A Consensual Coherent Priority Vector of Pairwise Comparison Matrices in Group Decision-Making

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Abstract. The Analytic Hierarchy Process (AHP) is a method proposed to solve complex multi-criteria decision-making problems. Pairwise comparison methods are often used in AHP to derive the priorities of the successors of an element in the hierarchy. In this paper, we are concerned with group decision-making; that is, given n objects, such as criteria and/or variants, let m decision makers evaluate the n objects (pairwise) with respect to a criterion. The task is then to find a consensual priority vector of the m given $n \times n$ reciprocal pairwise comparison matrices. Recalling several desirable properties of the priority vector – consistency, intensity, and coherence – we consider the weakest one of the three, i.e. coherence, in the rest of the paper. In other words, given m coherent priority vectors, each provided by a decision maker of the group, the purpose is to find a single consensual priority vector of the group. To cope with this task, we propose a grade to measure the consensuality of a priority vector. We thus obtain an optimization problem, whose solution yields an optimal consensual ranking of the n given objects.

Keywords: multi-criteria group decision-making, pairwise comparison matrices, consensual priority vector, coherence, Analytic Hierarchy Process (AHP)

JEL Classification: C44, C65, C63, D79
AMS Classification: 90C29, 90C70

1 Introduction

The Analytic Hierarchy Process (AHP) is a popular and powerful tool to solve multi-criteria decision-making problems [9]. We consider the following main subproblem of the AHP, which is to be solved in every internal node of the hierarchy; that is, a node having some subnodes. Let n denote the number of these subnodes, which correspond to n objects $c_1, c_2, \dots, c_n$, i.e. criteria, subcriteria, and/or alternatives (variants). Notice that the internal node corresponds to some criterion, subcriterion, and/or the goal of the hierarchy. Henceforth, we shall use the single term criterion for simplicity. Given the information on the relative importance of the two items in each pair of the objects with respect to the given criterion (subcriterion, and/or the goal) in the form of an $n \times n$ pairwise comparison matrix A, the purpose is to calculate the priority vector, which is a vector of n weights $v_1, v_2, \dots, v_n$ assigned to the n objects $c_1, c_2, \dots, c_n$, respectively. The prominent methods to calculate the priority vector include Saaty's Eigenvector Method (EVM) and the Geometric Mean Method (GMM), see [9] and [8]. The priority vector provided by these methods, however, usually do not satisfy desirable properties – consistency, intensity, and/or coherence, in particular – see [10], [5], and [1].

For $i, j = 1, 2, \dots, n$, let a_{ij} be a value (i.e. quantity or number) that represents the decision maker's opinion how many times object c_i is more important or better than object c_j with respect to the given criterion. We thus obtain a (crisp) *pairwise comparison matrix* $A = \{a_{ij}\}$. This is a special case of the fuzzy case studied in [2], where the authors have proposed a new algorithm for computing priority vectors, satisfying desirable properties, of a fuzzy pairwise comparison matrix. In [3], the authors have improved and extended their new algorithm to the case when there are m decision makers (evaluators), and each of them assesses the relative importance of the two items in each pair of the objects with respect to the given criterion. In other words, given $n \times n$ pairwise comparison matrices $A^1, A^2, \dots, A^m$ such that the element a_{ij}^k of the k-th matrix represents the k-th decision maker's opinion how many times c_i is more important or better than c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$ and for $k = 1, 2, \dots, m$, the extended algorithm provides a *joint* priority vector, satisfying the desirable properties, of the m pairwise comparison matrices $A^1, A^2, \dots, A^m$. In this paper, it is our purpose to provide an algorithm to compute a *consensual* priority vector, satisfying the desirable property of coherence only.

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2 Abelian linearly ordered groups

In order to unify and generalize various approaches known from the literature, we use the elements of an Abelian linearly ordered group to evaluate the relative importance of the two items in each pair of the objects with respect to the given criterion, see [4] and [7]. Recall that an Abelian group is a pair $(G, \odot)$ where G is a non-empty set and $\odot$ is a commutative and associative binary operation on G satisfying also the existence of the neutral element $e \in G$ and the existence of the inverse element $a^{(-1)} \in G$ for each $a \in G$. We then have $a \odot e = a$ and $a \odot a^{(-1)} = e$ for every $a \in G$. We also put $a \div b = a \odot b^{(-1)}$ for all $a, b \in G$. An Abelian linearly ordered group (alo-group) is a triple $(G, \odot, \leq)$ such that $(G, \odot)$ is an Abelian group and $\leq$ is a binary relation of linear ordering on G such that $a \leq b$ implies $a \odot c \leq b \odot c$ for all $a, b, c \in G$. The well-known examples of alo-groups are the Multiplicative alo-group $\mathcal{R}_+ = (\mathbb{R}_+, \cdot, \leq)$ with the usual multiplication and the neutral element $e = 1$, the Additive alo-group $\mathcal{R} = (\mathbb{R}, +, \leq)$ with the usual addition and the neutral element $e = 0$, and the Fuzzy Multiplicative alo-group $\mathcal{F}_{]0;1[} = (]0;1[, \odot, \leq)$ with $a \odot b = ab/(ab + (1-a)(1-b))$ for $a, b \in]0;1[$ and the neutral element $e = \frac{1}{2}$, see [4], [7], and [8].

3 Desirable properties of the priority vector

Let us consider an alo-group $\mathcal{G} = (G, \odot, \leq)$ and let us denote the set of the first n positive natural numbers by $\mathcal{N}$; that is, we put $\mathcal{N} = \{1, 2, \dots, n\}$. Considering the set $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$, let $A = \{a_{ij}\}$ be an $n \times n$ matrix such that each of its element $a_{ij} \in G$ evaluates the relative importance of the objects c_i and c_j with respect to the given criterion. The matrix $A = \{a_{ij}\}$ is called a *pairwise comparison matrix*, or *PC matrix* for short, if it is *reciprocal*; that is, if the following two conditions hold for each $i, j \in \mathcal{N}$:

$$a_{ii} = e, \qquad \text{and} \qquad a_{ij} \odot a_{ji} = e. \tag{1}$$

Then the result of a pairwise comparison method based on the PC matrix $A = \{a_{ij}\}$ is a vector $v = (v_1, v_2, \dots, v_n)$ of the weights $v_1, v_2, \dots, v_n \in G$ of the objects $c_1, c_2, \dots, c_n \in \mathcal{C}$, respectively. In other words, the i-th component v_i of the priority vector v is the weight of the object c_i for $i \in \mathcal{N}$. We say the priority vector $v = (v_1, v_2, \dots, v_n)$ is *normalized* if $\bigodot_{i=1}^{n} v_i = e$.

Based upon the ideas that have already appeared in the literature ([10], [1], [5], [6], [2] and [3]), we define the notions of desirable properties as follows.

Definition 1. Let $A = \{a_{ij}\}$ be a PC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$ and let $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, be a priority vector.

(i) We say that the vector v is a *consistent vector* (CsV) of the PC matrix A if the following condition holds:

$$a_{ij} = v_i \div v_j \qquad \text{for all} \quad i, j \in \mathcal{N}. \tag{2}$$

(ii) We say that the vector v is an *intensity vector* (InV) of the PC matrix A if the following condition holds:

$$a_{ij} > a_{kl} \quad \text{if and only if} \quad v_i \div v_j > v_k \div v_l \qquad \text{for all} \quad i, j, k, l \in \mathcal{N}. \tag{3}$$

(iii) We say that the vector v is a *coherent vector* (CoV) of the PC matrix A if the following condition holds:

$$a_{ij} > e \quad \text{if and only if} \quad v_i > v_j \qquad \text{for all} \quad i, j \in \mathcal{N}. \tag{4}$$

If there exists a consistent, intensity, or coherent vector of the PC matrix A, then A is called a *consistent*, *intensity*, or *coherent PC matrix*, respectively.

By reciprocity (1) and by Definition 1, the following result is easy to see. This is why we omit its proof.

Proposition 2. *Let $A = \{a_{ij}\}$ be a PC matrix on an alo-group $\mathcal{G} = (G, \odot, \leq)$ and let $v = (v_1, v_2, \dots, v_n)$, with $v_j \in G$, be a vector. Then:*

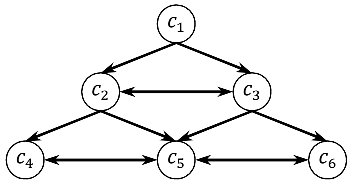
(i) *If v is a consistent priority vector of the PC matrix A, then it is an intensity priority vector of A.*
(ii) *If v is an intensity priority vector of the PC matrix A, then it is a coherent priority vector of A.*

A coherent matrix of pairwise comparisons of elements $c_1, \dots, c_6$ with respect to the criterion by the 1st expert:

$$A_1 = \begin{pmatrix} 1 & 5 & 5 & 5 & 5 & 5 \\ 1/5 & 1 & 1 & 5 & 5 & 5 \\ 1/5 & 1 & 1 & 5 & 5 & 5 \\ 1/5 & 1/5 & 1/5 & 1 & 1 & 1 \\ 1/5 & 1/5 & 1/5 & 1 & 1 & 1 \\ 1/5 & 1/5 & 1/5 & 1 & 1 & 1 \end{pmatrix}$$

The induced quasi-linear ordering:

$$\underbrace{c_1}_{\mathcal{K}_1} \succ \underbrace{c_2 \approx c_3}_{\mathcal{K}_2} \succ \underbrace{c_4 \approx c_5 \approx c_6}_{\mathcal{K}_3}$$



The matrix P_1 representing the induced quasi-linear ordering:

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

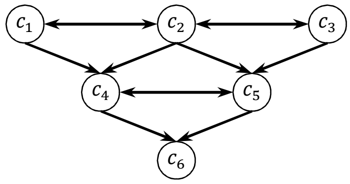
Figure 1 The 1st expert's judgements of 6 elements $c_1, c_2, c_3, c_4, c_5, c_6$ with respect to some criterion

A coherent matrix of pairwise comparisons of elements $c_1, \dots, c_6$ with respect to the criterion by the 2nd expert:

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 5 & 5 & 5 \\ 1 & 1 & 1 & 5 & 5 & 5 \\ 1 & 1 & 1 & 5 & 5 & 5 \\ 1/5 & 1/5 & 1/5 & 1 & 1 & 5 \\ 1/5 & 1/5 & 1/5 & 1 & 1 & 5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1 \end{pmatrix}$$

The induced quasi-linear ordering:

$$\underbrace{c_1 \approx c_2 \approx c_3}_{\mathcal{K}_1} \succ \underbrace{c_4 \approx c_5}_{\mathcal{K}_2} \succ \underbrace{c_6}_{\mathcal{K}_3}$$



The matrix P_2 representing the induced quasi-linear ordering:

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2 The 2nd expert's judgements of 6 elements $c_1, c_2, c_3, c_4, c_5, c_6$ with respect to some criterion

4 An algorithm to generate a consensual priority vector of coherent PC matrices in group decision-making

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Given an $n \times n$ pairwise comparison matrix $A = \{a_{ij}\}$ on the alo-group $\mathcal{G}$, we introduce binary relations $\succ$, $\approx$ and $\succcurlyeq$ on the set $\mathcal{N}$ as follows. For $i, j \in \mathcal{N}$, we define that $i \succ j$ if and only if $a_{ij} > e$, and we define that $i \approx j$ if and only if $a_{ij} = e$, where e is the neutral element of the alo-group $\mathcal{G}$. Finally, for $i, j \in \mathcal{N}$, we define that $i \succcurlyeq j$ if and only if $i \succ j$ or $i \approx j$. Notice that the PC matrix A is coherent if and only if the relation $\succcurlyeq$ is a quasi-linear ordering of the set $\mathcal{N}$, and also $\approx$ is a relation of equivalence on $\mathcal{N}$; that is, the relation $\succcurlyeq$ is complete ($i \succcurlyeq j$ or $j \succcurlyeq i$) and transitive ($i \succcurlyeq j \succcurlyeq k \implies i \succcurlyeq k$), and the relation $\approx$ is reflexive ($i \approx i$), symmetric ($i \approx j \implies j \approx i$), and transitive ($i \approx j \approx k \implies i \approx k$).

Assume in the sequel that the PC matrix $A = \{a_{ij}\}$ is coherent. Then the relation $\succcurlyeq$ of quasi-ordering of the set $\mathcal{N}$ can equivalently be represented by an $n \times n$ matrix $P = \{p_{ij}\}$ consisting of 0's and 1's as follows. Let $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_R$ be all the pairwise distinct equivalence classes of the relation $\approx$ on the set $\mathcal{N}$, and let the classes be ordered so that $r < s$ implies $k \succ l$ for all $k \in \mathcal{K}_r$, for all $l \in \mathcal{K}_s$, and for all $r, s = 1, 2, \dots, R$. Let $i_1 \coloneqq n$, and for $r = 1, 2, \dots, R$, let $i_{r+1} \coloneqq i_r - |\mathcal{K}_r|$, where $|\mathcal{K}_r|$ denotes the number of the elements of the class $\mathcal{K}_r$. Notice that $i_{R+1} = 0$. Then, for $r = 1, 2, \dots, R$, we put $p_{i_r j} = 1$ for $j \in \mathcal{K}_r$, and we put $p_{i_r j} = 0$ for $j \in \mathcal{N} \setminus \mathcal{K}_r$. Finally, we put $p_{ij} = 0$ for all $i \in \mathcal{N} \setminus \{i_1, i_2, \dots, i_R\}$ and for all $j = 1, 2, \dots, n$. Notice that P is a binary matrix, which consists of elements 0 and 1, and satisfies the following system of inequalities and conditions:

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad \text{for} \quad j \in \mathcal{N}, \tag{5}$$

$$\sum_{j=1}^{n} p_{ij} \leq i \times \max\{0,\ 1 + k - \sum_{j=1}^{n} p_{i+k,j}\} \qquad \text{for} \quad k = 1, \dots, n-i \qquad \text{for} \quad i \in \mathcal{N}, \tag{6}$$

$$p_{ij} \in \{0, 1\} \qquad \text{for} \quad i, j \in \mathcal{N}. \tag{7}$$

In words, the maximal elements are in the class $\mathcal{K}_1$ and the smaller elements are in the subsequent classes $\mathcal{K}_2, \dots \mathcal{K}_R$. The elements of the class $\mathcal{K}_r$ are represented in the i_r-th row of the matrix P. Moreover, there being $|\mathcal{K}_r|$ elements in the class $\mathcal{K}_r$, therefore $|\mathcal{K}_r|$ 1's in the i_r-th row, then the subsequent $(|\mathcal{K}_r| - 1)$ rows, i.e. rows $i_r - 1, \dots, i_r - |\mathcal{K}_r| + 1$, of the matrix P must be zero. Examples presented in Figures 1 and 2 illustrate this procedure.

In this paper, it is our purpose to consider the main subproblem of the AHP extended as follows. Given the alo-group $\mathcal{G} = (G, \odot, \leq)$ and the n objects $c_1, c_2, \dots, c_n$ to be judged with respect to the given criterion by m independent decision makers (evaluators), each of the decision makers assesses the relative importance of the two items in each pair of the objects with respect to the given criterion by using an element of the alo-group $\mathcal{G}$; that is, let $a_{ij}^k \in G$ represent the k-th decision maker's opinion how many times c_i is more important or better than c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$ and for $k = 1, 2, \dots, m$. Additionally, we assume that each of the

m decision makers is coherent; that is, let each PC matrix $A^k = \{a_{ij}^k\}$ be coherent and let $w^k \in G^n$ be a coherent priority vector of the PC matrix A^k for $k = 1, 2, \dots, m$. Now, given the coherent PC matrices $A^1, A^2, \dots, A^m$, or their coherent priority vectors $w^1, w^2, \dots, w^m \in G^n$, our purpose is to find a single *consensual* priority vector $v = (v_1, v_2, \dots, v_n) \in G^n$; that is, a priority vector $v \in G^n$ that is consensual with each of the priority vectors $w^1, w^2, \dots, w^m$ as much as possible.

Generally speaking, we define that two priority vectors $w^k \in G^n$, and $v \in G^n$ are *consensual* if it holds $w_i^k > w_j^k \Leftrightarrow v_i > v_j$ for every $i, j = 1, 2, \dots, n$. Actually, as all the PC matrices $A^1, A^2, \dots, A^m$ are coherent, they induce the quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ of the set $\mathcal{N}$, defined in the above given way. Therefore, we define that quasi-linear orderings $\succcurlyeq^k$ and $\succcurlyeq$ of $\mathcal{N}$ are *consensual* if it holds $i \succ^k j \iff i \succ j$ for every $i, j \in \mathcal{N}$. Recalling that $w^k \in G^n$ is a coherent priority vector of the PC matrix A^k, observe that $i \succ^k j$ implies $w_i^k > w_j^k$ for $i, j \in \mathcal{N}$ for $k = 1, 2, \dots, m$. Consequently, it is possible to simplify our task as follows: given the quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ of the set $\mathcal{N}$, our purpose is to find a single *consensual* quasi-linear ordering $\succcurlyeq$ of the set $\mathcal{N}$; that is, a quasi-linear ordering $\succcurlyeq$ of $\mathcal{N}$ that is consensual with each of the quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ as much as possible.

Given quasi-linear orderings $\succcurlyeq^k$ and $\succcurlyeq$ of the set $\mathcal{N}$, we define the grade of their non-consensuality as follows: Let $P^k = \{p_{ij}^k\}$ and $P = \{p_{ij}\}$ be the binary matrix representing the relation $\succcurlyeq^k$ and $\succcurlyeq$, respectively, defined in the above given way; notice that the matrix P satisfies relations (5)–(7). We then define the grade of their non-consensuality as

$$\delta(P^k, P) = \sum_{j=1}^{n} \sum_{\substack{i^k=1 \\ p_{i^k j}^k = 1}}^{n} \sum_{\substack{i=1 \\ p_{ij}=1}}^{n} |i^k - i|, \tag{8}$$

where $|i^k - i|$ denotes the absolute value of the difference $i^k - i$. (Alternatively, we could replace $|i^k - i|$ by its square $|i^k - i|^2$, cube $|i^k - i|^3$, or any other power of it.) The idea behind (8) is to penalize the change of the "level" of the element $j = 1, 2, \dots, n$ when transiting from one quasi-ordering (e.g. $\succcurlyeq^k$) to the other (e.g. $\succcurlyeq$). Then the total non-consensuality of the quasi-linear ordering $\succcurlyeq$ of $\mathcal{N}$ with the quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ is defined as $\delta(P^1, P^2, \dots, P^m, P) = \sum_{k=1}^{m} \delta(P^k, P)$; that is,

$$\delta(P^1, P^2, \dots, P^m, P) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \sum_{i^k=1}^{n} |i^k - i| \times p_{i^k j}^k \right) \times p_{ij}\,. \tag{9}$$

To meet our purpose; that is, to find a single quasi-linear ordering $\succcurlyeq$ of the set $\mathcal{N}$ that is consensual with each of the quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ as much as possible, we minimize (9) subject to (5)–(7).

We notice that the aforegiven optimization problem (minimize (9) subject to (5)–(7)) is integer and non-smooth, hence difficult to solve. For this reason, we may restrict $\succcurlyeq$ to be a consensual *linear* ordering of the set $\mathcal{N}$, so that the corresponding matrix P reduces to a simple *permutation* matrix. Constraints (6) then reduce to $\sum_{j=1}^{n} p_{ij} \le i$ for $i \in \mathcal{N}$, which, by taking (5) and (7) into account, can further be simplified to $\sum_{j=1}^{n} p_{ij} = 1$ for $i \in \mathcal{N}$. Then, to find a single linear ordering $\succcurlyeq$ of the set $\mathcal{N}$ that is consensual with each of the quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ as much as possible, we minimize (9) subject to

$$\textstyle\sum_{i=1}^{n} p_{ij} = 1 \qquad \text{for} \quad j \in \mathcal{N}, \tag{10}$$

$$\textstyle\sum_{j=1}^{n} p_{ij} = 1 \qquad \text{for} \quad i \in \mathcal{N}, \tag{11}$$

$$p_{ij} \in \{0, 1\} \qquad \text{for} \quad i, j \in \mathcal{N}, \tag{12}$$

which is an assignment problem actually. It is well-known that the matrix of the coefficients by the variables p_{ij} in (10) and (11) is totally unimodular, so that (12) can be relaxed to

$$0 \le p_{ij} \le 1 \qquad \text{for} \quad i, j \in \mathcal{N}, \tag{13}$$

yet the easy problem of continuous linear programming (minimize (9) subject to (10), (11), and (13)) has an *integer* optimal solution.

Once we find an optimal solution to the above optimization problem (minimize (9) subject to either (5)–(7), or (10), (11), and (13)), we construct the corresponding consensual quasi-linear or linear ordering $\succcurlyeq$ of the set $\mathcal{N}$ as follows. Let $i_1, i_2, \dots, i_R$ be all the pairwise distinct elements of the set $\{ i \in \mathcal{N} \mid p_{ij} = 1 \text{ for some } j \in \mathcal{N} \}$ and let them be ordered so that $i_1 > i_2 > \cdots > i_R$. For $r = 1, 2, \dots, R$, put $\mathcal{K}_r = \{ j \in \mathcal{N} \mid p_{i_r j} = 1 \}$. Finally, let $i \approx j$

An optimal solution to the problem:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The quasi-linear ordering represented by the optimal solution P:

$$\underbrace{c_1 \approx c_2 \approx c_3}_{\mathcal{K}_1} \succ \underbrace{c_4 \approx c_5}_{\mathcal{K}_2} \succ \underbrace{c_6}_{\mathcal{K}_3}$$

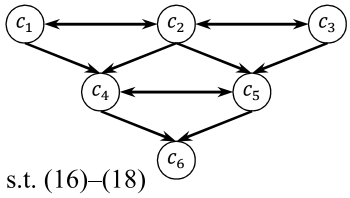


Figure 3 An optimal solution to the illustrative example min (15) s.t. (16)–(18)

An optimal solution to the problem:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear ordering represented by the optimal solution P:

$$\underbrace{c_1}_{\mathcal{K}_1} \succ \underbrace{c_2}_{\mathcal{K}_2} \succ \underbrace{c_3}_{\mathcal{K}_3} \succ \underbrace{c_4}_{\mathcal{K}_4} \succ \underbrace{c_5}_{\mathcal{K}_5} \succ \underbrace{c_6}_{\mathcal{K}_6}$$

Figure 4 An integer optimal solution to the simple illustrative example min (19) s.t. (20)–(22)

for all $i, j \in \mathcal{K}_r$ for $r = 1, 2, \ldots, R$, and let $i \succ j$ for all $i \in \mathcal{K}_r$ and for all $j \in \mathcal{K}_s$ for $r = 1, 2, \ldots, R - 1$ and for $s = r + 1, r + 2, \ldots, R$.

5 An illustrative example

Let the alo-group $\mathcal{G} = (G, \odot, \leq)$ be the usual multiplicative group $\mathcal{R}_+ = (\mathbb{R}_+, \cdot, \leq)$ of the field of the reals with the usual multiplication and usual linear ordering, and with the neutral element $e = 1$. We are given $n = 6$ objects $c_1, c_2, c_3, c_4, c_5, c_6$ to be judged with respect to some criterion by $m = 2$ independent decision makers (evaluators or experts). The experts have independently assessed the relative importance of the two items in each pair of the objects with respect to the criterion. Matrices A_1 and A_2 presented in Figures 1 and 2, respectively, present the opinions of the two experts. Both matrices are coherent. The corresponding coherent priority vectors of the matrices are, e.g. $w^1 = \left(\frac{3}{10}, \frac{2}{10}, \frac{2}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$ and $w^2 = \left(\frac{3}{14}, \frac{3}{14}, \frac{3}{14}, \frac{2}{14}, \frac{2}{14}, \frac{1}{14}\right)$, respectively. We can see the opinions of the experts are different: while the 1st expert considers elements c_2 and c_3 to be less important that element c_1, the 2nd expert considers all three elements c_1, c_2, c_3 to be equally important; while the 1st expert considers elements c_4, c_5, c_6 to be equally important, the 2nd expert considers elements c_4 and c_5 to be more important that element c_6. Now, our purpose is to find a single consensual priority vector $v = (v_1, v_2, v_3, v_4, v_5, v_6) \in \mathbb{R}_+^6$. To this end, we find an integer optimal solution to the above given linear programming problem to minimize (9) subject to either (5)–(7) or (10), (11), and (13). First, we construct the objective function; that is, the matrix of coefficients:

$$C = \begin{pmatrix} 5 & 4 & 4 & 3 & 3 & 3 \\ 4 & 3 & 3 & 2 & 2 & 2 \\ 3 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 5 & 5 & 2 & 2 & 0 \\ 4 & 4 & 4 & 1 & 1 & 1 \\ 3 & 3 & 3 & 0 & 0 & 2 \\ 2 & 2 & 2 & 1 & 1 & 3 \\ 1 & 1 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 3 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 9 & 9 & 5 & 5 & 3 \\ 8 & 7 & 7 & 3 & 3 & 3 \\ 6 & 5 & 5 & 1 & 1 & 3 \\ 4 & 3 & 3 & 1 & 1 & 3 \\ 2 & 1 & 1 & 3 & 3 & 5 \\ 0 & 1 & 1 & 5 & 5 & 7 \end{pmatrix} \qquad (14)$$

Next, we solve the non-smooth integer optimization problem (minimize (9) subject to (5)–(7)); that is,

$$\text{minimize} \quad \sum_{i=1}^{6} \sum_{j=1}^{6} c_{ij} p_{ij} \qquad (15)$$

subject to

$$\sum_{i=1}^{6} p_{ij} = 1 \qquad \text{for} \quad j = 1, 2, 3, 4, 5, 6, \qquad (16)$$

$$\sum_{j=1}^{6} p_{ij} \leq i \times \max\left\{0,\ 1 + k - \sum_{j=1}^{6} p_{i+k,j}\right\} \qquad \text{for} \quad k = 1, \ldots, 6 - i \qquad \text{for} \quad i = 1, 2, 3, 4, 5, 6, \qquad (17)$$

$$p_{ij} \in \{0, 1\} \qquad \text{for} \quad i, j = 1, 2, 3, 4, 5, 6. \qquad (18)$$

By using the Excel Solver, we find an optimal solution along with the respective consensual quasi-linear ordering of the set $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ presented in Figure 3.

We notice that the optimal solution $P = P_2$; that is, the opinion of the 2nd expert is consensual in this example. A consensual coherent priority vector is then, e.g., $v = w^2 = \left(\frac{3}{14}, \frac{3}{14}, \frac{3}{14}, \frac{2}{14}, \frac{2}{14}, \frac{1}{14}\right)$.

We also solve the simple linear programming problem (minimize (9) subject to (10), (11), and (13)); that is,

$$\text{minimize} \quad \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}p_{ij} \tag{19}$$

subject to

$$\sum_{i=1}^{6} p_{ij} = 1 \qquad \text{for} \quad j = 1, 2, 3, 4, 5, 6, \tag{20}$$

$$\sum_{j=1}^{6} p_{ij} = 1 \qquad \text{for} \quad i = 1, 2, 3, 4, 5, 6, \tag{21}$$

$$p_{ij} \geq 0 \qquad \text{for} \quad i, j = 1, 2, 3, 4, 5, 6. \tag{22}$$

By using the Excel Solver, we find an optimal solution along with the respective consensual linear ordering of the set $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ presented in Figure 4.

A consensual coherent priority vector is then, e.g., $v = \left(\frac{6}{21}, \frac{5}{21}, \frac{4}{21}, \frac{3}{21}, \frac{2}{21}, \frac{1}{21}\right)$.

6 Concluding remarks

In Section 4, we proposed the notion of consensuality of two priority vectors and also that of two quasi-linear orderings of n objects $c_1, c_2, \dots, c_n$. Subsequently, given two quasi-linear orderings, we proposed the notion of grade of their non-consensuality. Finally, given m quasi-linear orderings $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$, i.e. evaluations of the n objects by m evaluators (experts) with respect to some criterion, and yet an eventually consensual quasi-linear ordering $\succcurlyeq$ of the objects, we defined the total non-consensuality of $\succcurlyeq$ with $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ by formula (9), which is to be minimized. When constructing the objective function (9), we assumed that each of the m evaluators is equally skilled to judge the n objects $c_1, c_2, \dots, c_n$. Instead, we can express the importance and/or qualifications of the evaluators by weights $u_1, u_2, \dots, u_m > 0$ such that $u_1 + u_2 + \cdots + u_m = 1$. Then, using the general weighted power mean with $q \in (-\infty, +\infty) \setminus \{0\}$, of which the weighted arithmetic mean is a special case ($q = 1$), or the weighted geometric mean ($q = 0$), we can define the total non-consensuality of $\succcurlyeq$ with $\succcurlyeq^1, \succcurlyeq^2, \dots, \succcurlyeq^m$ as:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\sum_{k=1}^{m} u_k \times \left(\sum_{i^k=1}^{n} |i^k - i| \times p_{i^k j}^k\right)^q\right]^{1/q} \times p_{ij} \qquad \text{or} \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\prod_{k=1}^{m}\left(\sum_{i^k=1}^{n} |i^k - i| \times p_{i^k j}^k\right)^{u_k}\right] \times p_{ij}\,.$$

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