^{\nu_1}$, …, a non-negative $\boldsymbol{v}_n \in V^{\nu_n}$ such that $\iota\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{e}\iota = \iota w_1$, …, $\iota\boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{e}\iota = \iota w_n$, which equivalently means that $\iota\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{e} = w_1$, …, $\iota\boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{e} = w_n$, and also $\gamma = \iota\boldsymbol{u}^{\mathrm{T}}A + \iota\boldsymbol{v}_1^{\mathrm{T}}B_1 + \cdots + \iota\boldsymbol{v}_n^{\mathrm{T}}B_n$, which concludes the proof. ∎

4 Special cases of the universal theorem of the alternative

We discuss some special cases of the universal theorem of the alternative (Theorem 2) in this section.

4.1 Farkas' Lemma

In the previous section, we used Farkas' Lemma 1 to prove Theorem 2. At the same time, Farkas' Lemma 1 is also a special case of Theorem 2 when $n = 0$. In particular, if $n = 0$, then the empty sum $\iota w_1 \max\{\beta_{11}x, \dots, \beta_{1v_1}x\} + \cdots + \iota w_n \max\{\beta_{n1}x, \dots, \beta_{nv_n}x\} = 0$, the zero of the vector space V, by convention. The remaining terms '$\exists \boldsymbol{v}_j \in V^{\nu_j}$, $\boldsymbol{v}_j^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}}$, $\iota \boldsymbol{v}_j^{\mathrm{T}} \boldsymbol{e} = w_j$' and '$\iota \boldsymbol{v}_j^{\mathrm{T}} B_j$' vanish in Theorem 2 if $n = 0$, whence the conclusion is easy to see.

4.2 Dax's new theorem of the alternative

The version of Dax's new theorem of the alternative which was given in the Introduction can be found in Dax (1993, Sections 5.1 and 5.4). A generalized version can be found in Dax (1990). Here, we obtain the following generalization of Dax's new theorem of the alternative, where m and n are non-negative natural numbers, as a special case of Theorem 2.

Theorem 3 (Dax's new theorem of the alternative)**.** *Let F be a linearly ordered (commutative or skew) field, let W be a vector space over the field F, and let V be a linearly ordered vector space over the linearly ordered field F. Let $A: W \to F^m$ be a linear mapping, let $B_1: W \to F$, …, $B_n: W \to F$ be linear forms, let $w_1, \dots, w_n \in V$ be non-negative weights, and let $\gamma: W \to V$ be a linear mapping. It then holds*

$$\forall x \in W: \quad Ax \leq \mathbf{0} \implies \gamma x \preccurlyeq \iota w_1 |B_1 x| + \cdots + \iota w_n |B_n x|$$

if and only if

$$\begin{aligned} &\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}}, \\ &\exists v_1 \in V,\ -w_1 \preccurlyeq v_1 \preccurlyeq w_1,\ \dots,\ \exists v_n \in V,\ -w_n \preccurlyeq v_n \preccurlyeq w_n: \quad \gamma = \iota \boldsymbol{u}^{\mathrm{T}} A + \iota v_1 B_1 + \cdots + \iota v_n B_n\,. \end{aligned}$$

Proof. In Theorem 2, consider $\nu_1 = \cdots = \nu_n = 2$ with $\beta_{11} = B_1, \beta_{12} = -B_1, \dots, \beta_{n1} = B_n, \beta_{n2} = -B_n$. Observe that the absolute value $|B_j x| = \max\{B_j x, -B_j x\} = \max\{\beta_{j1}x, \beta_{j2}x\}$ for every $x \in W$ for $j = 1, \dots, n$. Moreover, we have for any $v_j \in V$ that $-w_j \preccurlyeq v_j \preccurlyeq w_j$ if and only if $v_j = v_{j1} - v_{j2}$ for some non-negative $v_{j1}, v_{j2} \in V$ such that $v_{j1} + v_{j2} = w_j$ for $j = 1, \dots, n$. Finally, notice that $\iota v_j B_j = \iota(v_{j1} - v_{j2}) B_j = \iota v_{j1} \beta_{j1} + \iota v_{j2} \beta_{j2}$ for $j = 1, \dots, n$. The equivalence is easy to see now. ■

The original version due to Dax (1993, Sections 5.1 and 5.4), presented in the Introduction, is obtained by considering the field $F := \mathbb{R}$ of the real numbers, the finite-dimensional vector space $W := \mathbb{R}^N$, the one-dimensional real line $V := \mathbb{R}^1$, and by letting $m := 0$.

4.3 Rohn's residual existence theorem for linear equations

Rohn's residual existence theorem for linear equations (Rohn, 2010, Theorem 2), which was presented in the Introduction, says in words that the system of linear equations $\boldsymbol{c}^{\mathrm{T}} = \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{Z}$ has a solution $\boldsymbol{\xi}^{\mathrm{T}} \in \mathbb{R}^{1\times p}$ in the convex hull of the set $\{\boldsymbol{b}_1^{\mathrm{T}}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}}\}$ if and only if $\boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \leq \max\{\boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{Z} \boldsymbol{x}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}} \boldsymbol{Z} \boldsymbol{x}\}$ for every $\boldsymbol{x} \in \mathbb{R}^N$. We obtain the following generalization of Rohn's residual existence theorem for linear equations, where m and ν is a non-negative and positive, respectively, natural number, as a special case of Theorem 2.

Theorem 4 (Rohn's residual existence theorem for linear equations)**.** *Let F be a linearly ordered (commutative or skew) field, let W be a vector space over the field F, and let V be a linearly ordered vector space over the linearly ordered field F. Let $A: W \to F^m$ be a linear mapping, let $\beta_1, \dots, \beta_\nu: W \to F$ be linear forms, let $w \in V$ be a non-negative weight, and let $\gamma: W \to V$ be a linear mapping. It then holds*

$$\forall x \in W: \quad Ax \leq \mathbf{0} \implies \gamma x \preccurlyeq \iota w \max\{\beta_1 x, \dots, \beta_\nu x\}$$

if and only if

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\ \exists v_1, \dots, v_\nu \in V,\ v_1, \dots, v_\nu \succcurlyeq 0,\ v_1 + \dots + v_\nu = w: \quad \gamma = \iota \boldsymbol{u}^{\mathrm{T}} A + \iota v_1 \beta_1 + \dots + \iota v_n \beta_n\,.$$

Proof. Consider Theorem 2 with $n = 1$. ■

The original version of Rohn's residual existence theorem for linear equations (Rohn, 2010, Theorem 2), presented in the Introduction, is obtained by considering the field $F \coloneqq \mathbb{R}$ of the real numbers, the finite-dimensional vector space $W \coloneqq \mathbb{R}^N$, the one-dimensional real line $V \coloneqq \mathbb{R}^1$, by letting $m \coloneqq 0$ and $n \coloneqq 1$, and by taking the weight $w \coloneqq 1$. Then, given the row vector $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$, the matrix $\boldsymbol{Z} \in \mathbb{R}^{p\times N}$, and the row vectors $\boldsymbol{b}_1^{\mathrm{T}}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}} \in \mathbb{R}^{1\times p}$, consider the mapping $\gamma: \mathbb{R}^N \to \mathbb{R}^1$ and the linear forms $\beta_1, \dots, \beta_\nu: \mathbb{R}^N \to \mathbb{R}$ defined by $\gamma: \boldsymbol{x} \mapsto \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ and $\beta_k: \boldsymbol{x} \mapsto \boldsymbol{b}_k^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x}$ for $k = 1, \dots, \nu$, respectively, for every $\boldsymbol{x} \in \mathbb{R}^N$.

More generally, given the row vector $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^{1\times N}$, the matrix $\boldsymbol{Z} \in \mathbb{R}^{p\times N}$, and the set $\{\boldsymbol{b}_1^{\mathrm{T}}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}}\} \subseteq \mathbb{R}^{1\times p}$, consider yet a set $\{\boldsymbol{a}_1^{\mathrm{T}}, \dots, \boldsymbol{a}_m^{\mathrm{T}}\} \subseteq \mathbb{R}^{1\times p}$. Then, by Theorem 4, the system of linear equations $\boldsymbol{c}^{\mathrm{T}} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{Z}$ has a solution $\boldsymbol{\xi}^{\mathrm{T}} \in \mathbb{R}^{1\times p}$ of the form $\boldsymbol{\xi}^{\mathrm{T}} = u_1\boldsymbol{a}_1^{\mathrm{T}} + \dots + u_m\boldsymbol{a}_m^{\mathrm{T}} + {} + v_1\boldsymbol{b}_1^{\mathrm{T}} + \dots + v_\nu\boldsymbol{b}_\nu^{\mathrm{T}}$ for some $u_1, \dots, u_m \geq 0$ and for some $v_1, \dots, v_\nu \geq 0$ such that $v_1 + \dots + {} + v_\nu = 1$, that is in the Minkowski sum (the convex hull of $\{\boldsymbol{b}_1^{\mathrm{T}}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}}\}$ plus the convex conical hull of $\{\boldsymbol{a}_1^{\mathrm{T}}, \dots, \boldsymbol{a}_m^{\mathrm{T}}\}$) if and only if $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x}, \dots, \boldsymbol{a}_m^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x} \leq 0$ implies $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \max\{\boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x}, \dots, \boldsymbol{b}_\nu^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{x}\}$ for every $\boldsymbol{x} \in \mathbb{R}^N$. See Bartl (2012b, Theorem 5) for further generalization in this direction.

4.4 Motzkin's Theorem of the alternative

Motzkin's Theorem of the alternative (Motzkin, 1934; Motzkin, 1952, Theorem D6 [in Chap. III, § 13, par. 73], p. 60; cf. Tucker, 1956, Corollary 2A part (i)) is used to establish optimality conditions in non-linear optimization, see, e.g., Mangasarian (1994), Birbil et al. (2007). Notice that, depending upon the approach, Farkas' Lemma can be used to establish optimality conditions in non-linear optimization directly, see, e.g., Franklin (2002). Below, we obtain the following generalization of Motzkin's Theorem of the alternative, where m and ν is a non-negative and positive, respectively, natural number and 0 denotes the zero vector of the space V, as a special case of Theorem 2; it is a special case of Theorem 4 actually.

Theorem 5 (Motzkin's Theorem of the alternative)**.** *Let F be a linearly ordered (commutative or skew) field, let W be a vector space over the field F, and let V be a linearly ordered vector space over the linearly ordered field F. Let $A: W \to F^m$ be a linear mapping, let $\beta_1, \dots, \beta_\nu: W \to F$ be linear forms, and let $w \in V$ be a non-negative weight. It then holds*

$$\nexists x \in W: \quad Ax \leq \mathbf{0} \wedge \iota w \max\{\beta_1 x, \dots, \beta_\nu x\} \prec 0$$

if and only if

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\ \exists v_1, \dots, v_\nu \in V,\ v_1, \dots, v_\nu \succcurlyeq 0,\ v_1 + \dots + v_\nu = w: \quad \iota \boldsymbol{u}^{\mathrm{T}} A + \iota v_1 \beta_1 + \dots + \iota v_n \beta_n = o\,.$$

Proof. There is no $x \in W$ such that $Ax \leq 0$ and $\iota w \max\{\beta_1 x, \dots, \beta_\nu x\} \prec 0$ if and only if $Ax \leq 0$ implies $0 \preccurlyeq \iota w \max\{\beta_1 x, \dots, \beta_\nu x\}$ for every $x \in W$. Now, conclude the proof by considering Theorem 4 with $\gamma = o$, the zero linear mapping $o: W \to V$. ■

Remark 1. It holds $\max\{\beta_1 x, \dots, \beta_\nu x\} < 0$ if and only if $Bx < \mathbf{0}$, cf. Motzkin's Theorem in the Introduction.

By identifying the vector space V with the one-dimensional line F^1, i.e. by taking $V \coloneqq F^1$, and by considering the weight $w \coloneqq 1$ and also Remark 1, we obtain the generalization of Motzkin's Theorem due to Bartl (2007, Theorem 5.1).

To obtain the original formulation of Motzkin's Theorem (Motzkin, 1934; Motzkin, 1952, Theorem D6, p. 60; see Birbil et al., 2007, Lemma 2.2, for another equivalent formulation), presented in the Introduction, consider the field $F := \mathbb{R}$ of the real numbers, the finite-dimensional vector space $W := \mathbb{R}^N$, the one-dimensional real line $V := \mathbb{R}^1$, the weight $w := 1$, and Remark 1.

4.5 Tucker's Theorem of the alternative

Tucker's Theorem of the alternative (Tucker, 1956, Corollary 2A part (ii)) is dual to Motzkin's Theorem of the alternative; compare both theorems in the Introduction to see this. The following generalization of Tucker's Theorem of the alternative, where m and n are non-negative natural numbers and 0 denotes the zero vector of the space V, is a special case of Theorem 2.

Theorem 6 (Tucker's Theorem of the alternative). *Let F be a linearly ordered (commutative or skew) field, let W be a vector space over the field F, and let V be a linearly ordered vector space over the linearly ordered field F. Let $A: W \to F^m$ be a linear mapping, let $B: W \to F^n$ be a linear mapping consisting of the linear forms $\beta_1, \dots, \beta_n: W \to F$, and let $\boldsymbol{w} \in V^n$ be a column vector of non-negative weights $w_1, \dots, w_n \in V$. It then holds*

$$\nexists x \in W: \quad Ax \leq \mathbf{0} \ \wedge \ Bx \leq \mathbf{0} \ \wedge \ \iota w_1\beta_1 x + \cdots + \iota w_n \beta_n x \prec 0$$

if and only if

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\ \exists \boldsymbol{v} \in V^n,\ \boldsymbol{v}^{\mathrm{T}} \succcurlyeq \boldsymbol{w}^{\mathrm{T}}: \quad \iota\boldsymbol{u}^{\mathrm{T}}A + \iota\boldsymbol{v}^{\mathrm{T}}B = o\,.$$

Proof. There is no $x \in W$ such that $Ax \leq \mathbf{0}$ and $Bx \leq \mathbf{0}$ and also $\iota w_1\beta_1 x + \cdots + \iota w_n\beta_n x \prec 0$ if and only if

$$\forall x \in W: \quad \binom{A}{B} x \leq \binom{\mathbf{0}}{\mathbf{0}} \ \Rightarrow \ 0 \preccurlyeq \iota w_1\beta_1 x + \cdots + \iota w_n \beta_n x\,.$$

By Theorem 2, with $v_1 = \cdots = v_n = 1$ and $\gamma = o$, the zero linear mapping $o: W \to V$, and also with the linear mapping $A: W \to F^m$ replaced with the linear mapping $\binom{A}{B}: W \to F^m \dot{\times} F^n$, it equivalently holds that

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}},\ \exists \bar{\boldsymbol{v}} \in V^n,\ \bar{\boldsymbol{v}}^{\mathrm{T}} \succcurlyeq \mathbf{0}^{\mathrm{T}}: \quad \iota\boldsymbol{u}^{\mathrm{T}}A + \iota\bar{\boldsymbol{v}}^{\mathrm{T}}B + \iota w_1\beta_1 + \cdots + \iota w_n\beta_n = o\,.$$

We have $\iota\bar{\boldsymbol{v}}^{\mathrm{T}}B + \iota w_1\beta_1 + \cdots + \iota w_n\beta_n = \iota\bar{\boldsymbol{v}}^{\mathrm{T}}B + \iota\boldsymbol{w}^{\mathrm{T}}B = \iota(\bar{\boldsymbol{v}} + \boldsymbol{w})^{\mathrm{T}}B$. By considering $\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{w}$, it is equivalent to say that $\iota\boldsymbol{u}^{\mathrm{T}}A + \iota\boldsymbol{v}^{\mathrm{T}}B = o$ for some non-negative $\boldsymbol{u} \in V^m$ and for some $\boldsymbol{v} \in V^n$ such that $\boldsymbol{v}^{\mathrm{T}} \succcurlyeq \boldsymbol{w}^{\mathrm{T}}$, which means we are done. ■

Remark 2. If the weights $\boldsymbol{w} \in V^n$ are positive, then $\boldsymbol{v}^{\mathrm{T}} \succcurlyeq \boldsymbol{w}^{\mathrm{T}} \succ \mathbf{0}^{\mathrm{T}}$, cf. Tucker's Theorem in the Introduction.

By identifying the vector space V with the one-dimensional line F^1, i.e. by taking $V := F^1$, and by considering the weights $w_1 := \cdots := w_n := 1$, we obtain the generalization of Tucker's Theorem due to Bartl (2007, Theorem 5.2).

To obtain the original formulation of Tucker's Theorem (Tucker, 1956, Corollary 2A part (ii)), presented in the Introduction, consider the field $F := \mathbb{R}$ of the real numbers, the finite-dimensional vector space $W := \mathbb{R}^N$, the one-dimensional real line $V := \mathbb{R}^1$, and the weights $w_1 := \cdots := w_n := 1$.

5 Concluding remarks

We presented a new universal theorem of the alternative (Theorem 2). We proved this result by using Farkas' Lemma 1. We then showed that many other theorems of the alternative (Farkas' Lemma itself, Dax's new theorem of the alternative, Rohn's residual existence theorem for linear equations, Motzkin's Theorem of the alternative and Tucker's Theorem of the alternative) are special cases of the universal theorem of the alternative (Theorem 2).

Fan (1956) and Chernikov (1968) also considered (finite) systems of linear inequalities and theorems of the alternative from the algebraic point of view, i.e. in a vector space of arbitrary dimension. Fan (1956) studies linear inequalities in a vector space over the field of real numbers $\mathbb{R}$ and also over the field of the complex numbers $\mathbb{C}$. Chernikov (1968) presents a theory of linear inequalities in a vector space over a linearly ordered commutative field F. In this paper, we consider any linearly ordered (commutative or skew) field F, a vector space W over the field F, and a linearly ordered vector space V over the linearly ordered field F.

Let the linear mappings $B_1: W \to F^{\nu_1}$, …, $B_n: W \to F^{\nu_n}$ and the non-negative weights $w_1, \dots, w_n \in V$ be as in Theorem 2. Notice that the mapping $p: W \to V$ defined by

$$p: x \longmapsto \iota w_1 \max\{\beta_{11}x, \dots, \beta_{1\nu_1}x\} + \dots + \iota w_n \max\{\beta_{n1}x, \dots, \beta_{n\nu_n}x\}$$

is sublinear, that is $p(\lambda x) = \lambda p(x)$ for all positive $\lambda \in F$ and for all $x \in W$ and also $p(x + y) \preccurlyeq \preccurlyeq p(x) + p(y)$ for all $x, y \in W$. It seems that Theorem 2 can be used to obtain optimality conditions for some optimization problems with a non-smooth objective function of a special form.

By using a discrete variant of Farkas' Lemma (Bartl, 2020), we could achieve analogous results in the discrete setting of a module over a non-trivial linearly ordered commutative ring.

Finally, by using an extended variant of Farkas' Lemma, which allows the system '$Ax \le \mathbf{0}$' to be infinite, we could generalize the results even further.

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