AN ALGORITHM TO COMPUTE A JOINT PRIORITY VECTOR OF PAIRWISE COMPARISON MATRICES WITH FUZZY ELEMENTS IN GROUP DECISION MAKING

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Abstract
We study the problem how to aggregate different opinions of m decision makers who evaluate n objects with respect to a criterion into a single opinion of the group. In particular, each of the m decision makers evaluates the n objects pairwise with respect to the given criterion. In order to unify various approaches, we assume that the decision makers use the elements of an Abelian linearly ordered group (alo-group) to assess the relative importance of the two items in each pair of the n objects. Moreover, a decision maker (DM) can use a fuzzy subset of the alo-group to assess the relative importance whenever the DM is uncertain about the exact value of the assessment. Thus, the task is to compute a joint priority vector of m given $n \times n$ reciprocal pairwise comparison matrices with fuzzy elements, i.e. fuzzy subsets of an alo-group. In this paper, we consider several desirable properties of the priority vector – consistency, intensity, and coherence – and we propose a new algorithm to compute priority vectors satisfying these desirable properties.

Keywords: multi-criteria group decision making, fuzzy pairwise comparison matrix, joint priority vector, consistency, intensity, coherence, alo-group, Analytic Hierarchy Process (AHP)
JEL codes: C60, C65, D79

1. Introduction

Entrepreneurs, as well as small and medium-sized enterprises (SMEs), frequently need to deal with various multi-criteria decision making (MCDM) problems, e.g., when acquiring new equipment (such as cars, machines, furniture, etc.), assessing investment opportunities, evaluating and improving the services they provide, and so on. Popular methods to solve such MCDM problems include the Analytic Hierarchy Process (AHP) and pairwise comparisons. Moreover, the computers, tablets, smartphones and other computing devices have spread among entrepreneurs and SMEs during the past two decades. Consequently, the popular methods are implemented on computers and the software has become easily available, see Górecki (2023) in these proceedings.

The Analytic Hierarchy Process (AHP) is a popular and powerful tool to solve multi-criteria decision making problems (Saaty, 1980). We consider the following main subproblem of the AHP, which is to be solved in every internal node of the hierarchy; that is, a node having some subnodes. Let n denote the number of these subnodes, which correspond to n objects $c_1, c_2, \dots, c_n$, i.e. criteria, subcriteria, and/or alternatives (variants). Notice that the internal node corresponds to some criterion, subcriterion, and/or the goal of the hierarchy. Henceforth, we shall use the single term criterion for simplicity. Given the information on the relative importance of the two items in each pair of the objects with respect to the given criterion (subcriterion, and/or the goal) in the form of an $n \times n$ pairwise comparison matrix A, the purpose is to calculate the priority vector, which is a vector of n weights $v_1, v_2, \dots, v_n$ assigned to the n objects $c_1, c_2, \dots, c_n$, respectively. The prominent methods to calculate the priority vector include Saaty's Eigenvector Method (EVM) and the Geometric Mean Method (GMM), see Saaty (1980) and Ramík (2020). The priority vector provided by these methods, however, usually do not satisfy desirable properties – consistency, intensity, and/or coherence, in particular – see Saaty and Vargas (1984), D'Apuzzo et al. (2007), and Bana e Costa and Vansnick (2008).

Additionally, the decision maker may not be sure about the exact value of the relative importance of objects c_i and c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$. It is then appropriate, instead of the exact, i.e. crisp, value a_{ij}, to use fuzzy value $\tilde{a}_{ij}$, which captures the decision maker's uncertainty and represents the decision maker's (fuzzy) opinion how many times c_i is better or more important than c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$.

Based on these premises, the authors have proposed a new algorithm for computing priority vectors, satisfying desirable properties, of an $n \times n$ fuzzy pairwise comparison matrix $\tilde{A}$, see Bartl and Ramík (2022).

In this paper, we consider the above main subproblem extended as follows. There are m decision makers (evaluators), and each of them assesses the relative importance of the two items in each pair of the objects with respect to the given criterion in a fuzzy way to capture the decision makers' uncertainty. The fuzzy element $\tilde{a}_{ij}^k$ represents the k-th decision maker's (fuzzy) opinion how many times c_i is better or more important than c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$ and for $k = 1, 2, \dots, m$. We obtain $n \times n$ fuzzy pairwise comparison matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ thus. Our purpose is to improve and extend the authors' new algorithm (Bartl and Ramík, 2022) to this case; that is, our purpose is to compute the priority vector of n weights $v_1, v_2, \dots, v_n$ assigned to the n objects $c_1, c_2, \dots, c_n$, respectively, yet satisfying the aforementioned desirable properties.

2. Preliminaries and notation

In order to unify and generalize various approaches known from the literature, we use the elements of an Abelian linearly ordered group to evaluate the relative importance of the two items in each pair of the objects with respect to the given criterion, see Cavallo and D'Apuzzo (2009) and Ramík (2015). Recall that an *Abelian group* is a pair $(G, \odot)$ where G is a non-empty set and $\odot$ is a commutative and associative binary operation on G satisfying also the existence of the identity element $e \in G$ and the existence of the inverse element $a^{(-1)} \in G$ for each $a \in G$. We then have $a \odot e = a$ and $a \odot a^{(-1)} = e$ for every $a \in G$. We also put $a \div b = a \odot b^{(-1)}$ for all $a, b \in G$. An *Abelian linearly ordered group* (*alo-group*) is a triple $(G, \odot, \leq)$ such that $(G, \odot)$ is an Abelian group and $\leq$ is a binary relation of linear ordering on G such that $a \leq b$ implies $a \odot c \leq b \odot c$ for all $a, b, c \in G$. The well-known examples of alo-groups are the Multiplicative alo-group $\mathcal{R}_+ = (\mathbb{R}_+, \cdot, \leq)$ with the usual multiplication and the neutral element $e = 1$, the Additive alo-group $\mathcal{R} = (\mathbb{R}, +, \leq)$ with the usual addition and the neutral element $e = 0$, and the Fuzzy Multiplicative alo-group $\mathcal{F}_{]0;1[} = (]0; 1[, \odot, \leq)$ with $a \odot b = ab/(ab + (1-a)(1-b))$ for $a, b \in]0; 1[$ and the neutral element $e = \frac{1}{2}$, see Cavallo and D'Apuzzo (2009), Ramík (2015), and Ramík (2020).

A *fuzzy subset* S of the non-empty set G (or a *fuzzy set* on G) is a family $\{S_\alpha\}_{\alpha \in [0;1]}$ of subsets of G such that $S_0 = G$, and $S_\beta \subseteq S_\alpha$ whenever $0 \leq \alpha \leq \beta \leq 1$, and also $S_\beta = \bigcap_{0 \leq \alpha < \beta} S_\alpha$ whenever $0 < \beta \leq 1$. The *membership function of* S is the function μ_S from G into the unit interval $[0; 1]$ defined by $\mu_S(x) = \sup\{\alpha \mid x \in S_\alpha\}$. Given an $\alpha \in]0; 1]$, the set $[S]_\alpha = \{x \in G \mid \mu_S(x) \geq \alpha\}$ is called the *α-cut of the fuzzy set S*. We say that a fuzzy subset S of G is a *fuzzy interval* whenever S is normal, closed, bounded, and convex; that is, for every $\alpha \in]0; 1]$, there exist two elements $a^{\mathrm{L}}(\alpha), a^{\mathrm{R}}(\alpha) \in G$ such that $a^{\mathrm{L}}(\alpha) \leq a^{\mathrm{R}}(\alpha)$ and the α-cut $[S]_\alpha = \{x \in G \mid a^{\mathrm{L}}(\alpha) \leq x \leq a^{\mathrm{R}}(\alpha)\}$. We refer the reader to Ramík (2020) for further details.

3. Desirable properties of the priority vector

Let us consider an alo-group $\mathcal{G} = (G, \odot, \leq)$ and let us denote the set of the first n positive natural numbers by $\mathcal{N}$; that is, we put $\mathcal{N} = \{1, 2, \dots, n\}$. Considering the set $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$, let $\tilde{A} = \{\tilde{a}_{ij}\}$ be an $n \times n$ matrix such that each of its element $\tilde{a}_{ij}$ is a fuzzy interval of G and evaluates the relative importance of the objects c_i and c_j with respect to the given criterion. Let $\alpha \in [0; 1]$, and let $[\tilde{a}_{ij}]_\alpha = [a^{\mathrm{L}}_{ij}(\alpha); a^{\mathrm{R}}_{ij}(\alpha)]$, with $a^{\mathrm{L}}_{ij}(\alpha), a^{\mathrm{R}}_{ij}(\alpha) \in G$ and $a^{\mathrm{L}}_{ij}(\alpha) \leq a^{\mathrm{R}}_{ij}(\alpha)$, be the α-cut of $\tilde{a}_{ij}$ for $i, j \in \mathcal{N}$. The matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is said to be *α-reciprocal* if the following two conditions hold for each $i, j \in \mathcal{N}$:

$$a^{\mathrm{L}}_{ii}(\alpha) = a^{\mathrm{R}}_{ii}(\alpha) = e, \qquad \text{and} \qquad a^{\mathrm{L}}_{ij}(\alpha) \odot a^{\mathrm{R}}_{ji}(\alpha) = e. \tag{1}$$

If $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-reciprocal for every $\alpha \in [0; 1]$, then it is called *reciprocal*. If $\tilde{A} = \{\tilde{a}_{ij}\}$ is reciprocal, then $\tilde{A} = \{\tilde{a}_{ij}\}$ is called a *fuzzy pairwise comparison matrix*, or *FPC matrix* for short.

Then the result of a pairwise comparison method based on the FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is a vector $v = (v_1, v_2, \dots, v_n)$ of the weights $v_1, v_2, \dots, v_n \in G$ of the objects $c_1, c_2, \dots, c_n \in \mathcal{C}$, respectively. In other words, the i-th component v_i of the priority vector v is the weight of the object c_i for $i \in \mathcal{N}$. We say the priority vector $v = (v_1, v_2, \dots, v_n)$ is *normalized* if $\bigodot_{i=1}^{n} v_i = e$.

Based upon the ideas that have already appeared in the literature (Saaty and Vargas, 1984, Bana e Costa and Vansnick, 2008, D'Apuzzo et al., 2007, and Kułakowski, 2015), we extend the notions of desirable properties to the fuzzy case as follows, cf. Bartl and Ramík (2022, Definition 6.1).

Definition 1. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$, let $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, be a priority vector, let $\alpha \in [0; 1]$, and let $[\tilde{a}_{ij}]_\alpha = [a^{\mathrm{L}}_{ij}(\alpha); a^{\mathrm{R}}_{ij}(\alpha)]$ be the α-cut of $\tilde{a}_{ij}$.

(i) We say that the vector v is an *α-consistent vector* (*α-CsV*) of the FPC matrix $\tilde{A}$ if the following condition holds:

$$a^{\mathrm{L}}_{ij}(\alpha) \leq v_i \div v_j \qquad \text{for all} \quad i, j \in \mathcal{N}. \tag{2}$$

Moreover, the vector v is a *consistent vector* (*CsV*) of the FPC matrix $\tilde{A}$ if condition (2) holds for all $\alpha \in [0; 1]$. If there exists an α-consistent vector or consistent vector of the FPC matrix $\tilde{A}$, then $\tilde{A}$ is called an α-consistent FPC matrix or consistent FPC matrix, respectively.

(ii) We say that the vector v is an *α-intensity vector* (*α-InV*) of the FPC matrix $\tilde{A}$ if the following condition holds:

$$a^{\mathrm{L}}_{ij}(\alpha) > a^{R}_{kl}(\alpha) \quad \text{implies} \quad v_i \div v_j > v_k \div v_l \qquad \text{for all} \quad i, j, k, l \in \mathcal{N}. \tag{3}$$

Moreover, the vector v is an *intensity vector* (*InV*) of the FPC matrix $\tilde{A}$ if condition (3) holds for all $\alpha \in [0; 1]$. If there exists an α-intensity vector or intensity vector of the FPC matrix $\tilde{A}$, then $\tilde{A}$ is called an α-intensity FPC matrix or intensity FPC matrix, respectively.

(iii) We say that the vector v is an *α-coherent vector* (*α-CoV*) of the FPC matrix $\tilde{A}$ if the following condition holds:

$$a^{\mathrm{L}}_{ij}(\alpha) > e \quad \text{implies} \quad v_i > v_j \qquad \text{for all} \quad i, j \in \mathcal{N}. \tag{4}$$

Moreover, the vector v is a *coherent vector* (*CoV*) of the FPC matrix $\tilde{A}$ if condition (4) holds for all $\alpha \in [0; 1]$. If there exists an α-coherent vector or coherent vector of the FPC matrix $\tilde{A}$, then $\tilde{A}$ is called an α-coherent FPC matrix or coherent FPC matrix, respectively.

Remark 1. Notice that by the reciprocity property (1) it is easy to see that $v = (v_1, v_2, \dots, v_n)$ is an α-consistent vector of the FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ if and only if

$$a^{\mathrm{L}}_{ij}(\alpha) \leq v_i \div v_j \leq a^{\mathrm{R}}_{ij}(\alpha) \qquad \text{for all} \quad i, j \in \mathcal{N}. \tag{5}$$

The following result, the proof of which can be found in Bartl and Ramík (2022, Proposition 6.4), turns out to be useful.

Proposition 1. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$, let $\alpha \in [0; 1]$, and let $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha); a^R_{ij}(\alpha)]$ be the α-cut of $\tilde{a}_{ij}$. A priority vector $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, satisfies*

$$a^{\mathrm{L}}_{ij}(\alpha) \div a^{\mathrm{R}}_{kl}(\alpha) \leq (v_i \div v_j) \div (v_k \div v_l) \qquad \text{for all} \quad i,j,k,l \in \mathcal{N} \tag{6}$$

if and only if v is an α-consistent vector of the FPC matrix $\tilde{A}$.

4. Measuring desirable properties of priority vectors

Given an alo-group $\mathcal{G} = (G, \odot, \leq)$, let $\bar{G} = G \cup \{-\infty, +\infty\}$ be the set G extended by adding the two infinity elements $+\infty$ and $-\infty$, and let us extend the ordering $\leq$ of the alo-group $\mathcal{G}$ by defining that $-\infty < x < +\infty$ for every $x \in G$.

Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$, let $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, be a vector, and let $\alpha \in [0; 1]$. Notice that, if v is an α-consistent priority vector of the FPC matrix $\tilde{A}$, then it is an α-intensity priority vector of $\tilde{A}$, yet, if v is an α-intensity priority vector of the FPC matrix $\tilde{A}$, then it is an α-coherent priority vector of $\tilde{A}$, see Bartl and Ramík (2022, Remark 6.5). In practice, FPC matrices are often inconsistent, even more, the intensity condition is not satisfied and/or the FPC matrices are incoherent. Therefore, it is useful to know "how much" these desirable properties of the priority vector are violated. This is why we measure the inconsistency, non-intensity, and incoherence of the given FPC matrix $\tilde{A}$ and priority vector v by special grades, see, e.g., Saaty (1980) and Ramík (2020); the following definition improves the earlier one given by Bartl and Ramík (2022, Definition 7.1).

Definition 2. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$. For every $i,j,k,l \in \mathcal{N}$, for a priority vector $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, and for $\alpha \in [0; 1]$, define:

(i) The *local α-inconsistency grade of two elements of FPC matrix $\tilde{A}$ and vector v*

$$\varepsilon^{\mathrm{Cs}}_{ijkl}(\tilde{A}, v, \alpha) = \left(a^{\mathrm{L}}_{ij}(\alpha) \div (v_i \div v_j)\right) \odot \left((v_k \div v_l) \div a^{\mathrm{R}}_{kl}(\alpha)\right), \tag{7}$$

and the *global α-inconsistency grade of FPC matrix $\tilde{A}$ and vector v*

$$E^{\mathrm{Cs}}(\tilde{A}, v, \alpha) = \max\{\, \varepsilon^{\mathrm{Cs}}_{ijkl}(\tilde{A}, v, \alpha) \mid i,j,k,l \in \mathcal{N}\,\}. \tag{8}$$

(ii) The *local α-non-intensity grade of two elements of FPC matrix $\tilde{A}$ and vector v*

$$\varepsilon^{\mathrm{In}}_{ijkl}(\tilde{A}, v, \alpha) = \begin{cases} (v_k \div v_l) \div (v_i \div v_j) & \text{if } a^{\mathrm{L}}_{ij}(\alpha) > a^{\mathrm{R}}_{kl}(\alpha), \\ -\infty & \text{otherwise}, \end{cases} \tag{9}$$

and the *global α-non-intensity grade of FPC matrix $\tilde{A}$ and vector v*

$$E^{\mathrm{In}}(\tilde{A}, v, \alpha) = \max\{\, \varepsilon^{\mathrm{In}}_{ijkl}(\tilde{A}, v, \alpha) \mid i,j,k,l \in \mathcal{N}\,\}. \tag{10}$$

(iii) The *local α-incoherence grade of an element of FPC matrix $\tilde{A}$ and vector v*

$$\varepsilon^{\mathrm{Co}}_{ij}(\tilde{A}, v, \alpha) = \begin{cases} v_j \div v_i & \text{if } a^{\mathrm{L}}_{ij}(\alpha) > e, \\ -\infty & \text{otherwise}, \end{cases} \tag{11}$$

and the *global α-incoherence grade of FPC matrix $\tilde{A}$ and vector v*

$$E^{\mathrm{Co}}(\tilde{A}, v, \alpha) = \max\{\, \varepsilon^{\mathrm{Co}}_{ij}(\tilde{A}, v, \alpha) \mid i,j \in \mathcal{N}\,\}. \tag{12}$$

Remark 2. The local α-inconsistency grade $\varepsilon_{ijkl}^{\mathrm{Cs}}(\tilde{A}, v, \alpha)$, defined by (7), can equivalently be written as

$$\varepsilon_{ijkl}^{\mathrm{Cs}}\big(\tilde{A}, v, \alpha\big) = \Big(a_{ij}^{\mathrm{L}}(\alpha) \div a_{kl}^{\mathrm{R}}(\alpha)\Big) \div \Big((v_i \div v_j) \div (v_k \div v_l)\Big)\,, \tag{13}$$

cf. inequality (6) in Proposition 1, by using which it is easy to see that v is an α-consistent vector of $\tilde{A}$ if and only if $\varepsilon_{ijkl}^{\mathrm{Cs}}(\tilde{A}, v, \alpha) \leq e$ for all $i, j, k, l \in \mathcal{N}$, where e is the neutral element of the alo-group $\mathcal{G}$.

The following proposition is a direct consequence of Definitions 1 and 2 and Remark 2. A detailed proof can be found in Bartl and Ramík (2022, Proposition 7.3).

Proposition 2. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$, let $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, be a vector, and let $\alpha \in [0; 1]$. Then:*

(i) $E^{\mathrm{Co}}(\tilde{A}, v, \alpha) \leq e$ *if and only if the vector v is α-consistent,*
(ii) $E^{\mathrm{In}}(\tilde{A}, v, \alpha) < e$ *if and only if the vector v is an α-intensity vector,*
(iii) $E^{\mathrm{Cs}}(\tilde{A}, v, \alpha) < e$ *if and only if the vector v is α-coherent.*

The subsequent relations (14) reflect the fact that, if v is an α-consistent priority vector of the FPC matrix $\tilde{A}$, then it is an α-intensity priority vector of $\tilde{A}$, yet, if v is an α-intensity priority vector of the FPC matrix $\tilde{A}$, then it is an α-coherent priority vector of $\tilde{A}$, which we have already remarked above. A detailed proof of the next proposition can be found in Bartl and Ramík (2022, Proposition 7.3).

Proposition 3. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$, let $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, be a vector, and let $\alpha \in [0; 1]$. Then:*

$$E^{\mathrm{Co}}(\tilde{A}, v, \alpha) \leq E^{\mathrm{In}}(\tilde{A}, v, \alpha) < E^{\mathrm{Cs}}(\tilde{A}, v, \alpha)\,. \tag{14}$$

5. An algorithm to generate a joint priority vectors of FPC matrices in group decision making

In this paper, we consider the main subproblem of the AHP extended as follows. There are n objects $c_1, c_2, \dots, c_n$ that are to be judged with respect to the given criterion by m independent decision makers (evaluators). Given an alo-group $\mathcal{G} = (G, \odot, \leq)$, each of the decision makers assesses the relative importance of the two items in each pair of the objects with respect to the given criterion by using a fuzzy interval on the alo-group $\mathcal{G}$; that is, let the fuzzy subset $\tilde{a}_{ij}^k$ of G be a fuzzy interval and represent the k-th decision maker's (fuzzy) opinion how many times c_i is better or more important than c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$ and for $k = 1, 2, \dots, m$. We are thus given $n \times n$ fuzzy pairwise comparison matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. The purpose is to find a single (joint) priority vector $v = (v_1, v_2, \dots, v_n) \in G^n$ having the aforedefined desirable properties – consistency, intensity, and coherence – with respect to the FPC matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. To this end, we improve and extend the authors' new algorithm (Bartl and Ramík, 2022) to this case. Our enhanced Algorithm consists of the following six steps:

STEP 1. Set $\alpha := 0$. Find an optimal solution v^α to PROBLEM 1:

$$\max\big\{\, E^{\mathrm{Cs}}\big(\tilde{A}^1, v, \alpha\big),\ E^{\mathrm{Cs}}\big(\tilde{A}^2, v, \alpha\big),\ \dots,\ E^{\mathrm{Cs}}\big(\tilde{A}^m, v, \alpha\big) \big\} \ \longrightarrow\ \min \tag{15}$$

subject to

$$v \in G^n.$$

If the minimal value of the objective function is greater than the neutral element e of the alo-group $\mathcal{G}$, then there is no 0-consistent priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$, see Proposition 2.(i); go to Step 3. Otherwise, if the minimal value of the objective function is less than or equal to e, then the optimal solution v^0 is a 0-consistent priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. Look for an α-consistent vector with the maximal $\alpha \in [0; 1]$, i.e. proceed with Step 2.

STEP 2. Find an optimal solution α^*, v^{α^*} to PROBLEM 2:

$$\alpha \longrightarrow \max \tag{16}$$

subject to

$$E^{\mathrm{Cs}}(\tilde{A}^k, v, \alpha) \leq e \quad \text{for } k = 1, 2, \dots, m, \qquad v \in G^n, \qquad \alpha \in [0;1].$$

The optimal solution v^{α^*} is an α^*-consistent priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ such that $\alpha^* \in [0;1]$ is maximal. At the same time, it is an α^*-intensity vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. Look for an α-intensity vector with the maximal $\alpha \in [0;1]$, i.e. go to Step 4.

STEP 3. Set $\alpha \coloneqq 0$. Find an optimal solution v^α to PROBLEM 3:

$$\max\{E^{\mathrm{In}}(\tilde{A}^1, v, \alpha), E^{\mathrm{In}}(\tilde{A}^2, v, \alpha), \dots, E^{\mathrm{In}}(\tilde{A}^m, v, \alpha)\} \longrightarrow \min \tag{17}$$

subject to

$$v \in G^n.$$

If the minimal value of the objective function is greater than or equal to the neutral element e of the alo-group $\mathcal{G}$, then there is no 0-intensity priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$, see Proposition 2.(ii); go to Step 5. Otherwise, if the minimal value of the objective function is less than e, then the optimal solution v^0 is a 0-intensity priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. Look for an α-intensity vector with the maximal $\alpha \in [0;1]$, i.e. proceed with Step 4.

STEP 4. Find an optimal solution $\alpha^{**}, v^{\alpha^{**}}$ to PROBLEM 4:

$$\alpha \longrightarrow \max \tag{18}$$

subject to

$$E^{\mathrm{In}}(\tilde{A}^k, v, \alpha) < e \quad \text{for } k = 1, 2, \dots, m, \qquad v \in G^n, \qquad \alpha \in [0;1].$$

The optimal solution $v^{\alpha^{**}}$ is an α^{**}-intensity priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ such that $\alpha^{**} \in [0;1]$ is maximal. At the same time, it is an α^{**}-coherent vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. Look for an α-coherent vector with the maximal $\alpha \in [0;1]$, i.e. go to Step 6.

STEP 5. Set $\alpha \coloneqq 0$. Find an optimal solution v^α to PROBLEM 5:

$$\max\{E^{\mathrm{Co}}(\tilde{A}^1, v, \alpha), E^{\mathrm{Co}}(\tilde{A}^2, v, \alpha), \dots, E^{\mathrm{Co}}(\tilde{A}^m, v, \alpha)\} \longrightarrow \min \tag{19}$$

subject to

$$v \in G^n.$$

If the minimal value of the objective function is greater than or equal to the neutral element e of the alo-group $\mathcal{G}$, then there is no 0-coherent priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$, see Proposition 2.(iii); go to End. Otherwise, if the minimal value of the objective function is less than e, then the optimal solution v^0 is a 0-coherent priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$. Look for an α-coherent vector with the maximal $\alpha \in [0;1]$, i.e. proceed with Step 6.

STEP 6. Find an optimal solution $\alpha^{***}, v^{\alpha^{***}}$ to PROBLEM 6:

$$\alpha \longrightarrow \max \tag{20}$$

subject to

$$E^{\mathrm{Co}}(\tilde{A}^k, v, \alpha) < e \quad \text{for } k = 1, 2, \dots, m, \qquad v \in G^n, \qquad \alpha \in [0;1].$$

The optimal solution $v^{\alpha^{***}}$ is an α^{***}-intensity priority vector common to all matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ such that $\alpha^{***} \in [0;1]$ is maximal. Go to End.

END.

If the minimal value of the objective function is greater than or equal to e in Step 5, then the Algorithm does not provide any joint priority vector with desirable properties; that is, some of the initial FPC matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ is too incoherent. The decision makers may wish to revise some of their pairwise comparisons; that is, change some elements of the FPC matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ and run the Algorithm again. Otherwise, the proposed Algorithm provides up to three joint priority vectors with the desirable properties, i.e. the α^*-consistent vector v^{α^*}, the α^{**}-intensity vector $v^{\alpha^{**}}$, and the α^{***}-intensity vector $v^{\alpha^{***}}$, such that their membership grades α^*, α^{**}, and α^{***} are maximal and non-decreasing ($0 \le \alpha^* \le \alpha^{**} \le \alpha^{***} \le 1$).

6. An improvement of the Algorithm

In this section, we propose an improvement of the Algorithm consisting in that we consider weights $u_1, u_2, \dots, u_m \in G$, such that $u_1, u_2, \dots, u_m \ge e$, of the m independent decision makers (evaluators) in order to express their importance and/or expertise to judge the objects $c_1, c_2, \dots, c_n$ with respect to the given criterion. Then, in problems (15), (17), and (19), we minimize the respective weighted maximum

$$\max\{ u_1 \odot E^{\mathrm{Cs/In/Co}}(\tilde{A}^1, v, \alpha),\, u_2 \odot E^{\mathrm{Cs/In/Co}}(\tilde{A}^2, v, \alpha),\, \dots,\, u_m \odot E^{\mathrm{Cs/In/Co}}(\tilde{A}^m, v, \alpha) \}$$

of the global grades of α-inconsistency / α-non-intensity / α-incoherence, respectively, and we replace the constraints

$$E^{\mathrm{Cs}}(\tilde{A}^k, v, \alpha) \le e\,, \qquad E^{\mathrm{In}}(\tilde{A}^k, v, \alpha) < e\,, \qquad \text{and} \qquad E^{\mathrm{Co}}(\tilde{A}^k, v, \alpha) < e\,,$$

in problems (16), (18), and (20), by

$$u_k \odot E^{\mathrm{Cs}}(\tilde{A}^k, v, \alpha) \le e\,, \qquad u_k \odot E^{\mathrm{In}}(\tilde{A}^k, v, \alpha) < e\,, \qquad \text{and} \qquad u_k \odot E^{\mathrm{Co}}(\tilde{A}^k, v, \alpha) < e\,,$$

respectively. Notice that, if $e \le u \in G$ and $E \in G$ is such that $u \odot E \le e$ or $u \odot E < e$, where e is the neutral element of the alo-group $\mathcal{G} = (G, \odot, \le)$, then $E \le e$ or $E < e$, respectively, too, so that the respective properties (consistency, intensity, and coherence) of the joint priority vector v are preserved, see Proposition 2.

7. Conclusion

In this paper, we have extended our previous algorithm (Bartl and Ramík, 2022) for generating crisp priority vectors of desirable properties – consistency, intensity, and coherence – of a fuzzy pairwise comparison matrix (provided by a single decision maker or evaluator) to generate joint crisp priority vectors of the desirable properties of fuzzy pairwise comparison matrices (provided by a group of independent decision makers or evaluators), addressing the situation of multi-criteria decision making thus. Another topic is left for further research, however.

Given the fuzzy pairwise comparison matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$, the Algorithm proposed in Sections 5 and 6 finds a joint priority vector v. Another approach is to apply our previous algorithm (Bartl and Ramík, 2022) to each of the FPC matrices $\tilde{A}^1, \tilde{A}^2, \dots, \tilde{A}^m$ separately and obtain m priority vectors $w^1, w^2, \dots, w^m$; that is, each of the m decision makers has their own priority vector. Now, given two priority vectors $v = (v_1, v_2, \dots, v_n) \in G^n$ and $w = (w_1, w_2, \dots, w_n) \in G^n$, we define that the two vectors are *consensual* if $v_i > v_j \Leftrightarrow w_i > w_j$ for all $i, j = 1, 2, \dots, n$. The goal is then to find a single consensual priority vector v, or rather (since such a vector may not exist) the goal is to find a vector v such that it is pairwise consensual with respect to each of the m priority vectors $w^1, w^2, \dots, w^m$ as much as possible.

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