On the Non-Emptiness of the Core of a Cooperative Fuzzy Game

David Bartl[1]

Department of Informatics and Mathematics, School of Business Administration in Karviná, Silesian University in Opava, Univerzitní náměstí 1934/3, 733 40 Karviná, Czechia
bartl@opf.slu.cz, david.bartl@post.cz

Abstract. We introduce the concept of a fuzzy coalition structure on a finite set of players. Then, we propose a new model of a cooperative fuzzy game with transferable utility: an existing coalition is assumed to endeavour in a branch of industry, and a deviation of a new coalition from the coalition structure is seen as an opportunity of the coalition. Based on these premisses, we introduce the concept of the core of the cooperative fuzzy TU-game with respect to a general fuzzy coalition structure. Finally, we define the concept of balancedness and formulate a generalization of the Bondareva-Shapley Theorem.

Keywords: Cooperative fuzzy TU-game · Core · Balanced game · Bondareva-Shapley theorem.

1 Introduction

Consider a classical cooperative game of n players with transferable utility. The *coalition* is any subset of the set $N = \{1, 2, \dots, n\}$ of the players, and the potency set $\mathcal{P}(N) = \{\, K : K \subseteq N \,\}$ of the set N is the collection of all coalitions $K \subseteq N$ that can potentially emerge. Finally, if a coalition $K \subseteq N$ emerges, then it will achieve its total profit of $v(K)$ units of some transferable utility (e.g. money); it is assumed that $v(\emptyset) = 0$. In other words, the cooperative game is given by its *coalition function*, which is a mapping $v\colon \mathcal{P}(N) \to \mathbb{R}$ such that $v(\emptyset) = 0$.

The *coalition structure* is any partition of the set N of the players; that is, the coalition structure is any collection $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ of coalitions such that $\bigcup_{p=1}^{r} S_p = N$ and $S_p \cap S_q = \emptyset$ whenever $p \neq q$ for $p, q = 1, 2, \dots, n$, and also $\emptyset \notin \mathcal{S}$.

Assume that a coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ has crystallized. It means that the coalitions $S_1, S_2, \dots, S_r$ have emerged, they exist now, and they will achieve the profits $v(S_1), v(S_2), \dots, v(S_r)$, respectively. Now, the purpose is that the players within each coalition $S_1, S_2, \dots, S_r$ divide the total profit of their coalition among themselves. The division of the profit among the players is described by the payoff vector.

The *payoff vector* is any vector $\boldsymbol{a} = (a_i)_{i=1}^{n} \in \mathbb{R}^n$, where a_i is the profit apportioned to the i-th player for $i = 1, 2, \dots, n$. It is usual to require that the payoff vector belongs to a certain solution concept of the cooperative game. Informally speaking, the solution concept is a mapping that assigns a certain set of payoff vectors (i.e. a subset of $\mathbb{R}^n$) to the coalition function $v\colon \mathcal{P}(N) \to \mathbb{R}$ and to the coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$. The core [8, 6, 7] is an example of the solution concept.

The *core* of the cooperative game with transferable utility (TU-game) given by the coalition function v with respect to the coalition structure $\mathcal{S}$ is the set

$$\mathcal{C} = \left\{ \boldsymbol{a} \in \mathbb{R}^n : \sum_{i\in S} a_i = v(S) \text{ for } S \in \mathcal{S} \text{ and } \sum_{i\in K} a_i \geq v(K) \text{ for } K \in \mathcal{P}(N) \setminus \mathcal{S} \right\},$$

see [1]. In words, the core is the set of all the payoff vectors $\boldsymbol{a} \in \mathbb{R}^n$ that satisfy the conditions of feasibility ($\sum_{i\in S} a_i \leq v(S)$ for $S \in \mathcal{S}$), efficiency or group rationality ($\sum_{i\in S} a_i \geq v(S)$ for $S \in \mathcal{S}$), and group stability ($\sum_{i\in K} a_i \geq v(K)$ for $K \in \mathcal{P}(N) \setminus \mathcal{S}$). Now, the key question is whether the core is non-empty.

The next classical result provides an answer to the question:

Bondareva-Shapley Theorem [3, 9]**.** *The core $\mathcal{C}$ of the cooperative TU-game given by the coalition function v with respect to the coalition structure $\mathcal{S} = \{N\}$ is non-empty if and only if the game is balanced.*

As we can see, the classical Bondareva-Shapley Theorem provides the answer in the special case when the coalition structure consists of the grand coalition ($\mathcal{S} = \{N\}$) only. We ask whether we can define the concept of balancedness with respect to a general coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ and prove the respective generalization of the Bondareva-Shapley Theorem. Regarding the generalization in the case of cooperative crisp TU-games, see [2]. Now, our purpose is to extend the results further to the case of cooperative fuzzy TU-games.

2 The core and balancedness of fuzzy TU-games

Consider again a cooperative game of n players with transferable utility. Now, the *fuzzy coalition* is any fuzzy subset $\tilde{K}$ of the set $N = \{1, 2, \dots, n\}$ of the players; we denote this fact by writing $\tilde{K} \mathrel{\tilde{\subseteq}} N$. Recall that any fuzzy subset $\tilde{K} \mathrel{\tilde{\subseteq}} N$ is given by its *membership vector* $\boldsymbol{\kappa} \in [0,1]^N$, which is here understood as a row vector $\boldsymbol{\kappa} = (\kappa_1 \ \kappa_2 \ \dots \ \kappa_n)$ with $0 \leq \kappa_i \leq 1$ for $i \in N$. Notice that if the membership vector is restricted so that $\boldsymbol{\kappa} \in \{0,1\}^N$; that is, $\kappa_i \in \{0,1\}$ for $i \in N$, then it corresponds to the crisp coalition $K \subseteq N$, with $i \in K$ if and only if $\kappa_i = 1$ for $i \in N$. The membership vector corresponding to the empty coalition $\emptyset$ and to the grand coalition N is $\boldsymbol{\chi}^\emptyset$ and $\boldsymbol{\chi}^N$, with $\chi_i^\emptyset = 0$ and $\chi_i^N = 1$, respectively, for $i \in N$.

The collection $\tilde{\mathcal{P}}(N) = \{\, \tilde{K} : \tilde{K} \mathrel{\tilde{\subseteq}} N \,\}$ of all fuzzy subsets of the set N contains all the fuzzy coalitions $\tilde{K} \mathrel{\tilde{\subseteq}} N$ that can potentially emerge. This collection is identified with the aforementioned set $[0,1]^N$ of all the membership vectors $\boldsymbol{\kappa}$.

The *fuzzy coalition structure* is any indexed collection $\tilde{\mathcal{S}} = (\tilde{S}_p)_{p\in\mathcal{R}}$ of fuzzy coalitions $\tilde{S}_p \mathrel{\tilde{\subseteq}} N$ with membership vectors $\boldsymbol{\sigma}_p \in [0,1]^N$ for $p \in \mathcal{R}$, where $\mathcal{R}$ is an index set, such that $\sum_{p\in\mathcal{R}} \boldsymbol{\sigma}_p = \boldsymbol{\chi}^N$ and $\boldsymbol{\sigma}_p \neq \boldsymbol{\chi}^\emptyset$ for $p \in \mathcal{R}$. Notice that, even though the set N of the players is finite, the index set $\mathcal{R}$ may be infinite and a fuzzy coalition $\tilde{S} \tilde{\subseteq} N$ may be present several times in the fuzzy coalition structure $\tilde{\mathcal{S}}$; that is, we may have $\tilde{S}_p = \tilde{S}_q$ for distinct $p, q \in \mathcal{R}$. Moreover, if the membership vectors are restricted so that $\boldsymbol{\sigma}_p \in \{0,1\}^N$, then the index set $\mathcal{R}$ is finite, let $\mathcal{R} = \{1, 2, \ldots, r\}$, say, and the fuzzy coalition structure $\tilde{\mathcal{S}}$ reduces to the crisp coalition structure $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ with $S_p = \{\, i \in N : (\boldsymbol{\sigma}_p)_i = 1 \,\}$ for $p = 1, 2, \ldots, r$. We obviously have $\bigcup_{p=1}^{r} S_p = N$ and $S_p \cap S_q \neq \emptyset$ iff $p = q$ for $p, q = 1, 2, \ldots, r$.

Assume that a fuzzy coalition structure $\tilde{\mathcal{S}} = (\tilde{S}_p)_{p \in \mathcal{R}}$ has crystallized. It means that the fuzzy coalitions $\tilde{S}_p \tilde{\subseteq} N$, for $p \in \mathcal{R}$, have emerged and exist. We interpret the fact that $0 \leq (\boldsymbol{\sigma}_p)_i \leq 1$ for $p \in \mathcal{R}$ so that the player i is involved in the coalition $\tilde{S}_p$ for "part-time job" in general; that is, the player is not involved in the coalition at all if $(\boldsymbol{\sigma}_p)_i = 0$, the player is involved for "full-time job" if $(\boldsymbol{\sigma}_p)_i = 1$, and the player is involved for "part-time job" in the remaining cases. Moreover, we understand the fact that formally the same coalition $\tilde{S}_p = \tilde{S}_q$, for $p, q \in \mathcal{R}$ with $p \neq q$, can be present several times in the coalition structure $\tilde{\mathcal{S}}$ so that the coalitions $\tilde{S}_p$ and $\tilde{S}_q$ are actually *distinct* and they endeavour in different branches of industry in general. Given this interpretation, it follows that the total profits achieved by the distinct coalitions $\tilde{S}_p$ and $\tilde{S}_q$, both of which exist at the same time, may be distinct too in general.

Based on these considerations, we propose a new model of cooperative fuzzy game with transferable utility. We propose that the cooperative fuzzy game is given by a pair of functions $V\colon \mathcal{R} \to \mathbb{R}$ and $v\colon [0,1]^N \to \mathbb{R}$ with $v(\boldsymbol{\chi}^{\emptyset}) = 0$. The first function V assigns the total profit of $V(p)$ units of some transferable utility to any fuzzy coalition $\tilde{S}_p$ of the present fuzzy coalition structure $\tilde{\mathcal{S}}$ for $p \in \mathcal{R}$; that is, the total profit $V(p)$ is assigned to any coalition $\tilde{S}_p$ that presently exists and is active and endeavouring in some branch of industry. (This approach loosely resembles that of Thrall and Lucas [10].) Now, a new fuzzy coalition $\tilde{K} \tilde{\subseteq} N$ may take the opportunity and form, leave the present coalition structure $\tilde{\mathcal{S}}$, and start to endeavour in a new branch of industry. This is the reason why we consider the second function v. It assigns the total profit of $v(\boldsymbol{\kappa})$ units of the transferable utility to the fuzzy coalition $\tilde{K} \tilde{\subseteq} N$ that decides to take the opportunity and leave the present coalition structure $\tilde{\mathcal{S}}$.

(We remark that the above model can easily be adapted to include the case of restricted cooperation: Let $\mathcal{A} \subseteq [0,1]^N$ be the collection of the membership vectors that correspond to the feasible fuzzy coalitions. We then define the function v on the collection $\mathcal{A}$ only ($v\colon \mathcal{A} \to \mathbb{R}$) and adapt the below given considerations accordingly.)

Now, again, the purpose is that the players within each fuzzy coalition $\tilde{S}_p$ divide the total profit $V(p)$ of their coalition among themselves for $p \in \mathcal{R}$. The division of the profit will be described by the *payoff matrix* which is any matrix $\boldsymbol{A} \in \mathbb{R}^{N \times \mathcal{R}}$, where a_{ip} is the profit apportioned to player i in coalition $\tilde{S}_p$ for $i \in N$ and for $p \in \mathcal{R}$. Moreover, we set $a_{ip} := 0$ for $i \in N$ and for $p \in \mathcal{R}$ such that $(\boldsymbol{\sigma}_p)_i = 0$; that is, the player i is not involved in the fuzzy coalition $\tilde{S}_p$ at all. (The total profit of player i achieved via all the player's involvements in the coalitions is the row sum $\pi_i = \sum_{p\in\mathcal{R}} a_{ip}$ for $i \in N$.) Our purpose is to extend the classical concept of the core to the present setting. Thus, consider a payoff matrix $\boldsymbol{A} \in \mathbb{R}^{N\times\mathcal{R}}$. We agree that, if $\boldsymbol{A}$ belongs to the core, then the equations $\sum_{i\in N} a_{ip} = V(p)$, which express the feasibility and efficiency or group rationality, must hold for all $p \in \mathcal{R}$. Regarding the group stability, assume that a fuzzy coalition $\tilde{K} \tilde{\subseteq} N$ with membership vector $\boldsymbol{\kappa} \in [0,1]^N$ takes the opportunity and deviates from the present coalition structure $\tilde{\mathcal{S}}$. Then the coalition $\tilde{K}$ endeavouring in a new branch industry will achieve its total profit of $v(\boldsymbol{\kappa})$ units of the utility. We stipulate that each player $i \in N$ must have left some coalitions so that the sum of the players "part-time jobs" exceeds κ_i. Mathematically speaking, we stipulate that there exists an index subset $\mathcal{K} \subseteq \mathcal{R}$ such that $\sum_{p\in\mathcal{K}} \boldsymbol{\sigma}_p \geq \boldsymbol{\kappa}$. Though the index subset $\mathcal{K} \subseteq \mathcal{R}$ could be infinite in general, we shall assume that the index subset $\mathcal{K}$ is finite to obtain a simple definition of balancedness below. Then the inequalities which prevent the fuzzy coalition $\tilde{K} \tilde{\subseteq} N$ from the deviation from the coalition structure $\tilde{\mathcal{S}}$ are $\sum_{i\in N}\sum_{p\in\mathcal{K}} a_{ip} \geq v(\boldsymbol{\kappa})$ for every finite $\mathcal{K} \subseteq \mathcal{R}$ such that $\sum_{p\in\mathcal{K}} \boldsymbol{\sigma}_p \geq \boldsymbol{\kappa}$.

To conclude, we define the *core* of the cooperative fuzzy TU-game given by its fuzzy coalition structure $\tilde{\mathcal{S}} = (\tilde{S}_p)_{p\in\mathcal{R}}$, the coalition of this fuzzy coalition structure function $V\colon \mathcal{R} \to \mathbb{R}$ and the fuzzy coalition function $v\colon [0,1]^N \to \mathbb{R}$ with $v(\boldsymbol{\chi}^{\emptyset}) = 0$ to be the set

$$\begin{aligned}
\mathcal{C} = \big\{\, \boldsymbol{A} \in \mathbb{R}^{N\times\mathcal{R}} : (\boldsymbol{\sigma}_p)_i = 0 \implies a_{ip} = 0 \qquad & \text{for} \quad i \in N \text{ and for } p \in \mathcal{R}, \\
\textstyle\sum_{i\in N} a_{ip} = V(p) \qquad & \text{for} \quad p \in \mathcal{R}, \\
\textstyle\sum_{p\in\mathcal{K}}\sum_{i\in N} a_{ip} \geq v(\boldsymbol{\kappa}) \qquad & \text{for} \quad \boldsymbol{\kappa} \in [0,1]^N \text{ and} \\
& \text{for finite } \mathcal{K} \subseteq \mathcal{R} \text{ such that } \textstyle\sum_{p\in\mathcal{K}} \boldsymbol{\sigma}_p \geq \boldsymbol{\kappa} \,\big\}
\end{aligned}$$

Notice that, if $\boldsymbol{A} \in \mathcal{C}$, then each of the variables a_{ip} is bounded below and above for $i \in N$ and for $p \in \mathcal{R}$. Indeed, if $i \in N$ and $p \in \mathcal{R}$ are such that $(\boldsymbol{\sigma}_p)_i = 0$, then $a_{ip} = 0$. Consider now $i \in N$ and $p \in \mathcal{R}$ are such that $(\boldsymbol{\sigma}_p)_i > 0$. Take the membership vector $\boldsymbol{\kappa} \in [0,1]^N$ such that $\kappa_i = (\boldsymbol{\sigma}_p)_i$ and $\kappa_j = 0$ for $j \in N \setminus \{i\}$. Then $a_{ip} \geq v(\boldsymbol{\kappa})$, which is a lower bound. Let $\underline{a}_{ip}$ be a lower bound of a_{ip} for $i \in N$ and for $p \in \mathcal{R}$. Consider again $i \in N$ and $p \in \mathcal{R}$ such that $(\boldsymbol{\sigma}_p)_i > 0$. We then have $a_{ip} + \sum_{j\in N\setminus\{i\}} \underline{a}_{jp} \leq \sum_{j\in N} a_{jp} = V(p)$, whence $a_{ip} \leq V(p) - \sum_{j\in N\setminus\{i\}} \underline{a}_{jp}$, which is an upper bound. Let $\overline{a}_{ip}$ be an upper bound of a_{ip} for $i \in N$ and for $p \in \mathcal{R}$.

Let us suppose wlog that $\underline{a}_{ip} \leq \overline{a}_{ip}$ for $i \in N$ and for $p \in \mathcal{R}$. (Should we have $\underline{a}_{ip} > \overline{a}_{ip}$, then let $\underline{a}_{ip} := \overline{a}_{ip}$, say.) Then the closed interval $[\underline{a}_{ip}, \overline{a}_{ip}]$, endowed with the usual Euclidean topology, is compact, therefore the product $\mathcal{X} = \prod_{i\in N}\prod_{p\in\mathcal{R}} [\underline{a}_{ip}, \overline{a}_{ip}]$, endowed with the product topology, is a compact topological space by Tychonoff's Theorem. Notice that the core $\mathcal{C} \subseteq \mathcal{X}$.

It is easy to see that the core $\mathcal{C}$ is non-empty if and only if the following system of linear inequalities, where a_{ip} are variables, has a solution:

$$\begin{aligned}
\textstyle\sum_{i\in N, (\boldsymbol{\sigma}_p)_i > 0} a_{ip} &\leq V(p) \qquad \text{for} \quad p \in \mathcal{R}, \\
-\textstyle\sum_{i\in N, (\boldsymbol{\sigma}_p)_i > 0} a_{ip} &\leq -V(p) \qquad \text{for} \quad p \in \mathcal{R},
\end{aligned} \tag{1}$$

$$-\sum_{p\in\mathcal{K}}\sum_{i\in N,(\boldsymbol{\sigma}_p)_i>0} a_{ip} \le -v(\boldsymbol{\kappa}) \qquad \text{for} \quad \boldsymbol{\kappa}\in[0,1]^N \text{ and for finite } \mathcal{K}\subseteq\mathcal{R} \text{ such that } \sum_{p\in\mathcal{K}}\boldsymbol{\sigma}_p \ge \boldsymbol{\kappa}. \tag{2}$$

Notice that there is a finite number of variables on the left-hand side of each inequality in (1)–(2). Moreover, it is easy to see that, for any finite subset $\mathcal{I}\subseteq N\times\mathcal{R}$ and for any constant $c\in\mathbb{R}$, the halfspace $F=\left\{\boldsymbol{A}\in\mathbb{R}^{N\times\mathcal{R}} : \sum_{(i,p)\in\mathcal{I}} a_{ip} \le c\right\}$ is a closed set in the product topology of the space $\mathcal{X}$. It follows that the core $\mathcal{C}$ is the intersection of (possibly infinitely many) closed halfspaces. Since the space $\mathcal{X}$ is compact, we conclude that the core $\mathcal{C}$ is non-empty if and only if every finite subsystem of (1)–(2) has a solution; that is, for any natural numbers $r,s\in\mathbb{N}$, for any $p_1,p_2,\dots,p_r\in\mathcal{R}$, for any $\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\dots,\boldsymbol{\kappa}_s\in[0,1]^N$ and for any finite $\mathcal{K}_1,\mathcal{K}_2,\dots,\mathcal{K}_s\subseteq\mathcal{R}$ such that $\sum_{p\in\mathcal{K}_q}\boldsymbol{\sigma}_p\ge\boldsymbol{\kappa}_q$ for $q=1,\,2,\,\dots,\,s$, the following system of linear inequalities, where a_{ip} are variables, has a solution:

$$\begin{aligned}
\textstyle\sum_{i\in N,(\boldsymbol{\sigma}_{p_\rho})_i>0} a_{ip} &\le V(p_\rho) && \text{for} \quad \rho=1,2,\dots,r,\\
-\textstyle\sum_{i\in N,(\boldsymbol{\sigma}_{p_\rho})_i>0} a_{ip} &\le -V(p_\rho) && \text{for} \quad \rho=1,2,\dots,r,\\
-\textstyle\sum_{p\in\mathcal{K}_q}\sum_{i\in N,(\boldsymbol{\sigma}_p)_i>0} a_{ip} &\le -v(\boldsymbol{\kappa}_q) && \text{for} \quad q=1,2,\dots,s.
\end{aligned} \tag{3}$$

The following result is useful:

Gale's Theorem of the alternative [4,5]**.** *Let $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ be a matrix and let $\boldsymbol{b}\in\mathbb{R}^m$ be a vector. Then there exists a solution $\boldsymbol{x}\in\mathbb{R}^n$ to the system of linear inequalities*

$$\boldsymbol{A}\boldsymbol{x}\le\boldsymbol{b} \tag{4}$$

if and only if

$$\forall\boldsymbol{\lambda}^{\mathrm{T}}\in\mathbb{R}^{1\times m},\ \boldsymbol{\lambda}^{\mathrm{T}}\ge\boldsymbol{0}^{\mathrm{T}}\colon\ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A}=\boldsymbol{0}^{\mathrm{T}} \implies \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b}\ge 0\,. \tag{5}$$

By identifying system (4) with (3), the condition (5) and some calculations yield the concept of balancedness of the cooperative fuzzy TU-game.

It will be useful to introduce the operation of rounding up. A number $\sigma\in[0,1]$ is rounded up as follows: we let $\lceil\sigma\rceil=0$ if $\sigma=0$, and $\lceil\sigma\rceil=1$ if $\sigma>0$. Given a row membership vector $\boldsymbol{\sigma}\in[0,1]^N$, the operation $\lceil\cdot\rceil$ is applied to the vector componentwise; that is, we have $\lceil\boldsymbol{\sigma}\rceil\in\{0,1\}^N$ and $\lceil\boldsymbol{\sigma}\rceil_i=0$ or $\lceil\boldsymbol{\sigma}\rceil_i=1$ if $\sigma_i=0$ or $\sigma_i>0$, respectively, for $i\in N$.

Recall that the fuzzy coalition structure $\tilde{\mathcal{S}}=(\tilde{S}_p)_{p\in\mathcal{R}}$ consists of fuzzy coalitions $\tilde{S}_p\,\tilde{\subseteq}\,N$ with membership vectors $\boldsymbol{\sigma}_p\in[0,1]^N$ for $p\in\mathcal{R}$. We say that a collection $\{\tilde{K}_1,\tilde{K}_2,\dots,\tilde{K}_s\}$ of fuzzy coalitions $\tilde{K}_1,\tilde{K}_2,\dots,\tilde{K}_s\,\tilde{\subseteq}\,N$ with membership vectors $\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\dots,\boldsymbol{\kappa}_s\in[0,1]^N$ along with a collection $\{\mathcal{K}_1,\mathcal{K}_2,\dots,\mathcal{K}_s\}$ of finite index sets $\mathcal{K}_1,\mathcal{K}_2,\dots,\mathcal{K}_s\subseteq\mathcal{R}$ such that $\sum_{p\in\mathcal{K}_q}\boldsymbol{\sigma}_p\ge\boldsymbol{\kappa}_q$ for $q=1,\,2,\,\dots,\,s$ is *balanced* with respect to the fuzzy coalition structure $\tilde{\mathcal{S}}=(\tilde{S}_p)_{p\in\mathcal{R}}$ if and only if

$$\sum_{q=1}^{s}\sum_{p\in\mathcal{K}_q}\lambda_q\lceil\boldsymbol{\sigma}_p\rceil=\sum_{\rho=1}^{r}\mu_{p_\rho}\lceil\boldsymbol{\sigma}_{p_\rho}\rceil$$

for some balancing weights $\lambda_1, \lambda_2, \ldots, \lambda_s \geq 0$, for some natural number $r \in \mathbb{N}$, for some indices $p_1, p_2, \ldots, p_r \in \mathcal{R}$, and for some $\mu_{p_1}, \mu_{p_2}, \ldots, \mu_{p_r} \geq 0$ such that $\mu_{p_1} + \mu_{p_2} + \cdots + \mu_{p_r} = 1$.

Finally, we say that the given cooperative fuzzy TU-game is *balanced* with respect to the fuzzy coalition structure $\tilde{\mathcal{S}} = (\tilde{S}_p)_{p \in \mathcal{R}}$ if and only if

$$\sum_{q=1}^{s} \sum_{p \in \mathcal{K}_q} \lambda_q v(\boldsymbol{\kappa}_q) \leq \sum_{\rho=1}^{r} \mu_{p_\rho} V(p_\rho)$$

for every balanced collection $\{\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_s\}$ of fuzzy coalitions along with the corresponding collection $\{\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_s\}$ of the finite index sets.

By combining all the facts together, we come to the main result of this paper:

Bondareva-Shapley Theorem, generalized version. *Let a fuzzy cooperative TU-game; that is, a fuzzy coalition structure* $\tilde{\mathcal{S}} = (\tilde{S}_p)_{p \in \mathcal{R}}$, *a function* $V\colon \mathcal{R} \to \mathbb{R}$ *of the coalition of this fuzzy coalition structure and a fuzzy coalition function* $v\colon [0,1]^N \to \mathbb{R}$ *with* $v(\boldsymbol{\chi}^{\emptyset}) = 0$ *be given. Then the core* $\mathcal{C} = \big\{ \boldsymbol{A} \in \mathbb{R}^{N \times \mathcal{R}} : \sum_{i \in N} a_{ip} = V(p)$ for $p \in \mathcal{R}$, and $\sum_{p \in \mathcal{K}} \sum_{i \in N} a_{ip} \geq v(\boldsymbol{\kappa})$ for $\boldsymbol{\kappa} \in [0,1]^N$, and also $a_{ip} = 0$ if $(\boldsymbol{\sigma}_p)_i = 0$ for $i \in N$ and for $p \in \mathcal{R} \big\}$ *is non-empty if and only if the game is balanced.*

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References

1. Aumann, R. J., Dreze, J. H.: Cooperative Games with Coalition Structures. International Journal of Game Theory **3**(4), 217–237 (1974)
2. Bartl, D.: On the Non-Emptiness of the Core of a Cooperative Game. In: Proceedings of the 25th International Conference on Mathematical Methods in Economics 2007, pp. 9-11. Ostrava: VŠB–Technical University of Ostrava. Faculty of Economics (2007) ISBN 978-80-248-1457-5
3. Bondareva, O. N.: Some Applications of Linear Programming Methods to the Theory of Cooperative Games (in Russian). Problemy Kybernetiki **10**, 119–139 (1963)
4. Fan, K.: On Systems of Linear Inequalities. In: Kuhn, H. W., Tucker, A. W. (eds.) Linear Inequalities and Related Systems, Annals of Mathematics Studies, vol. 38, pp. 99–156. Princeton: Princeton University Press (1956)
5. Gale, D.: The Theory of Linear Economic Models. New York: McGraw-Hill (1960)
6. Gillies, D. B.: Solutions to General Non-zero-sum Games. In: Tucker, A. W., Luce, R. D. (eds.) Contributions to the Theory of Games, Vol. IV, Annals of Mathematics Studies, vol. 38, pp. 47–85. Princeton: Princeton University Press (1959)
7. Kannai, Y.: The Core and Balancedness. In: Aumann, R. J., and Hart, S. (eds.) Handbook of Game Theory with Economic Applications, Vol. I, pp. 591–667. Amsterdam: North-Holland (1992)
8. Shapley, L. S.: Markets as Cooperative Games. RAND Corporation Paper P-629 (1955)
9. Shapley, L. S.: On Balanced Sets and Cores. Naval Research Logistics Quarterly **14**, 453–460 (1967)
10. Thrall, R. M., Lucas, W. F.: N-person games in partition function form. Naval Research Logistics Quarterly **10**, 281–298 (1963)