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On backward attractors of interval maps[*]

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Abstract

Special α-limit sets ($s\alpha$-limit sets) combine together all accumulation points of all backward orbit branches of a point x under a noninvertible map. The most important question about them is whether or not they are closed. We challenge the notion of $s\alpha$-limit sets as backward attractors for interval maps by showing that they need not be closed. This disproves a conjecture by Kolyada, Misiurewicz, and Snoha. We give a criterion in terms of Xiong's attracting centre that completely characterizes which interval maps have all $s\alpha$-limit sets closed, and we show that our criterion is satisfied in the piecewise monotone case. We apply Blokh's models of solenoidal and basic ω-limit sets to solve four additional conjectures by Kolyada, Misiurewicz, and Snoha relating topological properties of $s\alpha$-limit sets to the dynamics within them. For example, we show that the isolated points in a $s\alpha$-limit set of an interval map are always periodic, the non-degenerate components are the union of one or two transitive cycles of intervals, and the rest of the $s\alpha$-limit set is nowhere dense. Moreover, we show that $s\alpha$-limit sets in the interval are always both F_σ and G_δ. Finally, since $s\alpha$-limit sets need not be closed, we propose a new notion of β-limit sets to serve as backward attractors. The β-limit set of x is the smallest closed set to

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which all backward orbit branches of x converge, and it coincides with the closure of the $s\alpha$-limit set. At the end of the paper we suggest several new problems about backward attractors.

1. Introduction

Let a *discrete dynamical system* be defined as an ordered pair (X, f) where X is a compact metric space and f is a continuous map acting on X. To understand the dynamical properties of such a system it is necessary to analyse the behaviour of the trajectories of any point $x \in X$ under the iteration of f. Limit sets of trajectories are a helpful tool for this purpose since they can be used to understand the long term behaviour of the dynamical system.

The ω-*limit sets* ($\omega(x)$ for short), i.e. the sets of limit points of forward trajectories, were deeply studied by many authors. For instance, one can ask for a criterion which determines whether a given closed invariant subset of X is an ω-limit set of some point $x \in X$. The question is very hard in general, however the answer for ω-limit sets of a continuous map acting on the compact interval was provided by Blokh *et al* in [5]. A closely related question is that of characterizing all those dynamical systems which may occur as restrictions of some system to one of its ω-limit sets. These abstract ω-limit sets were studied by Bowen [6] and Dowker and Frielander [12]. It was also proved that each ω-limit set of a continuous map of the interval is contained in a maximal one by Sharkovsky [24].

Backward limit sets were introduced as a dual concept to ω-limit sets in order to capture the 'source' of the trajectory of a point. For invertible systems they are defined quite simply: one just reverses the direction of time and considers the ω-limit sets of the inverse system. These so-called α-limit sets are very important in the study of flows, where they are used to define unstable manifolds, homoclinic and heteroclinic trajectories, and the Morse decompositions at the heart of Conley index theory [7, 14]. But when we study continuous mappings $f : X \to X$ (not necessarily invertible), a point x may have many preimages (or none at all), and we must clarify what kind of backward limit set we wish to speak of. Several definitions have been proposed including conical limit sets, α-limit sets, branch α-limit sets, special α-limit sets, and others [1, 7, 11, 15, 16, 20].

One of the most classic applications of *forward limit sets* in dynamics is due to Birkhoff. There are many notions of recurrence in topological dynamics (such as periodicity, non-wandering behaviour, chain-recurrence, etc), but the term *recurrent point* has been reserved for those points x which belong to their own ω-limit sets. Birkhoff showed that these points can be used to identify the *Birkhoff centre* (Birkhoff called it the 'set of central motions') of a topological dynamical system (X, f), which is obtained by restricting f to its non-wandering set, then restricting that system to its non-wandering set, and so on through transfinite induction (taking intersections at limit ordinals) until reaching some countable ordinal (the 'depth') at which the sequence stabilizes. Birkhoff's result is that the centre of the system obtained in this way is the same as the closure of the set of recurrent points [2].

In light of Birkhoff's work, one can ask the analogous question, what is the significance of a point belonging to its own *backward limit set*? If we consider homeomorphisms than Birkhoff's results already apply (just using the inverse map), so if we wish to get something new we must consider general continuous mappings $f : X \to X$. In one-dimensional dynamics

some very good answers to this question have appeared for two kinds of backward limit sets, namely, the α-limit set and the special α-limit set. Coven and Nitecki showed that a point x is non-wandering for a continuous interval map $f : [0, 1] \to [0, 1]$ if and only if x belongs to its own α-limit set [9]. But there is a deeper result related to the *attracting centre* of an interval map $f : [0, 1] \to [0, 1]$, defined by Xiong as the set of all points x such that x is in the ω-limit set of some point x_1, which itself is in the ω-limit set of some point x_2, and so on for some infinite sequence $\{x_i\}_{i=1}^{\infty}$ of points in the interval [29]. Xiong showed that the attracting centre is a subset of the Birkhoff centre (they can coincide) and that if x_1, x_2 can be found as above, then x is already in the attracting centre (so the 'depth' here is at most 2) [29]. The connection to backward limit sets was made in 1992 by Hero, who defined special α-limit sets ($s\alpha$-limit sets, for short) and showed that a point x belongs to the attracting centre of a continuous interval map if and only if it belongs to its own $s\alpha$-limit set [15]. Partial generalizations of this result to graph maps and some dendrite maps have appeared since then [26–28].

We now recall Hero's definition of a special α-limit set. A *backward orbit branch* of a point x is any sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for all i. The corresponding *branch α-limit set* is defined as the set of all limits of convergent subsequences x_{i_j} (analogously as ω-limit sets are defined from forward trajectories). Then the *special α-limit set* of a point x, denoted $s\alpha(x)$, is defined as the union of all branch α-limit sets over all backward orbit branches of x [15].

Several studies in recent years have focused on branch α-limit sets, and in light of the definition it is easy to deduce corresponding properties for $s\alpha$-limit sets, for example:

- Branch α-limit sets are always closed and strictly invariant, and therefore each $s\alpha$-limit set is strictly invariant (i.e. $f(s\alpha(x)) = s\alpha(x)$) and contains the orbit closure of each of its points [18].
- Branch α-limit sets are always internally chain transitive [16], and therefore $s\alpha$-limit sets are internally chain recurrent (but not necessarily chain transitive, see example 9 below).
- All the recurrent points of a system are contained in the union of its $s\alpha$-limit sets, since this property holds for the union of the branch α-limit sets [13].
- For interval maps $f : [0, 1] \to [0, 1]$, each branch α-limit set is locally expanding and hence coincides with an ω-limit set of the same map f [1]. It follows that each $s\alpha$-limit set of an interval map f is a union of some of its ω-limit sets.

Notably lacking in the list above are purely topological properties. For example, it seems that Hero did not consider the basic question whether all $s\alpha$-limit sets are closed. Outside the realm of one-dimensional dynamics the situation is even more complicated. It has been shown that $s\alpha$-limit sets are always analytic, but not necessarily closed or even Borel [17, 18]. Therefore it seems prudent to study more closely the properties of $s\alpha$-limit sets in one-dimensional dynamics, and especially in the most important one-dimensional space where Hero's work began, the unit interval.

Kolyada, Misiurewicz, and Snoha began this study as a systematic programme in [18]. They investigated special α-limit sets of interval maps and proved that for interval maps with a closed set of periodic points, every special α-limit set has to be closed. This result led to the following conjecture:

Conjecture 1   [18]. For all continuous maps of the unit interval all special α-limit sets are closed.

We disprove the conjecture in theorem 45 by showing a counterexample of a continuous interval map with a special α-limit set which is not closed and give the properties of continuous interval maps that determine if all special α-limit sets are closed in theorem 41. In corollaries

42–44 we identify three classes of continuous interval maps for which all $s\alpha$-limit sets are closed, namely, piecewise monotone maps, zero entropy maps with a closed set of recurrent points and maps which are not Li–Yorke chaotic. On the other hand, we show that for all continuous maps of the unit interval all special α-limit sets are both F_σ and G_δ in theorem 40. We give further topological properties of special α-limit sets of interval maps. If $s\alpha(x)$ is not closed, then it is uncountable and nowhere dense by theorem 39. If $s\alpha(x)$ is closed, then it is the union of a nowhere dense set and finitely many (perhaps zero) closed intervals by theorem 24, and in section 4.2 we prove some amount of transitivity of f on those intervals. Since $s\alpha$-limit sets need not be closed, we propose a new notion of β-limit sets to serve as backward attractors in definition 49. The β-limit set of x is the smallest closed set to which all backward orbit branches of x converge, and it coincides with the closure of the $s\alpha$-limit set.

Kolyada *et al* also made the following conjecture.

Conjecture 2 [18]. The isolated points in a special α-limit set for a continuous interval map are always periodic.

We verify this conjecture in theorem 21. We also show that a countable special α-limit set for an interval map is a union of periodic orbits. These results are opposite to the case of ω-limit sets. The ω-limit sets of a general dynamical system do not posses any periodic isolated points unless $\omega(x)$ is a single periodic orbit [23].

The authors of [18] also investigated the properties of special α-limit sets of *transitive* interval maps and stated the following conjecture:

Conjecture 3 [18]. Let $f : [0, 1] \to [0, 1]$ be a continuous map and $x, y \in [0, 1]$.

- If $x \neq y$ and $s\alpha(x) = s\alpha(y) = [0, 1]$, then f is transitive.
- If $s\alpha(x) = [0, 1]$ then either f is transitive or there is $c \in (0, 1)$ such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are transitive.

We show in theorem 26 that f is transitive if there are three distinct points $x, y, z \in [0, 1]$ with $s\alpha(x) = s\alpha(y) = s\alpha(z) = [0, 1]$. If f has one or two points with special α-limit sets equal to $[0, 1]$, but not more, then $[0, 1]$ is the union of two transitive cycles of intervals by corollary 27.

It is known that if two ω-limit sets of an interval map contain a common open set, then they are equal. The last conjecture in [18] suggested that a similar property holds for special α-limit sets:

Conjecture 4 [18]. Let f be a continuous map $f : [0, 1] \to [0, 1]$ and $x, y \in [0, 1]$. If $\text{Int}(s\alpha(x) \cap s\alpha(y)) \neq \emptyset$ then $s\alpha(x) = s\alpha(y)$.

We correct this conjecture by showing that at most three distinct special α-limit sets of f can contain a given nonempty open set in corollary 29.

One additional motivation for studying $s\alpha$-limit sets is that they provide more information about α-limit sets (see section 2 for the definition), since there is always the containment $sa(x) \subset \alpha(x)$. For transitive interval maps this containment is in fact an equality $s\alpha(x) = \alpha(x)$ for all $x \in [0, 1]$ (this can be deduced from [18, proposition 3.10] or theorem 33). The question then arises whether this is the typical situation, or perhaps typically $s\alpha(x) = \alpha(x)$ at least for 'most' points x. We show in section 4.5 that for the generic interval map $f : [0, 1] \to [0, 1]$ (in the topology of uniform convergence) there is a whole interval of points $x \in [0, 1]$ for which $\alpha(x, f) \neq s\alpha(x, f)$.

To summarize, the key properties of limit sets as they apply to continuous maps of the interval are as follows:

$$x \text{ is recurrent} \iff x \in \omega(x),$$

$$x \text{ is nonwandering} \iff x \in \alpha(x),$$
$$x \text{ is in the attracting centre} \iff x \in s\alpha(x).$$

We conjecture additionally in this paper (see conjecture 53) that

$$x \text{ is in the Birkhoff centre} \iff x \in \overline{s\alpha(x)}.$$

The paper is organised as follows. Sections 1 and 2 are introductory. Section 3 investigates the relation of maximal ω-limit sets to special α-limit sets and provides tools necessary for proving the main results. It also contains a simple example showing that, unlike ω-limit sets, the special α-limit sets of an interval map need not be contained in maximal ones. Section 4 is devoted to the above mentioned results on various properties of special α-limit sets of interval maps. Section 5 studies properties of special α-limit sets which are not closed. The paper closes with open problems and related questions in section 6.

2. Terminology

Let X be a compact metric space and $f : X \to X$ a continuous map. A sequence $\{x_n\}_{n=0}^{\infty}$ is called

- The *forward orbit* of a point x if $f^n(x) = x_n$ for all $n \geqslant 0$,
- A *preimage sequence* of a point x if $f^n(x_n) = x$ for all $n \geqslant 0$,
- A *backward orbit branch* of a point x if $x_0 = x$ and $f(x_{n+1}) = x_n$ for all $n \geqslant 0$.

A point y belongs to the *ω-limit set of a point x*, denoted by $\omega(x)$, if and only if the forward orbit of x has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \to y$. A point y belongs to the *α-limit set of a point x*, denoted by $\alpha(x)$, if and only if some preimage sequence of x has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \to y$. And a point y belongs to the *special α-limit set of a point x*, also written as the *$s\alpha$-limit set* and denoted $s\alpha(x)$, if and only if some backward orbit branch of x has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \to y$. If we wish to emphasize the map, we will write $\omega(x, f)$, $\alpha(x, f)$ and $s\alpha(x, f)$.

To summarize, the ω, α, and $s\alpha$-limit sets of a point x are defined as follows. The set $\omega(x)$ is the set of all accumulation points of its forward orbit and $\alpha(x)$ (resp. $s\alpha(x)$) is the set of all accumulation points of all its preimage sequences (resp. of all its backward orbit branches).

Let $T : X \to X$ and $F : Y \to Y$ be continuous maps of compact metric spaces. If there is a surjective map $\phi : X \to Y$ such that $\phi \circ T = F \circ \phi$ then it is said that ϕ semiconjugates T to F and ϕ is a *semiconjugacy*.

Let $f : X \to X$ be a continuous map. A set $A \subset [0, 1]$ is *invariant* if $f(A) \subset A$. The forward orbit of a point, regarded as a subset of X rather than a sequence, will be denoted by $\mathrm{Orb}(x) = \{f^n(x) : n \geqslant 0\}$. The *forward orbit of a set* is $\mathrm{Orb}(A) = \bigcup \{f^n(A) : n \geqslant 0\}$. We call f *transitive* if for any two nonempty open subsets $U, V \subset X$ there is $n \geqslant 0$ such that $f^n(U) \cap V \neq \emptyset$. We call f *topologically mixing* if for any two nonempty open subsets $U, V \subset X$ there is an integer $N \geqslant 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geqslant N$.

Now let $f : [0, 1] \to [0, 1]$ be an interval map. We say that a point x is *periodic* if $f^n(x) = x$ for some $n \geqslant 1$, x is *recurrent* if $x \in \omega(x)$ and x is *non-wandering* if for every neighbourhood U of x there is $m \geqslant 1$ such that $f^m(U) \cap U \neq \emptyset$. We write $\mathrm{Per}(f)$, $\mathrm{Rec}(f)$, and $\Omega(f)$ for the sets of periodic points, recurrent points, and non-wandering points of f, respectively. We say that x is *preperiodic* if $x \notin \mathrm{Per}(f)$ but $f^n(x) \in \mathrm{Per}(f)$ for some $n \geqslant 1$. We write $\Lambda^1(f) = \bigcup_{x \in [0,1]} \omega(x)$ for the union of all ω-limit sets of f and $\mathrm{SA}(f) = \bigcup_{x \in [0,1]} s\alpha(x)$ for the union of all $s\alpha$-limit sets. Following [29] we define the *attracting centre* of f as $\Lambda^2(f) = \bigcup_{x \in \Lambda^1(f)} \omega(x)$. The *Birkhoff centre* of f is the closure of the set of recurrent points $\overline{\mathrm{Rec}(f)}$ and coincides with $\overline{\mathrm{Per}(f)}$ [8].

If the map f is clear from the context, we may drop it from the notation. The relation of these sets is given by the following summary theorem from the works of Hero and Xiong [15, 29].

Theorem 5 [15, 29]. *For any continuous interval map $f : [0, 1] \to [0, 1]$, we have*

$$\text{Per} \subset \text{Rec} \subset \Lambda^2 = \text{SA} \subset \overline{\text{Rec}} \subset \Lambda^1 \subset \Omega.$$

If $K \subset [0, 1]$ is a non-degenerate closed interval such that the sets $K, f(K), \ldots, f^{k-1}(K)$ are pairwise disjoint and $f^k(K) = K$, then we call the set $M = \text{Orb}(K)$ a *cycle of intervals* and the *period* of this cycle is k. We may also call K an *n-periodic interval*. If $f|_M$ is transitive then we call M a *transitive cycle* for f.

3. Maximal ω-limit sets and their relation to special α-limit sets

An important property of the ω-limit sets of an interval map f is that each ω-limit set is contained in a maximal one. These maximal ω-limit sets come in three types: periodic orbits, basic sets, and solenoidal ω-limit sets.

A *solenoidal ω-limit set* is a maximal ω-limit set which contains no periodic points. Any solenoidal ω-limit set is uncountable and is contained in a nested sequence $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ of cycles of intervals with periods tending to infinity, also known as a *generating sequence* [4, assertion 4.2]. Here is the theorem relating $s\alpha$-limit sets to solenoidal ω-limit sets; the proof is given in section 3.1.

Theorem 6 (Solenoidal sets).   *Let* $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ *be a nested sequence of cycles of intervals for the continuous interval map f with periods tending to infinity. Let* $Q = \bigcap \text{Orb}(I_n)$ *and* $S = Q \cap \text{Rec}(f)$.

(a) *If $\alpha(y) \cap Q \neq \emptyset$, then $y \in Q$.*
(b) *If $y \in Q$, then $s\alpha(y) \supset S$ and $s\alpha(y) \cap Q = S$.*

A *basic set* is an ω-limit set which is infinite, maximal among ω-limit sets, and contains some periodic point. If B is a basic set then with respect to inclusion there is a minimal cycle of intervals M which contains it, and B may be characterized as the set of those points $x \in M$ such that $\overline{\text{Orb}(U)} = M$ for every relative neighbourhood U of x in M, see [4]. Conversely, if M is a cycle of intervals for f, then we will write

$$B(M, f) = \{x \in M : \text{for any relative neighbourhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\text{Orb}(U)} = M\},$$

and if this set is infinite, then it is a basic set [4]. Here is the theorem relating $s\alpha$-limit sets to basic sets; the proof is given in section 3.2.

Theorem 7 (Basic sets).   *Let f be a continuous interval map and fix $y \in [0, 1]$.*

(a) *If $\alpha(y)$ contains an infinite subset of a basic set $B = B(M, f)$, then $y \in M$ and $s\alpha(y) \supset B$.*
(b) *If $s\alpha(y)$ contains a preperiodic point x, then there is a basic set $B = B(M, f)$ such that $x \in B \subset s\alpha(y)$.*

The sharpness of the second claim of theorem 7 is illustrated in figure 1. The first map has two basic sets $B([0, 1], f)$ and $B(M, f)$, where M is the invariant middle interval. It is easy to see that set $B([0, 1], f)$ is a Cantor set and it is contained in $s\alpha(1)$. But $s\alpha(1)$ does not contain the basic set $B(M, f)$ although it includes the left endpoint of M, which is preperiodic. The second map shows that we cannot weaken the assumption to $\alpha(y)$. The α-limit set of 1 includes the preperiodic endpoint of M but $s\alpha(1)$ does not contain any basic set.
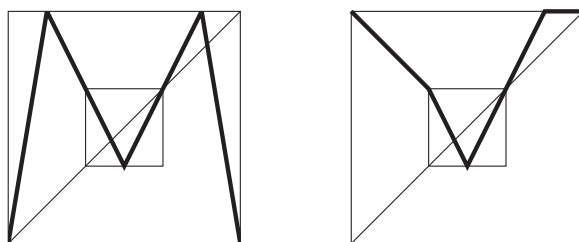
Figure 1. A map where the $s\alpha$-limit set of 1 (respectively, the α-limit set of 1) contains a preperiodic point from a basic set $B(M, f)$, but $s\alpha(1) \not\supset B(M, f)$.
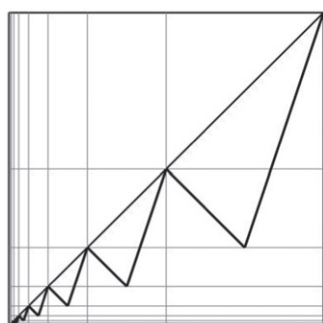

Figure 2. A map with an increasing nested sequence of $s\alpha$-limit sets not contained in any maximal one.

Periodic orbits may also be related to $s\alpha$-limit sets. The following result is one of the main theorems in [18]. Moreover, it holds for all periodic orbits of an interval map, even those which are not maximal ω-limit sets.

Theorem 8   [18, theorem 3.2]. *Let P be a periodic orbit for the continuous interval map f. If $\alpha(y) \cap P \neq \emptyset$, then $s\alpha(y) \supset P$.*

One additional observation is appropriate in this section. Unlike ω-limit sets, the $s\alpha$-limit sets of an interval map need not be contained in maximal ones.

Example 9.   Fix two sequences of real numbers $1 = a_1 > b_1 > a_2 > b_2 > \ldots$ both decreasing to 0 and consider the 'connect-the-dots' map $f : [0, 1] \to [0, 1]$ where

$$f(0) = 0, \quad f(a_i) = a_i, \quad f(b_i) = a_{i+2}, \quad (i = 1, 2, \ldots)$$

and f is linear on all the intervals $[a_{i+1}, b_i]$, $[b_i, a_i]$. The graph of such a function f is shown in figure 2. The $s\alpha$-limit sets of this map are

$$s\alpha(x) = \{a_1, \ldots, a_n\} \quad \text{for } x \in (a_{n+1}, a_n] \text{ and } s\alpha(0) = \{0\}.$$

In particular, we get a strictly increasing sequence of $s\alpha$-limit sets and no $s\alpha$-limit set containing them all.

3.1. Solenoidal sets

This section is devoted to the proof of theorem 6.

We start with a broader definition of solenoidal sets, taken from [4]. A *generating sequence* is any nested sequence of cycles of intervals $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ for f with periods tending to infinity. The intersection $Q = \bigcap_n \text{Orb}(I_n)$ is automatically closed and *strongly invariant*, i.e. $f(Q) = Q$, and any closed and strongly invariant subset S of Q (including Q itself) will be called a *solenoidal set*. Two examples described in [4] which we will need later are

(a) The set of all ω-limit points in Q, denoted $S_\omega = S_\omega(Q) = Q \cap \Lambda^1(f)$, and
(b) The set of all recurrent points in Q, denoted $S_{\text{Rec}} = S_{\text{Rec}}(Q) = Q \cap \text{Rec}(f)$.

Blokh showed that Q contains a perfect set S such that $S = \omega(x)$ for all $x \in Q$ [4, theorem 3.1]. Clearly $S = S_{\text{Rec}}$. We refer to S_{Rec} as a *minimal solenoidal set* both because it is the smallest solenoidal set in Q with respect to inclusion, and because the mapping $f|_{S_{\text{Rec}}}$ is minimal, i.e. all forward orbits are dense.

If $\omega(x)$ is a maximal ω-limit set for f and contains no periodic points (what Sharkovskii calls a maximal ω-limit set of genus 1), then it is in fact a solenoidal set [4, assertion 4.2]. Thus it has a generating sequence $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ of cycles of intervals and belongs to their intersection Q. If $Q' = \bigcap_n \text{Orb}(I'_n)$ is formed from another generating sequence for f, then it is well known (and an easy exercise) that Q and Q' are either identical or disjoint. This means that given any solenoidal set S, there is a unique maximal solenoidal set Q which contains it (so Q is uniquely determined, even if the generating sequence is not).

One can use a translation in a zero-dimensional infinite group as a model for the map f acting on a solenoidal set $Q = \bigcap \text{Orb}(I_j)$. Let $D = \{m_j\}_{j=0}^\infty$ where m_j is the period of $\text{Orb}(I_j)$ and let $H(D) = \{(r_0, r_1, \ldots) : r_{j+1} = r_j (\text{mod } m_j), \text{ for all } j \geqslant 0\}$ where r_j is an element of the group of residues modulo m_j, for every j. Denote by τ the translation in $H(D)$ by the element $(1, 1, \ldots)$.

Theorem 10 [4, theorem 3.1]. *Let* $\{I_j\}_{j=0}^\infty$ *be generating sequence with periods* $D = \{m_j\}_{j=0}^\infty$ *for a solenoidal set* $Q = \bigcap_{j \geqslant 0} \text{Orb} I_j$. *Then there exists a semiconjugacy* $\phi : Q \to H(D)$ *between* $f|_Q$ *and* τ *with the following properties:*

(a) *There exists a unique set* $S_{\text{Rec}} = Q \cap \text{Rec}(f)$ *such that* $\omega(x) = S_{\text{Rec}}$ *for every* $x \in Q$ *and* $f|_{S_{\text{Rec}}}$ *is minimal*
(b) *For every* $r \in H(D)$, *the set* $J = \phi^{-1}(r)$ *is a connected component of* Q *and it is either a singleton* $J = \{a\}, a \in S_{\text{Rec}}$, *or an interval* $J = [a, b], \emptyset \neq S_{\text{Rec}} \cap J \subset S_\omega \cap J \subset \{a, b\}$.

One lemma which we will need several times throughout the paper is the following

Lemma 11. *If A is invariant for f and* $\alpha(x) \cap \text{Int}(A) \neq \emptyset$, *then* $x \in A$. *In particular, if* $s\alpha(x) \cap \text{Int}(A) \neq \emptyset$, *then* $x \in A$.

Proof. Choose $a \in \alpha(x) \cap \text{Int}(A)$ and choose a neighbourhood U of a contained in A. There is $n \in \mathbb{N}$ and a point $x_{-n} \in U$ such that $f^n(x_{-n}) = x$. Since $U \subset A$ and A is invariant, x must belong to A. We get the same conclusion when $s\alpha(x) \cap \text{Int}(A) \neq \emptyset$, because $s\alpha(x) \subset \alpha(x)$. □

Now we are ready to give the proof of theorem 6.

Proof of theorem 6.

(a) Fix $z \in Q \cap \alpha(y)$ and let $S = S_{\text{Rec}} = Q \cap \text{Rec}(f)$ be the minimal solenoidal set in Q. Then by theorem 10 property (a) $S = \omega(z)$ and since $\alpha(y)$ is a closed invariant set it must contain S. In particular, $\alpha(y)$ contains infinitely many points from each cycle of intervals $\text{Orb}(I_n)$, and so by lemma 11 $y \in \text{Orb}(I_n)$, for all n. Therefore $y \in Q$.

(b) Fix $y \in Q$. Since $f(Q) = Q$ we can choose a backward orbit branch for y which never leaves Q. Therefore it has an accumulation point in Q, and so $s\alpha(y) \cap Q \neq \emptyset$. Let $w \in s\alpha(y) \cap Q$. By theorem 10 property (a), $\omega(w)$ is the minimal solenoidal set S. According to [15, lemma 1], if $s\alpha(y)$ contains a point, then it contains the whole ω-limit set of that point as well. Therefore $s\alpha(y) \supset S$ and $s\alpha(y) \cap Q \supset S$. To finish the proof it is enough to show the opposite inclusion $s\alpha(y) \cap Q \subset S$.

We will assume otherwise. Suppose there is a point $z \in s\alpha(y) \cap (Q \setminus S)$. Let $\phi(z) = r$, where ϕ is defined in theorem 10. Since $z \in s\alpha(y) \subset SA(f) \subset \Lambda^1(f)$, we can assume $z \in \Lambda^1(f) \cap (Q \setminus S) = S_\omega \setminus S$. By theorem 10 property (b), $\phi^{-1}(r)$ has to be an interval and z has to be one of its endpoints, say, the right endpoint, and $\phi^{-1}(r) = [x, z]$, $x \in S$. Since ϕ is a semiconjugacy we have $f^i([x, z]) \subset \phi^{-1}(\tau^i(r))$ for all $i \geqslant 0$. But the intervals $\phi^{-1}(\tau^i(r))$ are pairwise disjoint. This shows that $[x, z]$ is a wandering interval.

Claim. $z \in \text{Int}(\text{Orb}(I_j))$, for every $j \geqslant 0$.

We will assume otherwise. Let K be the connected component of $\text{Orb}(I_N)$, for some $N \geqslant 0$, where z is an endpoint of K. Let v be a point such that $z \in \omega(v)$. By theorem 10 property (a), $S = \omega(z) \subset \omega(v)$, we have $\omega(v) \cap \text{Orb}(I_N)$ infinite and necessarily $\text{Orb}(v) \cap \text{Int}(\text{Orb}(I_N)) \neq \emptyset$. This implies $f^k(v) \in \text{Orb}(I_N)$ for all sufficiently large k. It follows that $\text{Orb}(v)$ accumulates on z from the interior of K and we can find $k > 0$ such that $f^k(v) \in (x, z)$. But $[x, z]$ is a wandering interval, so $\text{Orb}(v)$ cannot accumulate on z which contradicts $z \in \omega(v)$.

Let $\{y_n\}_{n=0}^\infty$ be a backward orbit branch of y with a subsequence $\{y_{n_i}\}_{i=0}^\infty$ such that $\lim_{i \to \infty} y_{n_i} = z$. Since $z \in \text{Int}(\text{Orb} I_j)$ it follows from lemma 11 that $y_n \in \text{Orb}(I_j)$ for all $j, n \geqslant 0$. Therefore $\{y_n\}_{n=0}^\infty \subset Q$. For every $n \geqslant 1$, denote $\phi(y_n) = r_n$. Then by theorem 10 property (b) $\phi^{-1}(r_n)$ are connected, pairwise disjoint sets, each containing an element $s_n \in S$. Since $y_n \in \phi^{-1}(r_n)$, we have $\lim_{i \to \infty} s_{n_i} = z$. But S is a closed set and $z \notin S$, which is impossible. Therefore $s\alpha(y) \cap (Q \setminus S) = \emptyset$ and $s\alpha(y) \cap Q \subset S$. $\square$

Corollary 12.   *A $s\alpha$-limit set contains at most one solenoidal set.*

Proof.   Let $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ and $\text{Orb}(I_0') \supset \text{Orb}(I_1') \supset \ldots$ be nested sequences of cycles of intervals generating two solenoidal sets $Q = \bigcap \text{Orb}(I_n)$ and $Q' = \bigcap_n \text{Orb}(I_n')$. If $s\alpha(y) \cap Q \neq \emptyset$ and $s\alpha(y) \cap Q' \neq \emptyset$ then, by theorem 6, $y \in Q \cap Q'$. Since two solenoidal sets Q and Q' are either identical or disjoint we have $Q = Q'$. Then the only solenoidal set contained in $s\alpha(y)$ is $S = Q \cap \text{Rec}(f)$. $\square$

3.2. Basic sets

This section is devoted to the proof of theorem 7.

Let f be a continuous map acting on an interval I. We say that an endpoint y of I is *accessible* if there is $x \in \text{Int}(I)$ and $n \in \mathbb{N}$ such that $f^n(x) = y$. If y is not accessible, then it is called *non-accessible*. The following proposition is derived from [21, proposition 2.8].

Proposition 13   [21]. *Let f be a topologically mixing continuous map acting on an interval I. Let $x \in I$ and $\epsilon > 0$ be such that $[x - \epsilon, x + \epsilon] \subset I$ and any endpoints of I in $[x - \epsilon, x + \epsilon]$ are accessible. Then for every non degenerate interval $U \subset I$, there exists an integer N such that $f^n(U) \supset [x - \epsilon, x + \epsilon]$, for all $n \geqslant N$.*

An *m-periodic transitive map of a cycle of intervals* is a transitive map $g : M \to M$, where $M \subset \mathbb{R}$ is a finite union of pairwise disjoint compact intervals $I, g(I), \ldots, g^{m-1}(I)$, and $g^m(I) = I$.

We write $\text{End}(M)$ for the endpoints of the connected components of M and refer to these points simply as *endpoints of M*. The set of *exceptional points of g* is defined

$$E := M \backslash \bigcap_U \bigcup_{n=1}^{\infty} g^n(U),$$

where U ranges over all relatively open nonempty subsets of M. It is known that E is finite; it can contain some endpoints and at most one non-endpoint from each component of M. If $g^m|_I$ is topologically mixing, then $E = \bigcup_{i=0}^{m-1} E_i$, where E_i is the set of non-accessible endpoints of $g^m|_{g^i(I)}$ by proposition 13. If $g^m|_I$ is transitive but not mixing, then by [21, proposition 2.16] there is an m-periodic orbit $c_0, c_1, \ldots, c_{m-1}$ of points such that $c_i \in \text{Int}(g^i(I))$, $g^{2m}|_{[a_i,c_i]}$ and $g^{2m}|_{[c_i,b_i]}$ are topologically mixing interval maps, where $g^i(I) = [a_i, b_i]$, for $i = 0, \ldots, m-1$. Then $E = \bigcup_{i=0}^{m-1} E_i$, where E_i is the union of the sets of non-accessible endpoints of $g^{2m}|_{[a_i,c_i]}$ and $g^{2m}|_{[c_i,b_i]}$.

By [21, lemma 2.32], every point in E is periodic and therefore $g(E) = E$. By the definition of non-accessible points, $g^{-1}(E) \cap M = E$.

From [18, proposition 3.10] it follows that $s\alpha(x) \supseteq M$ for all $x \in M \backslash E$. On the other hand, if $x \in E$, then its $s\alpha$-limit set is disjoint from the interior of $M \backslash E$ by lemma 11. Therefore we have another characterization of E using $s\alpha$-limit sets,

$$E = \{x \in M : s\alpha(x) \not\supseteq M\}. \tag{1}$$

We use m-periodic transitive maps of cycles of intervals as models for maps acting on basic sets. We recall again Blokh's definition. If M is a cycle of intervals for the map $f : [0,1] \to [0,1]$, then we set

$$B(M,f) = \{x \in M : \text{for any relative neighbourhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\text{Orb}(U)} = M\},$$

and if $B(M,f)$ is infinite, then it is a basic set for f.

Theorem 14 [4, theorem 4.1]. *Let I be an m-periodic interval for f, $M = \text{Orb}(I)$ and $B = B(M,f)$ be a basic set. Then there is a transitive m-periodic map $g : M' \to M'$ and a monotone map $\phi : M \to M'$ such that ϕ semiconjugates $f|_M$ to g and $\phi(B) = M'$. Moreover, for any $x \in M'$, $1 \leqslant \#\{\phi^{-1}(x) \cap B\} \leqslant 2$ and $\text{Int}(\phi^{-1}(x)) \cap B = \emptyset$, and so $\phi^{-1}(x) \cap B \subset \partial\phi^{-1}(x)$. Furthermore, B is a perfect set.*

Lemma 15. *Let B be a basic set, M the smallest cycle of intervals for f which contains B, and $\phi : (M,f) \to (M',g)$ the semiconjugacy to the m-periodic transitive map g given by theorem 14. Let E be the set of exceptional points of the map g acting on M'. Suppose $y \in M$ and $\phi(y) \notin E \cup \text{End}(M')$. Then $s\alpha(y) \supset B$.*

Proof. Let $x \in B$. There is $\epsilon > 0$ such that $\phi|_{(x,x+\epsilon)}$ is not constant and $\phi((x,x+\epsilon)) \cap \text{End}(M') = \emptyset$ or $\phi|_{(x-\epsilon,x)}$ is not constant and $\phi((x-\epsilon,x)) \cap \text{End}(M') = \emptyset$. Otherwise x has a neighbourhood N such that $\phi|_N$ is constant which is in a contradiction with $x \in B$ by theorem 14. We can assume $\phi|_{(x,x+\epsilon)}$ is not constant and $\phi((x,x+\epsilon)) \cap \text{End}(M') = \emptyset$, and denote $V = (x,x+\epsilon)$. Then $U = \phi(V)$ is a non-degenerate interval in M'. Since $\phi(y) \notin E \cup \text{End}(M')$, there is $\delta > 0$ such that $(\phi(y) - \delta, \phi(y) + \delta) \subset M'$ and $(\phi(y) - \delta, \phi(y) + \delta) \cap E = \emptyset$. The set E equals the union of non-accessible endpoints of a topologically mixing map g^m (resp. g^{2m}) acting on the components of M', therefore we can apply the proposition 13 to the map g^m (resp. g^{2m}) acting on the component $I \subset M'$ such that $(\phi(y) - \delta, \phi(y) + \delta) \subset I$. There is an $N > 0$ such that $g^N(U) \supset (\phi(y) - \delta, \phi(y) + \delta)$. But $\phi(f^N(V)) = g^N(U)$, so $\phi(y)$ is in $\phi(f^N(V))$. Since

ϕ is monotone this means either $y \in f^N(V)$ or $\phi(y)$ is an endpoint of the interval $\phi(f^N(V))$. But we have seen that it is not an endpoint. Therefore $y \in f^N(V)$ and we can find $y_1 \in V$ and $N_1 = N$ such that $f^{N_1}(y_1) = y$. Notice that $\phi(y_1) \notin E$ since $g^{N_1}(\phi(y_1)) = \phi(y) \notin E$ and $g^{-N_1}(E) \cap M' = E$; and $\phi(y_1) \notin \text{End}(M')$ since $y_1 \in V$ and $\phi(V) \cap \text{End}(M') = \emptyset$. By the same procedure, we can find $y_2 \in (x, x + \epsilon/2) \cap M$ and $N_2 \in \mathbb{N}$ such that $f^{N_2}(y_2) = y_1$. By repeating this process we construct a sequence $\{y_n\}_{n=1}^{\infty}$ converging to x which is a subsequence of a backward orbit branch of y. Since $x \in B$ was arbitrary, this shows $B \subset s\alpha(y)$. $\quad\square$

Corollary 16. *For every basic set B, there is $y \in B$ such that $s\alpha(y) \supset B$.*

Proof. Let M, ϕ, g, M', E be as in the previous proof. Since the map $\phi|_B$ is at most 2-to-1 and E is a finite set, there are uncountably many points $y \in B$ such that $\phi(y) \notin E \cup \text{End}(M')$. The result follows by lemma 15. $\quad\square$

Before we proceed to the proof of theorem 7 we need to recall the definition of a *prolongation set* and its relation to basic sets. Let M be a cycle of intervals. Let the side T be either the left side $T = L$ or the right side $T = R$ of a point $x \in M$ and $W_T(x)$ be a *one-sided neighbourhood* of x from the T-hand side, i.e. $W_T(x)$ contains for some $\epsilon > 0$ the interval $(x, x + \epsilon)$ (resp. $(x - \epsilon, x)$) when $T = R$ (resp. $T = L$). We do not consider the side $T = R$ (resp. $T = L$) when x is a right endpoint (resp. left endpoint) of a component of M. Now let

$$P_M^T(x) = \bigcap_{W_T(x)} \bigcap_{n \geqslant 0} \overline{\bigcup_{i \geqslant n} f^i(W_T(x) \cap M)},$$

where the intersection is taken over the family of all one-sided neighbourhoods $W_T(x)$ of x. We will write $P^T(x)$ instead of $P_{[0,1]}^T(x)$. The following auxiliary lemmas 17–19 are taken from [4].

Lemma 17 [4, *lemma 2.2*]. *Let $x \in [0, 1]$. Then $P^T(x)$ is a closed invariant set and only one of the following possibilities holds:*

- *There exists a wandering interval $W_T(x)$ with pairwise disjoint forward images and $P^T(x) = \omega(x)$.*
- *There exists a periodic point p such that $P^T(x) = \text{Orb}(p)$.*
- *There exists a solenoidal set Q such that $P^T(x) = Q$.*
- *There exists a cycle of intervals M such that $P^T(x) = M$.*

There is a close relation between prolongation sets and basic sets. If M is a cycle of intervals for f then we define

$$E(M, f) = \{x \in M : \text{there is a side } T \text{ of } x \text{ such that } P_M^T(x) = M\},$$

and, for $x \in E(M, f)$, we call this side T a *source side* of x. By [4, theorem 4.1] if there exists the basic set $B = B(M, f)$ then $E(M, f) = B$. In particular, if $E(M, f)$ is infinite then $E(M, f) = B(M, f)$ (see the discussion on page 48 in [4]).

Lemma 18 [4, *lemma 4.5*]. *Let M be a cycle of intervals. If $E(M,f)$ is a finite set then $E(M,f) = \text{Orb}(p)$ where p is a periodic point. If $E(M,f)$ is infinite then $E(M,f) = B(M,f)$.*

The next property of basic sets follows from step B7 on page 47 in [4]:

Lemma 19 [4]. *Let $B(M,f)$ be a basic set. Then for any $x \in B(M,f)$ with a source side T and any one-sided neighbourhood from the T-hand side $W_T(x)$, we have $W_T(x) \cap B(M,f) \neq \emptyset$.*

Proof of theorem 7.

(a) Let ϕ and g be the maps given in theorem 14 and E be the set of exceptional points of the map g acting on M'. Since $\phi|_B$ is an at most 2-to-1 map, $\phi(\alpha(y) \cap B)$ is an infinite subset of M'. But $E \cup \text{End}(M')$ is a finite set, so we can find a point $z \in \alpha(y) \cap B$ such that $\phi(z) \notin E \cup \text{End}(M')$ and therefore $z \notin \phi^{-1}(E \cup \text{End}(M'))$. The set $\phi^{-1}(E \cup \text{End}(M'))$ is a union of finitely many, possibly degenerate, closed intervals in M. Since $z \in (\alpha(y) \cap B) \backslash \phi^{-1}(E \cup \text{End}(M'))$, there is a pre-image $y' \in M$ of y, $y = f^k(y')$, for some $k \geqslant 0$, and simultaneously $y' \notin \phi^{-1}(E \cup \text{End}(M'))$, which implies $\phi(y') \notin E \cup \text{End}(M')$. Then $y \in M$ by the invariance of M. By lemma 15 applied to y', $s\alpha(y') \supset B$. But the containment $s\alpha(y) \supset s\alpha(y')$ is clear from the definition of $s\alpha$-limit sets, and so $s\alpha(y) \supset B$.

(b) Let $\{y_i\}_{i=0}^{\infty}$ be the backward orbit branch of y accumulating on x. Since x is not a periodic point, it is not contained in $\{y_i\}_{i=0}^{\infty}$ more then one time and we can assume that $\{y_i\}_{i=0}^{\infty}$ accumulates on x from one side T. Consider the prolongation set $P^T(x)$. Clearly $\{y_i\}_{i=0}^{\infty} \subset P^T(x)$. Since $P^T(x)$ is closed and invariant, x and $\text{Orb}(x)$ belong to $P^T(x)$, we see that $P^T(x)$ contains both periodic and non-periodic points. By lemma 17, there is only one possibility $P^T(x) = M$, where M is a cycle of intervals. The other possibilities are ruled out—$\text{Orb}(p)$, where p is a periodic point and $\omega(x)$, where x is a preperiodic point, can not contain a non-periodic point; and a solenoidal set Q can not contain a periodic point. Since $P^T(x) = M$ contains $\{y_i\}_{i=0}^{\infty}$ it must contain a one-sided neighbourhood of x from the T-hand side and therefore $P_M^T(x) = P^T(x) = M$. Let $E(M, f) = \{z \in M : \text{there is a side } S \text{ of } z \text{ such that } P_M^S(z) = M\}$. Since $x \in E(M, f)$ and x is not periodic, by lemma 18, $E(M, f) = B(M, f)$ and T is a source side of x in $B(M, f)$. Let ϕ and g be the maps given in theorem 14 and E be the set of exceptional points of the map g acting on M'. By lemma 19, $B(M, f)$ accumulates on x from the T-hand side and thus the map ϕ is not constant on some one-sided neighbourhood of x from the T-hand side $W_T(x)$ (otherwise we can find a point $y \in B(M, f) \cap W_T(x)$ such that $y \in \text{Int}(\phi^{-1}(\phi(x)))$ which is in the contradiction with the properties of ϕ from theorem 14). Since $\{y_i\}_{i=0}^{\infty}$ accumulates on x from the T-hand side and ϕ is not constant on $W_T(x)$, we can find $j > 0$ such that $y_j \in W_T(x)$ and $\phi(y_j) \notin E \cup \text{End}(M')$. Then $s\alpha(y_j) \supset B(M, f)$ by lemma 15, and $s\alpha(y) \supset s\alpha(y_j)$ since y_j is a preimage of y. We conclude that $s\alpha(y) \supset B(M, f)$.   □

We record here one corollary which we will need several times in the rest of the paper.

Corollary 20. *If $s\alpha(x)$ contains infinitely many points from a transitive cycle M, then $x \in M$ and $s\alpha(x) \supseteq M$.*

Proof. In this case M is itself a basic set, so we may apply theorem 7.   □

4. General properties of special α-limit sets for interval maps

4.1. Isolated points are periodic

Unless an ω-limit set is a single periodic orbit, its isolated points are never periodic [23]. The opposite phenomenon holds for the $s\alpha$-limit sets of an interval map.

Theorem 21. *Isolated points in a $s\alpha$-limit set for a continuous interval map are periodic.*

Proof. Let $z \in s\alpha(y)$ such that z is neither periodic nor preperiodic. Then z is a point of an infinite maximal ω-limit set, i.e. a basic set or a solenoidal set. This follows from Blokh's decomposition theorem, that $\Lambda^1(f)$ is the union of periodic orbits, solenoidal sets and basic sets,

and from theorem 5, $z \in \mathrm{SA}(f) \subset \Lambda^1(f)$. According to [15, lemma 1], when $s\alpha(y)$ contains a point z, it contains its orbit $\mathrm{Orb}(z)$ as well. If z is in a basic set B then, $\mathrm{Orb}(z) \subset B \cap s\alpha(y)$ is infinite and by theorem 7 (a), $B \subset s\alpha(y)$. Then the point z is not isolated in $s\alpha(y)$ since B is a perfect set. If z is in a solenoidal set $Q = \bigcap_n \mathrm{Orb}(I_n)$ then, by theorem 6, $s\alpha(y) \cap Q = S$. Again, the point z is not isolated in $s\alpha(y)$ since S is a perfect set.

Let $z \in s\alpha(y)$ such that z is a preperiodic point. By theorem 7 (b), there is a basic set B such that $z \in B \subset s\alpha(y)$. Then the point z is not isolated in $s\alpha(y)$ since B is a perfect set. $\square$

In the previous proof, we have shown that if a point $z \in s\alpha(y)$ is not periodic then $s\alpha(y)$ contains either a minimal solenoidal set S or a basic set B. In both cases, $s\alpha(y)$ has to be uncountable. Therefore we have the following corollary.

Corollary 22. *A countable $s\alpha$-limit set for a continuous interval map is a union of periodic orbits.*

4.2. The interior and the nowhere dense part of a special α-limit set

A well-known result by Sharkovsky says that each ω-limit set of an interval map is either a transitive cycle of intervals or a closed nowhere dense set [24]. What can we say in this regard for $s\alpha$-limit sets of interval maps? When $\mathrm{Int}(s\alpha(x))$ is nonempty, Kolyada, Misiurewicz, and Snoha showed that $M = \overline{\mathrm{Int}s\alpha(x)}$ is a cycle of intervals containing x, see [18, proposition 3.6]. We strengthen this result by showing that the non-degenerate components of $s\alpha(x)$ are in fact closed, and the rest of $s\alpha(x)$ is nowhere dense. We also get some amount of transitivity.

The following lemma is simple and we leave the proof to the reader.

Lemma 23. *Let M be a cycle of intervals for f of period k and let K be any of its components. Then*

(a) *$f|_M$ is transitive if and only if $f^k|_K$ is transitive.*
(b) *If $L \subset K$ is a cycle of intervals for f^k, then $\bigcup_{i<k} f^i(L)$ is a cycle of intervals for f.*
(c) *For $x \in K$ and $y \in K \backslash \mathrm{End}(K)$, we have $y \in s\alpha(x, f)$ if and only if $y \in s\alpha(x, f^k|_K)$.*

Theorem 24. *A $s\alpha$-limit set for a continuous interval map f is either nowhere dense, or it is the union of a cycle of intervals M for f and a nowhere dense set. Moreover, M is either a transitive cycle, or it is the union of two transitive cycles.*

Proof. Consider a limit set $s\alpha(x)$ for an interval map $f : [0, 1] \to [0, 1]$. Let M be the union of the non-degenerate components of $\overline{s\alpha(x)}$. If $M = \emptyset$ then $s\alpha(x)$ is nowhere dense. Otherwise M must be a finite or countable union of closed intervals, and since M contains the interior of the closure of the $s\alpha$-limit set we know that $s\alpha(x) \backslash M$ is nowhere dense.

Let K be any component of M. By theorem 5 we have $K \subset \overline{\mathrm{Per}(f)}$, and therefore periodic points are dense in K. Let $n \geqslant 1$ be minimal such that $f^n(K) \cap K \neq \emptyset$. Since $f^n(K)$ is connected and K is a component of the invariant set $\overline{s\alpha(x)}$, we know that $f^n(K) \subseteq K$. Since periodic points are dense in K we must have $f^n(K) = K$. Therefore $\mathrm{Orb}(K)$ is a cycle of intervals, and by lemma 11 we get $x \in \mathrm{Orb}(K)$. Since this holds for every component K of M, we must have $\mathrm{Orb}(K) = M$, i.e. M is a cycle of intervals and $x \in M$.

From now on we take K to be the component of M containing x. Put $g = f^n|_K$. Then $g : K \to K$ is an interval map with a dense set of periodic points. There is a structure theorem

for interval maps with a dense set of periodic points [21, theorem 3.9][3] which tells us that K is a union of the transitive cycles and periodic orbits of g,

$$K = \left(\bigcup \{L : L \text{ is a transitive cycle for } g\} \right) \cup \operatorname{Per}(g)$$

By lemma 23, $s\alpha(x, g)$ contains a dense subset of K. By corollary 20 each transitive cycle $L \subseteq K$ for g must contain x. Since transitive cycles have pairwise disjoint interiors, g has at most two transitive cycles. If their union is not K, then K must contain a non-degenerate interval of periodic points of g. But by an easy application of lemma 11, no $s\alpha$-limit set can contain a dense subset of an interval of periodic points. Therefore K is the union of one or two transitive cycles for g. By lemma 23, M is the union of one or two transitive cycles for f.

Finally, if L is one of the (at most two) transitive cycles for f that compose M, then by corollary 20 we have $s\alpha(x) \supseteq L$. Therefore $M \subseteq s\alpha(x)$.   □

Remark 25.   If $s\alpha(x)$ contains a cycle of intervals M, then $s\alpha(x)$ is in fact a closed set, but we are not yet ready to prove this fact. See theorem 39 below.

4.3. Transitivity and points with $s\alpha(x) = [0, 1]$

Let $f : [0, 1] \to [0, 1]$ be an interval map. We say that the point x has a *full* $s\alpha$-limit set if $s\alpha(x) = [0, 1]$. Kolyada, Misiurewicz, and Snoha proved that when f is transitive, all points $x \in [0, 1]$ with at most three exceptions have a full $s\alpha$-limit set. Conversely, they conjectured that if at least two points $x \in [0, 1]$ have full $s\alpha$-limit sets, then f is transitive. The conjecture was not quite right; the correct result is as follows:

Theorem 26.   *A continuous interval map* $f : [0, 1] \to [0, 1]$ *is transitive if* $s\alpha(x_i) = [0, 1]$ *for at least three distinct points* x_1, x_2, x_3.

Proof.   Suppose that f is not transitive. We will prove that at most two points have a full $s\alpha$-limit set. Suppose there is at least one point x with $s\alpha(x) = [0, 1]$. By theorem 24 there are two transitive cycles L, L' for f such that $[0, 1] = L \cup L'$, and by corollary 20 every point with a full $s\alpha$-limit set belongs to $L \cap L'$. We will show that the cardinality of $L \cap L'$ is at most two.

Let $A_1, A_2, \ldots, A_n$ be the components of L, numbered from left to right in $[0, 1]$. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the cyclic permutation defined by $f(A_i) = A_{\sigma(i)}$. If $n \geqslant 3$ then there must exist i such that $|\sigma(i) - \sigma(i + 1)| \geqslant 2$, so there is some A_j strictly between $A_{\sigma(i)}, A_{\sigma(i+1)}$. Let B be the component of L' between A_i and A_{i+1}, see figure 3. By the intermediate value theorem, $f(B) \supset A_j$. This contradicts the invariance of L'. Therefore L has at most two components. For the same reason L' has at most two components. Moreover, L, L' cannot both have two components; otherwise the middle two of those four components have a point in common, but their images do not, again see figure 3.

There are two cases remaining. L, L' can have one component each, and then $L \cap L'$ has cardinality one. Otherwise, one of the cycles, say L, has two components, while L' has only one, and then $L \cap L'$ has cardinality two.   □

In the course of the proof we have also shown the following result:

Corollary 27.   *If a continuous map* $f : [0, 1] \to [0, 1]$ *has one or two points with full $s\alpha$-limit sets, but not more, then* $[0, 1]$ *is the union of two transitive cycles of intervals.*

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[3] In fact, [21, theorem 3.9] tells us that all the transitive cycles for g have period at most 2, and the periodic orbits not contained in transitive cycles also have period at most 2. Some of this extra information can be shown quite easily; it comes up again in our proof of theorem 26.
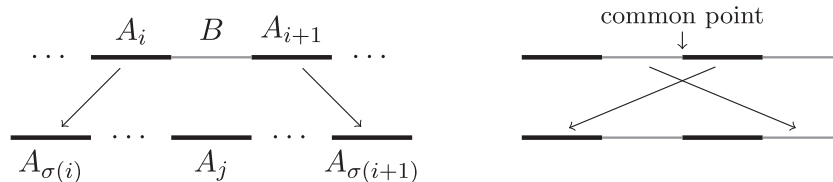
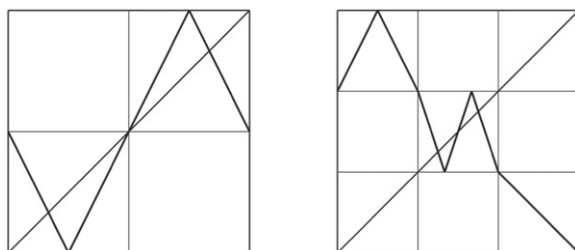**Figure 3.** Diagrams for the proof of theorem 26.

Figure 4. Maps for which $s\alpha(x) = [0, 1]$ for only 1 or 2 points x.

Corollary 27 corrects the second part of conjecture 3 (originally [18, conjecture 4.14]) to allow for two points with a full $s\alpha$-limit set.[4] Both possibilities from the corollary are shown in figure 4. One of the interval maps shown has exactly one point with a full $s\alpha$-limit set, and the other has exactly two.

4.4. Special α-limit sets containing a common open set

Now we study the $s\alpha$-limit sets that contain a given transitive cycle of intervals. We get a sharpening of theorem 7 in the case when $B = M$, i.e. when our basic set is itself a transitive cycle of intervals.

Let $M \subset [0, 1]$ be a transitive cycle of intervals for f. For the reader's convenience, we recall some definitions from section 3.2. We write $\text{End}(M)$ for the endpoints of the connected components of M and refer to these points simply as *endpoints of M*. The main role in our analysis is played by the set of *exceptional points of M*

$$E := M \Big\backslash \bigcap_U \bigcup_{n=1}^{\infty} f^n(U),$$

where U ranges over all relatively open nonempty subsets of M. It is known that E is finite; it can contain some endpoints and at most one non-endpoint from each component of M. It is also known that E and $M \backslash E$ are both invariant under f, see [21]. Endpoints of M in E are called *non-accessible endpoints*, as explained in section 3.2. Recall from equation (1) that $E = \{x \in M : s\alpha(x) \not\supseteq M\}$.

Theorem 28. *Let M be a transitive cycle of intervals for $f : [0, 1] \to [0, 1]$ and let E be its set of exceptional points.*

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[4] Incidentally, when there is exactly one point with a full $s\alpha$-limit set, the conclusion of the conjecture holds as stated in [18]: there is $c \in (0, 1)$ such that such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are both transitive.
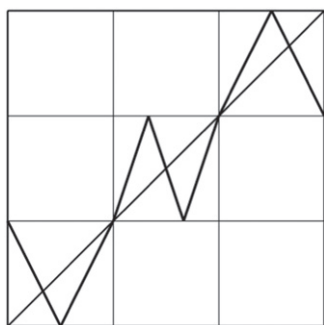
Figure 5. An example where three distinct $s\alpha$-limit sets contain a common open interval.

(a) *Each point $x \in M \setminus (E \cup \text{End}(M))$ has the same $s\alpha$-limit set.*
(b) *At most three distinct $s\alpha$-limit sets of f contain M.*

We will see in the course of the proof that if $s\alpha(x') \supseteq M$ is distinct from the $s\alpha$-limit set described in part (a), then x' belongs to a periodic orbit contained in $\text{End}(M)$. Since there are at most two such periodic orbits in $\text{End}(M)$ we get part (b).

Before giving the proof we discuss some consequences of this theorem.

Corollary 29. *At most three distinct special alpha-limit sets of a continuous map f can contain a given nonempty open set.*

Proof. Let U be a nonempty open set in $[0, 1]$. If a $s\alpha$-limit set of f contains U, then by theorem 24, U contains a whole subinterval from some transitive cycle M. Applying corollary 20, we see that any $s\alpha$-limit set which contains U also contains M. By theorem 28 there are at most three such $s\alpha$-limit sets. $\qquad\square$

This corollary corrects conjecture 4 (originally [18, conjecture 4.10]), in which it was conjectured that two $s\alpha$-limit sets which contain a common open set must be equal. For comparison, note that if two ω-limit sets of an interval map contain a common open set, then they are in fact equal, since an ω-limit set with nonempty interior is itself a transitive cycle [24]. We also remark that the number three in corollary 29 cannot be improved, as is shown by the following example.

Example 30. Let $f : [0, 1] \to [0, 1]$ be an interval map for which $[0, \frac{1}{3}]$ is a full two-horseshoe, $[\frac{1}{3}, \frac{2}{3}]$ is a full three-horseshoe, and $[\frac{2}{3}, 1]$ is a full two-horseshoe, as shown in figure 5. Then $\frac{1}{2}$ belongs to only one of the three transitive invariant intervals for f and $s\alpha(\frac{1}{2}) = [\frac{1}{3}, \frac{2}{3}]$. But both $\frac{1}{3}$ and $\frac{2}{3}$ are accessible endpoints of adjacent transitive intervals and so $s\alpha(\frac{1}{3}) = [0, \frac{2}{3}]$ and $s\alpha(\frac{2}{3}) = [\frac{1}{3}, 1]$.

In what follows it is necessary to allow for a weaker notion of a cycle of intervals for f. An interval is called *non-degenerate* if it contains more than one point. If U is a non-degenerate interval (not necessarily closed) such that $U, f(U), \ldots, f^{n-1}(U)$ are pairwise disjoint non-degenerate intervals and $f^n(U) \subseteq U$ (not necessarily equal), then we will call $\text{Orb}(U)$ a *weak cycle of intervals* of period n. The next lemma records one of the standard ways to produce a weak cycle of intervals. Similar lemmas appear in [4] and several other papers, but since we were unable to find the exact statement we needed, we chose to give our own formulation here.

Lemma 31. *If a subinterval U contains three distinct points from some orbit of f, then* $\mathrm{Orb}(U) = \bigcup_{i=0}^{\infty} f^i(U)$ *is a weak cycle of intervals for f.*

Proof. Let $x, f^n(x), f^m(x)$ be three distinct points in $U, 0 < n < m$. Clearly $\mathrm{Orb}(U)$ is invariant. Since the intervals $U, f^n(U)$ both contain $f^n(x)$ we see that $U \cup f^n(U)$ is connected, i.e. it is an interval. Then also $f^n(U) \cup f^{2n}(U)$ is connected, and so on inductively. Therefore the set $A = \bigcup_{j=0}^{\infty} f^{jn}(U)$ is connected. Then $\mathrm{Orb}(U) = \bigcup_{i=0}^{n-1} f^i(A)$ has at most n connected components. Let $B \supseteq A$ be the component of $\mathrm{Orb}(U)$ containing U, and let $k \leqslant n$ be minimal such that $B \cap f^k(B) \neq \emptyset$. Then $f^k(B)$ is a connected subset of $\mathrm{Orb}(U)$, so $f^k(B) \subseteq B$. For $0 \leqslant i < j < k$ if $f^i(B) \cap f^j(B) \neq \emptyset$, then $f^{i+k-j}(B) \cap f^k(B) \neq \emptyset$, so $f^{i+k-j}(B) \cap B \neq \emptyset$, contradicting the choice of k. This shows that $B, f(B), \ldots, f^{k-1}(B)$ are pairwise disjoint. It remains to show that they are all non-degenerate. Clearly all three points $x, f^n(x), f^m(x)$ are in B. From the disjointness of $B, f(B), \ldots, f^{k-1}(B)$ it follows that n, m are multiples of k. And since $x, f^n(x), f^m(x)$ are distinct we get $n = j_1 k, m = j_2 k$ with $0 < j_1 < j_2$. If any $f^i(B)$ is a singleton, $i \leqslant k$, then so also is $f^k(B)$. Then using $B \supseteq f^k(B) \supseteq f^{2k}(B) \supseteq \ldots$, we see that all the sets $f^{jk}(B), j \geqslant 1$, are the same singleton. But in that case $f^{j_1 k}(B) = \{f^n(x)\}, f^{j_2 k}(B) = \{f^m(x)\}$ are not the same singleton, which is a contradiction. $\square$

Lemma 32. *Let N be a weak cycle of intervals for f and M a transitive cycle of intervals. Let E be the set of exceptional points for M. If $N \cap M$ is nonempty, then the following hold:*

(a) *Either $N \supseteq M \backslash E$ or else $N \cap M$ is a periodic orbit contained in* $\mathrm{End}(M)$, *and*
(b) *The period of N is at most twice the period of M.*

Proof. Let n, m be the periods of N, M, respectively. Let N_i, M_j, be the components of N, M with the temporal ordering, so $f(N_i) \subset N_{i+1 \bmod n}$ for all $i < n$ and $f(M_j) = M_{j+1 \bmod m}$ for all $j < m$.

Suppose first that $N \cap M$ is infinite. Then N contains a non-degenerate interval $U \subset M$. We have $N \supseteq \mathrm{Orb}(U) \supseteq M \backslash E$, where the first containment comes from the invariance of N and the second from the definition of the exceptional set E. Since E contains at most one non-endpoint of each component M_j, it follow that each component M_j meets at most two components N_i. So in this case $n \leqslant 2m$.

For the rest of the proof we suppose that $N \cap M$ is finite. We no longer need transitivity and the sets N, M will play symmetric roles. Clearly each nonempty intersection $N_i \cap M_j$ is at common endpoints. This shows that $N \cap M \subseteq \mathrm{End}(M)$. It also shows that each component N_i contains at most two points from M, and conversely each component M_j contains at most two points from N.

Claim 1. Each component N_i contains the same number of points of M. Conversely, each component M_j contains the same number of points of N. Suppose first that some N_i contains two distinct points a, b from M. Then a, b belong to distinct components of M, and therefore $f(a), f(b)$ also belong to distinct components of M. But they both belong to $N_{i+1 \bmod n}$. Continuing in this way we see that each each component of N contains two points from M. Now suppose instead that each component of N contains at most one point from M. Surely some N_i contains at least one point $a \in M$. Then $N_{i+1 \bmod n}$ contains the point $f(a) \in M$. Continuing in this way we see that each component N_i contains exactly 1 point from M. Moreover, the whole argument still works if we reverse the roles of N and M. This concludes the proof of claim 1.

Claim 2. *The intersection $N \cap M$ is a periodic orbit.* Suppose first that each component of N contains two points from M. We reuse an argument from the proof of theorem 26. Let $A_1, A_2, \ldots, A_m$ be the components of M in the spatial order, i.e. numbered from left to right in $[0, 1]$. Let $\sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}$ be the cyclic permutation defined by $f(A_i) \subseteq A_{\sigma(i)}$.

If $m \geqslant 3$ then there must exist i such that $|\sigma(i) - \sigma(i+1)| \geqslant 2$, so there is some A_j strictly between $A_{\sigma(i)}, A_{\sigma(i+1)}$. Let N_k be the component of N which intersects both A_i and A_{i+1}. By the intermediate value theorem, $f(N_k) \supset A_j$. This contradicts the fact that $M \cap N$ is finite. Therefore M has only two components, and N has only one, i.e. $N = N_1$. The two endpoints of N_1 belong to A_1, A_2, respectively, and are therefore interchanged by f. So in this case $M \cap N$ is a periodic orbit of period 2.

The symmetric situation arises if each component of M contains two points from N. Then N has two components and M has only 1, and again $M \cap N$ is a periodic orbit of period 2.

Now suppose that each N_i contains exactly one point from M, and each M_j contains exactly one point from N. Then $m = n$ and we may assume the components are indexed such that $N_i \cap M_j$ is nonempty if and only if $i = j$. Let x_i be the unique point of intersection of N_i and M_i. Since these intersection points are unique we get $f(x_i) = x_{i+1 \bmod n}$ for all i. Thus $N \cap M$ is a periodic orbit of period $m = n$. $\qquad\square$

Proof of theorem 28.   We have already seen that the set of exceptional points in M may be characterized as $E = \{x \in M : s\alpha(x) \not\supseteq M\}$. Combined with lemma 11 this shows that

$$s\alpha(x) \supseteq M \iff x \in M \backslash E.$$

Our task is to compare the $s\alpha$-limit sets of the various points $x \in M \backslash E$. Since they all contain M it is enough to check whether or not they coincide outside of M. For any point $y \in [0, 1]$ let us write $s\alpha \mathrm{Ba} \sin(y) = \{z : y \in s\alpha(z)\}$.

Step 1: if $x \in M \backslash E$ and $y \in s\alpha(x) \backslash M$, then there is a weak cycle of intervals N for f such that $x \in N \subseteq s\alpha \mathrm{Ba} \sin(y)$. To prove this claim, let (x_i) be a backward orbit branch of x accumulating on y. The sequence (x_i) cannot contain y twice, for otherwise y would be a periodic point containing x in its orbit, contradicting the invariance of M. Therefore there is a subsequence (x_{i_j}) which converges to y monotonically from one side. We will suppose $x_{i_j} \searrow y$ from the right. The proof when $x_{i_j} \nearrow y$ from the left is similar.

For $k = 1, 2, \ldots$ let $U_k = (y, y + \frac{1}{k})$. Then U_k contains the points x_{i_j} for large enough j. By lemma 31, the set $V_k = \mathrm{Orb}(U_k)$ is a weak cycle of intervals. Since V_k is invariant and contains points x_i for arbitrarily large natural numbers i, we must have the whole backward orbit branch (x_i) contained in each V_k. Moreover, we have nesting $V_1 \supseteq V_2 \supseteq \ldots$. Letting v_k denote the period (i.e. the number of connected components) of V_k this implies $v_1 \leqslant v_2 \leqslant \ldots$. But each V_k contains the point $x \in M$, so by lemma 32 (b) each $v_k \leqslant 2m$, where m is the period of M. A bounded increasing sequence of natural numbers is eventually constant. So fix k' such that $v_{k'} = v_{k'+1} = \ldots$.

For each k let V_k^0 be the connected component of V_k which contains U_k. Choose some i such that $x_i \in V_{k'}^0$, $x_i > y$. For $k \geqslant k'$, V_k is a weak cycle of intervals contained in $V_{k'}$ and with the same number of components. It follows that V_k^0 is the only component of V_k which meets $V_{k'}^0$. Therefore V_k^0 must contain x_i as well. In particular, setting $\delta = x_i - y$ we find that

$$\forall_{\epsilon>0}, \mathrm{Orb}((y, y+\epsilon)) \supseteq (y, y+\delta).$$

Now let $N = \mathrm{Orb}((y, y+\delta)) = \bigcap_{\epsilon>0} \mathrm{Orb}((y, y+\epsilon))$. It follows easily that $N \backslash \{y\} \subseteq s\alpha \mathrm{Ba} \sin(y)$. For if $z \in N$, $z \neq y$, then taking $\epsilon_1 < \min(\delta, |z - y|)$ we find $z_1 \in (y, y+\epsilon_1)$ and $n_1 \geqslant 1$ such that $f^{n_1}(z_1) = z$. Then taking $\epsilon_2 < z_1 - y$ we find $z_2 \in (y, y+\epsilon_2)$ and $n_2 \geqslant 1$ such that $f^{n_2}(z_2) = z_1$. Continuing inductively, we get a subsequence of a backward orbit branch of z which accumulates on y.

Since $(y, y + \delta)$ contains x_{i_j} for sufficiently large j, lemma 31 also implies that N is a weak cycle of intervals, and since it is forward invariant we have $x \in N$. This concludes step 1.

Step 2: if $x, x' \in M \backslash E$ *and* $y \in s\alpha(x) \backslash M$, *then* $s\alpha(x) \subseteq s\alpha(x')$. To prove this claim, fix an arbitrary point $y \in s\alpha(x)$. We need to show that $y \in s\alpha(x')$ as well. If $y \in M$ then there is nothing to prove, since $M \subset s\alpha(x')$. So suppose $y \notin M$. By step 1 there is a weak cycle of intervals N such that $x \in N \subseteq s\alpha \mathrm{Ba} \sin(y)$. Now we apply lemma 32 (a), noting that $M \cap N$ contains the point $x \notin \mathrm{End}(M)$. Therefore $M \backslash E \subseteq N$. We now have $x' \in M \backslash E \subseteq N \subseteq s\alpha \mathrm{Ba} \sin(y)$, from which it follows that $y \in s\alpha(x')$. This concludes step 2.

Step 3: e*ach point in* $M \backslash (E \cup \mathrm{End}(M))$ *has the same sα-limit set*. Suppose $x, x' \in M \backslash (E \cup \mathrm{End}(M))$. Then we may apply step 2 to get both containments $s\alpha(x) \subseteq s\alpha(x')$ and $s\alpha(x') \subseteq s\alpha(x)$. This concludes step 3 and the proof of theorem 28 (a). From now on we will refer to this common $s\alpha$-limit set as S.

Step 4: if *the sα-limit set of* x' *contains M and is distinct from S, then* x' *belongs to a periodic orbit contained in* $\mathrm{End}(M)$. The hypothesis $M \subseteq s\alpha(x')$ implies that $x' \in M \backslash E$. Fix $x \in M \backslash (E \cup \mathrm{End}(M))$ and apply step 2 to conclude that $S = s\alpha(x) \subseteq s\alpha(x')$. Our hypothesis is that this containment is strict. So choose $y \in s\alpha(x') \backslash s\alpha(x)$. Clearly $y \notin M$. Applying step 1 to x' we get a weak cycle of intervals N such that $x' \in N \subseteq s\alpha \mathrm{Ba} \sin(y)$. Moreover $x \notin N$ since $y \notin s\alpha(x)$. Now $M \cap N$ contains x' but not x, so we may apply lemma 32 (a) and conclude that $M \cap N$ is a periodic orbit contained in $\mathrm{End}(M)$. This concludes step 4.

Step 5: at *most three distinct sα-limit sets of* f *contain M*. One of these sets is S from step 3. By step 4, the only other sets to consider are the $s\alpha$-limit sets of those periodic points whose whole orbits are contained in $\mathrm{End}(M)$. Since M is a cycle of intervals it is clear that $\mathrm{End}(M)$ contains at most two periodic orbits. And from the definitions it is clear that all points in a periodic orbit have the same $s\alpha$-limit set. This concludes step 5 and the proof of theorem 28 (b). $\qquad\square$

4.5. Comparison of α-limit sets with special α-limit sets

One motivation for studying $s\alpha$-limit sets is that they provide more information about α-limit sets, since there is always the containment $sa(x) \subset \alpha(x)$. For transitive interval maps this containment is in fact an equality $s\alpha(x) = \alpha(x)$ for all $x \in [0, 1]$ (this can be deduced from [18, proposition 3.10] or the theorem below). The question then arises whether this is the typical situation, or perhaps typically $s\alpha(x) = \alpha(x)$ at least for 'most' points x.

If (X, f) is a topological dynamical system and $W \subset X$ is open and wandering (for all $n \geqslant 1$, $f^n(W) \cap W = \emptyset$), then each backward orbit branch visits W at most once, so W is disjoint from all $s\alpha$-limit sets of f. But a wandering set can contain α-limit points. In example 34 below we use this kind of wandering behaviour to show that for the generic interval map $f : [0, 1] \to [0, 1]$ (in the topology of uniform convergence) there is a whole interval of points $x \in [0, 1]$ for which $\alpha(x, f) \neq s\alpha(x, f)$. We show in theorem 33 that if there are no wandering intervals then $\alpha(x, f) = s\alpha(x, f)$ for all $x \in [0, 1]$.

Theorem 33. *Let* $f : [0, 1] \to [0, 1]$ *be a continuous map. If* $\Omega(f) = [0, 1]$ *then* $\alpha(x) = s\alpha(x)$ *for all* $x \in [0, 1]$.

Proof. By [29, theorem 1] the set $\Omega(f) \backslash \overline{\mathrm{Rec}(f)}$ is at most countable and since $\overline{\mathrm{Rec}(f)}$ is a closed set we have $\overline{\mathrm{Per}(f)} = [0, 1]$. Thus f has a dense set of periodic points. By [21, theorem 3.9] (see also the discussion on page 40 in [21]) every non-periodic point of f belongs to the interior of some transitive cycle of intervals L. Fix $x \in [0, 1]$. We will show that $\alpha(x) \subseteq s\alpha(x)$. Let $y \in \alpha(x)$. If y is periodic then by theorem 8 $y \in s\alpha(x)$. Suppose that y

is not periodic and let L be the transitive cycle of intervals for f with $y \in \text{Int}(L)$ where interior is taken relative to $[0, 1]$. Let $\{x_i\}_{i=1}^{\infty}$ be a preimage sequence of x with some subsequence $x_{i_j} \to y$ as $j \to \infty$. The points from $\{x_i\}_{i=1}^{\infty}$ are all distinct since x is not periodic and thus there are infinitely many preimages of x in L. In particular, one of the preimages is a non-exceptional point of L and $y \in L \subset s\alpha(x)$ by equation (1). $\qquad \square$

Let $C^0([0, 1])$ be the complete metric space of all maps $f : [0, 1] \to [0, 1]$ with the usual uniform metric $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. If some comeager subset of maps in $C^0([0, 1])$ all have some property, then we call that property *generic*.

Example 34. Let $\xi : [0, 9] \to [0, 9]$ be the 'connect-the-dots' map with $\xi(0) = 0$, $\xi(1) = 7$, $\xi(2) = \xi(5) = 4$, $\xi(6) = \xi(8) = 7$, $\xi(9) = 9$, and which is linear (affine) on each of the intervals $[0, 1]$, $[1, 2]$, $[2, 5]$, $[5, 6]$, $[6, 8]$, and $[8, 9]$. Every map in $C^0([0, 1])$ has a fixed point. By perturbation, we can replace that fixed point with a small invariant interval on which we insert a miniature homeomorphic copy of ξ. Call the resulting map f and let A, C, W denote the images under the homeomorphism of $[5, 6]$, $[3, 8]$, and $[1, 2]$, respectively, so that $A \subset f(A) = f(C) = f(W) \subset C$. These containments are stable under further perturbation, that is, for all g sufficiently close to f the three intervals $g(A), g(C), g(W)$ all contain A and are contained in C. It follows that $A \subset g^n(W) \subset C$ for all $n = 1, 2, \ldots$. Then each $x \in A$ has g-preimages of arbitrarily high order in W, so by compactness $\alpha(x, g) \cap W \neq \emptyset$. But since W is disjoint from $\Omega(g)$, it is disjoint from all $s\alpha$-limit sets of g, see theorem 5. This shows that $\alpha(x, g) \neq s\alpha(x, g)$ for all x in the non-degenerate interval A. We have identified an open dense set of maps $g \in C^0([0, 1])$ for which the set $\{x | \alpha(x, g) \neq s\alpha(x, g)\}$ contains a whole interval.

5. On special α-limit sets which are not closed

5.1. Points in the closure of a special α-limit set

We start section 5 with two theorems which are important for understanding any non-closed $s\alpha$-limit sets of an interval map. Theorem 35 relates the closure of a $s\alpha$-limit set to the three kinds of maximal ω-limit sets. Theorem 36 determines precisely which points do or do not have a closed $s\alpha$-limit set, and which points from the closure are not present in the limit set. In section 5.2 we apply these results to establish some topological properties of non-closed $s\alpha$-limit sets for interval maps, showing that they are always uncountable, nowhere dense, and of type F_σ and G_δ. In section 5.3 we address the question of which interval maps have all of their $s\alpha$-limit sets closed. Most importantly, this holds in the piecewise monotone case. In section 5.4 we give a concrete example of an interval map with a non-closed $s\alpha$-limit set.

Recall that a *generating sequence* is any nested sequence of cycles of intervals $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ for an interval map f with periods tending to infinity. In light of the discussion at the beginning of section 3.1 we may define a *maximal solenoidal set* as the intersection $Q = \bigcap \text{Orb}(I_n)$ of a generating sequence. Recall that a *solenoidal ω-limit set* is an infinite ω-limit set containing no periodic points, and a solenoidal ω-limit set is always contained in a maximal solenoidal set. Recall also that the *Birkhoff centre* of f is the closure of the set of recurrent points.

Theorem 35. *Let f be a continuous interval map, let $x \in \overline{s\alpha(y)}$, and suppose that any of the following conditions holds:*

(a) x is periodic,

(*b*) *x belongs to a basic set B, or*

(*c*) *x is a recurrent point in a solenoidal ω-limit set.*

Then $x \in s\alpha(y)$.

Theorem 36. *Let* $f: [0, 1] \to [0, 1]$ *be a continuous interval map and* $y \in [0, 1]$. *The set* $s\alpha(y)$ *is not closed if and only if y belongs to a maximal solenoidal set Q which contains a nonrecurrent point from the Birkhoff centre of f. In this case*

$$\overline{s\alpha(y)} \backslash s\alpha(y) = Q \cap \left(\overline{\mathrm{Rec}(f)} \backslash \mathrm{Rec}(f) \right).$$

The rest of section 5.1 is devoted to the proofs of these two theorems. For theorem 35 the main idea is that $\overline{s\alpha(y)} \subset \alpha(y)$, so we can apply theorems 6, 7, and 8. However, some extra care is needed for preperiodic points. Recall that *x* is *preperiodic* for *f* if $x \notin \mathrm{Per}(f)$ but there exists $n \geqslant 1$ such that $f^n(x) \in \mathrm{Per}(f)$.

Lemma 37. *Let f be a continuous interval map. If* $x \neq f(x) = f^2(x)$ *and* $x \in \overline{s\alpha(y)}$, *then* $x \in s\alpha(y)$.

Proof. We may suppose without loss of generality that $x < f(x)$. A set *U* is called a *right-hand neighbourhood* (resp. *left-hand neighbourhood*) of *x* if it contains an interval $(x, x + \epsilon)$ (resp. $(x - \epsilon, x)$) for some $\epsilon > 0$. By hypothesis either $x \in s\alpha(y)$, or every right-hand neighbourhood of *x* contains points from $s\alpha(y)$, or every left-hand neighbourhood of *x* contains points from $s\alpha(y)$. In the first case there is nothing to prove. The remaining two cases will be considered separately.

Suppose that every right-hand neighbourhood of *x* contains points from $s\alpha(y)$. We will construct inductively a sequence of points $y_n \to x$ and times $k_n > 0$ such that $f^{k_0}(y_0) = y$ and $f^{k_n}(y_n) = y_{n-1}$ for $n \geqslant 1$. We also construct points $a_n \in s\alpha(y)$ and open intervals $U_n \ni a_n$ that are compactly contained (i.e. their closure is contained) in $(x, f(x))$. The points a_n will decrease monotonically to *x*, the intervals U_n will be pairwise disjoint, and the inequalities $\sup U_n < y_{n-1} < \sup U_{n-1}$ will hold for all $n \geqslant 1$.

For the base case, choose any point $a_0 \in s\alpha(y)$ with $x < a_0 < f(x)$. Choose a small open interval $U_0 \ni a_0$ which is compactly contained in $(x, f(x))$. There exists $y_0 \in U_0$ and $k_0 > 0$ such that $f^{k_0}(y_0) = y$. Now we make the induction step. Suppose that $x < y_{n-1} < \sup U_{n-1}$ and choose $a_n \in s\alpha(y)$ with $x < a_n < \min\{y_{n-1}, \inf U_{n-1}, x + \frac{1}{n}\}$. Choose a small open interval $U_n \ni a_n$ compactly contained in $(x, f(x))$ with $\sup U_n < \min\{y_{n-1}, \inf U_{n-1}\}$. By theorem 5 we know that a_n is a non-wandering point, so there exist $b, c \in U_n$ and $k_n > 0$ such that $f^{k_n}(b) = c$. Thus $f^{k_n}([x, b])$ contains both $f(x), c$, but $c < y_{n-1} < f(x)$, so by the intermediate value theorem there is $y_n \in (x, b)$ with $f^{k_n}(y_n) = y_{n-1}$. Clearly $y_n < \sup U_n$, so we are ready to repeat the induction step. This completes the proof in the case when every right-hand neighbourhood of *x* contains points from $s\alpha(y)$.

Now suppose that every left-hand neighbourhood of *x* contains points from $s\alpha(y)$. Consider the set

$$W = \bigcap_{\epsilon > 0} W_\epsilon, \quad \text{where } W_\epsilon = \bigcup_{t=1}^{\infty} f^t((x - \epsilon, x]).$$

For $\epsilon > 0$ the set W_ϵ is invariant for *f*. It is connected because the intervals $f^t((x - \epsilon, x])$ all contain the common point $f(x)$. Now choose a point $a \in s\alpha(y) \cap (x - \epsilon, x)$. Since *a* is non-wandering there are points $b, c \in (x - \epsilon, x)$ such that $f^t(b) = c$ for some $t > 0$. Thus $c \in W_\epsilon$, so by connectedness we get $[x, f(x)] \in W_\epsilon$. Since these properties hold for arbitrary ϵ, we see that *W* is invariant, connected, and contains $[x, f(x)]$.

We claim that W contains a left-hand neighbourhood of x. Assume to the contrary that $x = \min W$. By continuity there is a point $z > f(x)$ such that $f([f(x), z]) \subset (x, 1]$. Again by continuity there is a point $w < x$ such that $f([w, x]) \subset (x, z)$. Fix $\epsilon > 0$ arbitrary. Put $U = (\max\{w, x - \epsilon\}, x)$. Choose $a \in s\alpha(y) \cap U$. Since a is non-wandering there are points $b, c \in U$ such that $f^s(b) = c$ for some $s > 0$. But $f(b) > x$, so there is some $t \geqslant 1$ such that $f^t(b) > x$ and $f^{t+1}(b) < x$. Since W is invariant, $f^t(b) \notin W$. Therefore $f^t(b) > z$, so it follows that $z \in W_\epsilon$. Since $\epsilon > 0$ was arbitrary, this shows that $z \in W$. Again fix $\epsilon > 0$ arbitrary and let U, a, b, c, s be as they were before. We have $f(b) \in (x, z) \subset W$ and W is invariant, so $f^s(b) = c \in W$, contradicting the assumption that $x = \min W$.

Finally, we construct inductively a monotone increasing sequence of points $y_n \to x$ and a sequence of times $k_n > 0$ such that $f^{k_0}(y_0) = y$ and $f^{k_n}(y_n) = y_{n-1}$ for $n \geqslant 1$. Let $\delta > 0$ be such that W contains the left-hand neighbourhood $(x - \delta, x)$. For the base case we find $a \in s\alpha(y) \cap (x - \delta, x)$. Then there is $y_0 \in (x - \delta, x)$ and $k_0 > 0$ such that $f^{k_0}(y_0) = y$. For the induction step, suppose we are given $y_{n-1} \in (x - \delta, x)$. Choose a positive number $\epsilon < \min\{|x - y_n|, \frac{1}{n}\}$. Since $(0, \delta) \subset W \subset W_\epsilon$, we see from the definition of W that there exist $y_n \in (x - \epsilon, x]$ and $k_n > 0$ such that $f^{k_n}(y_n) = y_{n-1}$. Clearly $y_n \neq x$, since $y_{n-1} \neq f(x)$. Therefore $y_n \in (x - \delta, x)$ and the induction can continue. $\qquad\square$

Proposition 38. *Let f be a continuous interval map. If x is preperiodic and $x \in \overline{s\alpha(y)}$, then $x \in s\alpha(y)$.*

Proof. Find n such that $x \neq f^n(x) = f^{2n}(x)$ and apply [18, proposition 2.9], which says $sa(y, f) = \bigcup_{j=0}^{n-1} s\alpha(f^j(y), f^n)$. Since the closure of a finite union is the union of the closures, we find j such that $x \in \overline{s\alpha(f^j(y), f^n)}$. By lemma 1, $x \in s\alpha(f^j(y), f^n)$. Again applying [18, proposition 2.9] we conclude $x \in s\alpha(y, f)$. $\qquad\square$

Proof of theorem 35. Let $x \in \overline{s\alpha(y)}$. Since $\alpha(y)$ is a closed set containing $s\alpha(y)$, we have $x \in \alpha(y)$ as well. If x is periodic, then by theorem 8, $x \in s\alpha(y)$. If x is a recurrent point in a solenoidal ω-limit set, then by theorem 6 we again have $x \in s\alpha(y)$. If x belongs to a basic set B, then it must be periodic, preperiodic, or have $\text{Orb}(x)$ infinite. In the periodic case theorem 8 applies. In the preperiodic case proposition 38 shows that $x \in s\alpha(y)$. And when $\text{Orb}(x)$ is infinite, then it must be contained in both B and $\alpha(y)$ since those sets are both invariant. So by theorem 7 we again have $x \in s\alpha(y)$. $\qquad\square$

Proof of theorem 36. Suppose first that $s\alpha(y)$ is not closed. Pick any point $x \in \overline{s\alpha(y)} \setminus s\alpha(y)$. By theorem 5 we have $x \in \overline{\text{Rec}(f)}$ and in particular $x \in \Lambda^1(f)$. Let $\omega(z)$ be a maximal ω-limit set of f containing x. By theorem 35, x is not recurrent and $\omega(z)$ is a solenoidal ω-limit set. Then $\omega(z)$ is contained in some maximal solenoidal set Q. By theorem 6 we know that $y \in Q$. But Q also contains x, so we have shown that y belongs to a maximal solenoidal set which contains a nonrecurrent point from the Birkhoff centre. Now if x' is any other point from $\overline{s\alpha(y)} \setminus s\alpha(y)$, then by the same argument, x' is a nonrecurrent point in the Birkhoff centre and belongs to a maximal solenoidal set Q' which also contains y. Since two maximal solenoidal sets are either disjoint or equal, we get $Q = Q'$. This shows that

$$\overline{s\alpha(y)} \setminus s\alpha(y) \subseteq Q \cap \left(\overline{\text{Rec}(f)} \setminus \text{Rec}(f) \right).$$

To prove the converse part of the theorem, suppose that Q is any maximal solenoidal set for f which contains a nonrecurrent point from the Birkhoff centre, and fix $y \in Q$. We will show that $s\alpha(y)$ is not closed, and that

$$\overline{s\alpha(y)}\backslash s\alpha(y) \supseteq Q \cap \left(\overline{\mathrm{Rec}(f)}\backslash\mathrm{Rec}(f)\right).$$

To that end, let x be any point from $Q \cap (\overline{\mathrm{Rec}(f)}\backslash\mathrm{Rec}(f))$. Choose a generating sequence $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \dots$ with $Q = \bigcap \mathrm{Orb}(I_n)$. Without loss of generality we may assume that $x \in I_n$ for each n. Since x is not recurrent we have $x \notin s\alpha(y)$ by theorem 6. To finish the proof it suffices to show that $x \in \overline{s\alpha(y)}$. We do so by finding periodic points arbitrarily close to x which are in the α-limit set of y, and then applying theorem 8.

Let K be the connected component of Q containing x. By theorem 10 we know that the singleton components of Q are recurrent points, and in the non-degenerate components of Q, only the endpoints can appear in ω-limit sets. Since x is in an ω-limit set but is not recurrent, we know that K is a non-degenerate interval with x as one of its endpoints. Without loss of generality we may assume that x is the left endpoint of K and write $K = [x, b]$.

Again using theorem 10 (and the fact that the set S_{Rec} referred to there is perfect) we know that each component of Q has at least one recurrent endpoint, and the endpoint of a component is recurrent if and only if it is the limit of points of other components of Q. Since x is not recurrent, we know that b is. We conclude that Q does not accumulate on x from the left and it does accumulate on b from the right.

Since x is in the Birkhoff centre $x \in \overline{\mathrm{Rec}(f)} = \overline{\mathrm{Per}(f)}$ and K contains no periodic points, we know that the periodic points of f accumulate on x from the left. In particular, x is not the left endpoint of $[0, 1]$. That means we can find non-empty left-hand neighbourhoods of x in $[0, 1]$, i.e. open intervals with right endpoint x.

Since Q does not accumulate on x from the left, we can find a left-hand neighbourhood of x which contains no points from Q, and then there must be some n such that I_n does not contain that left-hand neighbourhood. It follows that $x = \min(Q \cap I_n)$. At this moment we do not know if x is the left endpoint of I_n or not, only that it is the left-most point of Q in I_n.

Let m be the period of the cycle of intervals $\mathrm{Orb}(I_n)$ and let $g = f^m$. Then I_n is an invariant interval for g. Also f and g have the same periodic points (but not necessarily with the same periods). Since $g^i(x) \in I_n$ for all i and $x \in Q$ is neither periodic nor preperiodic, there must be $i \leqslant 2$ such that $g^i(x)$ is not an endpoint of I_n. By continuity there is $\delta > 0$ such that $g^i((x - \delta, x]) \subset I_n$. Then by invariance we get

$$g^j((x - \delta, x]) \subset I_n \quad \text{for all } j \geqslant i. \tag{2}$$

Choose an arbitrary positive real number $\epsilon < \delta$. Choose a periodic point p in $(x - \epsilon, x)$. Let k be the period of p under the map g. Then $g^{ik}(p) = p$ and by (2) $g^{ik}(p) \in I_n$. This shows that $p \in I_n$. (Incidentally, we see that x is not the left endpoint of I_n.)

The set of fixed points of g^k is closed, so let p' be the largest fixed point of g^k in the interval $[p, x]$. Let h be the map $h = g^k = f^{mk}$. We know that $h(x)$ is an element of Q in I_n, and therefore it lies to the right of x. We also know $h(x) \notin [x, b]$ because no component of Q returns to itself. So the order of our points is $p' = h(p') < x < b < h(x)$, and the graph of $h|_{(p',x]}$ lies above the diagonal, since there are no other fixed points there, see figure 6.

Since Q accumulates on b from the right we can find $n' > n$ such that one of the components L of $\mathrm{Orb}(I_{n'}, f)$ lies between b and $h(x)$. By the intermediate value theorem, $h([p', x])$ contains L.

Let U be any right-hand neighbourhood of p', that is $U = (p', z)$ with $z - p' > 0$ as small as we like. We consider the orbit of U. Since the graph of h lies above the diagonal on $(p', x]$ we get a monotone increasing sequence $(h^j(z))_{j=0}^l$ with $l \geqslant 1$ minimal such that $h^l(z) \geqslant x$ (if there is no such l, then $h^j(z)$ converges to a fixed point of h in the interval $[z, x]$, which contradicts the choice of p'). Then $h^{l+1}(U) \supset L$. In particular, $\mathrm{Orb}(U, f) \supset \mathrm{Orb}(L, f) \supset \mathrm{Orb}(I_{n'}, f) \supset Q$.
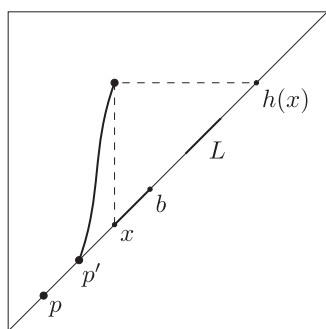
Figure 6. The graph of $h|_{I_n}$. Since $\mathrm{Orb}(L, f) \supset Q$ and each right-hand neighbourhood of p' eventually covers L, we see that $p' \in \alpha(y)$. Note: we do not claim that h is monotone on $(p', x]$, only that the graph stays above the diagonal.

We see that for every right-hand neighbourhood U of p' there is a point $y' \in U$ and a time t such that $f^t(y') = y$. As the neighbourhood U shrinks, the value of t must grow without bound, because p' is periodic. This shows that $p' \in \alpha(y) = \alpha(y, f)$. By theorem 8 it follows that $p' \in s\alpha(y)$.

Since p' was chosen from the ϵ-neighbourhood $(x - \epsilon, x)$ and $\epsilon > 0$ can be arbitrarily small, we conclude that $x \in \overline{s\alpha(y)}$. This completes the proof. $\qquad\square$

5.2. Properties of a non-closed special α-limit set

As an application of theorems 36 and 35, we get the following results.

Theorem 39.   *If $s\alpha(y)$ is not closed, then it is uncountable and nowhere dense.*

Proof.   Suppose $s\alpha(y)$ is not closed. By theorem 36 we have $y \in Q$ for some solenoidal set $Q = \bigcap \mathrm{Orb}(I_n)$. By theorem 6 we know that $s\alpha(y)$ contains $Q \cap \mathrm{Rec}(f)$. This set is perfect [4, theorem 3.1], and therefore uncountable.

Let M be a transitive cycle of intervals for f. If $Q \cap M \neq \emptyset$, then by lemma 32, each cycle of intervals $\mathrm{Orb}(I_n)$ has period at most twice the period of M. This contradicts the fact that the periods of the cycles of intervals tend to infinity. Therefore Q does not intersect any transitive cycle of intervals M for f. In particular, y does not belong to any transitive cycle of intervals M. By corollary 20 we conclude that $s\alpha(y)$ does not contain any transitive cycle. By theorem 24 it follows that $s\alpha(y)$ is nowhere dense. $\qquad\square$

A subset of a compact metric space X is called F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of open sets. These classes of sets make up the second level of the Borel hierarchy. Closed sets and open sets make up the first level of the Borel hierarchy and they are always both F_σ and G_δ. The next result shows that the $s\alpha$-limit sets of an interval map can never go past the second level of the Borel hierarchy in complexity.

Theorem 40.   *Each $s\alpha$-limit set for a continuous interval map f is both F_σ and G_δ.*

Proof.   We write $\mathrm{Bas}(f)$ for the union of all basic ω-limit sets of f and $\mathrm{Sol}(f)$ for the union of all solenoidal ω-limit sets of f. We continue to write $\mathrm{Per}(f)$ for the union of all periodic orbits of f.

To prove that $s\alpha(y)$ is of type F_σ we express it as the following union

$$s\alpha(y) = (s\alpha(y) \cap \mathrm{Per}(f)) \cup (s\alpha(y) \cap \mathrm{Bas}(f)) \cup (s\alpha(y) \cap \mathrm{Sol}(f)),$$

and show that each of the three sets in the union is of type F_σ.

The set $\mathrm{Per}(f) = \bigcup_n \{x : f^n(x) = x\}$ is clearly of type F_σ. By theorem 35, $s\alpha(y) \cap \mathrm{Per}(f)$ is a relatively closed subset of $\mathrm{Per}(f)$, and is therefore of type F_σ.

Since an interval map has at most countably many basic sets [4, lemma 5.2], their union $\mathrm{Bas}(f)$ is of type F_σ. By theorem 35, we know that $s\alpha(y) \cap \mathrm{Bas}(f)$ is a relatively closed subset of $\mathrm{Bas}(f)$, and is therefore of type F_σ.

By theorem 6 and corollary 12, we know that $s\alpha(y) \cap \mathrm{Sol}(f)$ is either the empty set, or a single minimal solenoidal set S, and minimal solenoidal sets are closed. Closed sets are trivially of type F_σ.

To prove that $s\alpha(y)$ is of type G_δ it is enough to show that $\overline{s\alpha(y)}\backslash s\alpha(y)$ is at most countable. By theorem 36 we know that this set is of the form

$$\overline{s\alpha(y)}\backslash s\alpha(y) = Q \cap (\overline{\mathrm{Rec}(f)}\backslash \mathrm{Rec}(f))$$

for some maximal solenoidal set Q. But by theorem 10 the only points in Q which can be in the Birkhoff centre but not recurrent are endpoints of non-degenerate components of Q. Since $Q \subset [0,1]$ has at most countably many non-degenerate components, this completes the proof.   □

5.3. Maps which have all special α-limit sets closed

In [18] it was proved that all $s\alpha$-limit sets of f are closed if $\mathrm{Per}(f)$ is closed. This is a very strong condition; it implies in particular that f has zero topological entropy. In this section we give necessary and sufficient criteria to decide if all $s\alpha$-limit sets of f are closed. We show in particular that this is the case for piecewise monotone maps. Note that an interval map $f : [0,1] \to [0,1]$ is called *piecewise monotone* if there are finitely many points $0 = c_0 < c_1 < \cdots < c_n = 1$ such that for each $i < n$, the restriction $f|_{[c_i, c_{i+1}]}$ is monotone, i.e. non-increasing or non-decreasing.

Theorem 41. *Let $f : [0,1] \to [0,1]$ be a continuous interval map. The following are equivalent:*

(a) *For some $y \in [0,1]$, $s\alpha(y)$ is not closed.*

(b) *The attracting centre $\Lambda^2(f)$ is not closed.*

(c) *The attracting centre is strictly contained in the Birkhoff centre $\Lambda^2(f) \subsetneq \overline{\mathrm{Rec}(f)}$.*

(d) *Some solenoidal ω-limit set of f contains a non-recurrent point in the Birkhoff centre.*

Proof of theorem 41. (a) $\Rightarrow$ (b): suppose there is a point $y \in [0,1]$ with $s\alpha(y)$ not closed. Choose $x \in \overline{s\alpha(y)}\backslash s\alpha(y)$. By theorem 36 we know that x is a non-recurrent point in a solenoidal set. By theorem 6 it follows that no $s\alpha$-limit set of f contains x. This shows that $x \in \overline{\mathrm{SA}(f)}\backslash \mathrm{SA}(f)$. Thus $\mathrm{SA}(f)$ is not closed. But $\Lambda^2(f) = \mathrm{SA}(f)$ by theorem 5.

(b) $\Rightarrow$ (c): this follows immediately from the containments $\mathrm{Rec}(f) \subset \Lambda^2(f) \subset \overline{\mathrm{Rec}(f)}$ in theorem 5.

(c) $\Rightarrow$ (d): suppose $\mathrm{SA}(f) = \Lambda^2(f) \neq \overline{\mathrm{Rec}(f)}$. Then we can find $x \in \overline{\mathrm{Rec}(f)}\backslash \mathrm{SA}(f)$. By theorem 5 we know that $x \in \Lambda^1(f)$. Because x is in an ω-limit set, it must belong to a periodic orbit, a basic set, or a solenoidal ω-limit set.

Each periodic orbit is contained in the $s\alpha$-limit set of any one of its points. Each basic set is also contained in a $s\alpha$-limit set by corollary 16. Since we supposed that x is not in any $s\alpha$-limit set, we must conclude that x is not in a periodic orbit or a basic set.

Now we know that x belongs to a solenoidal ω-limit set. Since x is not in any $s\alpha$-limit set we may use theorem 5 to conclude that x is not recurrent.

(d) $\Rightarrow$ (a): suppose a solenoidal ω-limit set $\omega(z)$ contains a non-recurrent point x in the Birkhoff centre $x \in \overline{\mathrm{Rec}(f)}$. Since $\omega(z)$ is solenoidal it has a generating sequence, i.e. a nested sequence of cycles of intervals $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ with $\omega(z) \subseteq Q = \bigcap_n \mathrm{Orb}(I_n)$. By theorem 36 for any $y \in Q$ the set $s\alpha(y)$ is not closed. $\qquad\square$

Corollary 42. *If f is a piecewise monotone continuous interval map, then all $s\alpha$-limit sets of f are closed.*

Proof. By [4, lemma PM2] each point in a solenoidal ω-limit set for a piecewise monotone map f is recurrent, so condition (d) of theorem 41 can never be satisfied. $\qquad\square$

We remark that in general the conditions of theorem 41 may be difficult to verify. Even condition (d) is difficult, since the non-recurrent points in a solenoidal ω-limit set need not belong to the Birkhoff centre. For an example, see [3]. But for maps with zero topological entropy, the whole picture simplifies considerably. For the definitions of topological entropy and Li–Yorke chaos we refer the reader to any of the standard texts in topological dynamics, e.g. [21].

Corollary 43. *Let $f\colon [0,1] \to [0,1]$ be a continuous interval map with zero topological entropy. Then all $s\alpha$-limit sets of f are closed if and only if $\mathrm{Rec}(f)$ is closed.*

Proof. For a continuous interval map f with zero topological entropy, the set of recurrent points is equal to $\Lambda^2(f)$ by [29]. Now apply theorem 41. $\qquad\square$

Corollary 44. *Suppose $f\colon [0,1] \to [0,1]$ is not Li–Yorke chaotic. Then all $s\alpha$-limit sets of f are closed.*

Proof. When f is not Li–Yorke chaotic, Steele showed that $\mathrm{Rec}(f)$ is closed [25, corollary 3.4]. Moreover, it is well known that such a map has zero topological entropy, see e.g. [21]. Now apply corollary 43. $\qquad\square$

5.4. Example of a non-closed special α-limit set

In 1986 Chu and Xiong constructed a map $f : [0,1] \to [0,1]$ with zero topological entropy such that $\mathrm{Rec}(f)$ is not closed [10]. This example appeared six years before the definition of $s\alpha$-limit sets [15], but by corollary 43 it provides an example of a continuous interval map whose $s\alpha$-limit sets are not all closed.

In this section we give a short direct proof that one of the $s\alpha$-limit sets of Chu and Xiong's map f is not closed. Here are the key properties of the map f from [10].

(a) There is a nested sequence of cycles of intervals $[0,1] = M_0 \supset M_1 \supset M_2 \supset \ldots$ for f, where $M_n = \mathrm{Orb}(J_n)$ has period 2^n.
(b) For each $n \in \mathbb{N}$ the interval J_n is the connected component of M_n which appears farthest to the left in $[0,1]$.
(c) For each $n \in \mathbb{N}$ we can express $J_n = A_n \cup J_{n+1} \cup B_n \cup K_{n+1} \cup C_n$ as a union of five closed non-degenerate intervals with disjoint interiors appearing from left to right in the order $A_n < J_{n+1} < B_n < K_{n+1} < C_n$.
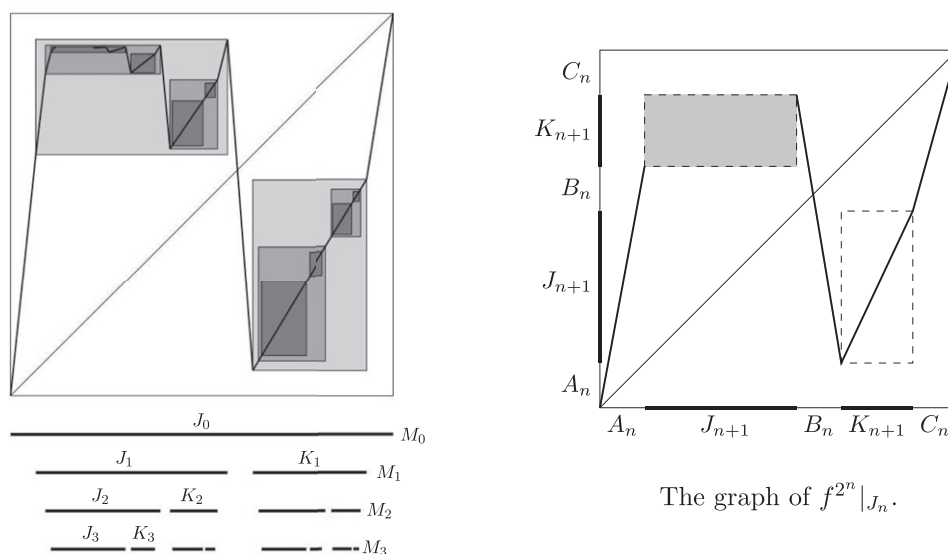
The graph of $f^{2^n}|_{J_n}$.

Figure 7. A map with a $s\alpha$-limit set which is not closed.

(d) For each $n \in \mathbb{N}$ the map $f^{2^n} : J_n \to J_n$ has the following properties:

   (1) $f^{2^n}|_{A_n} : A_n \to A_n \cup J_{n+1} \cup B_n$ is an increasing linear bijection,
   (2) $f^{2^n}|_{J_{n+1}} : J_{n+1} \to K_{n+1}$ is surjective,
   (3) $f^{2^n}|_{B_n} : B_n \to K_{n+1} \cup B_n \cup J_{n+1}$ is a decreasing linear bijection,
   (4) $f^{2^n}|_{K_{n+1}} : K_{n+1} \to J_{n+1}$ is an increasing linear bijection, and
   (5) $f^{2^n}|_{C_n} : C_n \to B_n \cup K_{n+1} \cup C_n$ is an increasing linear bijection.

(e) The nested intersection $J_\infty = \bigcap_{n=0}^{\infty} J_n$ is a non-degenerate interval $J_\infty = [x, y]$.

   The graph of the map $f : [0, 1] \to [0, 1]$ is shown in figure 7. Chu and Xiong showed that the left endpoint x of J_∞ is not recurrent, but it is a limit of recurrent points. We give a short direct proof that $s\alpha(x)$ is not closed.

Theorem 45. *Let f be the continuous interval map defined in* [10] *and let x be the left endpoint of J_∞ as defined above. Then $x \in \overline{s\alpha(x)} \setminus s\alpha(x)$, and therefore $s\alpha(x)$ is not closed.*

Proof. Fix $n \in \mathbb{N}$ and let a_n be the left endpoint of the interval A_n. Property (d) tells us that $f^{2^n} : A_n \to A_n \cup J_{n+1} \cup B_n$ is linear, say, with slope λ_n. By property (e) we have $x \in J_{n+1}$. Therefore there is a backward orbit branch $\{x_i\}_{i=0}^{\infty}$ of x such that $x_{k \cdot 2^n} = a_n + \frac{x - a_n}{\lambda_n^k}$ for all $k \in \mathbb{N}$. This shows that $a_n \in s\alpha(x)$. But $n \in \mathbb{N}$ was arbitrary. If we let $n \to \infty$, then $a_n \to x$. This shows that $x \in \overline{s\alpha(x)}$.

   Now we will show that $x \notin s\alpha(x)$. Let $\{x_i\}_{i=0}^{\infty}$ be any backward orbit branch of x. Let $Q = \bigcap_{n=0}^{\infty} M_n$. We distinguish two cases. First suppose that $x_i \in Q$ for all i. For any given $i \geqslant 1$ we can choose n with $2^n > i$. Since $x_i \in M_n$, $f^i(x_i) \in J_n$, and M_n is a cycle of intervals of period 2^n, we know that $x_i \notin J_n$. Since J_n is the left-most component of M_n in $[0, 1]$, it follows that $x_i > y > x$ (recall that $J_\infty = \bigcap J_n = [x, y]$ is non-degenerate). Since this holds for all $i \geqslant 1$ we see that the backward orbit branch does not accumulate on x.

   Now suppose there is i_0 with $x_{i_0} \notin Q$. For each $i \geqslant i_0$ there is $n(i) \in \mathbb{N}$ such that $x_i \in M_{n(i)} \setminus M_{n(i)+1}$. Since $f(x_{i+1}) = x_i$ and each M_n is invariant, we get $n(i + 1) \leqslant n(i)$.

A non-increasing sequence of natural numbers must eventually reach a minimum, say, $n(i_1) = n(i_1 + 1) = \cdots = n$. For $i \geqslant i_1$, $x_i \notin M_{n+1}$, so in particular $x_i \notin J_{n+1}$. But by properties (c) and (e) we know that J_{n+1} is a neighbourhood of x. This shows that this backward orbit branch does not accumulate on x either.   □

6. Open problems

Only one problem concerning $s\alpha$-limit sets of interval maps in [18] remains open:

Problem 46 [18]. Characterize all subsets A of [0, 1] for which there exists a continuous map $f : [0, 1] \to [0, 1]$ and a point $x \in [0, 1]$ such that $s\alpha(x, f) = A$.

We have seen that even for interval maps, $s\alpha$-limit sets need not be closed. If we want to work with closed limit sets, then there are several possible solutions. The first one, suggested to us by Snoha, is to answer the following question: *what are some sufficient conditions on a topological dynamical system (X, f) so that all of its $s\alpha$-limit sets are closed?* In this regard we state one conjecture which we were not able to resolve.

Conjecture 47. If $f : [0, 1] \to [0, 1]$ is continuously differentiable, then all $s\alpha$-limit sets of f are closed.

Another possibility is to ask whether the 'typical' continuous interval map has all $s\alpha$-limit sets closed. Let $C^0([0, 1])$ be the complete metric space of all maps $f : [0, 1] \to [0, 1]$ with the usual uniform metric $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. If some comeager subset of maps in $C^0([0, 1])$ all have some property, then we call that property *generic*.

Problem 48. Is the property of having all $s\alpha$-limit sets closed a generic property in $C^0([0, 1])$?

Another possible solution is to work with the closures of $s\alpha$-limit sets. Therefore we propose the following definition.

Definition 49. Let (X, f) be a discrete dynamical system (i.e. a continuous self-map on a compact metric space) and let $x \in X$. The β-*limit set of* x, denoted $\beta(x)$ or $\beta(x, f)$, is the smallest closed set such that $d(x_n, \beta(x)) \to 0$ as $n \to \infty$ for every backward orbit branch $\{x_n\}_{n=0}^{\infty}$ of the point x.

The letter β here means 'backward', since β-limit sets serve as attractors for backward orbit branches. It is clear from the definition that $\beta(x) = \overline{s\alpha(x)}$.

Proposition 50. *If (X, f) is a discrete dynamical system and $x \in X$, then $\beta(x)$ is closed and strongly invariant, i.e. $f(\beta(x)) = \beta(x)$. Additionally, $\beta(x)$ is nonempty if and only if $x \in \bigcap_{n=0}^{\infty} f^n(X)$. In particular, $\beta(x)$ is nonempty for every $x \in X$ when f is surjective.*

Proof. By [18, corollary 2.7] we have $f(s\alpha(x)) = s\alpha(x)$. Taking closures, we get $f(\beta(x)) = \beta(x)$. Additionally, $\beta(x)$ is nonempty if and only if $s\alpha(x)$ is. But by [18, proposition 2.3] we have $s\alpha(x) \neq \emptyset$ if and only if $x \in \bigcap_{n=0}^{\infty} f^n(X)$.   □

We can simplify problem 46 by working with β-limit sets, since they are always closed.

Problem 51. Characterize all subsets $A \subseteq [0, 1]$ for which there exists a continuous map $f : [0, 1] \to [0, 1]$ and a point $x \in [0, 1]$ such that $\beta(x, f) = A$.

If X is a compact metric space, then the space $K(X)$ consisting of all nonempty compact subsets of X can be topologized with the Hausdorff metric and it is again compact. The map

$X \ni x \mapsto \omega(x, f) \in K(X)$ associated to a dynamical system (X, f) is usually far from continuous, but its Baire class can be useful, see e.g. [25]. Therefore we propose the following question.

Problem 52. Of what Baire class (if any) is the function $[0, 1] \ni x \mapsto \beta(x, f) \in K([0, 1])$ when f is a surjective continuous interval map?

Hero used $s\alpha$-limit sets to characterize the attracting centre $\Lambda^2(f)$ of a continuous interval map f, and some work has been done to extend his results to trees and graphs [15, 26, 27]. We conjecture that for graph maps, the β-limit sets can be used to characterize the Birkhoff centre $\overline{\text{Rec}(f)}$ as follows:

Conjecture 53. Let $f : X \to X$ be a graph map, and let $x \in X$. The following are equivalent:

(a) $x \in \overline{\text{Rec}(f)}$
(b) $x \in \beta(x)$
(c) There exists $y \in X$ such that $x \in \beta(y)$.

The coexistence of periodic orbits for interval maps was studied by Sharkovsky [22]. A special case of his theorem, proved independently by Li and Yorke, says that if $f : [0, 1] \to [0, 1]$ has a periodic orbit of period three, then it has periodic orbits of all periods [19].

This suggests the problem of studying the coexistence of periodic orbits within special α-limit sets.[5] In this spirit, we offer one conjecture and one open problem.

Conjecture 54. Let $f : [0, 1] \to [0, 1]$ and $x \in [0, 1]$. If $s\alpha(x)$ contains a periodic orbit of period 3, then for every positive integer n, $s\alpha(x)$ contains a periodic orbit of period n or $2n$.

Problem 55. For which subsets $A \subseteq \mathbb{N}$ is there a map $f : [0, 1] \to [0, 1]$ and a point $x \in [0, 1]$ such that A is the set of periods of all periodic orbits of f contained in $s\alpha(x)$?

In support of conjecture 54, we show that the conclusion holds for $n = 1$.

Lemma 56. *Let* $f : [0, 1] \to [0, 1]$ *and* $x \in [0, 1]$. *If* $s\alpha(x)$ *contains a periodic orbit of period 3, then it contains a periodic orbit of period 1 or 2 as well.*

Proof. Let $s\alpha(x)$ contain the periodic orbit $\{a, b, c\}$ for the interval map f with $a < b < c$. We may assume without loss of generality that $f(a) = b$, $f(b) = c$, and $f(c) = a$. By continuity we may choose a closed interval $U = [b - \epsilon, b + \epsilon]$ with $\epsilon > 0$ small enough that $f^2(U) < U < f(U)$, that is to say, $\max f^2(U) < \min U < \max U < \min f(U)$. Now find $x_1 \in U$ and $n_1 \geqslant 1$ such that $f^{n_1}(x_1) = x$. By the intermediate value theorem we can find $x_2 \in (\max U, \min f(U))$ such that $f(x_2) = x_1$. Again, by the intermediate value theorem we can find $x_3 \in (x_1, x_2)$ such that $f(x_3) = x_2$. In the next step we find $x_4 \in (x_3, x_2)$ such that $f(x_4) = x_3$. Continuing inductively we find a whole sequence (x_i) such that $f(x_{i+1}) = x_i$, $i \geqslant 1$, arranged in the following order,

$$x_1 < x_3 < x_5 < \cdots < \cdots < x_6 < x_4 < x_2.$$

Since a bounded monotone sequence of real numbers has a limit, we may put $x_\infty^- = \lim_{i \to \infty} x_{2i+1}$ and $x_\infty^+ = \lim_{i \to \infty} x_{2i+2}$, and we have $x_\infty^- \leqslant x_\infty^+$. Then $f(x_\infty^-) = \lim f(x_{2i+1}) = \lim x_{2i} = x_\infty^+$ and similarly $f(x_\infty^+) = x_\infty^-$. This shows that $\{x_\infty^+, x_\infty^-\}$

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[5] Or equivalently, within β-limit sets, since by theorem 8 a periodic orbit of an interval map f is contained in $s\alpha(x)$ if and only if it is contained in $\beta(x)$.

is a periodic orbit contained in $s\alpha(x)$. The period is 2 if these points are distinct and 1 if they coincide.   □

Example 57. Let $f : [0,5] \to [0,5]$ be the 'connect-the-dots' map with $f(0) = 1$, $f(1) = 5$, $f(4) = 2$, $f(5) = 0$, and which is linear (affine) on each of the intervals $[0,1]$, $[1,4]$, and $[4,5]$. Then $s\alpha(0)$ contains the period-three orbit $\{0,1,5\}$ and the period-two orbit $\{2,4\}$, but not the unique period-one orbit $\{3\}$.

Example 58. Let $f : [0,8] \to [0,8]$ be the 'connect-the-dots' map with $f(0) = 4$, $f(4) = 8$, $f(5) = 3$, $f(8) = 0$, and which is linear (affine) on each of the intervals $[0,4]$, $[4,5]$, and $[5,8]$. Then $s\alpha(0)$ contains the period-three orbit $\{0,4,8\}$, the period-four orbit $\{1,5,3,7\}$ and the period-one orbit $\{\frac{14}{3}\}$, but not the unique period-two orbit $\{2,6\}$.

Acknowledgments

References

[1] Balibrea F, Dvorníková G, Lampart M and Oprocha P 2012 On negative limit sets for one-dimensional dynamics *Nonlinear Anal. Theory Methods Appl.* **75** 3262–7

[2] Birkhoff G 1927 *Dynamical Systems (Colloquium Publications vol 9)* (Providence, RI: American Mathematical Society)

[3] Block L and Coven E M 1986 ω-limit sets for maps of the interval *Ergod. Theor. Dynam. Syst.* **6** 335–44

[4] Blokh A M 1995 The 'spectral' decomposition for one-dimensional maps *Dynamics Reported (Expositions in Dynamical Systems vol 4)* (Berlin: Springer) pp 1–59

[5] Blokh A, Bruckner A M, Humke P D and Smítal J 1996 The space of ω-limit sets of a continuous map of the interval *Trans. Am. Math. Soc.* **348** 1357–72

[6] Bowen R 1975 ω-Limit sets for Axiom *A* diffeomorphisms *J. Differ. Equ.* **18** 333–9

[7] Conley C 1978 *Isolated Invariant Sets and the Morse Index (Regional Conference Series in Mathematics vol 38)* (Providence, RI: American Mathematical Society)

[8] Coven E M and Hedlund G A 1980 $\overline{P} = \overline{R}$ for maps of the interval *Proc. Am. Math. Soc.* **79** 316

[9] Coven E M and Nitecki Z 1981 Non-wandering sets of the powers of maps of the interval *Ergod. Theor. Dynam. Syst.* **1** 9–31

[10] Chu H and Xiong J C 1986 A counterexample in dynamical systems of the interval *Proc. Am. Math. Soc.* **97** 361

[11] Cui H and Ding Y 2010 The α-limit sets of a unimodal map without homtervals *Topol. Appl.* **157** 22–8

[12] Dowker Y N and Friedlander F G 1954 On limits sets in dynamical systems *Proc. London Math. Soc.* **3–4** 168–76

[13] Foryś-Krawiec M, Hantáková J and Oprocha P 2021 On the structure of α-limit sets of backward trajectories for graph maps (arXiv:2106.05539 [math.DS])

[14] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Applied Mathematical Sciences vol 42)* (Berlin: Springer)

[15] Hero M W 1992 Special α-limit points for maps of the interval *Proc. Am. Math. Soc.* **116** 1015

[16] Hirsch M W, Smith H L and Zhao X-Q 2001 Chain transitivity, attractivity, and strong repellors for semidynamical systems *J. Dyn. Differ. Equ.* **13** 107–31
[17] Jackson S, Mance B and Roth S 2021 A non-Borel special alpha-limit set in the square *Ergod. Theor. Dynam. Syst.* 1–11
[18] Kolyada S, Misiurewicz M and Snoha L 2020 Special α-limit sets *Dynamics: Topology and Numbers (Contemporary Mathematics vol 744)* (Providence, RI: American Mathematical Society) pp 157–73
[19] Li T-Y and Yorke J A 1975 Period three implies chaos *Am. Math. Mon.* **82** 985–92
[20] Przytycki F 1999 Conical limit set and Poincaré exponent for iterations of rational functions *Trans. Am. Math. Soc.* **351** 2081–99
[21] Ruette S 2017 *Chaos on the Interval (University Lecture Series vol 67)* (Providence, RI: American Mathematical Society)
[22] Sharkovskii A N 1995 Co-existence of cycles of a continuous mapping of the line into itself *Internationl Journal of Bifurcation and Chaos* **5** 1263–73
[23] Sharkovskii A N 1965 On attracting and attracted sets *Dokl. Akad. Nauk SSSR* **160** 1036–8
[24] Sharkovskii A N 1966 Continuous maps on the set of limit points of an iterated sequence *Ukr. Math. J.* **18** 127–30
[25] Steele T H 2003 Chaos and the recurrent set *Real Anal. Exch.* **29** 79–87
[26] Sun T, Xi H, Chen Z and Zhang Y 2004 The attracting centre and the topological entropy of a graph map *Advances in Mathematics (China)* **33** 540–6
[27] Sun T, Xi H and Liang H 2011 Special α-limit points and unilateral γ-limit points for graph maps *Sci. China Math.* **54** 2013–8
[28] Sun T, Tang Y, Su G, Xi H and Qin B 2018 Special α-limit points and γ-limit points of a dendrite map *Qual. Theory Dyn. Syst.* **17** 245–57
[29] Xiong J C 1988 The attracting centre of a continuous self-map of the interval *Ergod. Theor. Dynam. Syst.* **8** 205–13