Indexes of Desirable Properties of a Pairwise Comparison Matrix with Fuzzy Elements

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Abstract. In the Analytic Hierarchy Process (AHP), pairwise comparisons are used to quantify the relative importance of the elements, i.e. the criteria and/or alternatives. Fuzzy elements are appropriate whenever the decision maker is uncertain about the value of his/her evaluation of the relative importance of the elements in question. In this paper, we deal with the general case when the elements of the pairwise comparison matrix are fuzzy subsets of an Abelian linearly ordered group (alo-group). We then propose some desirable properties – consistency, intensity, and coherence – of the fuzzy pairwise comparison matrix and we also propose indexes to measure these desirable properties. Based on these indexes, a new solution algorithm to find the priority vector satisfying these desirable properties can be formulated.

Keywords: multi-criteria optimization, Analytic Hierarchy Process (AHP), pairwise comparison matrix, fuzzy elements, consistency, intensity, coherence, priority vector, alo-group

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1 Introduction

The main subproblem of the Analytic Hierarchy Process (AHP) is to calculate the priority vectors, i.e. the weights assigned to the elements of the hierarchy (criteria, subcriteria, and alternatives or variants), by using the information provided in the form of a pairwise comparison matrix. Given a set of elements and corresponding pairwise comparison matrix, whose entries evaluate the relative importance of the elements with respect to a given criterion, the purpose is to calculate the priority vector characterizing the ranking of the elements. There are various methods for calculating the vector of weights, e.g. Saaty's Eigenvector Method, the Geometric Mean Method, and others, see [3, 4].

Fuzzy sets as the elements of the pairwise comparison matrix can be applied whenever the decision maker is not sure about the preference degree of his/her evaluations of the pairs in question. Such an approach is also well known in the Fuzzy Analytic Hierarchy Process (FAHP) originated by Thomas Saaty in [6]. Recent development of the problem can be found in [2]. Comparing to [4], here, we propose newly reformulated desirable properties – consistency, intensity, and coherence – of the priority vector and we also propose indexes to measure these desirable properties of the given fuzzy pairwise comparison matrix. Then, a new algorithm for deriving the priority vector satisfying the desirable properties can be devised.

2 Preliminaries

The reader can find the corresponding basic definitions, concepts and results, e.g. in [4]. Here, we summarize some necessary concepts. For detailed information, we refer to [5].

A *fuzzy subset* S of a nonempty set X (or a *fuzzy set* on X) is a family $\{S_\alpha\}_{\alpha\in[0;1]}$ of subsets of X such that $S_0 = X$, and $S_\beta \subset S_\alpha$ whenever $0 \le \alpha \le \beta \le 1$, and also $S_\beta = \bigcap_{0\le\alpha<\beta} S_\alpha$ whenever $0 < \beta \le 1$. The *membership function of* S is the function μ_S from X into the unit interval $[0; 1]$ defined by $\mu_S(x) = \sup\{\,\alpha \mid x \in S_\alpha\,\}$. Given an $\alpha \in]0; 1]$, the set $[S]_\alpha = \{\,x \in X \mid \mu_S(x) \ge \alpha\,\}$ is called the *α-cut of the fuzzy set S*. In order to unify various approaches and to prepare a more flexible presentation, we apply alo-groups, see [1]. Recall that an Abelian group is a set, G, together with an operation $\odot$ and corresponding "group axioms" that combine any two elements

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$a, b \in G$ to form another element in G denoted by $a \odot b$. The well known examples of alo-groups can be found in [4, 3], or, [1].

3 Fuzzy pairwise comparison matrices, reciprocity and consistency

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group over an open interval G of $\mathbf{R}$, see [1]. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be an $n \times n$ matrix where each element is a bounded fuzzy interval of the alo-group $\mathcal{G}$, let $\alpha \in [0;1]$, and let $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha); a^R_{ij}(\alpha)]$ be the α-cut of $\tilde{a}_{ij}$. The matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is said to be *α-$\odot$-reciprocal* if the following two conditions hold for each $i, j \in \{1, \ldots, n\}$:

$$a^L_{ii}(\alpha) = a^R_{ii}(\alpha) = e, \qquad \text{and} \qquad a^L_{ij}(\alpha) \odot a^R_{ji}(\alpha) = e. \tag{1}$$

If $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-reciprocal for all $\alpha \in [0;1]$, then it is called *$\odot$-reciprocal*. If $\tilde{A} = \{\tilde{a}_{ij}\}$ is $\odot$-reciprocal, then $\tilde{A} = \{\tilde{a}_{ij}\}$ is called a *fuzzy pairwise comparison matrix*, *fuzzy PC matrix*, *FPC matrix*, or, shortly, *FPCM*.

Now, we turn to the concept of consistency of FPC matrices.

Definition 1. Let $\alpha \in [0;1]$. A FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is said to be *α-$\odot$-consistent* if the following condition holds: There exists a crisp matrix $A' = \{a'_{ij}\}$ with $a'_{ij} \in [\tilde{a}_{ij}]_\alpha$ such that $A' = \{a'_{ij}\}$ is consistent, i.e. for each $i, j, k \in \{1, \ldots, n\}$ it holds

$$a'_{ik} = a'_{ij} \odot a'_{jk}. \tag{2}$$

The FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is said to be *$\odot$-consistent* if $\tilde{A}$ is α-$\odot$-consistent for all $\alpha \in [0;1]$. If for some $\alpha \in [0;1]$ the FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is not α-$\odot$-consistent, then $\tilde{A}$ is called *α-$\odot$-inconsistent*. If for all $\alpha \in [0;1]$ the FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-inconsistent, then $\tilde{A}$ is called *$\odot$-inconsistent*.

The next proposition gives an equivalent condition for a FPC matrix to be α-$\odot$-consistent, see, e.g. [4]. The proof of the following proposition is easy, or can be found in [4].

Proposition 1. *Let $\alpha \in [0;1]$, let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix, and let $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha); a^R_{ij}(\alpha)]$ be an α-cut of $\tilde{a}_{ij}$. Then $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-consistent iff there exists a vector $w = (w_1, \ldots, w_n)$ with $w_i \in G$, for $i \in \{1, \ldots, n\}$, such that for each $i, j \in \{1, \ldots, n\}$, it holds:*

$$a^L_{ij}(\alpha) \leq w_i \div w_j \leq a^R_{ij}(\alpha). \tag{3}$$

Notice that another, i.e. stronger, concept of α-$\odot$-consistency has been defined in [4, 5].

4 Desirable properties of the priority vector

In this section, we shall investigate PC matrices with fuzzy elements, and extend some concepts and properties of PC matrices on alo-group with crisp elements.

From now on we shall assume that $\mathcal{G} = (G, \odot, \leq)$ is a continuous alo-group in $\mathbf{R}$. Then G is an open interval of $\mathbf{R}$, see [1]. We denote the set of the first n positive integers by $\mathcal{N}$, i.e. we put $\mathcal{N} = \{1, 2, \ldots, n\}$. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be an $n \times n$ fuzzy pairwise comparison matrix (FPCM), which is $\odot$-reciprocal.

The result of the pairwise comparisons method based on the FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is a rating of the set $C = \{c_1, c_2, \ldots, c_n\}$ of the elements, i.e. a mapping that assigns real values to the elements (criteria or alternatives). Formally, it can be introduced as follows.

The *ranking function* for C (or the *ranking* of C) is a function $w\colon C \to G$ that assigns to every element from $C = \{c_1, c_2, \ldots, c_n\}$ a value from the linearly ordered set G of the alo-group $\mathcal{G} = (G, \odot, \leq)$.

Here, $w(c)$ represents the ranking value for $c \in C$. The function w is usually written in the form of a vector of *weights*, i.e. $w = (w(c_1), w(c_2), \ldots, w(c_n))$, or, simply $w = (w_1, w_2, \ldots, w_n)$, and it is called the *priority vector*. Also, we say that the priority vector w is associated with the FPC matrix $\tilde{A}$, or that the priority vector w is generated by a priority generating method based on the FPC matrix $\tilde{A}$.

The priority vector $w = (w(c_1), w(c_2), \ldots, w(c_n))$ is *$\odot$-normalized*, if $\bigodot_{i=1}^{n} w(c_i) = e$.

We start with definitions of three various concepts of priority vectors: consistent, intensity, and coherent ones.

Definition 2. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on the alo-group $\mathcal{G} = (G, \odot, \leq)$, let $w = (w_1, w_2, \ldots, w_n)$, with $w_j \in G$, be a priority vector, let $\alpha \in [0;1]$, and let $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha); a^R_{ij}(\alpha)]$ be the α-cut of $\tilde{a}_{ij}$.

(i) We say that the vector w is an *α-consistent vector* (α-CsV) of the FPC matrix $\tilde{A}$ if the following condition holds:

$$a^L_{ij}(\alpha) \leq w_i \div w_j \qquad \text{for all} \quad i,j \in \mathcal{N}. \tag{4}$$

Moreover, the vector w is a *consistent vector* (CsV) of the FPC matrix $\tilde{A}$ if condition (4) holds for all $\alpha \in [0;1]$. If there exists an α-consistent vector or consistent vector of the FPC matrix $\tilde{A}$, then $\tilde{A}$ is called an α-consistent FPC matrix or consistent FPC matrix, respectively.

(ii) We say that the vector w is an *α-intensity vector* (α-InV) of the FPC matrix $\tilde{A}$ if the following condition holds:

$$a^L_{ij}(\alpha) > a^R_{kl}(\alpha) \quad \text{implies} \quad w_i \div w_j > w_k \div w_l \qquad \text{for all} \quad i,j,k,l \in \mathcal{N}. \tag{5}$$

Moreover, the vector w is an *intensity vector* (InV) of the FPC matrix $\tilde{A}$ if condition (5) holds for all $\alpha \in [0;1]$. If there exists an α-intensity vector or intensity vector of the FPC matrix $\tilde{A}$, then $\tilde{A}$ is called an α-intensity FPC matrix or intensity FPC matrix, respectively.

iii) We say that the vector w is an *α-coherent vector* (α-CoV) of the FPC matrix $\tilde{A}$ if the following condition holds:

$$a^L_{ij}(\alpha) > e \quad \text{implies} \quad w_i > w_j \qquad \text{for all} \quad i,j \in \mathcal{N}. \tag{6}$$

Moreover, the vector w is a *coherent vector* (CoV) of the FPC matrix $\tilde{A}$ if condition (6) holds for all $\alpha \in [0;1]$. If there exists an α-coherent vector or coherent vector of the FPC matrix $\tilde{A}$, then $\tilde{A}$ is called an α-coherent FPC matrix or coherent FPC matrix, respectively.

Remark 1. In definition (4), it looks like the concept of α-consistent FPC matrix depends only on the left sides $a^L_{ij}(\alpha)$ of the α-cuts of the elements of the matrix $\tilde{A}$, and not on the right sides. Notice that by the reciprocity property of the elements it is easy to see that w is an α-CsV of $\tilde{A} = \{\tilde{a}_{ij}\}$ if and only if

$$a^L_{ij}(\alpha) \leq w_i \div w_j \leq a^R_{ij}(\alpha) \qquad \text{for all} \quad i,j \in \mathcal{N}. \tag{7}$$

We obtain the following modification.

Proposition 2. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPCM on $\mathcal{G} = (G, \odot, \leq)$, let $\alpha \in [0;1]$, and let $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha); a^R_{ij}(\alpha)]$. A priority vector $w = (w_1, w_2, \ldots, w_n)$, with $w_j \in G$, satisfies*

$$a^L_{ij}(\alpha) \div a^R_{kl}(\alpha) \leq (w_i \div w_j) \div (w_k \div w_l) \qquad \text{for all} \quad i,j,k,l \in \mathcal{N} \tag{8}$$

if and only if w is an α-consistent vector of the FPC matrix $\tilde{A}$.

Remark 2. Notice that if $w = (w_1, w_2, \ldots, w_n)$ satisfies (8), then w is also an α-intensity vector of the FPC matrix $\tilde{A}$. To see that, let $a^L_{ij}(\alpha) > a^R_{kl}(\alpha)$, which is equivalent to $a^L_{ij}(\alpha) \div a^R_{kl}(\alpha) > e$. In other words, if w is an α-consistent vector, then w is an α-intensity vector of the FPC matrix $\tilde{A}$. Notice also that if $w = (w_1, w_2, \ldots, w_n)$ is an α-intensity vector, then it is also an α-coherent vector of the FPC matrix $\tilde{A}$. Indeed, condition (6) follows from (5) easily by considering $k = l$.

5 Measuring desirable properties of FPC matrices

The condition of consistency of FPC matrices is the strongest condition of the three: consistency condition, intensity condition, and coherence one. In practice, FPC matrices are often inconsistent, even more, the intensity condition is not satisfied and/or they are incoherent. Hence, it is useful to know "how much" these desirable conditions are satisfied. This is why we measure the inconsistency, non-intensity, or incoherence of a given FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ by special indexes, see, e.g., [6, 4]. Let $\mathcal{G} = (G, \odot, \leq)$ be a continuous alo-group in the set of the real numbers. Then $G =]g^-; g^+[$ is an open interval in $\mathbf{R}$.

For $a \in \mathcal{G} = (G, \odot, \leq)$, define

$$a^+ = \max\{a, e\}, \qquad a^- = \max\{a^{-1}, e\}, \tag{9}$$

the *absolute value* $|a|$ of a as

$$|a| = a^+ \odot a^- = \max\{a^{-1}, a\}. \tag{10}$$

Moreover, for a $w = (w_1, \ldots, w_n) \in G^n$, define the *norm* $\|w\|$ of w as

$$\|w\| = \max\{\, |w_i| \mid i \in \mathcal{N} \,\}. \tag{11}$$

Finally, for an $r \in G$, set the *ball* $B(r)$ as

$$B(r) = \{\, w \in G^n \mid \|w\| \leq r \,\}. \tag{12}$$

Definition 3. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$. For each pair $i, j \in \mathcal{N}$, for a priority vector $w = (w_1, w_2, \ldots, w_n)$, with $w_j \in G$, and for $r \in G$ and $\alpha \in [0; 1]$, define:

(i) Let the *local α-inconsistency grade of an element of FPC matrix $\tilde{A}$ and vector w* be defined as

$$\varepsilon_{ij}^{Cs}(\tilde{A}, w, \alpha) = a_{ij}^{L}(\alpha) \div (w_i \div w_j), \tag{13}$$

and define the *global α-inconsistency grade of FPC matrix $\tilde{A}$ and vector w* as

$$E^{Cs}(\tilde{A}, w, \alpha) = \max\{\, \varepsilon_{ij}^{Cs}(\tilde{A}, w, \alpha) \odot \varepsilon_{kl}^{Cs}(\tilde{A}, w, \alpha) \mid i, j, k, l \in \mathcal{N} \,\}. \tag{14}$$

Now, we define the *α-inconsistency index* of $\tilde{A}$ as

$$I^{Cs}(\tilde{A}, \alpha) = \inf\{\, E^{Cs}(\tilde{A}, w, \alpha) \mid w \in B(r) \,\}. \tag{15}$$

The smaller the α-inconsistency index $I^{Cs}(\tilde{A}, \alpha)$ is, the higher the α-consistency of the matrix $\tilde{A}$ is, i.e. the less α-inconsistent the matrix $\tilde{A}$ is.

(ii) Let the *local α-non-intensity grade of two elements of FPC matrix $\tilde{A}$ and vector w* be defined as

$$\varepsilon_{ijkl}^{In}(\tilde{A}, w, \alpha) = \begin{cases} (w_k \div w_l) \div (w_i \div w_j) & \text{if } a_{ij}^{L}(\alpha) > a_{kl}^{R}(\alpha), \\ g^{-} & \text{otherwise,} \end{cases} \tag{16}$$

and define the *global α-non-intensity grade of FPC matrix $\tilde{A}$ and vector w* as

$$E^{In}(\tilde{A}, w, \alpha) = \max\{\, \varepsilon_{ijkl}^{In}(\tilde{A}, w, \alpha) \mid i, j, k, l \in \mathcal{N} \,\}. \tag{17}$$

Now, we define the *α-non-intensity index* of $\tilde{A}$ as

$$I^{In}(\tilde{A}, \alpha) = \inf\{\, E^{In}(\tilde{A}, w, \alpha) \mid w \in B(r) \,\}. \tag{18}$$

The smaller the α-non-intensity index $I^{In}(\tilde{A}, \alpha)$ is, the higher the α-intensity of the matrix $\tilde{A}$ is, i.e. the less α-non-intensive the matrix $\tilde{A}$ is.

(iii) Let the *local α-incoherence grade of an element of FPC matrix $\tilde{A}$ and vector w* be defined as

$$\varepsilon_{ij}^{Co}(\tilde{A}, w, \alpha) = \begin{cases} w_j \div w_i & \text{if } a_{ij}^{L}(\alpha) > e, \\ g^{-} & \text{otherwise,} \end{cases} \tag{19}$$

and define the *global α-incoherence grade of FPC matrix $\tilde{A}$ and vector w* as

$$E^{Co}(\tilde{A}, w, \alpha) = \max\{\, \varepsilon_{ij}^{Co}(\tilde{A}, w, \alpha) \mid i, j \in \mathcal{N} \,\}. \tag{20}$$

Now, we define the *α-incoherence index* of $\tilde{A}$ as

$$I^{Co}(\tilde{A}, \alpha) = \inf\{\, E^{Co}(\tilde{A}, w, \alpha) \mid w \in B(r) \,\}. \tag{21}$$

The smaller the α-incoherence index $I^{Co}(\tilde{A}, \alpha)$ is, the higher the α-coherence of the matrix $\tilde{A}$ is, i.e. the less α-incoherent the matrix $\tilde{A}$ is.

Remark 3. As every FPCM is reciprocal, we can obtain a slightly different formulas for the above indexes for a FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$, replacing $i, j \in \mathcal{N}$ by $1 \leq i < j \leq n$.

Proposition 3. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix, let $r \in G$, and let $\alpha \in [0;1]$. Then it holds:*

A vector $w = (w_1, \ldots, w_n) \in B(r)$ is α-consistent if and only if

$$I^{Cs}(\tilde{A}, \alpha) \leq E^{Cs}(\tilde{A}, w, \alpha) \leq e. \tag{22}$$

A vector $w = (w_1, \ldots, w_n) \in B(r)$ is an α-intensity vector if and only if

$$I^{In}(\tilde{A}, \alpha) \leq E^{In}(\tilde{A}, w, \alpha) < e. \tag{23}$$

A vector $w = (w_1, \ldots, w_n) \in B(r)$ is α-coherent if and only if

$$I^{Co}(\tilde{A}, \alpha) \leq E^{Co}(\tilde{A}, w, \alpha) < e. \tag{24}$$

Proposition 4. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix, let $w = (w_1, \ldots, w_n)$, with $w_j \in G$, and let $\alpha \in [0;1]$. Then*

$$E^{Co}(\tilde{A}, w, \alpha) \leq E^{In}(\tilde{A}, w, \alpha) < E^{Cs}(\tilde{A}, w, \alpha). \tag{25}$$

For any vector $w = (w_1, \ldots, w_n)$, with $w_j \in G$, and for any element $r \in G$, consider the vector $w' = (w'_1, \ldots, w'_n)$ with $w'_j = w_j \odot r \div \|w\|$, where the norm $\|w\|$ of w is defined by (11), and observe that $w' \in B(r)$, where the ball is defined by (12). Consider yet a FPC matrix $\tilde{A}$ and $\alpha \in [0;1]$. By Definition 3, it is easy to see that the global α-inconsistency / α-non-intensity / α-incoherence grade of $\tilde{A}$ and w is equal to the global α-inconsistency / α-non-intensity / α-incoherence grade of $\tilde{A}$ and w', respectively. Hence, as a clear consequence of Proposition 4, particularly (25), we obtain the following result.

Proposition 5. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix, let $r \in G$, let $\alpha \in [0;1]$, and let $w = (w_1, \ldots, w_n) \in B(r)$. Then*

$$I^{Co}(\tilde{A}, \alpha) \leq I^{In}(\tilde{A}, \alpha) < I^{Cs}(\tilde{A}, \alpha). \tag{26}$$

Moreover, if $\alpha, \alpha' \in [0;1]$ are such that $\alpha \leq \alpha'$, then

$$I^{Co}(\tilde{A}, \alpha) \leq I^{Co}(\tilde{A}, \alpha'), \qquad I^{In}(\tilde{A}, \alpha) \leq I^{In}(\tilde{A}, \alpha'), \qquad I^{Cs}(\tilde{A}, \alpha) \leq I^{Cs}(\tilde{A}, \alpha'). \tag{27}$$

The proofs of Propositions 3, 4 and 5 are not difficult, they are left to the reader.

Example 1. Consider the multiplicative alo-group $\mathcal{R}_+$. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a 4×4 FPC matrix given by the α-cut representation, see the definition of fuzzy sets in Section 2, for $\alpha \in [0;1]$ as follows:

$$\tilde{A} = \begin{bmatrix} [\,1\,;\,1\,] & [\,1+\frac{1}{2}\alpha\,;\,2-\frac{1}{2}\alpha\,] & [\,1+\alpha\,;\,4-2\alpha\,] & [\,1+\alpha\,;\,3-\alpha\,] \\ [\,\frac{2}{4-\alpha}\,;\,\frac{2}{2+\alpha}\,] & [\,1\,;\,1\,] & [\,1+\alpha\,;\,4-2\alpha\,] & [\,2+6\alpha\,;\,9-\alpha\,] \\ [\,\frac{1}{4-2\alpha}\,;\,\frac{1}{1+\alpha}\,] & [\,\frac{1}{4-2\alpha}\,;\,\frac{1}{1+\alpha}\,] & [\,1\,;\,1\,] & [\,1+\alpha\,;\,3-\alpha\,] \\ [\,\frac{1}{3-\alpha}\,;\,\frac{1}{1+\alpha}\,] & [\,\frac{1}{9-\alpha}\,;\,\frac{1}{2+6\alpha}\,] & [\,\frac{1}{3-\alpha}\,;\,\frac{1}{1+\alpha}\,] & [\,1\,;\,1\,] \end{bmatrix}.$$

The α-cuts, for $\alpha = 1$, of the elements of the FPC matrix $\tilde{A}$ consist of the crisp elements of the PC matrix A. Notice that the elements located above the main diagonal of the FPC matrix $\tilde{A}$ are triangular elements with the piece-wise linear membership functions, whereas the elements located under the diagonal are reciprocal, hence the corresponding membership functions are non-linear.

Consider the priority vector $w^* = (1.515, 1.344, 0.975, 0.504)$. Evidently, $\tilde{A}$ is α-·-consistent FPCM for $\alpha = 0$. Moreover, it can be verified that $\tilde{A}$ is α-·-consistent FPCM, if and only if $\alpha \in [0; 0.117]$, otherwise, $\tilde{A}$ is not α-·-consistent. It can be easily verified that $\tilde{A}$ is α-·-coherent for all $\alpha \in [0;1]$, i.e. by Definition 3, $\tilde{A}$ is ·-coherent.

6 Deriving priority vectors of FPC matrices with the desirable properties

In this section, we present an Algorithm to generate a *crisp* priority vector, therefore no defuzzification is necessary for the final ranking of the elements, i.e. criteria or alternatives $c_1, c_2, \ldots, c_n \in C$. The Algorithm consists of the following six steps:

STEP 1. Choose radius $r \in G$, for example $r := \max\{\|a^R_{ij}(0)\| \mid i,j \in \mathcal{N}\}$, and set $\alpha := 0$. Find an optimal solution w^α to Problem 1:

$$E^{Cs}(\tilde{A}, w, \alpha) \longrightarrow \min \qquad \text{subject to} \quad w \in B(r). \tag{28}$$

If the minimal value of the objective function $E^{Cs}(\tilde{A}, w^\alpha, \alpha) > e$, then there is no 0-consistent vector; go to Step 3. Otherwise, if the minimal value of the objective function $E^{Cs}(\tilde{A}, w^\alpha, \alpha) \leq e$, then there exists a 0-consistent vector. Look for an α-consistent vector with the maximal $\alpha \in [0;1]$, i.e. proceed with Step 2.

STEP 2. Find an optimal solution α^*, w^{α^*} to Problem 2:

$$\alpha \longrightarrow \max \qquad \text{subject to} \quad E^{Cs}(\tilde{A}, w, \alpha) \leq e, \quad w \in B(r), \quad \alpha \in [0;1]. \tag{29}$$

The optimal solution $w^{\alpha^*} \in B(r)$ is an α^*-consistent priority vector such that $\alpha^* \in [0;1]$ is maximal. At the same time, it is an α^*-intensity vector, such that $E^{In}(\tilde{A}, w^{\alpha^*}, \alpha^*) < e$. Look for an α-intensity vector with the maximal $\alpha \in [0;1]$, i.e. go to Step 4.

STEP 3. Set $\alpha := 0$. Find an optimal solution w^α to Problem 3:

$$E^{In}(\tilde{A}, w, \alpha) \longrightarrow \min \qquad \text{subject to} \quad w \in B(r). \tag{30}$$

If the minimal value of the objective function $E^{In}(\tilde{A}, w^\alpha, \alpha) \geq e$, then there is no 0-intensity vector; go to Step 5. Otherwise, if the minimal value of the objective function $E^{In}(\tilde{A}, w^\alpha, \alpha) < e$, then there exists a 0-intensity vector. Look for an α-intensity vector with the maximal $\alpha \in [0;1]$, i.e. proceed with Step 4.

STEP 4. Find an optimal solution $\alpha^{**}, w^{\alpha^{**}}$ to Problem 4:

$$\alpha \longrightarrow \max \qquad \text{subject to} \quad E^{In}(\tilde{A}, w, \alpha) < e, \quad w \in B(r), \quad \alpha \in [0;1]. \tag{31}$$

The optimal solution $w^{\alpha^{**}} \in B(r)$ is an α^{**}-intensity priority vector such that $\alpha^{**} \in [0;1]$ is maximal. At the same time, it is an α^{**}-coherent vector, such that $E^{Co}(\tilde{A}, w^{\alpha^{**}}, \alpha^{**}) < e$. Look for an α-coherent vector with the maximal $\alpha \in [0;1]$, i.e. go to Step 6.

STEP 5. Set $\alpha := 0$. Find an optimal solution w^α to Problem 5:

$$E^{Co}(\tilde{A}, w, \alpha) \longrightarrow \min \qquad \text{subject to} \quad w \in B(r). \tag{32}$$

If the minimal value of the objective function $E^{Co}(\tilde{A}, w^\alpha, \alpha) \geq e$, then there is no 0-coherent vector; change some elements of the FPCM $\tilde{A}$ and go to Step 1. Otherwise, if the minimal value of the objective function $E^{Co}(\tilde{A}, w^\alpha, \alpha) < e$, then there exists a 0-coherent vector. Look for an α-coherent vector with the maximal $\alpha \in [0;1]$, i.e. proceed with Step 6.

STEP 6. Find an optimal solution $\alpha^{***}, w^{\alpha^{***}}$ to Problem 6:

$$\alpha \longrightarrow \max \qquad \text{subject to} \quad E^{Co}(\tilde{A}, w, \alpha) < e, \quad w \in B(r), \quad \alpha \in [0;1]. \tag{33}$$

The optimal solution $w^{\alpha^{***}} \in B(r)$ is an α^{***}-coherent priority vector such that $\alpha^{***} \in [0;1]$ is maximal.

END.

The proposed Algorithm provides three priority vectors with the desirable properties, i.e. the α^*-consistent vector w^{α^*}, the α^{**}-intensity vector $w^{\alpha^{**}}$, and the α^{***}-coherent vector $w^{\alpha^{***}}$, such that their membership grades α^*, α^{**}, and α^{***} are maximal and non-decreasing ($0 \leq \alpha^* \leq \alpha^{**} \leq \alpha^{***} \leq 1$).

7 Conclusion

In this paper we propose the method for deriving the priority vector consisting of six steps. With respect to natural logical requirements, we reformulated “desirable properties” of FPC matrix when compared with [4]. Then, an algorithm was proposed to obtain the priority vectors with the newly formulated properties. Such an approach is more natural from the DM perspectives and enables us also to extend various MCDM approaches known from the literature.

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