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O dlouhodobém chování trajektorií v diskrétních dynamických systémech

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To my husband Petr.

Thanks to you, I have never had to choose between family and mathematics.
I have it all.

Contents

1. Introduction

Many types of scientific and engineering problems are connected with nonlinear processes. The extensive development of nonlinear dynamics nowadays is explained not only by practical needs but also by new possibilities in the analysis. In this connection, a main role was played by simple nonlinear systems, which, on the one hand, are characterized by quite complicated dynamics but, on the other hand, admit profound qualitative analysis. The mathematical environment for modeling nonlinear phenomena is called a *discrete dynamical system.* It is usually defined as an ordered pair (X, f) where X is a compact metric space and f is a continuous map acting on X. To understand the dynamical properties of a system is necessary to analyze the trajectories of any point $x \in X$ under the iteration of f. *Limit sets of trajectories* are a helpful tool for these purposes since they can be used to understand long time behaviour of discrete dynamical systems in the future or the past. It is known that rendering long-term prediction of their behaviour by computer is impossible in general, even though these systems are deterministic. Limit sets can be very complicated, this is often the case when the dynamics is *chaotic.* Chaotic behaviour exists in many natural dynamical systems, continuous or discrete, such as weather and climate or biological model of population growth, and chaos theory has applications in several disciplines, including meteorology, physics, computer science, engineering, economics and biology.

The thesis studies the qualitative properties of limit dynamics when we assume that arbitrarily large amount of time (hence, infinite) has passed. In the two main sections of the thesis ”Limit sets of backward trajectories” and ”Chaotic behaviour” we recapitulate main results of 8 papers accompanied by a commentary. For proofs and details we refer reader to the corresponding paper in the attachments.

List of papers concerning the thesis

T1. M. Foryś-Krawiec, J. Hantáková, P. Oprocha, On the structure of α-limit sets of backward trajectories for graph maps, Discrete and Continuous Dynamical Systems (2021), doi:10.3934/dcds.2021159.

T2. J. Hantáková, S. Roth, On backward attractors of interval maps, Nonlinearity 34 (2021), 7415-7445.

T3. J. Hantáková, Li-Yorke sensitivity does not imply Li-Yorke chaos, Ergodic Theory and Dynamical Systems 39 (2019), 3066-3074.

T4. J. Doleželová, Distributionally scrambled invariant sets in a compact metric space, Nonlinear Analysis 79 (2013), 80–84.

T5. J. Doleželová, Scrambled and distributionally scrambled n-tuples, J. Difference Eq. Appl. 20 (2014), 1169–1177.

T6. J. Doleželová-Hantáková, Distributional chaos and factors, J. Difference Eq. Appl. 22 (2016), 99-106.

T7. J. Doleželová-Hantáková, S. Roth and Z. Roth, On the weakest version of distributional chaos, Int. J. Bifur. Chaos 26 (2016), 1650235, 13 pp.

T8. J. Hantáková, Iteration problem for distributional chaos, Int. J. Bifur. Chaos 27, no. 12, (2017), 1750183, 10 pp.

2. Limit sets of backward trajectories

2.1 Introduction and state-of-art

The *ω-limit sets* ($\omega(x)$ for short), i.e. the sets of limit points of a forward trajectories, were deeply studied by many authors. For instance, one can ask for a criterion allowing to decide whether a given closed invariant subset of X is an ω-limit set of some point $x \in X$. The question is very hard in general, however some cases are known. For example, the answer for ω-limit sets of a continuous map acting on the compact interval was provided by Blokh et al. in [8]. A closely related question is that of characterizing all those dynamical systems which may occur as restrictions of some system to one of its ω-limit sets. These abstract ω-limit sets were studied by Bowen [12] and Dowker and Frielander [16]. It was also proved that each ω-limit set of a continuous map of the interval is contained in the maximal one by Sharkovsky [33]. While the aforementioned results about ω-limit sets were first obtained for interval maps, some of them hold for maps acting on graphs, dendrites, Cantor space (for example, in work of Chudziak et al. [13], Kočan et al. [24], Barwell et al. [5]). In general compact metric spaces only partial results are known (e.g. recent work on ω-limit sets in topologically hyperbolic systems by Barwell et al. [6]).

α-limit sets ($\alpha(x)$ for short) were introduced as a dual concept to ω-limit sets and they should be regarded as a source of the trajectory of a point. While for invertible maps α-limit sets are well defined, for noninvertible maps there are many possibilities how to construct the limit of the backward trajectory. One possibility is to take as an α-limit set the set of all accumulation points of the set of pre-images $f^{-n}(x)$. This approach was used by Coven and Nitecki [14], who showed that for an interval map, a point x is nonwandering if and only if $x \in \alpha(x)$. Other approach used by Balibrea et al. in [3] propose instead of looking at all possible preimages to pick one backward branch and check accumulation points of this sequence. The union of the sets of accumulation points over all backward branches of the map was called a *special α-limit set* ($s\alpha(x)$ for short) by Hero [20]. While α-limit sets seems to be similar to ω-limit sets, they were not much explored so far. The reason for this is that they may have very rich structure, and also it is very hard to control the dynamics backward.

To study special α-limit sets is more complicated than to study ordinary α-limit sets or ω-limit sets. While it is clear that α-limit sets or ω-limit sets are always closed, the situation of special α-limit sets is unclear. By definition, those sets are in general uncountable unions of closed sets, so a priori their structure may be very complicated. Recent study by Kolyada et al. [23] provided some answers. In particular, they showed that a special α-limit set needs not to be closed in general setting and conjectured that, for all continuous maps of the unit interval all special α-limit sets are closed.

2.2 Main results

2.2.1 On the structure of α-limit sets of backward trajectories for graph maps

Recall that a *graph* G is a continuum which can be written as an union of finitely many arcs such that any two of them can intersect only in their endpoints. A *backward branch* of a point $x \in G$ is any sequence $\{x_i\}_{i\leq 0} \subset G$ such that $x_0 = x$ and $f(x_i) = x_{i+1}$ for each $i < 0$. In this section we study backward branches and their accumulation points forming *α-limit sets of a backward branch.* A point y belongs to the α-limit set of a backward branch $\{x_i\}_{i\leq 0}$, denoted by $\alpha(\{x_i\}_{i\leq 0})$, if and only if there is a strictly decreasing sequence of negative integers $\{n_i\}_{i\geq 0}$ such that $x_{n_i} \to y$ as $i \to \infty$. For interval maps, every α-limit set of a backward branch is an ω-limit set while the converse is not true by results in [3]. Our research in the article (T1) is motivated by the following question:

Question 1. *Let $A = \alpha(\{x_j\}_{j\leq 0})$ be an α-limit set of a backward branch $\{x_j\}_{j\leq 0}$ of a map f on topological graph. Is A an ω-limit set? How many different sets A can be generated using backward branches starting at x_0?*

We give full characterization under some additional conditions on f, and almost complete picture in general case. Complete answer to Question 1 is provided for topologically mixing $f\colon G \to G$ acting on a topological graph G.

Theorem 2 (T1)**.** *For every ω-limit set $\omega(y)$ in G and every accessible point x in G, there is a backward branch starting at x with the α-limit set being equal to $\omega(y)$. Conversely, every α-limit set of any backward branch in G is an ω-limit set of some point in G.*

Quite different, still complete, picture is obtained for maps $f\colon G \to G$ with zero topological entropy.

Theorem 3 (T1)**.** *For zero entropy graph maps, the family of α-limit sets of backward branches coincides with the family of minimal sets. Moreover, the collection of all α-limit sets of backward branches starting at a point x is thin - it contains at most one infinite set.*

When considering maps with positive entropy, some uncertainty enters our description. In this case, we may observe phenomena specific both for zero entropy maps and for mixing maps, however tools we use do not allow us to completely reveal the structure of some α-limit sets. We prove in (T1) that for all but at most countably many points x from a basic set D and every infinite ω-limit set $\omega(y) \subset D$, there exists a backward branch $\{z_j\}_{j\leq 0}$ starting at x such that $\alpha(\{z_j\}_{j\leq 0}) = \omega(y) \cup R$ where R is at most countable subset of isolated points of $\alpha(\{z_j\}_{j\leq 0})$. This shows that for a typical point x from a basic set the collection of all α-limit sets of backward branches starting at x is abundant. By results of A. Blokh [9, 10, 11], graph maps with positive entropy must contain a basic set, but it may contain also other maximal ω-limit sets which are typical for zero entropy maps, that is solenoidal sets, circumferential sets and periodic orbits. Thus we may detect this kind of α-limit sets of backward branches for maps with positive entropy as well.

As was stated above, our tools do not allow us to answer whether the at most countable set R is empty or not, however its possible existence is a result of incomplete control of backward trajectory in the construction rather than a fact. In practice it may happen that these α-limit sets behave exactly the same as for other backward trajectories, that is they always coincide with ω-limit sets and all ω-limit sets in basic sets can appear as α-limit sets (recall results of [3] that some ω-limit sets are never α-limit sets for zero entropy maps). These aspects of Question 1 remain as open problem for further research.

2.2.2 On backward attractors of interval maps

One of the most classic applications of *limit sets of forward trajectories* in dynamics is due to Birkhoff. There are many notions of recurrence in topological dynamics (such as periodicity, non-wandering behaviour, chain-recurrence, etc.), but the term *recurrent point* has been reserved for those points x which belong to their own ω-limit sets. Birkhoff showed that these points can be used to identify the *Birkhoff center* (Birkhoff called it the "set of central motions") of a topological dynamical system (X, f), which is obtained by restricting f to its non-wandering set, then restricting that system to its non-wandering set, and so on through transfinite induction (taking intersections at limit ordinals) until reaching some countable ordinal (the "depth") at which the sequence stabilizes. Birkhoff's result is that the center of the system obtained in this way is the same as the closure of the set of recurrent points [7].

In light of Birkhoff's work, one can ask the analogous question, what is the significance of a point belonging to its own *limit set of backward trajectory*? In one-dimensional dynamics good answers to this question have appeared for the α-limit set and the special α-limit set. Coven and Nitecki showed that a point x is non-wandering for a continuous interval map f if and only if x belongs to its own *α-limit set* [14]. But there is a deeper result related to the *attracting center* of an interval map $f : [0, 1] \to [0, 1]$, defined by Xiong as the set of all points x such that x is in the ω-limit set of some point x_1, which itself is in the ω-limit set of some point x_2, and so on for some infinite sequence $\{x_i\}_{i=1}^{\infty}$ of points in the interval [38]. Xiong showed that the attracting center is a subset of the Birkhoff center (they can coincide) and that if x_1, x_2 can be found as above, then x is already in the attracting center (so the "depth" here is at most 2) [38]. The connection to limit sets of backward trajectories was made in 1992 by Hero, who showed that a point x belongs to the attracting center of a continuous interval map if and only if it belongs to its own *$s\alpha$-limit set* [20].

We now recall Hero's definition of a special α-limit set. A *backward branch* of a point x is any sequence $\{x_i\}_{i\leq 0}$ such that $x_0 = x$ and $f(x_i) = x_{i+1}$ for each $i < 0$. The corresponding *α-limit set of a backward branch* is defined as the set of all limits of convergent subsequences x_{i_j} (analogously as ω-limit sets are defined from forward trajectories). Then the *special α-limit set* of a point x, denoted $s\alpha(x)$, is defined as the union of all α-limit sets over all backward branches of x [20].

Kolyada, Misiurewicz, and Snoha began a systematic study of $s\alpha$-limit sets in [23]. They investigated special α-limit sets of interval maps and proved that for interval maps with a closed set of periodic points, every special α-limit set has to be closed. This result led to the following conjecture:

Conjecture 4. *[23] For all continuous maps of the unit interval all special α-limit sets are closed.*

In the article (T2) we disprove the conjecture by showing a counterexample of a continuous interval map with a special α-limit set which is not closed and give the properties of continuous interval maps that determine if all special α-limit sets are closed in the following theorem.

Theorem 5 (T2)**.** *Let $f : [0,1] \to [0,1]$ be a continuous interval map. The following are equivalent:*

1. *For some $y \in [0,1]$, $s\alpha(y)$ is not closed.*
2. *The attracting center $\Lambda^2(f)$ is not closed.*
3. *The attracting center is strictly contained in the Birkhoff center $\Lambda^2(f) \subsetneq \overline{Rec(f)}$.*
4. *Some solenoidal ω-limit set of f contains a non-recurrent point in the Birkhoff center.*

We identify three classes of continuous interval maps for which all $s\alpha$-limit sets are closed - piecewise monotone maps, zero entropy maps with a closed set of recurrent points and maps which are not Li-Yorke chaotic. On the other hand, we show that for all continuous maps of the unit interval all special α-limit sets are both F_σ and G_δ. We give further topological properties of special α-limit sets of interval maps.

Theorem 6 (T2)**.** *If $s\alpha(x)$ is not closed, then it is uncountable and nowhere dense. If $s\alpha(x)$ is closed, then it is the union of a nowhere dense set and finitely many (perhaps zero) closed intervals.*

We verify the following conjecture by Kolyada et al. from [23].

Theorem 7 (T2)**.** *The isolated points in a special α-limit set for a continuous interval map are always periodic.*

We also show that a countable special α-limit set for an interval map is a union of periodic orbits. These results are opposite to the case of ω-limit sets. The ω-limit sets of a general dynamical system do not posses any periodic isolated points unless $\omega(x)$ is a single periodic orbit.

The authors of [23] also investigated the properties of special α-limit sets of *transitive* interval maps and stated the following conjecture:

Conjecture 8. *[23] Let $f : [0,1] \to [0,1]$ be a continuous map and $x, y \in [0,1]$.*

- *If $x \neq y$ and $s\alpha(x) = s\alpha(y) = [0,1]$, then f is transitive.*
- *If $s\alpha(x) = [0,1]$ then either f is transitive or there is $c \in (0,1)$ such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are transitive.*

We show in (T2) that f is transitive if there are three distinct points $x, y, z \in [0,1]$ with $s\alpha(x) = s\alpha(y) = s\alpha(z) = [0,1]$. If f has one or two points with special α-limit sets equal to $[0,1]$, but not more, then $[0,1]$ is the union of two transitive cycles of intervals.

It is known that if two ω-limit sets of an interval map contain a common open set, then they are equal. The last conjecture in [23] suggested that a similar property holds for special α-limit sets. We correct this conjecture in the following theorem.

Theorem 9 (T2)**.** *At most three distinct special α-limit sets of a continuous interval map f can contain a given nonempty open set.*

Since $s\alpha$-limit sets need not be closed, we propose a new notion of *β-limit sets* ($\beta(x)$ for short) to serve as backward attractors. The β-limit set of x is the smallest closed set to which all backward orbit branches of x converge, and it coincides with the closure of the $s\alpha$-limit set.

To summarize, the key properties of limit sets as they apply to continuous maps of the interval are as follows:

$$\begin{aligned}
&x \text{ is recurrent} && \iff && x \in \omega(x),\\
&x \text{ is nonwandering} && \iff && x \in \alpha(x),\\
&x \text{ is in the attracting center} && \iff && x \in s\alpha(x).
\end{aligned}$$

We conjecture additionally in (T2) that

$$x \text{ is in the Birkhoff center} \quad \iff \quad x \in \beta(x).$$

3. Chaotic behaviour

3.1 Introduction and state-of-art

The term ”chaos” in connection with a function was introduced in [28] by Li and Yorke in 1975, since then several different definitions of what it means for a function to be chaotic have been proposed. One could say ”as many authors, as many definitions of chaos”; most of them are based on the idea of unpredictability of the behaviour of trajectories or sensitive dependence on initial conditions. The idea of chaos emerged from experiments in physics. Physicists expressed their opinion of what mathematical property could describe chaotic behaviour and then mathematicians began using the word ”chaos” as a label for this property. More recently, people from other fields have started to think about chaos - in computer science, chaos is connected to computational complexity. Biology and economics have also produced their own concepts of disorder. This creates some confusion in contemporary mathematical literature and therefore the word ”chaos” should always be understood in the right context. The chaotic properties considered in this thesis are limited to Li-Yorke chaos, various types of distributional chaos and Li-Yorke sensitivity.

It often occurs that in a restricted class of topological dynamical systems several chaotic properties or definitions of chaos coincide, or one implies another, while this is false when looking at the whole category of dynamical systems - among interval maps distributional chaos and positive topological entropy are equivalent properties, something which is false in general. That each smaller class of dynamical systems like interval maps or graph maps has its own theory of chaos is important. The picture of chaos changes completely when one extends the scope from interval maps to the whole field of topological dynamics. This thesis consider maps acting on general compact metric spaces and examine chaotic properties in this more universal setting.

3.2 Main results

3.2.1 Li-Yorke sensitivity does not imply Li-Yorke chaos

Li-Yorke sensitivity and Li-Yorke chaos are well-known properties of dynamical systems, where by a dynamical system we mean a phase space X endowed with an evolution map T. In this section we require that the phase space (X, d) is a compact metric space and the evolution map is a continuous surjective mapping $T : X \to X$. The definition of Li–Yorke sensitivity is a combination of sensitivity and Li–Yorke chaos. *Li–Yorke chaos* was introduced in 1975 by Li and Yorke in [28]. A dynamical system (X, d) is Li–Yorke chaotic if there is an uncountable scrambled set. A set S is scrambled if any two distinct points $x, y \in S$ are proximal (i.e. trajectories of x and y are arbitrarily close for some times) but not asymptotic, which means that

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(T^n(x), T^n(y)) > 0. \tag{3.1}$$

We will call a pair of points $(x, y) \in X^2$ with the property (3.1) as a *scrambled pair*. Recall that $(x, y) \in X^2$ is a *distal pair* if $\liminf_{n\to\infty} d(T^n(x), T^n(y)) > 0$. A dynamical system (X, T) is distal if every pair of distinct points in (X, T) is distal.

The initial idea of *sensitivity* goes back to Lorenz [29], but it was firstly used in topological dynamics by Auslander and Yorke in [2] and popularized later by Devaney in [15]. A map T is sensitive if there is $\epsilon > 0$ such that, for each $x \in X$ and each $\delta > 0$, there is $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) > \epsilon$. By Huang and Ye in [21], T is sensitive if and only if there is $\epsilon > 0$ with the property that any neighbourhood of any $x \in X$ contains a point y such that trajectories of x and y are separated by for infinitely many times, that is, $\limsup_{n\to\infty} d(T^n(x), T^n(y)) > \epsilon$. Inspired by the above results, Akin and Kolyada introduced *Li–Yorke sensitivity* in [1]. A map T is Li–Yorke sensitive if there is $\epsilon > 0$ with the property that any neighbourhood of any $x \in X$ contains a point y proximal to x, such that trajectories of x and y are separated by ϵ for infinitely many times. Thus,

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(T^n(x), T^n(y)) > \epsilon.$$

Akin and Kolyada proved, among other things, that weak mixing systems are Li–Yorke sensitive and stated five conjectures concerning Li–Yorke sensitivity. Three of them were disproved in [39] and [40], one was confirmed recently in [32]. Only one problem remained open until now, as follows.

Question 10. *[1] Are all Li–Yorke sensitive systems Li–Yorke chaotic?*

We show in (T3) that the answer is negative. We construct an infinite-dimensional compact metric space X, which is a closed subset of $S \times H$, where S is the unit circle and H is the Hilbert cube, and a skew-product map F, which is a combination of a rotation on S and a contraction on H, such that (X, F) is Li–Yorke sensitive but possesses at most countable scrambled sets. Moreover, the mapping F can be continuously extended to get a connected dynamical system with the same properties.

Theorem 11 (T3)**.** *The Li–Yorke sensitive dynamical system (X, F) which is not chaotic in the Li–Yorke sense.*

The mapping F from Theorem 11 is not minimal (it is even not transitive). In the case of minimal maps, we still have the following open question.

Question 12 (T3)**.** *Are all Li-Yorke senstive minimal systems Li-Yorke chaotic?*

The space X from Theorem 11 is infinite-dimensional. We can examine the relation between Li– Yorke sensitivity and Li–Yorke chaos for low-dimensional dynamical systems. It is known that in the case of graph mappings (in particular, interval mappings) Li–Yorke sensitivity implies Li–Yorke chaos, since, for graph mappings, the existence of a single scrambled pair implies the existence of an uncountable scrambled set. But this is not true for other classes of dynamical systems—shifts, maps on dendrites, triangular maps of the square.

Question 13 (T3)**.** *Are all Li-Yorke sensitive shifts, maps on dendrites or triangular maps of the square Li-Yorke chaotic?*

3.2.2 Distributional chaos

It was a long standing problem, what are the relations between positive topological entropy and the existence of Li-Yorke chaos. Topological entropy is a measure of the complexity of a dynamical system - for systems with positive topological entropy, the number of distinguishable orbits grows exponentially with time. A theorem by Misiurewicz [30] which characterizes positive topological entropy of interval maps in terms of topological horseshoes provided a tool for proving that positive topological entropy implies the existence of an uncountable scrambled set [22]. Since Xiong [37] and Smítal [34] constructed some interval maps with zero topological entropy which are Li-Yorke chaotic, Li-Yorke chaos was found to be a necessary but not sufficient condition for positivity of topological entropy.

Schweizer and Smítal [31] introduced the related concept of a distributionally chaotic pair, which means, roughly speaking, that the statistical distribution of distances between the orbits does not converge. They discovered that the existence of a single distributionally chaotic pair is equivalent to positivity of topological entropy when restricted to the interval case. This fact, combined with the characterization of maps with positive entropy by S. Li in [27], shows that the existence of a distributionally chaotic pair forces a very strong chaotic behaviour. In particular, distributional chaos, positive topological entropy, ω-chaos and chaos in the sense of Devaney are all equivalent properties for interval maps.

Two more chaotic properties of interval maps are equivalent to the positivity of topological entropy - invariant and multivariant chaos. By invariant chaos we mean existence of an invariant chaotic set (see [18]), by multivariant chaos we mean existence of scrambled n-tuples. Existence of one scrambled triple for $f \in \mathcal{C}(I)$ implies f having positive topological entropy by [25]. In particular, existence of a scrambled triple for the interval map f implies existence of an uncountable scrambled set (in the sense of pairs) and therefore scrambled triples are always accompanied by scrambled pairs.

Later the definition of distributional chaos was split into three versions of distributional chaos (briefly, DC1 – DC3), equivalent for the interval case but distinct for a general dynamical system. One can easily see from the definitions that DC1 implies DC2 and DC2 implies DC3. On the other hand, there are examples which show that DC1 is stronger than DC2 and DC2 is stronger than DC3. It is also obvious that either DC1 or DC2 implies Li-Yorke chaos. For maps acting on a general compact metric space, positive topological entropy implies DC2 by [17].

We proceed with the definition of distribution functions for a pair $(x_1, x_2) \in X^2$ whose value at δ may be interpreted as lower and upper asymptotic densities of times when the distance between the trajectories of x_1 and x_2 is less than δ.

Definition 1. *For a pair (x_1, x_2) of points in X, define the* lower distribution function *generated by f as*

$$\Phi_{(x_1,x_2)}(\delta) = \liminf_{m\to\infty} \frac{1}{m} \#\{0 \le k \le m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

and the upper distribution function *as*

$$\Phi^*_{(x_1,x_2)}(\delta) = \limsup_{m\to\infty} \frac{1}{m} \#\{0 \le k \le m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A.
A pair $(x_1, x_2) \in X^2$ is called distributionally chaotic of type 1 (briefly DC1) *if*

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \ \textit{for every } 0 < \delta \leq \textit{diam } X,$$

$$\Phi_{(x_1,x_2)}(\epsilon) = 0, \ \textit{for some } 0 < \epsilon \leq \textit{diam } X,$$

distributionally chaotic of type 2 (briefly DC2) *if*

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \ \textit{for every } 0 < \delta \leq \textit{diam } X,$$

$$\Phi_{(x_1,x_2)}(\epsilon) < 1, \ \textit{for some } 0 < \epsilon \leq \textit{diam } X,$$

distributionally chaotic of type 3 (briefly DC3) *if*

$$\Phi_{(x_1,x_2)}(\delta) < \Phi^*_{(x_1,x_2)}(\delta), \ \textit{for every } \delta \in (a,b), \ \textit{where } 0 \leq a < b \leq \textit{diam } X.$$

We can define both distribution functions at 0 as the limit $\Phi_{(x_1,x_2)}(0) = \lim_{\delta\to 0^+} \Phi_{(x_1,x_2)}(\delta)$ and $\Phi^*_{(x_1,x_2)}(0) = \lim_{\delta\to 0^+} \Phi^*_{(x_1,x_2)}(\delta)$. Then (x_1, x_2) being DC1 is equivalent to

$$\Phi^*_{(x_1,x_2)}(0) = 1, \quad \Phi_{(x_1,x_2)}(\epsilon) = 0, \ \text{for some } 0 < \epsilon \leq \text{diam } X;$$

and DC2 is equivalent to

$$\Phi^*_{(x_1,x_2)}(0) = 1, \quad \Phi_{(x_1,x_2)}(0) < 1.$$

A subset S of X is called *distributionally scrambled* of type i if every pair of distinct points in S is distributionally chaotic of type i. There are two ways to define distributional chaos - either as the existence of at least one distributionally chaotic pair or the existence of an uncountable distributionally scrambled set. We say that the dynamical system (X, f) is *distributionally chaotic* of type i (DCi for short), where $i = 1, 2, 3$, if there is at least one distributionally chaotic pair of type i in X. When we use the other way of defining distributional chaos, we will emphasize the fact that an uncountable distributionally scrambled set is concerned. We call a dynamical system *strictly DCi* if it is DCi but possesses no DCj pairs for $j < i$.

This section contains results of 5 separate articles (T4) - (T8) unified by the same subject - distributional chaos. In the first article (T4) we state a sufficient condition for invariant distributional chaos in general compact metric spaces.

Theorem 14 (T4)**.** *Let (X, d) be a compact metric space, $\#X > 1$, and let $f : X \to X$ be a continuous mapping with weak specification property which has a fixed point and infinitely many periodic points with mutually different periods. Then there is a point $x \in X$ such that $(f^i(x), f^j(x))$ is a DC1 pair for all $i \neq j$, i.e., the forward orbit of x is a distributionally scrambled set of type 1.*

Remark Later, authors in [35] found that the condition of having infinitely many periodic points can be omitted. The assumption about a fixed point is natural - if x belongs to an invariant scrambled set, then $(x, f(x))$ is proximal. By the compactness of X there is an increasing sequence k_i and a point $p \in X$ such that $\lim_{i\to\infty} f^{k_i}(x) = p$ and simultaneously $\lim_{i\to\infty} d(f^{k_i}(x), f^{k_i}(f(x))) = 0$, which by continuity of f implies that $f(p) = p$.

In the previous theorem we obtained a countable distributionally scrambled invariant set. Another interesting question is how large can this set be.

Theorem 15 (T4)**.** *Let (X, d) be a compact metric space, $\#X > 1$, and let $f : X \to X$ be a continuous mapping with weak specification property which has a fixed point and infinitely many periodic points with mutually different periods. Then there is a dense invariant distributionally scrambled (of type 1) Mycielski set (i.e. countable union of Cantor sets).*

We say that a continuous map from the k-dimensional unit cube I^k into itself exhibits invariant distributional chaos of type 1 almost everywhere (briefly invariant DC1 a.e.) if there exists a distributionally scrambled set $D \subset I^k$ of type 1 such that $\lambda(D) = 1$ and D is invariant, where λ denotes the Lebesgue measure on I^k.

It is known that if $D \subset I^k$ is a dense union of perfect sets then D is homeomorphic to a set of full Lebesgue measure. An appropriate homeomorphism is obtained by application of the Oxtoby-Ulam theorem. This fact together with Theorem 15 implies the following corollary.

Corollary. Every map $f \in \mathcal{C}(I^k)$ with weak specification property, a fixed point, and infinitely many periodic points with mutually different periods is conjugate to some map $g \in \mathcal{C}(I^k)$ which exhibits invariant DC1 a.e.

In the next article (T5) we investigate the relation between distributional chaos and existence of scrambled triples. The concept of scrambled pairs from Li-Yorke definition of chaos given by equation (3.1) can be generalized to *scrambled n-tuples.*

Definition 2. *A tuple $(x_1, x_2, ..., x_n) \in X^n$ is called* scrambled *if*

$$\liminf_{k\to\infty} \max_{1\le i<j\le n} d(f^k(x_i), f^k(x_j)) = 0 \tag{3.2}$$

and

$$\limsup_{k\to\infty} \min_{1\le i<j\le n} d(f^k(x_i), f^k(x_j)) > 0. \tag{3.3}$$

In the interval case, distributional chaos is equivalent to the positivity of topological entropy, which was shown in [25] to be equivalent to the existence of a scrambled triple. Hence distributionally chaotic pairs are always accompanied by scrambled triples. We show that it is no longer true for mappings acting on a general compact metric space. Both counterexamples from the following theorems are shift spaces, their construction is based on the well-known Thue-Morse sequence.

Theorem 16 (T5)**.** *There exists a dynamical system with an infinite extremal distributionally scrambled set but without any scrambled triple.*

Theorem 17 (T5)**.** *There exists a noncompact dynamical system with an uncountable extremal distributionally scrambled set but without any scrambled triple.*

The third article (T6) examines whether the distributional chaos is preserved under *semiconjugacy.* A dynamical system (X, f) is semiconjugate to an extension (Y, F) if there is a surjective continuous map $\pi : Y \to X$ such that $f \circ \pi = \pi \circ F$. Semiconjugacy is used as a common tool for proving topological chaos or positive topological entropy. The usual technique is to find a semiconjugacy π with a chaotic system and transfer the chaos to the extension. For instance, by continuity of π, the topological entropy of the extension is not smaller than the entropy of the factor system. Unfortunately, semiconjugacy may not automatically guarantee the distributional chaos in the extension. The system from the following theorem is a skew-product map acting on a converging sequence of unit intervals which is extended to a three-dimensional union of countably many homocentric cylinders with unit height and converging radius.

Theorem 18 (T6)**.** *There exists a distributionally chaotic system of type 1 (possessing an uncountable DC1 set) which is semiconjugate to an extension with no distributionally chaotic pair (of type 1 or 2).*

The last two articles (T7) and (T8) study the weakest version of distributional chaos. We show in (T7) that, unlike its stronger relatives, DC3 chaos does not imply Li-Yorke chaos, and DC3 chaos is not an invariant for topological conjugacy. In a weak sense, these two results were already stated in [4]. However, it should be noticed that the distributional chaos in [4] was defined as the existence of a single DC3 pair, but nowadays it is generally assumed that distributional chaos means the existence of an uncountable distributionally scrambled set. Moreover, the proof of Theorem 2 in [4] is unfortunately in error - the authors constructed a conjugacy which destroys a DC3 pair, but they overlooked many other DC3 pairs which persist.

The dynamical system in Theorem 19 is a skew-product map acting on the Cartesian product of Cantor set Ω and unit circle, where the base space is Ω and in the fibers we rotate the unit circle by an angle depending on $\omega \in \Omega$.

Theorem 19 (T7)**.** *There exists a distal dynamical system which possesses an uncountable DC3 scrambled set. Thus, distributional chaos of type 3 does not imply Li-Yorke chaos.*

The counterexample implying Theorem 20 has the phase space consisting of 2 concentric columns of rings, in each column the rings are accumulating on the bottom-most ring. The map carries each ring down to the next lower ring with some rotation and fixes the bottom ring.

Theorem 20 (T7)**.** *Distributional chaos of type 3 (assuming the existence of an uncountable scrambled set) is not preserved by conjugacy.*

Our second goal in (T7) is to strengthen the definition of the DC3 pair in such a way that it is preserved under conjugacy and implies Li-Yorke chaos – we denote the new definition by DC$2\frac{2}{1}$. DC$2\frac{1}{2}$ is stronger than DC3 (any distal DC3 system must be without DC$2\frac{1}{2}$ pairs) and weaker than DC2 (see the example of a strictly DC$2\frac{1}{2}$ system in (T7)). By results in [17], positive topological entropy implies existence of an uncountable DC2 set, hence strictly DC$2\frac{1}{2}$ systems must have zero topological entropy.

Definition 3 (T7)**.** *A pair* $(x_1, x_2) \in X^2$ *is called distributionally chaotic of type* $2\frac{1}{2}$ *if there are positive numbers* c *and* s *such that, for any* $0 < \delta < s$*,*

$$\Phi_{(x_1,x_2)}(\delta) < c < \Phi^*_{(x_1,x_2)}(\delta).$$

It follows immediately from the definition that $(x_1, x_2) \in X^2$ being DC$2\frac{1}{2}$ is equivalent to

$$\Phi_{(x_1,x_2)}(0) < \Phi^*_{(x_1,x_2)}(0).$$

By simple observation we can see that if $(x_1, x_2) \in X^2$ is DC$2\frac{1}{2}$ then it satisfies equation (3.1) and thus it is a scrambled pair. Since $c < \Phi^*_{(x_1,x_2)}(\delta)$, for arbitrary small δ, (x_1, x_2) must be proximal. Similarly, (x_1, x_2) is not asymptotic since for asymptotic pairs $\Phi_{(x_1,x_2)}(\delta)) = 1$ for every $\delta > 0$.

Like DC1 and DC2, DC$2\frac{1}{2}$ is *conjugacy invariant.*

Theorem 21 (T7)**.** *Let f and g be topologically conjugate continuous maps of a compact metric space X, i.e. there is a continuous map π on X such that π is one-to-one and onto and $\pi \circ f = g \circ \pi$. Then f is DC$2\frac{1}{2}$ if and only if g is DC$2\frac{1}{2}$.*

We claim in (T8) that DC$2\frac{1}{2}$ is also *iteration invariant* while DC3 is not. We say that a property P is an iteration invariant if, for any dynamical system (X, f) and any $n \in \mathbb{N}$, (X, f) has the property P if and only if (X, f^n) has P as well. Li in [26] and Wang et al. in [36] proved that DC1 and DC2 are iteration invariants and posed an open question whether DC3 is also preserved under iteration. Dvořáková proved in [19] one implication - if a function f is distributionally chaotic of type 3, then f^n is distributionally chaotic of type 3, for every $n \in N$, and conjectured that the opposite implication also holds. We disprove this conjecture by finding a dynamical system (X, f) which has a DC3 pair with respect to f^2 but no DC3 pairs with respect to f. That legitimates the attempt to replace DC3 by its slightly strengthened version denoted by DC$2\frac{1}{2}$.

Theorem 22 (T8)**.** *For any integer $N > 1$, the function f^N is distributionally chaotic of type $2\frac{1}{2}$ if and only if f is as well.*

The counterexample implying Theorem 23 is a disjoint union of three oscillatoric dynamical systems, where points regularly move from the right endpoint of some interval to the left endpoint (and back), and two of these oscillators are mirror reflections of each other with a line symmetry.

Theorem 23 (T8)**.** *Distributional chaos of type 3 with respect to f^2 doesn't imply distributional chaos of type 3 with respect to f. Thus the distributional chaos of type 3 is not iteration invariant.*

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Attachments

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T2. J. Hantáková, S. Roth, On backward attractors of interval maps, Nonlinearity 34 (2021), 7415-7445.

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ON THE STRUCTURE OF α-LIMIT SETS OF BACKWARD TRAJECTORIES FOR GRAPH MAPS

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Abstract. In the paper we study what sets can be obtained as α-limit sets of backward trajectories in graph maps. We show that in the case of mixing maps, all those α-limit sets are ω-limit sets and for all but finitely many points x, we can obtain every ω-limits set as the α-limit set of a backward trajectory starting in x. For zero entropy maps, every α-limit set of a backward trajectory is a minimal set. In the case of maps with positive entropy, we obtain a partial characterization which is very close to complete picture of the possible situations.

1. Introduction and main results.

Let a *dynamical system* be defined as a pair (X, f) where X is a compact metric space and f is a continuous map acting on X. To understand the dynamical properties of a system it is necessary to analyze the behavior of the trajectories of any point $x \in X$ under the iteration of f. Limit sets of trajectories are a helpful tool for understanding of qualitative properties of dynamics. The *ω-limit set* (set of limit points of forward trajectory of a point x; denoted $\omega(x)$), is among fundamental objects in theory of dynamical systems. The first question that comes to mind, is whether a given closed invariant subset of X is the ω-limit set of some point $x \in X$. Finding the answer is hard in general, however some cases are known. For example, a characterization of ω-limit sets of a continuous map acting on the compact interval was provided by Blokh et al. in [10]. A closely related question asks which dynamical systems may occur as ω-limit

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sets in larger systems. These abstract ω-limit sets were studied by Bowen [11] and Dowker and Frielander [15]. Of particular interest are invariant sets obtained as limits (in the Hausdorff metric) of ω-limit sets. Sharkovsky proved in [31] that every ω-limit sets of continuous map on the interval is contained in the maximal one, and later Blokh et al. in [10] showed that the family of all ω-limit sets of f endowed with the Hausdorff metric is compact. While the aforementioned results about ω-limit sets were first obtained for interval maps, some of them hold for maps acting on graphs, dendrites, Cantor space and others, e.g. see [12, 24, 5]. In general compact metric spaces only partial results are known and are usually hard to obtain (e.g. see [6]).

The properties of ω-limit sets for *graph maps* are to some extent similar to the interval case (however the proofs are usually much harder). Every ω-limit set is contained in a maximal one since the family of ω-limit sets of a graph map is closed with respect to the Hausdorff metric by result of Mai and Shao [26]. By Blokh's Decomposition Theorem [7], there are only four types of maximal ω-limit sets: basic sets, solenoidal sets, circumferential sets and periodic orbits. For a graph map f, the topological entropy of f is positive if and only if it possesses a basic set (i.e. an infinite maximal ω-limit set containing a periodic point; see Hric and Málek [21]). The topological characterization of ω-limit sets of graph maps [21] shows that an ω-limit set is a finite set, or an infinite compact nowhere dense set, or a cycle of connected subgraphs. Conversely, whenever a set A is of one of the above forms then there is a graph map f such that A is an ω-limit set for f.

As a dual concept to ω-limit sets the *α-limit sets* (denoted $\alpha(x)$) were introduced. Intuitively, they represent a "source" of the trajectory of a point. While for invertible maps α-limit sets can be defined as ω-limits sets of dynamical system with reversed time, for noninvertible maps there are a few possibilities how to construct the limit along the backward trajectory which can not be uniquely defined. One possibility is to take as an α-limit set the set of all accumulation points of the set of pre-images $f^{-n}(x)$. This approach was used by Coven and Nitecki [13], who showed that for an interval map, a point x is non-wandering if and only if $x \in \alpha(x)$. This approach attracted some attention, e.g. see Cui and Ding [14] on α-limit sets of unimodal interval maps. Another approach (see [4]), which is studied in the present paper, instead of complete preimages considers a fixed backward branch and its accumulation points forming an *α-limit set of a backward branch*. By results of [4] for interval maps, every α-limit set of a backward branch is an ω-limit set while the converse is not true. The third approach to α-limit sets, proposed by Hero in [19], falls somewhere between two possibilities mentioned above. It considers the union of α-limit sets over all backward branches starting at a point x, and call obtained set the *special α-limit set* (denoted $s\alpha(x)$). Recent studies by Kolyada et al. [25] and Hantáková and Roth [16] provided basic properties of special α-limit sets for interval maps. For instance, $s\alpha(x)$ does not need to be closed and its isolated points are always periodic, which is in some contrast to the properties of $\omega(x)$. Outside the realm of one-dimensional dynamics the situation is even more complicated. It has been shown that $s\alpha$-limit sets are always analytic, but not necessarily Borel [22]. If we denote by $\mathrm{SA}(f)$ (respectively, $\omega(f)$) the union of α-limit sets of all backward branches (respectively, all ω-limit sets) of a map f and by $\mathrm{Rec}(f)$ the set of all recurrent points of f, then $\mathrm{Rec}(f) \subseteq \mathrm{SA}(f) \subseteq \overline{\mathrm{Rec}(f)} \subseteq \omega(f)$, for every map f on the topological graph (see [34], cf. [19], [4]). It was shown in [33] that $\mathrm{Rec}(f) \subseteq \mathrm{SA}(f)$ holds in the special case of maps acting on dendrites with countable set of endpoints and that there are dendrite maps with $\mathrm{SA}(f) \not\subset \overline{\mathrm{Rec}(f)}$. We show that $\mathrm{Rec}(f) \subseteq \mathrm{SA}(f)$ holds for general dynamical systems in Corollary 1.

Our research is motivated by the following question:

Question 1.1. Let $A = \alpha(\{x_j\}_{j\leq 0})$ be an α-limit set of a backward branch $\{x_j\}_{j\leq 0}$ of a map f on topological graph. Is A an ω-limit set? How many different sets A can be generated using backward branches starting at x_0?

In the paper we provide full characterization under some additional conditions on f, and almost complete picture in general case.

Complete answer to Question 1.1 in the case of topologically mixing $f\colon G \to G$ on topological graph G is provided in Section 3. Strictly speaking we prove for these maps the following:

1. for every ω-limit set $\omega(y)$ in G and every accessible point x in G, there is a backward branch starting at x with the α-limit set being equal to $\omega(y)$,
2. every α-limit set of any backward branch in G is an ω-limit set of some point in G.

Quite different, still complete, picture is obtained for maps $f\colon G \to G$ with zero topological entropy. For these maps we prove in Section 5 that

1. family of α-limit sets of backward branches coincides with the family of minimal sets,
2. the collection of all α-limit sets of backward branches starting at x is rather thin - it contains at most one infinite set.

We also provide an example that in the above case, beyond one infinite minimal set, α-limit sets of backward branches starting at x can form quite large family of periodic orbits.

When considering maps with positive entropy, some uncertainty enters our description. In this case, we may observe phenomena specific both for zero entropy maps and for mixing maps, however tools we use (in Section 6) do not allow us to completely reveal the structure of some α-limit sets. We prove that for all but at most countably many points x from a basic set D and every infinite ω-limit set $\omega(y) \subset D$, there exists a backward branch $\{z_j\}_{j\leq 0}$ starting at x such that $\alpha(\{z_j\}_{j\leq 0}) = \omega(y) \cup R$ where R is at most countable subset of isolated points of $\alpha(\{z_j\}_{j\leq 0})$. This shows that for a typical point x from a basic set the collection of all α-limit sets of backward branches starting at x is abundant. By results mentioned earlier, graph maps with positive entropy must contain a basic set, but it may contain also other maximal ω-limit sets which are not limited to zero entropy maps only, that is solenoidal sets, circumferential sets and periodic orbits. Thus we may detect this kind of α-limit sets of backward branches starting at a point x in maps with positive entropy.

As was stated above, our tools do not allow us to answer whether the at most countable set R is empty or not, however its possible existence is a result of incomplete control of backward trajectory in the construction rather than a fact. In practice it may happen that these α-limit sets behave exactly the same as for other backward trajectories, that is they always coincide with ω-limit sets and all ω-limit sets in basic sets can appear as α-limit sets (recall results of [4] that some ω-limit sets are never α-limit sets for zero entropy maps). These aspects of Question 1.1 remain as open problem for further research.

2. Preliminaries.

2.1. Topological graphs.
Let $u_1, u_2, \dots, u_{k+1}$ be $k+1$ affinely independent points in $\mathbb{R}^n$. A *k-simplex* σ is the convex hull of $\{u_1, u_2, \dots, u_{k+1}\}$ where the convex hull is the set of all convex combinations of points u_i. We use special labels in the first 2 dimensions: *vertex* for 0-simplex and *edge* for 1-simplex. A face of a k-simplex σ is the convex hull of a non-empty subset of $\{u_1, u_2, \dots, u_{k+1}\}$. A *simplicial complex* is a finite collection of simplices K such that, for every face τ of any simplex $\sigma \in K$, $\tau \in K$, and, for every $\sigma, \eta \in K$, $\sigma \cap \eta$ is either empty or a face of both σ, η. The dimension of K is the maximum dimension of any of its simplices. The underlying space, denoted by $|K|$, is the union of its simplices together with the topology inherited from $\mathbb{R}^n$. A *triangulation* of a topological space X is a simplicial complex K together with a homeomorphism between X and $|K|$. Finally, a *(topological) graph* is a continuum $G \subset \mathbb{R}^3$ such that there is a triangulation of G to a one-dimensional simplicial complex K. Any subset of G which is a graph itself is called a *subgraph* of G. Without loss of generality we can assume that each graph is not just homeomorphic to a complex, but it is a simplicial complex embedded in the Euclidean space $\mathbb{R}^3$. In particular, it comes equipped with some fixed triangulation, and all triangulations are just subdivisions of the fixed one. Recall that a simplicial complex L is a *subdivision* of another simplicial complex K if $|L| = |K|$ and every simplex in L is contained in a simplex in K. We assume that G is endowed with the taxicab metric, that is, the distance between any two points of G is equal to the length of the shortest arc in G joining these points. If G is a graph, and K is a triangulation of G, then we denote the set of all vertices of K by $\mathcal{V}$ and the set of all edges by $\mathcal{E}$. The *star* of a vertex v, denoted by $St(v)$, is the union of all the edges that contain the vertex v. For every $x \in G$ we define the *degree* of x in the following way: if $x \in \mathcal{V}$ then $deg(x)$ is equal to the number of connected components of $St(x) \setminus \{x\}$, and $deg(x) = 2$ otherwise. Points $x \in G$ with $deg(x) = 1$ are called endpoints of the graph G and points with $deg(x) > 2$ are called branching points. We denote the set of all endpoints in G by $End(G)$ and the set of all branching points in G by $Br(G)$. An arc $J \subset G$ is free if and only if $J \cap Br(G) = \emptyset$. If $J \subset G$ is an arc with endpoints x and y, then it is convenient to write $J = [x, y]$, which means that we identify J with interval $[0, 1]$ by a homeomorphism $\pi\colon J \to [0, 1]$ with $\pi(x) = 0, \pi(y) = 1$. This way, we may use standard ordering on $[0, 1]$ in J. In particular, for $b \in [x, y] \setminus \{x, y\}$ we may write $x < b < y$ (using ordering of J) and also $[b, y] \subset [x, y]$ is defined in a natural way.

The next fact follows directly from the definition of the topological graph.

Fact 2.1. Let G be a topological graph.

1. Any non-degenerate subcontinuum $H \subset G$ is a subgraph of G.
2. For any subgraphs $H_1, H_2 \subset G$, the intersection $H_1 \cap H_2$ is either empty or has finitely many connected components.

Proof. (1) Let K be the simplicial complex for G. Then there is a subdivision L of K with at most two new vertices in every simplex $\sigma \in K$ such that H is triangulated by L.
(2) Let K_1, K_2 be the simplicial complexes for H_1, H_2. There is a subdivision of the complex for G in which both K_1 and K_2 are subcomplexes and so the intersection is a subcomplex. Any subcomplex has only finitely many connected components. □

2.2. **Dynamics on graphs.** By a *(graph) map* we mean a dynamical system on a graph, that is, a continuous map $f\colon G \to G$. The *orbit of a point* $x \in G$ is the set $\mathrm{Orb}_f(x) = \{f^n(x) : n \geq 0\}$, while the *orbit of a set* $A \subset G$ is the set $\mathrm{Orb}_f(A) = \cup_{n=0}^{\infty} f^n(A)$. If the function f in the above definitions is clear from the context, we use the notation $\mathrm{Orb}(x)$ and $\mathrm{Orb}(A)$. By $\mathrm{Per}(f)$ we denote the set of *periodic points* of f, that is points with the property that $f^p(x) = x$ for some $p > 0$. The smallest such p is the *period* of a point $x \in \mathrm{Per}(f)$. A set A is *invariant* if $f(A) \subseteq A$ and it is *strongly invariant* if $f(A) = A$.

A point y belongs to the *ω-limit set of a point* x, denoted by $\omega_f(x)$, if and only if there is a strictly increasing sequence of natural numbers $\{n_i\}_{i\geq 0}$ such that $f^{n_i}(x) \to y$ as $i \to \infty$. Otherwise stated $\omega_f(x) = \cap_{n\geq 1}\overline{\mathrm{Orb}(f^n(x))}$.

Fact 2.2. For every $\varepsilon > 0$ and every $x \in G$ there is $N > 0$ such that $\cup_{n=1}^{N} B(f^n(x), \varepsilon) \supset \omega_f(x)$.

Proof. Let $\{B(y_i, \frac{\varepsilon}{2})\}$ be a finite cover of $\omega_f(x)$ such that $y_i \in \omega_f(x)$ for every i. Then for every i we can choose $k_i \in \mathbb{N}$ such that $f^{k_i}(x) \in B(y_i, \frac{\varepsilon}{2})$ and the desired N is the maximal k_i. $\square$

We denote $\omega(f) = \bigcup_{x\in G} \omega_f(x)$. We say that x is *recurrent* if $x \in \omega_f(x)$ and by $\mathrm{Rec}(f)$ we denote the set of recurrent points for map f. A *backward branch* of a point $x \in G$ is any sequence $\{x_i\}_{i\leq 0} \subset G$ such that $x_0 = x$ and $f(x_i) = x_{i+1}$ for each $i < 0$. A point y belongs to the *α-limit set of a backward branch* $\{x_i\}_{i\leq 0}$, denoted by $\alpha_f(\{x_i\}_{i\leq 0})$, if and only if there is a strictly decreasing sequence of negative integers $\{n_i\}_{i\geq 0}$ such that $x_{n_i} \to y$ as $i \to \infty$. It is easy to see that both ω-limit sets and α-limit sets of backward branches are closed strongly invariant sets. We denote by $\mathrm{SA}(f)$ the union of all α-limit sets of backward branches in G. If the function from the definition of ω-limit set or α-limit set of a backward branch is clear, we use the notation $\omega(x)$, $\alpha(\{x_j\}_{j\leq 0})$.

Map $f\colon G \to G$ is *transitive* if for every pair of nonempty open subsets $U, V \subset G$ there is some integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$ and *totally transitive* if f^n is transitive for all $n > 0$. If f is transitive then $\{x \in G : \omega_f(x) = G\}$ is dense in G. Observe that if f is a transitive homeomorphism then f^{-1} is transitive as well, as for any open $U, V \subset X$ we have:

$$f(U) \cap V \neq \emptyset \Leftrightarrow U \cap f^{-1}(V) \neq \emptyset.$$

Map f is *sensitive* if there is $\delta > 0$ such that for every nonempty open $U \subset G$ there is $n > 0$ such that $\operatorname{diam} f^n(U) > \delta$ and it is *mixing* if for every pair of nonempty open subsets $U, V \subset G$ there is an $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for $n \geq N$. A point $x \in G$ is *non-wandering* if for every neighborhood U of x and every $N > 0$ there is some $n > N$ such that $f^n(U) \cap U \neq \emptyset$. If the opposite holds then we say x is a *wandering* point.

For a mixing graph map $f\colon G \to G$ we define the set $\mathcal{I}(f)$ of *inaccesible points* of f as follows:

$$\mathcal{I}(f) = G \setminus \bigcap_{U\in\mathcal{G}} \bigcup_{k=0}^{\infty} \operatorname{Int} f^k(U), \tag{2.1}$$

where $\mathcal{G}$ is the family of all subgraphs of G. By the results of [7] we have that $\mathcal{I}(f)$ is a finite strongly invariant set and hence it consists of periodic points. We say that $x \in G$ is an *accessible point* if $x \in G \setminus \mathcal{I}(f)$.

We define the *sample path space* (or *natural extension*) as the space of infinite chains:

$$G_f = \{\mathbf{z} \in G^{\mathbb{Z}} : z_{i+1} = f(z_i) \text{ for all } i \in \mathbb{Z}\}.$$

This is a closed subset of $G^{\mathbb{Z}}$, invariant with respect to the shift homeomorphism $\tilde{f}$ on $G^{\mathbb{Z}}$ defined as $\tilde{f}(\mathbf{z})_i = f(z_i) = z_{i+1}$ for every $i \in \mathbb{Z}$. The subsystem $(G_f, \tilde{f})$ is invertible and $\pi\colon (G_f, \tilde{f}) \to (G, f)$ where $\pi(\mathbf{z}) = z_0$ is a semiconjugacy. Note that the projection of G_f to its non-positive coordinates is a homeomorphism to the space of all backward branches in G. In particular, $\alpha_f(\{x_i\}_{i\leq 0}) = \pi(\omega_{\tilde{f}^{-1}}(\mathbf{x}))$ where $\mathbf{x} \in G^{\mathbb{Z}}$ with $(\mathbf{x})_i = x_i$ for $i \leq 0$. It follows immediately that $SA(f) = \pi(\omega(\tilde{f}^{-1}))$. The following results hold for general dynamical systems.

Proposition 1. [2, Proposition 1.4, Theorem 2.3] *A dynamical system (X, f) is transtive if and only if the sample path system $(X_f, \tilde{f})$ is transitive.*

Proposition 2. *Let (X, f) be a transitive dynamical system. There is a dense subset $D \subset X$ such that for every $x \in D$ there is a backward branch $\{x_i\}_{i\leq 0}$ with $x_0 = x$ and $\alpha_f(\{x_i\}_{i\leq 0}) = X$.*

Proof. By Proposition 1 the homeomorphism $\tilde{f}$ acting on the sample path space X_f is transitive implying that $\tilde{f}^{-1}$ is transitive as well. Thus the set $\{\mathbf{x} \in X_f : \omega_{\tilde{f}^{-1}}(\mathbf{x}) = X_f\}$ is dense in X_f. The desired set $D \subset X$ is the image of $\{\mathbf{x} \in X_f : \omega_{\tilde{f}^{-1}}(\mathbf{x}) = X_f\}$ by the surjective continuous map π. □

For every recurrent point $x \in X$, the dynamical system $(\omega_f(x), f)$ is transitive and obviously $x \in \omega_f(x)$. Thus we have the following corollary.

Corollary 1. *Let (X, f) be a dynamical system. If $x \in \mathrm{Rec}(f)$ then there is a backward branch $\{x_i\}_{i\leq 0}$ such that $\alpha_f(\{x_i\}_{i\leq 0}) = \omega_f(x)$. In particular, $\mathrm{Rec}(f) \subset \mathrm{SA}(f)$.*

For fixed $n \geq 0$ and $\varepsilon > 0$ we define the *Bowen ball* as follows:

$$B_n(x, \varepsilon) = \{y \in G : d(f^i(x), f^i(y)) \leq \varepsilon \text{ for } i = 0, \dots, n\}$$

and by $B'_n(x, \varepsilon)$ we denote the connected component of the Bowen ball $B_n(x, \varepsilon)$ that contains x.

For any nonempty compact subsets $U, V \subset G$ we define their *Hausdorff distance* by:

$$d_H(U, V) = \max\{\mathrm{dist}(u, V), \mathrm{dist}(v, U) : u \in U, v \in V\},$$

where $\mathrm{dist}(x, U) = \inf\{d(x, y) : y \in U\}$. We denote by $K(G)$ be the set of all compact subsets of G equipped with the *Hausdorff metric* d_H. We call a set $A \subseteq G$ the *Hausdorff limit* of the sequence of compact sets $\{A_i\}_{i\geq 0}$ if $\{A_i\}_{i\geq 0}$ converges to A in the metric d_H.

A subset $Y \subset G$ is *internally chain transitive* if for every pair of points $u, v \in Y$ and every $\varepsilon > 0$ there is a finite sequence $z_0, \dots, z_n$ of points in Y such that $z_0 = u, z_n = v$ and $d(f(z_i), z_{i+1}) < \varepsilon$ for $i = 0, \dots, n-1$. It is well-known fact that in any dynamical system (X, f), every ω-limit set is internally chain transitive and the same holds for α-limit sets of a backward branch by [20, Lemma 2.1]. Therefore α-limit sets of a backward branch share the following property of ω-limit sets.

Lemma 2.1. *Let (X, f) be a dynamical system and $\alpha(\{x_j\}_{j\leq 0}) \subset X$ be an α-limit set of a backward branch $\{x_j\}_{j\leq 0}$. Then every periodic orbit that lies in $\alpha(\{x_j\}_{j\leq 0})$ but does not coincide with $\alpha(\{x_j\}_{j\leq 0})$ is not isolated in $\alpha(\{x_j\}_{j\leq 0})$. In particular, if*

$\alpha(\{x_j\}_{j\leq 0})$ consists of finitely many points, then these points form a single periodic orbit.

Proof. Assume that the statement does not hold, i.e. there is a periodic orbit $\operatorname{Orb}(p) \subset \alpha(\{x_j\}_{j\leq 0})$ isolated in $\alpha(\{x_j\}_{j\leq 0})$ and a point $q \in \alpha(\{x_j\}_{j\leq 0}) \setminus \operatorname{Orb}(p)$. Let $\varepsilon > 0$ be such that $\cup_{x\in \operatorname{Orb}(p)} B(x,\varepsilon)$ does not contain any point from $\alpha(\{x_j\}_{j\leq 0})\setminus \operatorname{Orb}(p)$. As $\alpha(\{x_j\}_{j\leq 0})$ is internally chain transitive, there is a chain $z_0, \ldots, z_n$ of points from $\alpha(\{x_j\}_{j\leq 0})$ joining p and q with $d(f(z_i), z_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. We have $z_i \in \operatorname{Orb}(p)$, for every $i = 0, \ldots, n-1$, by the choice of ε. But then $d(f(z_{n-1}), q) < \varepsilon$ implies $q \in \operatorname{Orb}(p)$ which contradicts the assumption. Therefore $\alpha(\{x_j\}_{j\leq 0}) \subseteq \operatorname{Orb}(p)$ and since $\alpha(\{x_j\}_{j\leq 0})$ is an invariant set, $\alpha(\{x_j\}_{j\leq 0}) = \operatorname{Orb}(p)$. □

Remark 1. Since every ω-limit set $\omega(y) \subset G$ is internally chain transitive, we obtain by the same reasoning as above that if $\omega(y)$ is finite then it is a single periodic orbit.

3. Mixing graph maps.

We will show that any mixing graph map $f\colon G \to G$ has the following properties:

1. for every ω-limit set $\omega(y)$ in G and every accessible point x in $G \setminus \mathcal{I}(f)$, there is a backward branch starting at x with the α-limit set being equal to $\omega(y)$,
2. every α-limit set of any backward branch in G is an ω-limit set of some point in G.

We start with the construction of the backward branch starting at any accessible point whose α-limit set equals $\omega(y)$ for a chosen point $y \in G$. Lemma 3.3 and Lemma 3.4 distinguish two cases depending on the cardinality of $\omega(y)$. In both cases, for infinite and finite $\omega(y)$, the idea of construction is based on the properties of Bowen balls expressed in the following two lemmas from [17].

Lemma 3.1. [17, Lemma 10.4] *Let $f\colon G \to G$ be a mixing graph map. If $0 < \varepsilon < \frac{1}{2}\operatorname{diam} G$ and $\delta > 0$ then there is an $N = N(\varepsilon, \delta) > 0$ such that $B'_n(x,\varepsilon) \subset B(x,\delta)$ for all $x \in G$ and all $n \geq N$.*

Lemma 3.2. [17, Lemma 10.5] *Let $f\colon G \to G$ be a mixing graph map. For every $\varepsilon > 0$ there is a constant $\eta = \eta(\varepsilon)$ such that*

$$0 < \eta \leq \operatorname{diam} f^n(B'_n(x,\varepsilon))$$

for every $n \geq 0$ and $x \in G$.

Lemma 3.3. *Let $f\colon G \to G$ be a mixing graph map and $y \in G$ such that $\omega(y)$ is finite. There exists an open connected set $U \subset G$ such that, for every $x_0 \in U$ there is a backward branch $\{\tilde{x}_j\}_{j\leq 0}$ with $\alpha(\{\tilde{x}_j\}_{j\leq 0}) = \omega(y)$.*

Proof. By Remark 1, every finite $\omega(y)$ is an orbit of some periodic point $p \in G$. First assume that $p \in G$ is a fixed point, i.e. $f^i(p) = p$ for all $i \geq 0$, and denote $\deg(p) = L$. Choose $\varepsilon > 0$ and let $\eta > 0$ be the constant from Lemma 3.2 such that for every $n \in \mathbb{N}$ and $x \in G$ we have:

$$0 < \eta \leq \operatorname{diam} f^n(B'_n(x,\varepsilon)).$$

Decreasing η if necessary, we can choose $\{S_i^{(1)}\}_{i\geq 0}, \ldots \{S_i^{(L)}\}_{i\geq 0}$ where each $\{S_i^{(l)}\}_{i\geq 0}$ is a nested sequence of closed arcs in G such that $\operatorname{diam} S_0^{(l)} = \frac{\eta}{2}$ and

$\operatorname{diam} S_{i+1}^{(l)} = \frac{1}{2} \operatorname{diam} S_i^{(l)}$ for every $1 \le l \le L$ and every $i \ge 0$, and the fixed point p is the unique element in the intersection $\bigcap_{l=1}^{L} S_i^{(l)}$, for every $i \ge 0$.

Choose $x_0^{(1)}, \dots, x_0^{(L)} \in G$ such that $x_0^{(l)} \in S_0^{(l)}$ and $d(x_0^{(l)}, p) = \frac{\eta}{4}$ for $1 \le l \le L$. By Lemma 3.1 there exists $N_0 = N(\varepsilon, \frac{\eta}{8}) > 0$ such that for $n > N_0$ and every $1 \le l \le L$ we have:

$$B_n'(x_0^{(l)}, \varepsilon) \subset B(x_0^{(l)}, \frac{\eta}{8}) \subset S_0^{(l)}.$$

Pick $x_1^{(l)} \in S_1^{(l)}$, $1 \le l \le L$ such that $d(x_1^{(l)}, p) = \frac{\eta}{2^3}$. Again by Lemma 3.1 we get $N_1 > 0$ such that for every $n > N_1$ we have:

$$B_n'(x_1^{(l)}, \varepsilon) \subset B(x_1^{(l)}, \frac{\eta}{2^4}) \subset S_1^{(l)}.$$

By Lemma 3.2 we know that $\eta \le \operatorname{diam} f^n(B_n'(x_1^{(l)}, \varepsilon))$ for $1 \le l \le L$, which implies $\eta \le \operatorname{diam} f^n(S_1^{(l)})$ as well for any $1 \le l \le L$ and $n > N_1$. As $S_0^{(l)} \cap S_1^{(l)} = S_1^{(l)} \ni p$, we have that for every $1 \le l \le L$ and any $n > N_1$ there exists some $m \in \{1, \dots, L\}$ such that $f^n(S_1^{(l)}) \supset S_0^{(m)}$. For any $l \in \{1, \dots, L\}$ we denote:

$$n_1^{(l)} = \min\{n \le N_1 + 1 : \text{ there exists } m \in \{1, \dots, L\} \text{ such that } f^n(S_1^{(l)}) \supset S_0^{(m)}\}.$$

Note that for $n < n_1^{(l)}$ we have $f^n\left(S_1^{(l)}\right) \subset \bigcup_{m=1}^{L} S_0^{(m)}$, so in particular the diameter of $f^n\left(S_1^{(l)}\right)$ is less than η.

Now for every $1 \le l \le L$ we are going to construct a sequence $\{n_i^{(l)}\}_{i>0}$, whose element $n_i^{(l)}$ indicates the first iteration of f such that $f^{n_i^{(l)}}(S_i^{(l)}) \supset S_{i-1}^{(m)}$ for some $m \in \{1, \dots, L\}$. Fix $l \in \{1, \dots, L\}$. Let $N_{k+1} > 0$ be the constant from Lemma 3.1 such that for $n > N_{k+1}$ we have:

$$B_n'(x_{k+1}^{(l)}, \varepsilon) \subset B(x_{k+1}^{(l)}, \frac{\eta}{2^{k+3}}) \subset S_{k+1}^{(l)}.$$

By Lemma 3.2 we have that $\eta \le \operatorname{diam} f^n(B_n'(x_{k+1}^{(l)}, \varepsilon))$ for $1 \le l \le L$, which implies $\eta \le \operatorname{diam} f^n(S_{k+1}^{(l)})$ as well for any $1 \le l \le L$ and $n > N_{k+1}$. As $S_{k+1}^{(l)} \cap S_k^{(l)} = S_{k+1}^{(l)} \ni p$, we have that for $n > N_{k+1}$ each $f^n(S_{k+1}^{(l)})$ covers $S_0^{(m)}$ for some $1 \le m \le L$. In other words, for each $n > N_{k+1}$ there exists some $m \in \{1, \dots, L\}$ such that :

$$f^n(S_{k+1}^{(l)}) \supset S_k^{(m)}. \tag{3.2}$$

Denote:

$$n_{k+1}^{(l)} = \min\{n \le N_{k+1} + 1 : \text{there exists } m \in \{1, \dots, L\} \text{such that} f^n(S_{k+1}^{(l)}) \supset S_k^{(m)}\}.$$

Note that for $n < n_{k+1}^{(l)}$ we have $f^n\left(S_{k+1}^{(l)}\right) \subset \bigcup_{m=1}^{L} S_k^{(m)}$, so in particular the diameter of $f^n\left(S_{k+1}^{(l)}\right)$ is less than $\eta/2^k$.

We are going to construct a nested sequence of closed sets $\{Z_i\}_{i \ge 0}$ such that $Z_i \subset \{1, \dots, L\}^{\mathbb{N}_0}$ for each $i \ge 0$ as follows. A point $z = \{z_n\}_{n \ge 0}$ belongs to Z_0 if the following holds:

$$z_1 = l, z_0 = m \text{ for } m, l \in \{1, \dots, L\} \text{ such that } S_0^{(z_0)} \subset f^{n_1^{(z_1)}}\left(S_1^{(z_1)}\right).$$

In particular z_1 can take any value from the set $\{1, \dots, L\}$ since in (3.2) we have already shown there always exists at least one $z_0 \in \{1, \dots, L\}$ with the above property, so Z_0 is nonempty.

Below we use the notation $z_{[0,K]}$ to denote a finite sequence of symbols $z_0 z_1 \dots z_K$ over the given alphabet $\{1, \dots, L\}$. For $i > 0$ assume Z_{i-1} is nonempty and for any letter $l \in \{1, \dots, L\}$ there is some $w \in Z_{i-1}$ such that $w_{i-1} = l$. A point $z = \{z_n\}_{n \geq 0}$ belongs to Z_i if there exists some $w \in Z_{i-1}$ such that $z_{[0,i-1]} = w_{[0,i-1]}$ and the following holds:

$$z_i \in \{1, \dots, L\} \text{ is such that } S_{i-1}^{(z_{i-1})} \subset f^{n_i^{(z_i)}}\left(S_i^{(z_i)}\right).$$

Pick any $l \in \{1, \dots, L\}$ and let m be provided by (3.2). By inductive assumption there is some $w \in Z_{i-1}$ such that $w_{i-1} = m$. Therefore there is some $z \in Z_i$ with $z_i = l$ and in particular Z_i is nonempty.

The intersection $Z = \bigcap_{i \geq 0} Z_i$ is non-empty, since $\{Z_i\}_{i \geq 0}$ is a nested sequence of nonempty compact sets, so fix some $z \in Z$. Depending on the first symbol of z put $U = \operatorname{Int} S_0^{(l)}$ provided that $z_0 = l$, $l \in \{1, \dots, L\}$. For every $i \geq 1$ denote :

$$k_i = n_i^{(l)} \text{ for such an } l \in \{1, \dots, L\} \text{ that } z_i = l$$

and

$$\tilde{S}_i = S_i^{(l)} \text{ for such an } l \in \{1, \dots, L\} \text{ that } z_i = l.$$

Then $f^{k_i}(\tilde{S}_i) \supset \tilde{S}_{i-1}$, for every $i > 1$. Let $\tilde{x}_0$ be an arbitrary point from U. The backward branch $\{\tilde{x}_j\}_{j \leq 0}$ starting at $\tilde{x}_0$ is defined as follows. For $j = -\sum_{i=1}^m k_i$ for consecutive $m = 1, 2, 3, \dots$, we pick

$$\tilde{x}_j \in \tilde{S}_m \text{ such that } f^{k_m}(\tilde{x}_j) = \tilde{x}_{j+k_m}$$

and for all other j we define

$$\tilde{x}_j = f(\tilde{x}_{j-1}) \text{ for } -\sum_{s=1}^{m} k_s < j < -\sum_{s=1}^{m-1} k_s, m > 1, \text{ or } -k_1 < j < 0.$$

The above conditions guarantee that $\tilde{x}_j$ is well defined for $j \leq 0$ and $\{\tilde{x}_j\}_{j \leq 0}$ is a backward branch of some point from U. By the construction for every $m > 0$ we will find some $t > 0$ such that $\{\tilde{x}_j\}_{j \leq -t} \subset B(p, \frac{\eta}{2^{m+1}})$, so altogether $\{p\} = \alpha(\{\tilde{x}_j\}_{j \leq 0})$.

Now assume that p is a point of period $K > 1$. We use the above result for map f^K for which p is a fixed point to get the backward branch $\{\tilde{x}_j\}_{j \leq 0}$ with $\tilde{x}_0 \in U$. Hence $\{\tilde{y}_j\}_{j \leq 0}$ defined as follows:

$$\begin{aligned} \tilde{y}_{Kj} &= \tilde{x}_j, j \leq 0, \\ \tilde{y}_s &= f(\tilde{y}_{s-1}) \text{ for } (K+1)j < s < Kj, j \leq 0 \end{aligned}$$

is the backward branch of $\tilde{y}_0 = \tilde{x}_0 \in U$ for map f. By continuity, and the fact that $\{p\} = \alpha_{f^K}(\{\tilde{x}_j\}_{j \leq 0})$ we obtain that $\operatorname{Orb}(p) = \alpha_f(\{\tilde{y}_j\}_{j \leq 0})$. □

Lemma 3.4. *Let $f\colon G \to G$ be a mixing graph map and $y \in G$ such that $\omega(y)$ is infinite. There exists an open connected set $U \subset G$ such that, for every $\tilde{x}_0 \in U$, there is a backward branch $\{\tilde{x}_j\}_{j \leq 0}$ with $\alpha(\{\tilde{x}_j\}_{j \leq 0}) = \omega(y)$.*

Proof. Let $a \in \omega(y)$ be an accumulation point of the infinite set $\omega(y)$ and $J = [a, b]$ be an arc such that a is an accumulation point of the set $\Lambda = \omega(y) \cap [a, b]$. We may assume that $J \setminus \{a, b\}$ is an open free arc, that means $(J \setminus \{a, b\}) \cap Br(G) = \emptyset$. Let $\{\varepsilon_n\}_{n \geq 0}$ be a sequence of positive numbers such that $\varepsilon_0 = \frac{1}{2} \operatorname{diam} J$ and $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$

for every $n \geq 0$. Now let $\{\eta_n\}_{n\in\mathbb{N}}$ be the sequence of constants from Lemma 3.2, such that for every $n, k \in \mathbb{N}$ and $x \in G$ we have:

$$0 < \eta_n \leq \operatorname{diam} f^k(B'_k(x, \varepsilon_n)).$$

Choose some $l_0, m_0, r_0 \in \Lambda$ such that $d(l_0, r_0) < \frac{\eta_1}{4}$ and:

$$a < l_0 < m_0 < r_0.$$

As $l_0, r_0 \in \omega(y)$ there exist $x_0, \hat{x}_0 \in \operatorname{Orb}(y) \cap (J \setminus \{a, b\})$ such that $d(x_0, l_0) < \frac{\eta_1}{8}$ and $d(\hat{x}_0, r_0) < \frac{\eta_1}{8}$. Define $\gamma_0 = \frac{1}{8}\min\{d(l_0, x_0), d(r_0, \hat{x}_0), \eta_2\}$ and note that in particular $\gamma_0 < \frac{\eta_1}{64}$. Let $N_0 = N(\varepsilon_0, \gamma_0) > 0$ be the constant from Lemma 3.1 and take $n_0, \hat{n}_0 > N_0$ such that $B'_{n_0}(x_0, \varepsilon_0) \subset B(x_0, \gamma_0)$ and $B'_{\hat{n}_0}(\hat{x}_0, \varepsilon_0) \subset B(\hat{x}_0, \gamma_0)$.

Choose $l_1, m_1, r_1 \in \Lambda$ such that $d(l_1, r_1) < \frac{\eta_2}{4}$ and:

$$a < l_1 < m_1 < r_1 < l_0.$$

Again there exist $x_1, \hat{x}_1 \in \operatorname{Orb}(y) \cap (J \setminus \{a, b\})$ with $d(l_1, x_1) < \gamma_0$, and $d(\hat{x}_1, r_1) < \gamma_0$. Denote $\gamma_1 = \frac{1}{8}\min\{d(l_1, x_1), d(r_1, \hat{x}_1), \eta_3\}$ and let $N_1 = N(\varepsilon_1, \gamma_1) > 0$ be the constant from Lemma 3.1. Using Fact 2.2 take $n_1, \hat{n}_1 > N_1$ such that:

$$B'_{n_1}(x_1, \varepsilon_1) \subset B(x_1, \gamma_1) \text{ and } d(f^{n_1}(x_1), m_0) < \gamma_1 \text{ and } \bigcup_{i=0}^{n_1-1} B(f^i(x_1), \varepsilon_1) \supset \omega(y),$$

$$B'_{\hat{n}_1}(\hat{x}_1, \varepsilon_1) \subset B(\hat{x}_1, \gamma_1) \text{ and } d(f^{\hat{n}_1}(\hat{x}_1), m_0) < \gamma_1 \text{ and } \bigcup_{i=0}^{\hat{n}_1-1} B(f^i(\hat{x}_1), \varepsilon_1) \supset \omega(y).$$

By Lemma 3.2 we have:

$$\eta_1 \leq \operatorname{diam} f^{n_1}(B'_{n_1}(x_1, \varepsilon_1)),$$
$$\eta_1 \leq \operatorname{diam} f^{\hat{n}_1}(B'_{\hat{n}_1}(\hat{x}_1, \varepsilon_1)).$$

Note that the following inequalities hold:

$$\begin{aligned} d(f^{n_1}(x_1), m_0) + d(x_0, m_0) + \operatorname{diam} B'_{n_0}(x_0, \varepsilon_0) &< \frac{\eta_1}{64} + \frac{\eta_1}{4} + \frac{\eta_1}{8} < \frac{\eta_1}{2}, \\ d(f^{n_1}(x_1), m_0) + d(\hat{x}_0, m_0) + \operatorname{diam} B'_{\hat{n}_0}(\hat{x}_0, \varepsilon_0) &< \frac{\eta_1}{64} + \frac{\eta_1}{4} + \frac{\eta_1}{8} < \frac{\eta_1}{2}, \end{aligned} \tag{3.3}$$

hence at least one of the following inclusions must hold:

$$f^{n_1}(B'_{n_1}(x_1, \varepsilon_1)) \supset B'_{n_0}(x_0, \varepsilon_0) \text{ or } f^{n_1}(B'_{n_1}(x_1, \varepsilon_1)) \supset B'_{\hat{n}_0}(\hat{x}_0, \varepsilon_0).$$

Analogously, we also have:

$$\begin{aligned} d(f^{\hat{n}_1}(\hat{x}_1), m_0) + d(x_0, m_0) + \operatorname{diam} B'_{n_0}(x_0, \varepsilon_0) &< \frac{\eta_1}{64} + \frac{\eta_1}{4} + \frac{\eta_1}{8} < \frac{\eta_1}{2}, \\ d(f^{\hat{n}_1}(\hat{x}_1), m_0) + d(\hat{x}_0, m_0) + \operatorname{diam} B'_{\hat{n}_0}(\hat{x}_0, \varepsilon_0) &< \frac{\eta_1}{64} + \frac{\eta_1}{4} + \frac{\eta_1}{8} < \frac{\eta_1}{2}. \end{aligned} \tag{3.4}$$

and therefore, as before:

$$f^{\hat{n}_1}(B'_{\hat{n}_1}(\hat{x}_1, \varepsilon_1)) \supset B'_{n_0}(x_0, \varepsilon_0) \text{ or } f^{\hat{n}_1}(B'_{\hat{n}_1}(\hat{x}_1, \varepsilon_1)) \supset B'_{\hat{n}_0}(\hat{x}_0, \varepsilon_0).$$

Now we are going to construct sequences of Bowen balls $\{B_{n_k}(x_k, \varepsilon_k)\}_{k\geq 0}$ and $\{B_{\hat{n}_k}(\hat{x}_k, \varepsilon_k)\}_{k\geq 0}$ with the properties as follows for every $k \geq 1$:

1. there are some properly chosen points $l_k, m_k, r_k \in \Lambda$ with $d(l_k, r_k) < \frac{\eta_{k+1}}{4}$ and $a < l_k < m_k < r_k < l_{k-1}$ for which we may find $x_k, \hat{x}_k \in \operatorname{Orb}(y) \cap (J \setminus \{a, b\})$ such that x_k is within γ_{k-1} distance from l_k and $\hat{x}_k$ is within γ_{k-1} distance from r_k, where $\gamma_{k-1} = \frac{1}{8}\min\{d(x_{k-1}, l_{k-1}), d(\hat{x}_{k-1}, r_{k-1})\}$.
2. $n_k, \hat{n}_k > N_k$

3. $f^{n_i}(B'_{n_i}(x_i,\varepsilon_i)) \supset B'_{n_{i-1}}(x_{i-1},\varepsilon_{i-1})$ or $f^{n_i}(B'_{n_i}(x_i,\varepsilon_i)) \supset B'_{\hat{n}_{i-1}}(\hat{x}_{i-1},\varepsilon_{i-1})$,
4. $f^{\hat{n}_i}(B'_{\hat{n}_i}(\hat{x}_i,\varepsilon_i)) \supset B'_{n_{i-1}}(x_{i-1},\varepsilon_{i-1})$ or $f^{\hat{n}_i}(B'_{\hat{n}_i}(\hat{x}_i,\varepsilon_i)) \supset B'_{\hat{n}_{i-1}}(\hat{x}_{i-1},\varepsilon_{i-1})$

Assume the above conditions are fulfilled for $i=0,\dots,k$ for some $k\geq 1$. We proceed with the construction to get $B_{n_{k+1}}(x_{k+1},\varepsilon_{k+1})$ and $B_{\hat{n}_{k+1}}(\hat{x}_{k+1},\varepsilon_{k+1})$. Choose $l_{k+1},m_{k+1},r_{k+1}\in\Lambda$ such that $d(l_{k+1},r_{k+1})<\frac{\eta_{k+2}}{4}$ and:

$$a<l_{k+1}<m_{k+1}<r_{k+1}<l_k.$$

Take some $x_{k+1},\hat{x}_{k+1}\in \operatorname{Orb}(y)\cap(J\setminus\{a,b\})$ such that $d(x_{k+1},l_{k+1})<\gamma_k$ and $d(\hat{x}_{k+1},r_{k+1})<\gamma_k$, let $\gamma_{k+1}=\frac{1}{8}\min\{d(l_{k+1},x_{k+1}),d(r_{k+1},\hat{x}_{k+1})),\eta_{k+3}\}$. Apply Lemma 3.1 to obtain $N_{k+1}=N(\varepsilon_{k+1},\gamma_{k+1})>0$ and pick $n_{k+1},\hat{n}_{k+1}>N_{k+1}$ by the Fact 2.2 such that:

$$B'_{n_{k+1}}(x_{k+1},\varepsilon_{k+1})\subset B(x_{k+1},\gamma_{k+1}) \text{ and } d(f^{n_{k+1}}(x_{k+1}),m_k)<\gamma_{k+1},$$
$$B'_{\hat{n}_{k+1}}(\hat{x}_{k+1},\varepsilon_{k+1})\subset B(\hat{x}_{k+1},\gamma_{k+1}) \text{ and } d(f^{\hat{n}_{k+1}}(\hat{x}_{k+1}),m_k)<\gamma_{k+1},$$
$$\bigcup_{i=0}^{n_{k+1}-1} B(f^i(x_{k+1}),\varepsilon_{k+1})\supset\omega(y),$$
$$\bigcup_{i=0}^{\hat{n}_{k+1}-1} B(f^i(\hat{x}_{k+1}),\varepsilon_{k+1})\supset\omega(y).$$

Next, by Lemma 3.2 we have:

$$\eta_{k+1}\leq \operatorname{diam} f^{n_{k+1}}(B'_{n_{k+1}}(x_{k+1},\varepsilon_{k+1})),$$
$$\eta_{k+1}\leq \operatorname{diam} f^{\hat{n}_{k+1}}(B'_{\hat{n}_{k+1}}(\hat{x}_{k+1},\varepsilon_{k+1})).$$

The estimations analogous to those in (3.3) and (3.4) imply that:

$$f^{n_{k+1}}(B'_{n_{k+1}}(x_{k+1},\varepsilon_{k+1}))\supset B'_{n_k}(x_k,\varepsilon_k) \text{ or } f^{n_{k+1}}(B'_{n_{k+1}}(x_{k+1},\varepsilon_{k+1}))\supset B'_{\hat{n}_k}(\hat{x}_k,\varepsilon_k)$$

and:

$$f^{\hat{n}_{k+1}}(B'_{\hat{n}_{k+1}}(\hat{x}_{k+1},\varepsilon_{k+1}))\supset B'_{\hat{n}_k}(x_k,\varepsilon_k) \text{ or } f^{\hat{n}_{k+1}}(B'_{\hat{n}_{k+1}}(\hat{x}_{k+1},\varepsilon_{k+1}))\supset B'_{\hat{n}_k}(\hat{x}_k,\varepsilon_k).$$

The induction is completed.

Now we will perform a construction similar to the one from Lemma 3.3. We are going to construct a nested sequence of closed sets $\{Z_i\}_{i\geq 0}$ such that $Z_i\subset\{l,r\}^{\mathbb{N}_0}$ for each $i\geq 0$ as follows. A point $z=\{z_n\}_{n\geq 0}$ from $\{l,r\}^{\mathbb{N}_0}$ is an element of Z_0 if one of the following holds:

$$z_0=l \text{ and } \left(B'_{n_0}(x_0,\varepsilon_0)\subset f^{n_1}(B'_{n_1}(x_1,\varepsilon_1)) \text{ or } B'_{n_0}(x_0,\varepsilon_0)\subset f^{\hat{n}_1}(B'_{\hat{n}_1}(\hat{x}_1,\varepsilon_1))\right),$$
$$z_0=r \text{ and } \left(B'_{\hat{n}_0}(\hat{x}_0,\varepsilon_0)\subset f^{n_1}(B'_{n_1}(x_1,\varepsilon_1)) \text{ or } B'_{\hat{n}_0}(\hat{x}_0,\varepsilon_0)\subset f^{\hat{n}_1}(B'_{\hat{n}_1}(\hat{x}_1,\varepsilon_1))\right).$$

For $i>0$ an infinite sequence a point w belongs to Z_i if there exists some $z\in Z_{i-1}$ such that $z_{[0,i-1]}=w_{[0,i-1]}$ and one of the following holds:

$$w_i=l, w_{i-1}=l \text{ and } B'_{n_{i-1}}(x_{i-1},\varepsilon_{i-1})\subset f^{n_i}(B'_{n_i}(x_i,\varepsilon_i)),$$
$$w_i=l, w_{i-1}=r \text{ and } B'_{n_{i-1}}(x_{i-1},\varepsilon_{i-1})\subset f^{\hat{n}_i}(B'_{\hat{n}_i}(\hat{x}_i,\varepsilon_i)),$$
$$w_i=r, w_{i-1}=l \text{ and } B'_{\hat{n}_{i-1}}(\hat{x}_{i-1},\varepsilon_{i-1})\subset f^{n_i}(B'_{n_i}(x_i,\varepsilon_i)),$$
$$w_i=r, w_{i-1}=r \text{ and } B'_{\hat{n}_{i-1}}(\hat{x}_{i-1},\varepsilon_{i-1})\subset f^{n_i}(B'_{\hat{n}_i}(\hat{x}_i,\varepsilon_i)).$$

The intersection $Z = \bigcap_{i\geq 0} Z_i$ is non-empty, since $\{Z_i\}_{i\geq 0}$ is nested sequence of compact sets, so fix some $z \in Z$. Depending on the first symbol of z put:

$$U = \begin{cases} \operatorname{Int} B'_{n_0}(x_0, \varepsilon_0) \text{ if } z_0 = l \\ \operatorname{Int} B'_{\hat{n}_0}(\hat{x}_0, \varepsilon_0) \text{ if } z_0 = r \end{cases}$$

and define:

$$k_i = \begin{cases} n_i \text{ if } z_i = l \\ \hat{n}_i \text{ if } z_i = r \end{cases}, i \geq 1.$$

and

$$B'_i = \begin{cases} B'_{k_i}(x_i, \varepsilon_i) \text{ if } z_i = l \\ B'_{k_i}(\hat{x}_i, \varepsilon_i) \text{ if } z_i = r \end{cases}, i \geq 1.$$

Let $\tilde{x}_0$ be an arbitrary point from U. The backward branch $\{\tilde{x}_j\}_{j\leq 0}$ starting at $\tilde{x}_0$ is defined as follows:

$$\tilde{x}_j \in B'_m \text{ such that } f^{k_m}(\tilde{x}_j) = \tilde{x}_{j+k_m} \text{ for } j = -\sum_{s=1}^{m} k_s, m \geq 1,$$

$$\tilde{x}_j = f(\tilde{x}_{j-1}) \text{ for } -\sum_{s=1}^{m} k_s < j < -\sum_{s=1}^{m-1} k_s, m > 1, \text{ or } -k_1 < j < 0.$$

By the definition of Z we have $f^{k_j}(B'_j) \supset B'_{j-1}$ for every $j \geq 1$, so $\tilde{x}_j$ is well defined for $j < 0$.

To prove that $\alpha(\{\tilde{x}_j\}_{j\leq 0}) \supset \omega(y)$ fix a point $q \in \omega(y)$ and an integer $m > 0$. Let $b \in \{x_m, \hat{x}_m\}$ be the point such that $B'_m = B'_{k_m}(b, \varepsilon_m)$. As $\omega(y) \subset \bigcup_{i=0}^{k_m-1} B(f^i(b), \varepsilon_m)$ we can find an integer $0 \leq s < k_m$ such that $d(f^s(b), q) < \varepsilon_m$. Simultaneously $d(f^s(b), f^s(\tilde{x}_j)) < \varepsilon_m$, for $j = -\sum_{i=1}^m k_i$, since $\tilde{x}_j \in B'_m$ by the definition of $\{\tilde{x}_j\}_{j\leq 0}$. Altogether we have $d(\tilde{x}_{j+s}, q) < 2\varepsilon_m$ so indeed $\alpha(\{\tilde{x}_j\}_{j\leq 0}) \supset \omega(y)$.

On the other hand, there exists an increasing sequence $\{s_k\}_{k\in\mathbb{N}}$ such that the orbit $\operatorname{Orb}(f^{s_k}(y))$ contains $f^l(x_k)$ for all $0 \leq l < n_k$ and $f^l(\hat{x}_k)$ for all $0 \leq l < \hat{n}_k$ and the orbit $\operatorname{Orb}(f^{s_k}(y))$ is ε_k-close from $\omega(y)$. Therefore for each $m \in \mathbb{N}$ there is $N \geq 0$ such that $\{\tilde{x}_j\}_{j\leq -N} \subset B(\overline{\operatorname{Orb}(f^{s_m}(y))}, \varepsilon_m)$ which yields that for any m we have $\alpha(\{\tilde{x}_j\}_{j\leq 0}) \subset B(\omega(y), 2\varepsilon_m)$ so $\alpha(\{\tilde{x}_j\}_{j\leq 0}) \subset \omega(y)$ indeed. □

Remark 2. In the proof of Lemma 3.4 by continuity of map f and the proper choice of the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with ε_n decreasing sufficiently fast, for any $\beta > 0$ and an arbitrarily large $s \in \mathbb{N}$, we will find a point following the orbits of each x_n and $\hat{x}_n$ for s iterations at distance at most β. To achieve this, numbers $n_k, \hat{n}_k$ in the construction must be very large, to overpass any fixed s, not only greater that N_k.

Theorem 3.5. *Let $f \colon G \to G$ be a mixing graph map. For every $z \in G \setminus \mathcal{I}(f)$ and every $y \in G$ there exists a backward branch $\{z_j\}_{j\leq 0}$ such that $z_0 = z$ and $\alpha(\{z_j\}_{j\leq 0}) = \omega(y)$.*

Proof. Depending on the cardinality of $\omega(y)$ we apply Lemma 3.3 or Lemma 3.4 to get an open set $U \subset G$ such that for every $\tilde{x}_0 \in U$, there is a backward branch $\{\tilde{x}_j\}_{j\leq 0}$ with $\alpha(\{\tilde{x}_j\}_{j\leq 0}) = \omega(y)$. Let $z \in G \setminus \mathcal{I}(f)$. Then there is $\varepsilon > 0$ such that $z \in G \setminus B(\mathcal{I}(f), \varepsilon)$. By [17, Theorem 4.6], we can find $k > 0$ such that $z \in \operatorname{Int} f^k(U)$ and thus there exists a preimage $z' \in U$ such that $z = f^k(z')$. Since z' is from U

we can find a backward branch $\{\tilde{z}_j\}_{j\leq 0}$ with $\tilde{z}_0 = z'$ and $\alpha(\{\tilde{z}_j\}_{j\leq 0}) = \omega(y)$. The desired backward branch $\{z_j\}_{j\leq 0}$ starting at z has the form $z_0 = z = f^k(z'), z_{-1} = f^{k-1}(z'), \dots, z_{-k+1} = z' = \tilde{z}_0, z_{-k} = \tilde{z}_{-1}, z_{-k-1} = \tilde{z}_{-2}, \dots$. □

Remark 3. By [17, Theorem 4.6] we know that inaccessible points are periodic and the set $\mathcal{I}(f)$ is finite and backward invariant, so the only α-limit sets of inaccessible points are periodic orbits contained in $\mathcal{I}(f)$.

Having proved that for mixing map every ω-limit set in G is an α-limit set of some backward branch, the natural question is whether it is also true that every α-limit set of a backward branch in G is the ω-limit set of some point from G at the same time. The answer to that problem is given below.

Theorem 3.6. *Let $f\colon G \to G$ be the mixing graph map. Then for every backward branch $\{x_j\}_{j\leq 0} \subset G$ the set $\alpha(\{x_j\}_{j\leq 0})$ is equal to an ω-limit set of some point in G.*

Proof. If $\alpha(\{x_j\}_{j\leq 0})$ is finite then by Lemma 2.1 it is a periodic orbit of a point $p \in G$ and obviously $\omega(p) = \alpha(\{x_j\})_{j\leq 0}$. Assume $\alpha(\{x_j\}_{j\leq 0})$ is infinite. Let $a \in \alpha(\{x_j\}_{j\leq 0})$ be an accumulation point of $\alpha(\{x_j\}_{j\leq 0})$ and $J = [a, b]$ be an arc such that the set $\Lambda = \alpha(\{x_j\}_{j\leq 0}) \cap [a, b]$ accumulates on a. We may assume that $J \setminus \{a, b\}$ is an open free arc, that means $(J \setminus \{a, b\}) \cap Br(G) = \emptyset$. We show that $\alpha(\{x_j\}_{j\leq 0})$ is approximated by an infinite sequence of periodic orbits. Let $\{\varepsilon_n\}_{n\geq 0}$ be a sequence of positive numbers such that $\varepsilon_0 = \frac{1}{2} \operatorname{diam} J$ and $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ for every $n \geq 0$. Let $\{\eta_n\}_{n\geq 0}$ be the sequence of constants from Lemma 3.2 such that for every $x \in G$ and every $k > 0$ we have:

$$0 < \eta_n \leq \operatorname{diam} f^k(B'_k(x, \varepsilon_n)).$$

For every $k > 0$ fix $l_k, m_k, r_k \in \Lambda$ such that $d(l_k, r_k) < \frac{\eta_k}{4}$ and:

$$a < l_1 < m_1 < r_1,$$
$$a < l_k < m_k < r_k < l_{k-1} \text{ for } k > 1.$$

Put $\gamma_k = \frac{1}{8} \min\{d(m_k, l_k), d(m_k, r_k)\}$ and let $N_k = N(\varepsilon_k, \gamma_k)$ be the constant from Lemma 3.1 implying that $B'_n(x, \varepsilon_k) \subset B(x, \gamma_k)$ for every $x \in G$ and $n > N_k$. Take a point $y_k \in \{x_j\}_{j\leq 0}$ from γ_k-neighborhood of m_k with the property that all points from the backward branch preceeding y_k are within ε_k-distance from $\alpha(\{x_j\}_{j\leq 0})$, that is:

$$\operatorname{dist}(x_i, \alpha(\{x_j\}_{j\leq 0})) < \varepsilon_k \text{ for } i < j_k \text{ where } y_k = x_{j_k}. \tag{3.5}$$

Choose $n_k, \hat{n}_k > N_k$ for which there exists $z_k, \hat{z}_k \in \{x_j\}_{j\leq 0}$ such that $d(l_k, z_k) < \gamma_k$, $d(r_k, \hat{z}_k) < \gamma_k$ and $f^{n_k}(z_k) = f^{\hat{n}_k}(\hat{z}_k) = y_k$ and note that increasing N_k when necessary, we may assume the following:

$$\bigcup_{i=0}^{N_k-1} B(x_{j_k-i}, \varepsilon_k) \supset \alpha(\{x_j\}_{j\leq 0}). \tag{3.6}$$

By the definition of η_k and N_k we have:

$$\eta_k \leq \operatorname{diam} f^{n_k}(B'_{n_k}(z_k, \varepsilon_k)),$$
$$\eta_k \leq \operatorname{diam} f^{\hat{n}_k}(B'_{\hat{n}_k}(\hat{z}_k, \varepsilon_k)).$$

and $B'_{n_k}(z_k,\varepsilon_k) \subset B(z_k,\gamma_k)$ and $B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k) \subset B(\hat{z}_k,\gamma_k)$. Moreover, y_k is the element of both $f^{n_k}(B'_{n_k}(z_k,\varepsilon_k))$ and $f^{\hat{n}_k}(B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k))$. Taking it all into consideration we get the following:

$$\begin{aligned} d(y_k,m_k)+d(m_k,l_k)+d(l_k,z_k)+&\operatorname{diam} B'_{n_k}(z_k,\varepsilon_k) \\ &< \frac{\eta_k}{32}+\frac{\eta_k}{4}+\frac{\eta_k}{32}+\frac{\eta_k}{16}<\frac{\eta_k}{2}, \\ d(y_k,m_k)+d(m_k,r_k)+d(r_k,\hat{z}_k)+&\operatorname{diam} B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k) \\ &< \frac{\eta_k}{32}+\frac{\eta_k}{4}+\frac{\eta_k}{32}+\frac{\eta_k}{16}<\frac{\eta_k}{2}, \end{aligned} \tag{3.7}$$

which implies that $f^{n_k}(B'_{n_k}(z_k,\varepsilon_k))$ covers either $(B'_{n_k}(z_k,\varepsilon_k))$ or $(B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k))$. In the first case there exists a point of period n_k inside $(B'_{n_k}(z_k,\varepsilon_k))$. In the second case we may take $f^{\hat{n}_k}(B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k))$ which, by (3.7), covers either $B'_{n_k}(z_k,\varepsilon_k)$ or $B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k)$. Then there exists a point of period $n_k+\hat{n}_k$ in $B'_{n_k}(z_k,\varepsilon_k)$ or a point of period $\hat{n}_k$ inside $B'_{\hat{n}_k}(\hat{z}_k,\varepsilon_k)$. Regardless of the case we found at least one periodic point in ε_k-neighborhood of the backward branch $\{x_j\}_{j\le 0}$. Some iteration of this periodic point denoted by p_k is contained in $B_{N_k}(x_{j_K-N_k},\varepsilon_k)$. Denote the period of p_k by d_k. The construction results in a set $\{p_k\}_{k>0}$ of periodic points and an increasing sequence $\{d_k\}_{k>0}$ of their periods. It assures that for $k>0$ the orbit of each p_k follows some finite segment starting at $x_{j_k-N_k}$ of length N_k of the backward branch ε_k-close, while at the same time all points from that segment stay ε_k-close to the set $\alpha(\{x_j\}_{j\le 0})$ by (3.5). Therefore by (3.5) and (3.6) we get that for every $k>0$ and every $q\in\alpha(\{x_j\}_{j\le 0})$ there exists some $i<N_k$ such that:

$$d(f^i(p_k),x_{j_k-i})+d(x_{j_k-i},q)<2\varepsilon_k.$$

Combining it with (3.5) we get that $d_H(\operatorname{Orb}(p_k),\alpha(\{x_j\}_{j\le 0}))<2\varepsilon_k$ for $k>0$. By [27, Theorem 3.1] we know that the set of all ω-limit sets is closed, so there exists a point $y\in G$ such that the sequence of orbits $\{\operatorname{Orb}(p_k)\}_{k>0}$ converges to $\omega(y)$ with respect to Hausdorff metric, which gives $\alpha(\{x_j\}_{j\le 0})=\omega(y)$ completing the proof. □

4. Relation of α-limit sets of backward branches to maximal ω-limit sets.

We will use the notation from a series of papers by A. Blokh [7, 8, 9]. In Blokh's papers, a "graph" (also called a one-dimensional branched manifold) is not assumed to be connected, and is actually a finite union of graphs with respect to the definition of a graph we use in the present paper. We reformulate his results for our purposes where necessary in a similar way as authors in [32]. Therefore we will provide references from both [7, 8, 9] and [32].

A subgraph K of G is called *periodic of period* k or k*-periodic* if $K, f(K),\dots,$ $f^{k-1}(K)$ are pairwise disjoint and $f^k(K)=K$. If, instead of $f^k(K)=K$, it is known only that $f^k(K)\subseteq K$, the subgraph K is called *weakly* k*-periodic*. Then the set $\operatorname{Orb}(K)=\cup_{i=0}^{k-1}f^i(K)$ is called a k*-cycle of graphs* if K is k-periodic and a *weak* k*-cycle of graphs* if K is weakly k-periodic. We will write just *cycle of graphs* and *weak cycle of graphs* when the period k is not relevant.

We start this section with simple facts about cycles of graphs containing an infinite α-limit set of a backward branch.

Lemma 4.1. *Let $f\colon G\to G$ be a graph map and $M\subseteq G$ be a weak n-cycle of graphs. Then there is an n-cycle of graphs $\hat{M}\subseteq M$. If M contains an infinite α-limit set of a backward branch $\alpha(\{y_i\}_{i\le 0})$ then $\hat{M}$ is non-degenerate and $\alpha(\{y_i\}_{i\le 0})\subseteq\hat{M}\subseteq M$.*

Proof. Let $M = \mathrm{Orb}(K)$, where K is an n-periodic subgraph of G. The set $\hat{K} := \cap_{k\geq 0} f^{kn}(K)$ is non-empty, compact and connected since it is a decreasing intersection of non-empty compact connected components, and $f^n(\hat{K}) = \hat{K}$. By Fact 2.1 (1) $\hat{K}$ is a subgraph of G. If there is an infinite $\alpha(\{y_i\}_{i\leq 0}) \subset M$, because $\alpha(\{y_i\}_{i\leq 0})$ is strongly invariant we have $\alpha(\{y_i\}_{i\leq 0}) \subseteq \mathrm{Orb}(\hat{K})$ and since $\alpha(\{y_i\}_{i\leq 0})$ is infinite, $\hat{K}$ is non-degenerate. The set $\hat{M} = \mathrm{Orb}(\hat{K})$ is an n-cycle of graphs. □

Let $f: G \to G$ be a graph map and $\alpha(\{y_i\}_{i\leq 0}) \subseteq G$ be an infinite α-limit set of a backward branch. Then define

$$C(\alpha(\{y_i\}_{i\leq 0})) := \{X | X \subseteq G \text{ is a cycle of graphs and } \alpha(\{y_i\}_{i\leq 0}) \subseteq X\}.$$

Since the graph G is weakly 1-periodic and $\alpha(\{y_i\}_{i\leq 0}) \subseteq G$, Lemma 4.1 implies that $C(\alpha(\{y_i\}_{i\leq 0}))$ is never empty.

Lemma 4.2. *Let $f: G \to G$ be a graph map and $\alpha(\{y_i\}_{i\leq 0}) \subseteq G$ be an infinite α-limit set of a backward branch. Let $X, Y \in C(\alpha(\{y_i\}_{i\leq 0}))$. Then there is $Z \subset X \cap Y$ which satisfies $Z \in C(\alpha(\{y_i\}_{i\leq 0}))$ and Z has period not smaller than the maximum of periods of X and Y.*

Proof. Since $\alpha(\{y_i\}_{i\leq 0})$ is infinite the intersection of $\alpha(\{y_i\}_{i\leq 0})$ with some connected component of X (resp. Y) is infinite. In fact, every connected component of X (resp. Y) contains infinite subset of $\alpha(\{y_i\}_{i\leq 0})$ since $\alpha(\{y_i\}_{i\leq 0})$ is strongly invariant and the preimage of an infinite set has to be infinite. By Fact 2.1 (2) $X \cap Y$ has finitely many connected components. Let $Z_1, \ldots, Z_n$ denote all the connected components of $X \cap Y$ intersecting $\alpha(\{y_i\}_{i\leq 0})$. For every $i \in \{1, \ldots, n\}$, there is $j \in \{1, \ldots, n\}$ such that $f(Z_i) \subseteq Z_j$ since $f(Z_i)$ is included in some component of $X \cap Y$ and meets $f(\alpha(\{y_i\}_{i\leq 0})) = \alpha(\{y_i\}_{i\leq 0})$. Therefore Z_i is weakly periodic with the period not greater than n. The set $\alpha(\{y_i\}_{i\leq 0})$ is internally chain transitive by [20, Lemma 2.1] and thus $\tilde{Z} = \{Z_1, \ldots, Z_n\}$ is one weak n-cycle of graphs, i.e. it cannot split into a few disjoint cycles. By Lemma 4.1 there is an n-cycle of graphs $Z \subset \tilde{Z}$ such that $\alpha(\{y_i\}_{i\leq 0}) \subset Z$, and clearly period of Z cannot decrease. □

Lemma 4.3. *Let $f: G \to G$ be a graph map and $\alpha(\{y_i\}_{i\leq 0}) \subseteq G$ be an infinite α-limit set of a backward branch such that the periods of the cycles in $C(\alpha(\{y_i\}_{i\leq 0}))$ are bounded. There exists a cycle of graphs $X \in C(\alpha(\{y_i\}_{i\leq 0}))$ such that $X \subseteq Y$ for every $Y \in C(\alpha(\{y_i\}_{i\leq 0}))$.*

Proof. Let j be the maximal period of the cycles in $C(\alpha(\{y_i\}_{i\leq 0}))$ and by $C_j \subseteq C(\alpha(\{y_i\}_{i\leq 0}))$ denote the family of j-cycles of graphs containing the set $\alpha(\{y_i\}_{i\leq 0})$. We will show that there exists $X \in C_j$, such that, for every $\tilde{X} \in C_j$, $\tilde{X} \subseteq X$ implies $\tilde{X} = X$. Let $(Y_\lambda)_{\lambda\in\Gamma}$ be a totally ordered family in C_j (that is, all elements in Γ are comparable and, if $\lambda \leq \mu$, then $Y_\lambda \subseteq Y_\mu$). Then $Y = \cap_{\lambda\in\Gamma} Y_\lambda$ is compact and has j connected components because this is a decreasing intersection of j-cycles, and $f(Y) = Y$. Moreover, $\alpha(\{y_i\}_{i\leq 0}) \subseteq Y$ and Y is non-degenerate since $\alpha(\{y_i\}_{i\leq 0})$ is infinite (at least one component of Y is non-degenerate and, by continuity of f, every component of Y is non-degenerate). Hence $Y \in C_j$. Thus Zorn's Lemma applies, and there exists a minimal (with respect to inclusion) element $X \in C_j$ that is, for every $\tilde{X} \in C_j$, $\tilde{X} \subseteq X$ implies $\tilde{X} = X$.

Let $Y \in C(\alpha(\{y_i\}_{i\leq 0}))$. Then by Lemma 4.2 there is $X \cap Y \supset Z \in C(\alpha(\{y_i\}_{i\leq 0}))$ which has period greater than or equal to the period of X . On the other hand, the period of Z is at most j by the definition. Hence $Z \in C_j$. Then $Z = X$ by the minimality of X, i.e., $X \subseteq Y$. □

A *generating sequence* or a *sequence generating a solenoidal set* is any nested sequence of cycles of graphs $M_1 \supset M_2 \supset \cdots$ for f with periods tending to infinity. The intersection $Q = \bigcap_n M_n$ is automatically closed and strongly invariant, i.e. $f(Q) = Q$, and any closed and strongly invariant subset of Q (including Q itself) will be called a *solenoidal set*. Blokh showed that Q contains a perfect minimal set $Q_{min} = Q \cap \overline{Perf}$ such that $Q_{min} = \omega(x)$, for all $x \in Q$, and a maximal ω-limit set (with respect to inclusion) Q_{max} such that $Q_{max} = Q \cap \omega(f)$ [7, Theorem 1].

If x is a point of a graph G, then by a *side* T *of the point* x we mean a family of open, non-degenerate arcs $\{V_T(x)\}$ containing no branching points, with one endpoint at x, such that $\bigcap_{V_T(x)\in T} \overline{V_T(x)} = \{x\}$ and if $V_T^1(x) \in T, V_T^2(x) \in T$, then either $V_T^1(x) \subseteq V_T^2(x)$ or $V_T^2(x) \subseteq V_T^1(x)$. Members of the family T are called T-sided neighborhoods of x.
Let $f\colon G \to G$ be a graph map and $M \subset G$ be a cycle of graphs. For every $x \in M$, we define the *prolongation set of* x *with respect to* $f|_M$:

$$P_M(x, f) = \bigcap_U \overline{\bigcup_{i=1}^{\infty} f^i(U)},$$

where U is a relative neighborhood of x in M. If it is clear which map is considered, then we will write $P_M(x)$, if $M = G$ then we will write $P(x, f)$ or $P(x)$. Observe that just as x being recurrent is equivalent to $x \in \omega_f(x)$, x being non-wandering is equivalent to $x \in P(x)$. Obviously, $P(x)$ is an invariant closed set and the map $f|_{P(x)}$ is surjective whenever x is a non-wandering point. Similarly, we define the *prolongation set of* x *with respect to a side* T:

$$P_M^T(x, f) = \bigcap_{V_T(x)} \overline{\bigcup_{i=1}^{\infty} f^i(V_T(x))},$$

where $V_T(x)$ is a relative T-sided neighborhood of x in M. We will call an arc $V \subseteq G$ *non-wandering* if there is an integer $m \geq 1$ such that $f^m(V) \cap V \neq \emptyset$. It is easy to see that if every $V_T(x) \in T$ is non-wandering then $f|_{P^T(x)}$ is surjective.

Lemma 4.4. *Let* $f\colon G \to G$ *be a graph map and* T *be a side of a point* $x \in G$. *If every set* $V_T(x) \in T$ *is non-wandering then* $P^T(x)$ *is one of the following:*
- $P^T(x)$ *is a periodic orbit,*
- $P^T(x)$ *is a cycle of graphs,*
- $P^T(x)$ *is a solenoidal set* Q.

Proof. By assumptions, for every $V_T(x) \in T$, there is $m \geq 1$ such that $f^m(V_T(x)) \cap V_T(x) \neq \emptyset$. Clearly, the set $J_k = \cup_{i=1}^{\infty} f^{mi+k}(V_T(x))$ is connected, for $0 \leq k < m$. Thus the set $\cup_{k=0}^{m-1}\overline{J_k} = \overline{\mathrm{Orb}(V_T(x))}$ has finitely many components. Let $I \supset V_T(x)$ be a component of $\overline{\mathrm{Orb}(V_T(x))}$ and n be the minimal integer such that $f^n(V_T(x)) \cap V_T(x) \neq \emptyset$. Then $f^n(I) \subseteq I$ and $\overline{\mathrm{Orb}(V_T(x))}$ is a weak cycle of graphs. Let us choose a family of arcs $\{W_m\}_{m=1}^{\infty}$ so that $W_m \in T$, $W_m \supset W_{m+1}$ and $\lambda(W_m) \to 0$ as $m \to \infty$, where $\lambda(A)$ denotes the length of the arc A. By the previous reasoning, $K_m := \overline{\mathrm{Orb}(W_m)}$ is a weak cycle of graphs, for every $m \geq 1$. Then $K_m \supset K_{m+1}$ and $P^T(x) = \cap_{m\geq 1} K_m$. If periods of K_m are bounded and the intersection is non-degenerate then $P^T(x)$ is a cycle of graphs, since $f|_{P^T(x)}$ is surjective. If periods of K_m are bounded and the intersection is degenerate then $P^T(x)$ is a periodic orbit. If periods of K_m are unbounded then we can find a generating sequence of cycles of graphs $K'_1 \supset K'_2 \supset \cdots$, where $K'_m := \cap_{k\geq 0} f^k(K_m)$, such that $P^T(x) = \cap_{m\geq 1} K'_m = Q$ is a solenoidal set. □

Let $M \subset G$ be a cycle of graphs. We define the following sets:

$$E(M, f) = \{x \in M : P_M(x, f) = M\}$$

and

$$E_S(M, f) = \{x \in M : \text{there is a side } T \text{ such that } P^T_M(x, f) = M\}.$$

Clearly, $E_S(M, f) \subseteq E(M, f)$. These sets are closed and invariant. If $E(M, f)$ is infinite then, by [7, Theorem 2], $E_S(M, f) = E(M, f)$. In general, $E_S(M, f) \neq E(M, f)$ and $f(E(X, f)) \neq E(X, f)$. See the following example from [32].

Example 4.1. Let $\mathbb{S}$ be a circle and decompose $\mathbb{S}$ as the union of "western half-circle" and "eastern half-circle". Let f restricted to any of these half-circles be topologically conjugate to the tent map, the "south pole" of $\mathbb{S}$ being a fixed point of f and the "north pole" being mapped to the "south pole". Then $E(\mathbb{S}, f)$ consists of the two "poles" but $f(E(\mathbb{S}, f))$ is a singleton containing just the "south pole" and $E_S(\mathbb{S}, f)$ is empty set.

Theorem 4.5. [7, 8] *Let $M \subset G$ be a cycle of graphs such that $E_S(M, f)$ is non-empty. If $E_S(M, f)$ is finite then it is a periodic orbit. Otherwise, $E_S(M, f) = E(M, f)$ and it is an infinite maximal ω-limit set.*

Let $E(M, f)$ be the infinite maximal ω-limit set from Theorem 4.5. Then we say that $E(M, f)$ is a *basic set* if $Per(f) \cap M \neq \emptyset$ and we denote it by $D(M, f)$, while for $Per(f) \cap M = \emptyset$ we say that $E(M, f)$ is a *circumferential set* and we denote it by $S(M, f)$. We will write just $D(M)$ and $S(M)$ in the case where f is clear from the context.

Remark 4. The set $D(M)$ (resp. $S(M)$) is contained in a minimal (with respect to inclusion) cycle of graphs if the periods of the cycles of graphs from the family $C(D(M))$ (resp. $C(S(M))$) are bounded according to Lemma 4.3. It was shown in [32, Remark 17] that for both $D(M)$ and $S(M)$ this is the case and the minimal cycle of graphs containing $D(M)$ (resp. $S(M)$) is exactly M.

Theorem 4.6. *Let $f: G \to G$ be a graph map and $\{y_i\}_{i\leq 0}$ be a backward branch starting at a point $y \in G$. Then $\alpha(\{y_i\}_{i\leq 0})$ is contained in a maximal ω-limit set.*

Proof. If $\alpha(\{y_i\}_{i\leq 0})$ is a periodic orbit, then it is an ω-limit set and therefore $\alpha(\{y_i\}_{i\leq 0})$ is contained in a maximal ω-limit set (recall that every ω-limit set of a graph map is contained in a maximal one by Mai and Shao [26]). If $\alpha(\{y_i\}_{i\leq 0})$ is not a periodic orbit, then $\{y_i\}_{i\leq 0}$ has to accumulate at every point $x \in \alpha(\{y_i\}_{i\leq 0})$ from at least one side T_x. For every $x \in \alpha(\{y_i\}_{i\leq 0})$, the prolongation set $P^{T_x}(x)$ contains $\overline{\{y_i\}_{i\leq 0}}$ and since $P^{T_x}(x)$ is a closed invariant set, $P^{T_x}(x) \supseteq \alpha(\{y_i\}_{i\leq 0}) \cup \{y_i\}_{i\leq 0} \cup \overline{\mathrm{Orb}(y)}$. By Lemma 4.4, $P^{T_x}(x)$ is either a cycle of graphs or a solenoidal set $Q(x)$. In the latter case, $Q(x) \supset \alpha(\{y_i\}_{i\leq 0})$ and, by results from [34], $\omega(f) \supset \alpha(\{y_i\}_{i\leq 0})$, therefore $\alpha(\{y_i\}_{i\leq 0})$ is contained in the ω-limit set $Q_{max} = Q(x) \cap \omega(f)$. Recall that Q_{max} is a maximal ω-limit set by [7, Theorem 1]. If there is no $x \in \alpha(\{y_i\}_{i\leq 0})$ such that $P^{T_x}(x)$ is a solenoidal set, then $P^{T_x}(x)$ is a cycle of graphs for every $x \in \alpha(\{y_i\}_{i\leq 0})$. The set $\alpha(\{y_i\}_{i\leq 0})$ is infinite and thus we can define the family $C(\alpha(\{y_i\}_{i\leq 0}))$. The next step of the proof depends on whether the periods of cycles of graphs in $C(\alpha(\{y_i\}_{i\leq 0}))$ are bounded or unbounded.

We show that if the periods of cycles of graphs in $C(\alpha(\{y_i\}_{i\leq 0}))$ are unbounded then there is a sequence of cycles of graphs $\{X_i\}_{i=1}^{\infty}$ with strictly increasing periods generating a solenoidal set $Q \supset \alpha(\{y_i\}_{i\leq 0})$ and therefore $\alpha(\{y_i\}_{i\leq 0})$ is again contained in a maximal solenoidal set $Q_{max} = Q \cap \omega(f)$. By the assumption there exists a sequence $\{Y_n\}_{n=1}^{\infty}$ of cycles of graphs in $C(\alpha(\{y_i\}_{i\leq 0}))$ with strictly increasing periods $\{l_n\}_{n=1}^{\infty}$. We define inductively a sequence $\{Y'_n\}_{n=1}^{\infty}$ as follows. Let $Y'_1 = Y_1$. If Y'_n is already defined then, according to Lemma 4.2, there exists a l'_{n+1}-cycle of graphs Y'_{n+1} such that $\alpha(\{y_i\}_{i\leq 0}) \subseteq Y'_{n+1} \subseteq Y'_n \cap Y_{n+1}$ and $l'_{n+1} \geq l_{n+1}$. Finally choose a subsequence $\{n_i\}_{i=1}^{\infty}$ such that $l'_{n_i+1} > l'_{n_i}$ for all $i \geq 1$ and set $X_i := Y'_{n_i}$.

If the periods of cycles of graphs in $C(\alpha(\{y_i\}_{i\leq 0}))$ are bounded then by Lemma 4.3 there exists an element $X \in C(\alpha(\{y_i\}_{i\leq 0}))$ such that $X \subseteq Y$ for every $Y \in C(\alpha(\{y_i\}_{i\leq 0}))$. Fix $x \in \alpha(\{y_i\}_{i\leq 0})$. We assumed that $P^{T_x}(x) \in C(\alpha(\{y_i\}_{i\leq 0}))$ and thus $P^{T_x}(x) \supset X$. We will show that the prolongation set $P^{T_x}(x)$ coincides with $P_X^{T_x}(x)$ and in consequence $P_X^{T_x}(x) = X$. Since $X \supseteq \alpha(\{y_i\}_{i\leq 0})$ and $\alpha(\{y_i\}_{i\leq 0})$ is infinite, $\operatorname{Int}(X) \cap \alpha(\{y_i\}_{i\leq 0})$ is nonempty. It follows that $\{y_i\}_{i\leq 0} \cap X$ is infinite and thus $\{y_i\}_{i\leq 0} \subset X$. Therefore X contains the T_x-sided neigborhood of x and $P^{T_x}(x) = P_X^{T_x}(x) = X$. By Theorem 4.5 the set $E_S(X, f)$ is finite iff it is a periodic orbit. But we have just showed that $\alpha(\{y_i\}_{i\leq 0}) \subseteq E_S(X, f)$ and $\alpha(\{y_i\}_{i\leq 0})$ is not a periodic orbit by the assumption. Therefore $E_S(X, f)$ is an infinite set and, by Theorem 4.5, it is a maximal ω-limit set. □

For any of the above-mentioned infinite maximal ω-limit sets we can find a model with which the ω-limit set is almost conjugated and this almost conjugacy is unique up to the homeomorphism. For basic sets, the model is a mixing map of a cycle of graphs as described in Corollary 2.

Definition 4.7. Let $f\colon X \to X$ and $g\colon Y \to Y$ be two continuous maps of compact metric spaces X, Y and $K \subseteq X$ be a closed invariant set. A continuous surjection $\phi\colon X \to Y$ is an almost conjugacy between $f|_K$ and g if $\phi \circ f = g \circ \phi$ and

1. $\phi(K) = Y$,
2. $\forall y \in Y, \phi^{-1}(y)$ is connected,
3. $\forall y \in Y, \phi^{-1}(y) \cap K = \partial\phi^{-1}(y)$, where ∂A denotes the boundary of A.

If X, Y are graphs or cycles of graphs, the conditions (2) and (3) imply $\phi^{-1}(y) \cap K$ is a set of endpoints of a subgraph of X and hence a finite set, for every $y \in Y$.

Lemma 4.8. *Let $f\colon X \to X$ and $g\colon Y \to Y$ be two continuous maps of cycles of graphs X, Y and $K \subseteq X$ be a closed invariant set. If there is an almost conjugacy ϕ between $f|_K$ and g, then ϕ is unique up to the homeomorphism.*

Proof. Let x be an arbitrary point from X and ϕ_1 and ϕ_2 be almost conjugacies between $f|_K$ and g. Then $\phi_1^{-1}(\phi_1(x))$ (respectively, $\phi_2^{-1}(\phi_2(x))$) is a closed connected set containing x and, by Fact 2.1 (1), it is a subgraph of X. We will show that $\phi_1^{-1}(\phi_1(x)) \equiv \phi_2^{-1}(\phi_2(x))$. Assume the contrary. Then ϕ_1 is not a constant function on $\operatorname{Int}(\phi_2^{-1}(\phi_2(x)))$ or ϕ_2 is not a constant function on $\operatorname{Int}(\phi_1^{-1}(\phi_1(x)))$. We can assume the first case without loss of generality. Then there are $y, z \in \operatorname{Int}(\phi_2^{-1}(\phi_2(x)))$ such that $\phi_1(y) \neq \phi_1(z)$. Since $\phi_1^{-1}(\phi_1(y))$ does not contain z, we have $\partial\phi_1^{-1}(\phi_1(y)) \cap \operatorname{Int}(\phi_2^{-1}(\phi_2(x))) \neq \emptyset$. This is impossible since, by (3) from Definition 4.7, $\partial\phi_1^{-1}(\phi_1(y)) \subset K$ and $\operatorname{Int}(\phi_2^{-1}(\phi_2(x))) \cap K = \emptyset$.

By [30, Corollary 22.3], the quotient spaces $\phi_1(X)$ and $\phi_2(X)$ are homeomorphic. □

Theorem 4.9. [7, 32] *Let $f\colon G \to G$ be a graph map and $X \subseteq G$ be a cycle of graphs. Suppose that $E(X, f)$ is infinite. Then there is a transitive map $g\colon Y \to Y$, where Y is a cycle of graphs, and $\phi\colon X \to Y$ which almost conjugates $f|_{E(X,f)}$ and g.*

A transitive graph map is either totally transitive or it can be decomposed into a totally transitive one.

Theorem 4.2. [17] Let $f\colon G \to G$ be a transitive graph map. Then exactly one of the following statements holds.

1. f is totally transitive,
2. there is $k > 1$ and non-degenerate subgraphs $G_0, \dots, G_{k-1}$ of G such that
   (a) $G = \cup_{i=0}^{k-1} G_i$,
   (b) $G_i \cap G_j = End(G_i) \cap End(G_j)$, for $i \neq j$,
   (c) $f(G_i) = G_{i+1 \mod k}$, for $i = 0, \dots, k-1$,
   (d) $f^k|_{G_i}$ is totally transitive, for $i = 0, \dots, k-1$.

It follows from [18, Corollary 4.3, Theorem 3.2] that every totally transitive map acting on a graph G where G is not the circle is mixing. Let $\mathbb{S}$ be the unit circle. If f acting on $\mathbb{S}$ is totally transitive and sensitive then f is mixing by [18, Theorem 4.2, Theorem 3.2]. If it is totally transitive and not sensitive then f is a transitive almost equicontinuous map, by Auslander-Yorke Dichotomy [3], and therefore f is a homeomorphism by [1]. Since a basic set $D(X)$ contains a periodic orbit, the transitive model map g with which $D(X)$ is almost conjugated contains a periodic orbit as well. By [23, Proposition 11.1.4, Proposition 11.2.2], a homeomorphism acting on $\mathbb{S}$ possessing periodic points is never transitive. Therefore g is not a homeomorphism of the circle and, by the reasoning above, if g is totally transitive then g is mixing. This fact together with Theorem 4.9 and Theorem 4.2 implies the following corollary.

Corollary 2. *Let $f\colon G \to G$ be a graph map and $X \subseteq G$ be a cycle of graphs. Suppose that $D(X)$ is a basic set. Then there is a transitive map $g\colon Y \to Y$, where Y is a cycle of graphs $Y_0, \dots Y_{n-1}$ with possibly non-empty intersection in the endpoints, and $\phi\colon X \to Y$ which almost conjugates $f|_{D(X)}$ and g. Moreover, $g^n|Y_i$ is mixing, for $i = 0, \dots, n-1$. The period n of Y is a multiple of the period of X and $Y_i \cap Y_j = End(Y_i) \cap End(Y_j) \neq \emptyset$ iff $i \neq j$ and i and j are congruent modulo the period of X.*

The next lemma will help us to transfer some constructions from the model space Y to the basic set $D(X)$ later in Section 6.

Lemma 4.10. *Let $X, f, D(X), \phi, g, Y$ be as in Corollary 2 and let $\delta > 0$. Then there is a finite set $P(\delta) = \{p_1, \dots, p_k\} \subset Y$ and $\gamma > 0$ such that, for every neigborhood $U(x)$ of any point $x \in Y \setminus P(\delta)$ with $\operatorname{diam}(U(x)) \leq \gamma$ and $U(x) \cap P(\delta) = \emptyset$, we have $\operatorname{diam}(\phi^{-1}(U(x))) < \delta$.*

Proof. Since $\operatorname{diam}(X)$ is finite, we have at most countably many points $y \in Y$ such that $\phi^{-1}(y)$ is not a singleton and we can arrange them into a sequence $\{p_i\}_{i\geq 1}$. We also include as first positions in the sequence all branching points of the graph Y. Since we have also $\sum_{i=1}^{\infty} \operatorname{diam}((\phi^{-1}(p_i)) < \infty$, there is $k \geq 1$ such that $\sum_{i=k+1}^{\infty} \operatorname{diam}((\phi^{-1}(p_i)) < \delta/8$ and k is larger than the number of branching points, so denote $P(\delta) = \{p_1, \dots, p_k\}$. Let $Z = X \setminus \bigcup_{i=1}^{k} \operatorname{Int} \phi^{-1}(p_i)$. Note that Z is a finite union of graphs, in particular it is compact. Let $V_1, \dots, V_s$ be an open cover of Z by connected sets of diameter $\delta/8$. Let $U_i = \phi(V_i)$ for each $i = 1, \dots, s$ and let $\gamma = \min_{1\leq i\leq s} \operatorname{diam} U_i$. Fix any $x \in Y \setminus P(\delta)$ with $\operatorname{diam}(U(x)) \leq \gamma$ and $U(x) \cap P(\delta) = \emptyset$. Take any $p, q \in \phi^{-1}(U(x))$ and consider $\hat{p} = \phi(p), \hat{q} = \phi(q) \in U(x)$. Since $U(x)$ is connected and does not contain branching points, there is $\hat{z} \in U(x)$ and i, j such that $\hat{p}, \hat{z} \in U_i$ and $\hat{q}, \hat{z} \in U_j$. But then

$$\begin{aligned} d(p,q) &\leq \operatorname{diam} \phi^{-1}(\hat{p}) + \operatorname{diam} V_i + \operatorname{diam} \phi^{-1}(\hat{z}) + \operatorname{diam} V_j + \operatorname{diam} \phi^{-1}(\hat{q}) \\ &\leq \frac{5\delta}{8} < \delta. \end{aligned}$$

The proof is complete. □

5. **Zero entropy graph maps.** We will show that the structure of the family of α-limit sets of backward branches for a graph map greatly depends on the entropy of the map. In particular, for zero entropy maps the family of α-limit sets of backward branches coincides with the family of minimal sets. The following well-known theorem shows that graph maps with zero topological entropy do not possess basic sets.

Theorem 5.1. [21] *Let f be a continuous graph map. Then the following conditions are equivalent:*

1. *$h(f) > 0$,*
2. *f has a basic set.*

Theorem 5.2. *Let f be a continuous map acting on a graph G with $h(f) = 0$. Then a set L is an α-limit set of a backward branch $\{x_j\}_{j\leq 0}$ if and only if L is a minimal set.*

Proof. Since a minimal set L is closed, for any backward branch $\{x_j\}_{j\leq 0} \subseteq L$ we have $\alpha(\{x_j\}_{j\leq 0}) \subseteq L$. But $\alpha(\{x_j\}_{j\leq 0})$ is a closed invariant set. By minimality of L, $\alpha(\{x_j\}_{j\leq 0}) = L$. By Blokh's Decomposition Theorem [7, Theorem 4] and Theorem 5.1, the maximal ω-limit sets of the system (G, f) are solenoidal sets, circumferential sets and periodic orbits which are maximal ω-limit sets with respect to inclusion. If L is an α-limit set of a backward branch $\{x_j\}_{j\leq 0}$ then, by Theorem 4.6, L is contained in one of these maximal ω-limit sets. If $L = \alpha(\{x_j\}_{j\leq 0})$ is contained in a periodic orbit, then L coincides with this periodic orbit. Assume that L is contained in a solenoidal maximal ω-limit set Q_{max} and let $M_1 \supset M_2 \supset \dots$ be the generating sequence of cycles of graphs with periods tending to infinity such that $Q_{max} \subseteq \cap_n M_n$. Since L is infinite, $\{x_j\}_{j\leq 0} \cap M_n \neq \emptyset$, for every n, and, by the invariance of M_n, $\{x_j\}_{j\leq 0} \subset M_n$, for every n. There is a cycle M_k with period $m(k)$ greater than $\#Br(G)$. Denote by I the connected component of M_k such that $M_k \cap B(G) = \emptyset$. The set $\{x_j\}_{j\leq 0} \cap I$ is infinite and forms a backward branch with respect to $f^{m(k)}$. Therefore $I \cap L$ is an α-limit set of the backward branch for the zero entropy interval map $f^{m(k)}|_I$ and, by Theorem 12 from [4], $I \cap L$ is a perfect set. If z is an isolated point of L, then z has a pre-image $\hat{z}$ in $I \cap L$. Since f is continuous and $\hat{z}$ is not isolated in $I \cap L$, there is a neighbourhood U of $\hat{z}$ in $I \cap L$ such that U is eventually mapped on z. This implies $f^{m(k)}(U)$ is a singleton and as a consequence U contains a periodic point. But it is impossible, since there are no periodic points in a solenoidal set Q_{max} and it follows that L is a perfect set. By [7, Theorem 1], Q_{max} has at most countable set of isolated points and the set of all limit points of Q_{max} is contained in the minimal set $Q_{min} = Q \cap \overline{Perf}$. Therefore $L \subseteq Q_{min}$ and, by minimality of Q_{min}, $L = Q_{min}$. Let L be contained in a circumferential set $S(X,f)$. The following result can be found in [7, Theorem 3] or [32] and we briefly recall it here. Let $X_1,\dots,X_n$ be the connected components of X. Then, either, for every i, $f^k|_{X_i}$ is conjugate to an irrational rotation (and in this case $S(X_i,f^k)=X_i$), or, for every i, there exists a semi-conjugacy ϕ_i between $f^k|_{X_i}$ and an irrational rotation which is an almost conjugacy on $f^k|_{S(X_i,f^k)}$. The latter case is called the Denjoy type of ω-limit set and it is described in detail in [27]. In both cases, $S(X,f)$ is the unique minimal set of the system (X,f) and all points in $X\setminus S(X,f)$ are wandering [27, Corollary 4.4]. By minimality of $S(X,f)$, $L\subseteq S(X,f)$ implies $L=S(X,f)$.

□

Remark 5. A minimal set for a graph map f with $h(f)=0$ is either a periodic orbit, a minimal solenoidal set or a circumferential set. Theorem 5.2 shows that these (and only these) sets can be realized as α-limit sets of backward branches for f.

Denote the family of all α-limit sets of backward branches starting at a point $x\in G$ by $\mathcal{A}(x)$,

$$\mathcal{A}(x)=\{L\in\mathcal{P}(G):\exists\{x_j\}_{j\le 0}\text{ such that }x_0=x\text{ and }L=\alpha(\{x_j\}_{j\le 0})\}.$$

Corollary 3. *Let f be a continuous map acting on a graph G with $h(f)=0$ and $x\in G$. Then $\mathcal{A}(x)$ contains at most one infinite set.*

Proof. Let $\{x_j\}_{j\le 0}$, $x_0=x$, be a backward branch with $L:=\alpha(\{x_j\}_{j\le 0})$ infinite. By Remark 5, L is either a minimal solenoidal set or a circumferential set. Assume the first case. Then there is a generating sequence of cycles of graphs $M_1\supset M_2\supset\dots$ with periods tending to infinity such that $L\subseteq Q$, where $Q=\cap_n M_n$. Since L is infinite, the backward branch $\{x_j\}_{j\le 0}$ intersects the cycle of graphs M_n, for every n. By the invariance of M_n, $x\in M_n$, for every n, and x belongs to the solenoidal set Q. It is a well known fact that two solenoidal sets $Q=\cap_n M_n$ and $Q'=\cap_n M_n'$ generated by different sequences $M_1\supset M_2\supset\dots$ and $M_1'\supset M_2'\supset\dots$ are either identical or disjoint. Since $x\notin Q'$, for any $Q'\neq Q$, and since there is only one minimal set in Q, L is the unique minimal solenoidal set in $\mathcal{A}(x)$.

It is easy to see that, for every circumferential set $S(X)$, where X is the minimal cycle of graphs containing $S(X)$, we have $M\supseteq X$ or $M\cap X=\emptyset$, for every cycle of graphs M. Clearly, $M\cap X\subsetneq X$ is impossible since $S(X)$ is the unique minimal set of the system (X,f) and, by Remark 4, X is the minimal cycle of graphs containing $S(X)$. Therefore $\cap_n M_n\cap X=\emptyset$ and $x\notin X$. Since every backward branch $\{x_j'\}_{j\le 0}$, $x_0'=x$, has empty intersection with X, $S(X)$ does not belong to $\mathcal{A}(x)$ and L is the unique infinite set in $\mathcal{A}(x)$. Assume that L is a circumferential set $S(X)$. Then $x\in X$. For any circumferential set $S(X')$, we have either $X\subseteq X'\wedge X'\subseteq X\implies X=X'$ and $S(X)=S(X')$, or the intersection $X\cap X'$ is empty and $x\notin X'$. Since every backward branch $\{x_j'\}_{j\le 0}$, $x_0'=x$, has empty intersection with X', for every $X'\neq X$, $S(X)$ is the unique circumferential set in $\mathcal{A}(x)$. By the reasoning above, there is no minimal solenoidal set in $\mathcal{A}(x)$ and L is the unique infinite set in $\mathcal{A}(x)$. □

In addition to one infinite α-limit set, the family $A(x)$ can contain many finite α-limit sets. Every finite α-limit set is a periodic orbit by Theorem 5.2. In the following example, we will construct $A(x)$ containing a circumferential set and uncountably many periodic orbits.

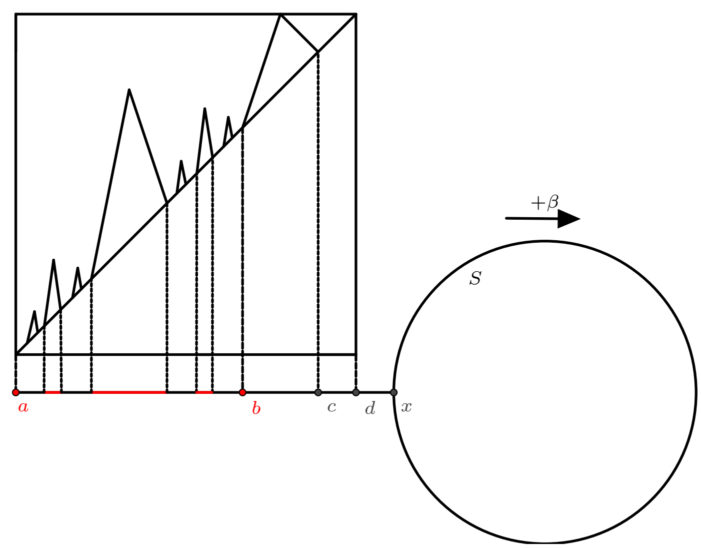

FIGURE 1. A map where the family $\mathcal{A}(x)$ is uncountable.

Example 5.1. Let G be the union of the circle S and the interval $[a, x]$ as shown on the Figure 1. The map $f : G \to G$ is defined such that $f|_S$ is the rotation by an irrational angle β and the graph of $f : [a, c] \to [a, d]$ is sketched on the Figure 1. The construction of the interval map $f|_{[a,b]}$ was previously used in [25, Example 4.8]. The remaining interval $[c, x]$ is mapped by f into $[c, x] \cup S$ continuously in such a way that $f(c) = c$ and $f(d) = x$. Then $\mathcal{A}(x)$ consists of a middle-third Cantor set of fixed points in the interval $[a, b]$ (drawn by red color on the Figure 1) and the circumferential set S.

6. **Positive entropy graph maps.** In this section, we will investigate α-limit sets of backward branches which are included in a basic set. By Theorem 5.1, every continuous graph map f with $h(f) > 0$ possess a basic set $D(X)$. The main goal is to use the model map $g : Y \to Y$ for the basic set $D(X)$ given by Corollary 2 to obtain a similar result as for mixing graph maps in Theorem 3.5 and 3.6. Recall that the model map g is almost conjugate to $f|_{D(X)}$ and $g^n|_{Y_i}$ is mixing on every component Y_i of Y. We introduce an equivalence relation on X as follows:

$$x \sim y \Leftrightarrow \phi(x) = \phi(y),$$

where $x, y \in X$ and ϕ is the almost conjugacy between $f|_{D(X)}$ and g from Corollary 2. The relation $\sim$ is well defined since ϕ is unique up to the homeomorphism by Lemma 4.8. Denote $[x]_\sim$ the equivalence class of a point $x \in X$ and $[A]_\sim = \bigcup_{x \in A} [x]_\sim$, for any $A \subset X$. Obviously $[x]_\sim = \phi^{-1}(\phi(x))$ and $[A]_\sim = \phi^{-1}(\phi(A))$. By the definition of almost conjugacy, $[x]_\sim$ is a subgraph of X such that $End([x]_\sim) = [x]_\sim \cap D(X)$ and $f([x]_\sim) \subseteq [f(x)]_\sim$. The last inclusion ensures that for every backward branch $\{\tilde{x}_j\}_{j \leq 0} \subset Y$ constructed with respect to the model map g and starting at $\phi(x)$ there is a backward branch $\{z_j\}_{j \leq 0} \subset X$ with respect to f such that $\phi(z_j) = \tilde{x}_j$, for every $j \geq 0$. Unfortunatelly, the oposite inclusion $f([x]_\sim) \supseteq [f(x)]_\sim$ may not hold in general for every $x \in X$. This makes our aim to use the model map g difficult since the backward branch $\{z_j\}_{j\leq 0}$ may not start at x but at some other point of $[x]_\sim$. Therefore in Theorem 6.2 we restrict ourselves to the case when $f([x]_\sim) = [f(x)]_\sim$ for every $x \in X$ or, equivalently, $f(\phi^{-1}(y)) = \phi^{-1}(g(y))$ for every $y \in Y$. Let $\mathcal{I}(g^n|_{Y_i})$ be the set of inacessible points of the mixing graph map $g^n : Y_i \to Y_i$ given by Corollary 2, for every $i = 0, 1, \dots, n-1$. Then we define the set of inaccesible points of X as the union of preimages of inacessible points of the model mixing map,

$$\mathcal{I}(X) = \bigcup_{0\leq i\leq n-1} \phi^{-1}(\mathcal{I}(g^n|_{Y_i})).$$

Lemma 6.1. *Let $V \subset X$ be a subgraph such that $\phi(V)$ is a non-degenerate subgraph of Y. Then $\bigcup_{k=0}^{\infty} f^k(V) \supseteq X \setminus \mathcal{I}(X)$. Consequently, for every point $x \in X \setminus \mathcal{I}(X)$ there is a preimage $z \in f^{-k}(x) \cap V$, for some $k > 0$.*

Proof. Notice that if $\phi(x) \in \mathrm{Int}(\phi(A))$, for some $x \in X$ and $A \subset X$, then $x \in A$. Since $\phi(V)$ is a non-degenerate subgraph of Y, there is a component Y_i of Y such that $\phi(V) \cap Y_i$ is non-degenerate. Since $g^n|_{Y_i}$ is mixing, we have by Equation 2.1,

$$\bigcup_{k=0}^{\infty} \mathrm{Int}(\phi(f^{n\cdot k}(V))) = \bigcup_{k=0}^{\infty} \mathrm{Int}(g^{n\cdot k}(\phi(V))) = Y_i \setminus \mathcal{I}(g^n|_{Y_i}).$$

The image of $\phi(V)$ by g is a non-degenerate subgraph of the component Y_{i+1} (otherwise $g^{n\cdot k}(\phi(V))$ is a singleton, for every $k > 1$, which is in a contradiction with the equation above) and the same holds for every $g^j(\phi(V))$, $j = 0, 1, \dots, n-1$. Again by Equation 2.1,

$$\bigcup_{k=0}^{\infty} \mathrm{Int}(\phi(f^k(V))) = \bigcup_{j=0}^{n-1}\bigcup_{k=0}^{\infty} \mathrm{Int}(g^{n\cdot k+j}(\phi(V))) = \bigcup_{j=0}^{n-1} Y_j \setminus \mathcal{I}(g^n|_{Y_j}) = \phi(X \setminus \mathcal{I}(X)).$$

Therefore $x \in X \setminus \mathcal{I}(X)$ implies $\phi(x) \in \mathrm{Int}(\phi(f^k(V)))$, for some $k > 0$, and we have $x \in f^k(V)$.

□

Theorem 6.2. *Let $D(X)$ be a basic set such that $f([x]_\sim) = [f(x)]_\sim$, for every $x \in X$. Then, for every $x \in X \setminus \mathcal{I}(X)$ and every ω-limit set $\omega_f(y)$ such that $\omega_f(y) \subset D(X)$ is infinite, there exists a backward branch $\{z_j\}_{j\leq 0}$ such that $z_0 = x$ and $\omega_f(y) \subseteq \alpha(\{z_j\}_{j\leq 0}) \subseteq [\omega_f(y)]_\sim \cap D(X)$.*

Proof. Assume first that $f|_{D(X)}$ is almost conjugate to a mixing graph map $g\colon Y \to Y$, i.e. the cycle of graphs Y has only one component and take $y \in D(X)$ such that $\omega_f(y) \subset D(X)$ is infinite. If $y \notin D(X)$ then we can replace it by a point from $[y]_\sim \cap D(X)$ since the diameter of sets $[f^i(y)]_\sim$ tends to 0 as $i \to \infty$ and $\omega_f(x) = \omega_f(y)$, for every $x \in [y]_\sim$. Note that $\phi(\omega_f(y)) = \omega_g(\phi(y))$. Image by ϕ of any limit point of $\mathrm{Orb}_f(y)$ is a limit point of $\mathrm{Orb}_g(\phi(y))$ and conversely, by compactness, any limit point of $\mathrm{Orb}_g(\phi(y))$ can be obtained as an image of a limit point in $\mathrm{Orb}_f(y)$. Below we use the notation from the proof of Lemma 3.4. We introduce some modification implied by Remark 2 to the construction in order to recover the desired backward branch $\{z_j\}_{j\leq 0}$. The construction from the proof of Lemma 3.4 applied for ω-limit set $\omega_g(\phi(y)) \subset Y$ and any x in an open set $U \subset Y$ gives us the backward brach $\{\tilde{x}_j\}_{j\leq 0}$ such that $\tilde{x}_0 = x$ and $\alpha_g(\{\tilde{x}_j\}_{j\leq 0}) = \omega_g(\phi(y))$. The modification in the proof of Lemma 3.4 is as follows. Fix $i \in \mathbb{N}$. Let $P(\epsilon_i)$ be the finite set from Lemma 4.10. Since $\mathrm{Orb}(\phi(y))$ is infinite ($\mathrm{Orb}(y)$

is infinite subset of $D(X)$ and $\phi|_{D(X)}$ is finite-to-one), we can find $N > 0$ such that $\phi(f^n(y)) \cap P(\epsilon_i) = \emptyset$, for $n > N$. In the proof of Lemma 3.4 we constructed a sequence $\{x_i\}_{i\in\mathbb{N}}$ and $\{\hat{x}_i\}_{i\in\mathbb{N}}$ such that $x_i, \hat{x}_i \in \mathrm{Orb}(\phi(y))$ for every $i \in \mathbb{N}$ and associated sequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{\hat{n}_i\}_{i\in\mathbb{N}}$ of times for which the orbits of points in the Bowen ball follows ϵ_i−close the orbit of x_i and $\hat{x}_i$ respectively. We may require that $x_i, \hat{x}_i \in \mathrm{Orb}(\phi(f^{N+1}(y)))$. Since none of the points $x_i, \hat{x}_i$ or their forward iterates belongs to $P(\epsilon_i)$, assuming that $n_i, \hat{n}_i$ are sufficiently large, we may require that set (which can be arbitrarily small) $B'_{n_i}(x_i, \epsilon_i)$ (resp. $B'_{\hat{n}_i}(\hat{x}_i, \epsilon_i)$) does not contain any points from $P(\epsilon_i)$. Furthermore, if we fix any s_i, then by continuity, if $n_i, \hat{n}_i$ are sufficiently large (in practice, much larger than s_i), also $g^j(B'_{n_i}(x_i, \epsilon_i))$, (resp. $g^j(B'_{\hat{n}_i}(\hat{x}_i, \epsilon_i))$) does not contain any points from $P(\epsilon_i)$ for $j = 0, \dots, s_i$. By Lemma 4.10, if we fix any q_i such that $\phi(q_i) = x_i$ and any $\tilde{q}_i$ such that $\phi(\tilde{q}_i) \in B'_{n_i}(x_i, \epsilon_i)$ then $d(f^j(q_i), f^j(\tilde{q}_i)) < \epsilon_i$ for $j = 0, \dots, s_i$. The same holds if $\phi(q_i) = \hat{x}_i$ and any $\tilde{q}_i$ such that $\phi(\tilde{q}_i) \in B'_{\hat{n}_i}(\hat{x}_i, \epsilon_i)$. In particular, we can take $q_i \in \mathrm{Orb}(f^{N+1}(y))$ obtaining that $\cup_{j\leq s_i} B(f^j(\tilde{q}_i), 2\epsilon_i) \supset \omega_f(y)$ provided that s_i was sufficiently large. This modification ensures that $\alpha(\{\tilde{q}_j\}_{j\leq 0}) \supset \omega_f(y)$ whenever $\{\tilde{q}_j\}_{j\leq 0}$ is a backward branch such that $\phi(\tilde{q}_j) = \tilde{x}_j$, for every $j \leq 0$. On the other hand, $\phi(\omega_f(y)) = \omega_g(\phi(y)) = \phi(\alpha(\{\tilde{q}_j\}_{j\leq 0}))$ which gives $\alpha(\{\tilde{q}_j\}_{j\leq 0}) \subset [\omega(y)]_\sim$. Since $\alpha(\{\tilde{q}_j\}_{j\leq 0})$ is a subset of a maximal ω-limit set by Theorem 4.6 and it contains points from $\omega_f(y) \subset D(X)$, we have also $\alpha(\{\tilde{q}_j\}_{j\leq 0}) \subset D(X)$.

It remains to show that $\{\tilde{q}_j\}_{j\leq 0}$ with $\tilde{q}_0 = x$ exists for every $x \in X \setminus \mathcal{I}(X)$. But the sets $\phi^{-1}(\tilde{x}_j)$ form an inverse sequence by assumption, that is $f(\phi^{-1}(\tilde{x}_{j-1})) = \phi^{-1}(\tilde{x}_j)$, for every $j \leq 0$, therefore we can construct $\{\tilde{q}_j\}_{j\leq 0}$ for every $\tilde{q}_0 \in \phi^{-1}(x)$, where x is an arbitrary point from an open connected set $U \subset Y$, hence for every $\tilde{q}_0 \in \phi^{-1}(U)$. Denote $V = \phi^{-1}(U)$. The result follows by Lemma 6.1.

If Y has n components then g^n is mixing on each of the periodic components Y_i, $i = 0, \dots, n-1$, and we can decompose $\omega_g(\phi(y))$ into infinite sets $\omega_g(\phi(y)) = \bigcup_{i=0}^{n-1} g^i(\omega_{g^n}(\phi(y)))$. Without loss of generality assume $\omega_{g^n}(\phi(y)) \subset Y_0$. The same construction as above gives us the backward brach $\{\tilde{x}_j\}_{j\leq 0}$ such that $\tilde{x}_0 = x$ and $\alpha_{g^n}(\{\tilde{x}_j\}_{j\leq 0}) = \omega_{g^n}(\phi(y))$, for every x in an open set $U \subset Y_0$. Set:

$$z_i = \begin{cases} \tilde{x}_j & \text{if } i = j \cdot n, \text{ for every } j \in \mathbb{N}_0, \\ g^k(\tilde{x}_j) & \text{if } i = j \cdot n + k, \text{ for every } 0 < k < n, j \in \mathbb{N}_0. \end{cases}$$

By continuity of g, $\alpha_g(\{z_i\}_{i\leq 0}) = \omega_g(\phi(y))$. We finish the proof in the same manner as above. □

Theorem 6.2 can be stated in various forms. We describe them in the following series of facts and remarks.

Fact 6.1. The set $\alpha_f(\{z_j\}_{j\leq 0}) \setminus \omega_f(y)$ is at most countable and consists from isolated points of $\alpha_f(\{z_j\}_{j\leq 0})$. Consequently, if every isolated point x of $\alpha_f(\{z_j\}_{j\leq 0})$ has $[x]_\sim = \{x\}$ then $\omega_f(y) = \alpha_f(\{z_j\}_{j\leq 0}$.

Proof. Since $[x]_\sim = \{x\}$, for all but countably many $x \in D(X)$, and $[x]_\sim \cap D(X) = End([x]_\sim)$ is a finite set, for every $x \in X$, we have $[\omega_f(y)]_\sim \cap D(X) \setminus \omega_f(y)$ countable. Moreover, points from $[\omega_f(y)]_\sim \cap D(X) \setminus \omega_f(y)$ are isolated in $[\omega_f(y)]_\sim \cap D(X)$. If $\{x_i\} \to \{x\}$ is a converging sequence such that $\{x_i\}_{i>0} \subset [\omega_f(y)]_\sim \cap D(X)$, then $x_i \in [y_i]_\sim$ where $y_i \in \omega_f(y)$, for every $i > 0$. If the sequence is not eventually constant, we can assume $[y_i]_\sim$ are pairwise disjoint subgraphs of X with diameter tending to 0 as $i \to \infty$ (we can pass to a subsequence if necessary since there is at most finitely many indeces k such that $[y_k]_\sim = [y_i]_\sim$, for any $i > 0$). Since $\omega_f(y)$ is closed, we have $y_i \to x$ and $x \in \omega_f(y)$. Therefore points from $([\omega_f(y)]_\sim \cap D(X)) \setminus \omega_f(y)$ are never accumulation points of $[\omega_f(y)]_\sim \cap D(X)$. The same holds for $\alpha_f(\{z_j\}_{j\leq 0}) \setminus \omega_f(y)$ since $\alpha_f(\{z_j\}_{j\leq 0}) \setminus \omega_f(y) \subseteq ([\omega_f(y)]_\sim \cap D(X)) \setminus \omega_f(y)$. □

Remark 6. If we omit the condition $f([x]_\sim) = [f(x)]_\sim$, for every $x \in X$, in the assumption of Theorem 6.2, then we obtain the following weaker result:
Let $D(X)$ be a basic set. Then, for every $x \in X \setminus \mathcal{I}(X)$ and every ω-limit set $\omega_f(y)$ such that $\omega_f(y) \subset D(X)$ is infinite, there exists a backward branch $\{z_j\}_{j\leq 0}$ such that $z_0 \in [x]_\sim$ and $\omega_f(y) \subseteq \alpha_f(\{z_j\}_{j\leq 0}) \subseteq [\omega_f(y)]_\sim \cap D(X)$.

Remark 7. The inaccesible points from $\mathcal{I}(X)$ have only finite α-limit sets of backward branches being a subset of $D(X)$. Nevertheless, they may have many infinite α-limit sets of backward branches being a subset of other basic set $B(Z)$ such that $D(X) \cap B(Z) \neq \emptyset$.

Combining together Fact 6.1, Remark 6 and the fact that $[x]_\sim = \{x\}$ for all but countably many points x from $D(X)$ we get the following corollary.

Corollary 4. *Let $D(X)$ be a basic set. Then for all but countably many points $x \in D(X)$ and every ω-limit set $\omega_f(y)$ such that $\omega_f(y) \subset D(X)$ is infinite, there exists a backward branch $\{z_j\}_{j\leq 0}$ such that $z_0 = x$ and $\alpha_f(\{z_j\}_{j\leq 0}) = \omega_f(y) \cup R$ where R is at most countable subset of isolated points of $\alpha_f(\{z_j\}_{j\leq 0})$. Moreover, if every isolated point x of $\alpha_f(\{z_j\}_{j\leq 0})$ has $[x]_\sim = \{x\}$ then R is empty.*

We leave the next question open for futher resaerch.

Question 6.2. Let $D(X)$ be a basic set. Is it true that for every $x \in D(X) \setminus \mathcal{I}(X)$ and every ω-limit set $\omega_f(y)$ such that $\omega_f(y) \subset D(X)$ and $\mathrm{Orb}(y)$ is infinite, there exists a backward branch $\{z_j\}_{j\leq 0}$ such that $z_0 = x$ and $\alpha_f(\{z_j\}_{j\leq 0}) = \omega_f(y)$?

The following example shows that Theorem 6.2 can not be applied when $\omega_f(y)$ is finite. In particular, we will show that there is a basic set $D(X)$ and a fixed point $p \in D(X)$ such that there is no backward branch $\{z_j\}_{j\leq 0}$ with $z_0 \in D(X)$ and $\{p\} \subset \alpha_f(\{z_j\}_{j\leq 0}) \subset [p]_\sim$ (with the exception of the constant backward branch $z_j = p$, for every $j \leq 0$).

Example 6.3. Let g be a mixing map of the unit interval I and p, q, r be points from Figure 2. Let $D(I, f)$ be a basic set such that there is an almost conjugacy ϕ between $f|_{D(I,f)}$ and g with the following properties: $\phi^{-1}(x)$ is not a singleton, for every $x \in \cup_{i\geq 0} f^{-i}(p)$, $\phi^{-1}(p)$ is an f-invariant interval $[p_1, p_2]$, where p_1, p_2 are fixed points with respect to f, $\phi^{-1}(q)$ is an interval $[q_1, q_2]$ such that $f(q_1) = p_1$ and $f(q_2) = p_2$ and $\phi^{-1}(r)$ is an interval $[r_1, r_2]$ such that $f(r_1) = p_2$ and $f(r_2) = p_1$. The fixed point p_1 can be reached only by a backward branch from the invariant interval $[p_1, p_2]$, since every backward branch $\{z_j\}_{j\leq 0}$ converging to p_1 from the left side or from both sides has $\alpha_f(\{z_j\}_{j\leq 0})$ containing q_1 or r_2.

Theorem 6.3. *Let $D(X)$ be a basic set. Then, for every backward branch $\{x_j\}_{j\leq 0} \subset X$ such that $\alpha_f(\{x_j\}_{j\leq 0}) \subset D(X)$ there is a point $y \in X$ such that $\omega_f(y) \subseteq [\alpha_f(\{x_j\}_{j\leq 0})]_\sim \cap D(X)$. Moreover, if the set $\{x \in \alpha_f(\{x_j\}_{j\leq 0}) : [x]_\sim = \{x\}\}$ is dense in $\alpha_f(\{x_j\}_{j\leq 0})$ then $\alpha_f(\{x_j\}_{j\leq 0}) \subseteq \omega_f(y) \subseteq [\alpha_f(\{x_j\}_{j\leq 0})]_\sim \cap D(X)$.*

Proof. If $\alpha_f(\{x_j\}_{j\leq 0})$ is finite then by Lemma 2.1 it is a periodic orbit of some point $y \in X$ and $\omega_f(y) = \alpha_f(\{x_j\})_{j\leq 0}$. Assume that $f|_{D(X)}$ is almost conjugate

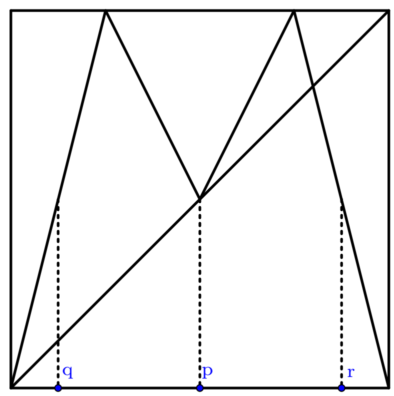

FIGURE 2. A map mixing interval map g with a fixed point p such that every backward branch $\{z_j\}_{j\leq 0}$ with $\{p\} = \alpha_g(\{z_j\}_{j\leq 0})$ converges to p only from the right side. Every backward branch $\{z_j\}_{j\leq 0}$ converging to p from the left side or from both sides has $\alpha_g(\{z_j\}_{j\leq 0}) \cap \{q, r\} \neq \emptyset$, where q, r are preimages of p.

to a mixing graph $g\colon Y \to Y$, i.e. the cycle of graphs Y has only one component. Let $\{x_j\}_{j\leq 0} \subset X$ be such that $\alpha_f(\{x_j\}_{j\leq 0}) \subset D(X)$ is infinite. Note that $\phi(\alpha_f(\{x_j\}_{j\leq 0}) = \alpha_g(\{\phi(x_j)\}_{j\leq 0})$.

First assume that the set $S := \{x \in \alpha_f(\{x_j\}_{j\leq 0}) : [x]_\sim = \{x\}\}$ is dense in $\alpha_f(\{x_j\}_{j\leq 0})$. Let $\delta > 0$ and $P(\delta), \gamma$ be from Lemma 4.10. We can find a finite set $M \subset S$ such that $\bigcup_{x\in M} B(x, \delta) \supset \alpha_f(\{x_j\}_{j\leq 0})$ and obviously $\phi(M) \cap P(\delta) = \emptyset$. Applying the construction from the proof of Theorem 3.6 to $\alpha_g(\{\phi(x_j)\}_{j\leq 0}) \subset Y$ we obtain a sequence of periodic orbits $\{\mathrm{Orb}(p_k)\}_{k>0} \subset Y$ with increasing periods $\{d_k\}_{k>0}$ such that $d_H(\mathrm{Orb}(p_k), \alpha_g(\{\phi(x_j)\}_{j\leq 0}) < 2\epsilon_k$, for every $k > 0$. Since ϵ_k goes to 0 as $k \to \infty$, we can find $k > 0$ such that $2\epsilon_k < \min\{\gamma, \mathrm{dist}(\phi(M), P(\delta))\}$. Then for every $x \in M$ there is $i \in \{0 \ldots, d_k - 1\}$ with $g^i(p_k) \in B(\phi(x), 2\epsilon_k)$. By Lemma 4.10, $\mathrm{diam}\, \phi^{-1}(B(\phi(x), 2\epsilon_k)) < \delta$, for every $x \in M$. Therefore $\phi^{-1}(B(\phi(x), 2\epsilon_k)) \subset B(x, \delta)$, for every $x \in M$. Since p_k has period d_k, we can find an f-periodic point $q_k \in \phi^{-1}(p_k)$ with period at least d_k and obviously $f^i(q_k) \in \phi^{-1}(g^i(p_k))$, for every $i \in \{0 \ldots, d_k - 1\}$. It follows that for every $x \in M$ there is $i \in \{0 \ldots, d_k - 1\}$ with $f^i(q_k) \in B(x, \delta)$. Since we have assumed that $\bigcup_{x\in M} B(x, \delta)$ covers $\alpha_f(\{x_j\}_{j\leq 0})$, we conclude $\cup_{z\in \mathrm{Orb}(q_k)} B(z, 2\delta) \supset \alpha_f(\{x_j\}_{j\leq 0})$ and $\alpha_f(\{x_j\}_{j\leq 0}) \subset \omega_f(y)$ where $\omega_f(y)$ is the Hausdorff limit of the sequence $\{\mathrm{Orb}(q_k)\}_{k>0}$.

On the other hand, we can use the sequence of periodic orbits $\{\mathrm{Orb}(p_k)\}_{k>0} \subset Y$ from the proof of Theorem 3.6 to construct $\omega_f(y)$ as the Hausdorff limit of the sequence $\{\mathrm{Orb}(q_k)\}_{k>0}$ with $\mathrm{Orb}(q_k) \subset \phi^{-1}(\mathrm{Orb}(p_k))$, for every $k > 0$, regardless of the existence of the set S. Then $\phi(\omega_f(y)) = \alpha_g(\{\phi(x_j)\}_{j\leq 0}) = \phi(\alpha_f(\{x_j\}_{j\leq 0})$ and at the same time $\omega_f(y) \subseteq D(X)$ which gives $\omega_f(y) \subseteq [\alpha_f(\{x_j\}_{j\leq 0})]_\sim \cap D(X)$.

If Y has n components $Y_0, \ldots, Y_{n-1}$ then g^n is mixing on each of the periodic components Y_i and $X = \cup_{i=0}^{n-1} \phi^{-1}(Y_i)$ where sets $\phi^{-1}(Y_i)$ are connected, pairwise disjoint with possibly non-empty intersection in the endpoints and they form a cycle of period n. We can decompose $\alpha_f(\{x_j\}_{j\leq 0})$ into infinite sets $\alpha_f(\{x_j\}_{j\leq 0}) = \bigcup_{i=0}^{n-1} f^i(\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})$. Without loss of generality assume $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0}) \subset \phi^{-1}(Y_0)$. First we show that if the set $S = \{x \in \alpha_f(\{x_j\}_{j\leq 0}) : [x]_\sim = \{x\}\}$ is dense in

$\alpha_f(\{x_j\}_{j\leq 0})$ then $S\cap\phi^{-1}(Y_0)$ is dense in $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})$. Since $S\cap \mathrm{Int}(\phi^{-1}(Y_0))$ is dense in the open (with respect to the subspace topology) set $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})\cap \mathrm{Int}(\phi^{-1}(Y_0))$ it suffices to show that $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})\cap End(\phi^{-1}(Y_0))$ is not isolated from $S\cap\phi^{-1}(Y_0)$. If $z\in\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})\cap End(\phi^{-1}(Y_0))$ has a pre-image in $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})\cap \mathrm{Int}(\phi^{-1}(Y_0))$ then z is either a limit point of S or belongs to S by continuity of f^n and invariance of S. If z has only pre-images in $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})\cap End(\phi^{-1}(Y_0))$ then z is a periodic point and consequently z being isolated from $S\cap\phi^{-1}(Y_0)$ implies that z is isolated from S. But this is impossible since $z\in\alpha_f(\{x_j\}_{j\leq 0})$.

Now we can apply the above procedure to the α-limit set $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})$ and obtain $\omega_{f^n}(y)$ with $\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})\subseteq\omega_{f^n}(y)\subseteq[\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})]_\sim\cap D(X)$ (in case S is not dense in $\alpha_f(\{x_j\}_{j\leq 0})$ we consider only the second inclusion). Obviously $f^i(\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0}))\subseteq f^i(\omega_{f^n}(y))\subseteq f^i([\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})]_\sim)\cap D(X)$, for $i=0,\dots,n-1$, and therefore:

$$\begin{aligned}\alpha_f(\{x_j\}_{j\leq 0}) = \bigcup_{i=0}^{n-1} f^i(\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0}))\subseteq \omega_f(y)=\bigcup_{i=0}^{n-1} f^i(\omega_{f^n}(y))\\ \subseteq [\alpha_f(\{x_j\}_{j\leq 0})]_\sim\cap D(X)=\bigcup_{i=0}^{n-1} f^i([\alpha_{f^n}(\{x_{n\cdot j}\}_{j\geq 0})]_\sim)\cap D(X).\end{aligned}\tag{6.8}$$

□

Repeating the arguments from Fact 6.1 we can show that $\omega_f(y)\setminus\alpha_f(\{x_j\}_{j\leq 0})$ consists of isolated points of $\omega_f(y)$ and the following corollary holds.

Corollary 5. *Let $D(X)$ be a basic set and $\{x_j\}_{j\leq 0}\subset X$ be a backward branch such that $\alpha(\{x_j\}_{j\leq 0})\subset D(X)$. If the set $\{x\in\alpha_f(\{x_j\}_{j\leq 0}) : [x]_\sim=\{x\}\}$ is dense in $\alpha_f(\{x_j\}_{j\leq 0})$ then there is a point $y\in X$ such that $\omega_f(y)=\alpha_f(\{x_j\}_{j\leq 0})\cup R$ where R is at most countable subset of isolated points of $\omega_f(y)$. Moreover, if every isolated point x of $\omega_f(y)$ has $[x]_\sim=\{x\}$ then R is empty.*

In the previous Section 5 we have proved that if $h(f)=0$ then every $\alpha_f(\{x_j\}_{j\leq 0})$ is a minimal set, hence it is an ω-limit set of any point from $\alpha_f(\{x_j\}_{j\leq 0})$. Clearly, the same holds for f with positive entropy and $\alpha_f(\{x_j\}_{j\leq 0})$ being a subset of one of the three maximal ω-limit sets which are in common to both zero entropy graph maps and positive entropy graph maps - solenoidal sets, circumferential sets and periodic orbits. In the light of Theorem 4.6 we can conclude that, for any graph map f, every α-limit set of a backward branch is an ω-limit set, providing the answer to the following question turns out positive.

Question 6.4. Let $D(X)$ be a basic set and $\alpha_f(\{x_j\}_{j\leq 0})\subset D(X)$, for a backward branch $\{x_j\}_{j\leq 0}$. Is $\alpha_f(\{x_j\}_{j\leq 0})=\omega_f(y)$, for some $y\in X$?

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On backward attractors of interval maps*

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Abstract

Special α-limit sets ($s\alpha$-limit sets) combine together all accumulation points of all backward orbit branches of a point x under a noninvertible map. The most important question about them is whether or not they are closed. We challenge the notion of $s\alpha$-limit sets as backward attractors for interval maps by showing that they need not be closed. This disproves a conjecture by Kolyada, Misiurewicz, and Snoha. We give a criterion in terms of Xiong's attracting centre that completely characterizes which interval maps have all $s\alpha$-limit sets closed, and we show that our criterion is satisfied in the piecewise monotone case. We apply Blokh's models of solenoidal and basic ω-limit sets to solve four additional conjectures by Kolyada, Misiurewicz, and Snoha relating topological properties of $s\alpha$-limit sets to the dynamics within them. For example, we show that the isolated points in a $s\alpha$-limit set of an interval map are always periodic, the non-degenerate components are the union of one or two transitive cycles of intervals, and the rest of the $s\alpha$-limit set is nowhere dense. Moreover, we show that $s\alpha$-limit sets in the interval are always both F_σ and G_δ. Finally, since $s\alpha$-limit sets need not be closed, we propose a new notion of β-limit sets to serve as backward attractors. The β-limit set of x is the smallest closed set to

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which all backward orbit branches of x converge, and it coincides with the closure of the $s\alpha$-limit set. At the end of the paper we suggest several new problems about backward attractors.

Keywords: interval map, transitivity, α-limit set, special α-limit set, β-limit set, backward attractor
Mathematics Subject Classification numbers: primary: 37E05, 37B20 secondary: 26A18.

1. Introduction

Let a *discrete dynamical system* be defined as an ordered pair (X, f) where X is a compact metric space and f is a continuous map acting on X. To understand the dynamical properties of such a system it is necessary to analyse the behaviour of the trajectories of any point $x \in X$ under the iteration of f. Limit sets of trajectories are a helpful tool for this purpose since they can be used to understand the long term behaviour of the dynamical system.

The ω*-limit sets* ($\omega(x)$ for short), i.e. the sets of limit points of forward trajectories, were deeply studied by many authors. For instance, one can ask for a criterion which determines whether a given closed invariant subset of X is an ω-limit set of some point $x \in X$. The question is very hard in general, however the answer for ω-limit sets of a continuous map acting on the compact interval was provided by Blokh *et al* in [5]. A closely related question is that of characterizing all those dynamical systems which may occur as restrictions of some system to one of its ω-limit sets. These abstract ω-limit sets were studied by Bowen [6] and Dowker and Frielander [12]. It was also proved that each ω-limit set of a continuous map of the interval is contained in a maximal one by Sharkovsky [24].

Backward limit sets were introduced as a dual concept to ω-limit sets in order to capture the 'source' of the trajectory of a point. For invertible systems they are defined quite simply: one just reverses the direction of time and considers the ω-limit sets of the inverse system. These so-called α-limit sets are very important in the study of flows, where they are used to define unstable manifolds, homoclinic and heteroclinic trajectories, and the Morse decompositions at the heart of Conley index theory [7, 14]. But when we study continuous mappings $f : X \to X$ (not necessarily invertible), a point x may have many preimages (or none at all), and we must clarify what kind of backward limit set we wish to speak of. Several definitions have been proposed including conical limit sets, α-limit sets, branch α-limit sets, special α-limit sets, and others [1, 7, 11, 15, 16, 20].

One of the most classic applications of *forward limit sets* in dynamics is due to Birkhoff. There are many notions of recurrence in topological dynamics (such as periodicity, non-wandering behaviour, chain-recurrence, etc), but the term *recurrent point* has been reserved for those points x which belong to their own ω-limit sets. Birkhoff showed that these points can be used to identify the *Birkhoff centre* (Birkhoff called it the 'set of central motions') of a topological dynamical system (X, f), which is obtained by restricting f to its non-wandering set, then restricting that system to its non-wandering set, and so on through transfinite induction (taking intersections at limit ordinals) until reaching some countable ordinal (the 'depth') at which the sequence stabilizes. Birkhoff's result is that the centre of the system obtained in this way is the same as the closure of the set of recurrent points [2].

In light of Birkhoff's work, one can ask the analogous question, what is the significance of a point belonging to its own *backward limit set*? If we consider homeomorphisms than Birkhoff's results already apply (just using the inverse map), so if we wish to get something new we must consider general continuous mappings $f : X \to X$. In one-dimensional dynamics some very good answers to this question have appeared for two kinds of backward limit sets, namely, the α-limit set and the special α-limit set. Coven and Nitecki showed that a point x is non-wandering for a continuous interval map $f : [0, 1] \to [0, 1]$ if and only if x belongs to its own α-limit set [9]. But there is a deeper result related to the *attracting centre* of an interval map $f : [0, 1] \to [0, 1]$, defined by Xiong as the set of all points x such that x is in the ω-limit set of some point x_1, which itself is in the ω-limit set of some point x_2, and so on for some infinite sequence $\{x_i\}_{i=1}^{\infty}$ of points in the interval [29]. Xiong showed that the attracting centre is a subset of the Birkhoff centre (they can coincide) and that if x_1, x_2 can be found as above, then x is already in the attracting centre (so the 'depth' here is at most 2) [29]. The connection to backward limit sets was made in 1992 by Hero, who defined special α-limit sets ($s\alpha$-limit sets, for short) and showed that a point x belongs to the attracting centre of a continuous interval map if and only if it belongs to its own $s\alpha$-limit set [15]. Partial generalizations of this result to graph maps and some dendrite maps have appeared since then [26–28].

We now recall Hero's definition of a special α-limit set. A *backward orbit branch* of a point x is any sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for all i. The corresponding *branch α-limit set* is defined as the set of all limits of convergent subsequences x_{i_j} (analogously as ω-limit sets are defined from forward trajectories). Then the *special α-limit set* of a point x, denoted $s\alpha(x)$, is defined as the union of all branch α-limit sets over all backward orbit branches of x [15].

Several studies in recent years have focused on branch α-limit sets, and in light of the definition it is easy to deduce corresponding properties for $s\alpha$-limit sets, for example:

- Branch α-limit sets are always closed and strictly invariant, and therefore each $s\alpha$-limit set is strictly invariant (i.e. $f(s\alpha(x)) = s\alpha(x)$) and contains the orbit closure of each of its points [18].
- Branch α-limit sets are always internally chain transitive [16], and therefore $s\alpha$-limit sets are internally chain recurrent (but not necessarily chain transitive, see example 9 below).
- All the recurrent points of a system are contained in the union of its $s\alpha$-limit sets, since this property holds for the union of the branch α-limit sets [13].
- For interval maps $f : [0, 1] \to [0, 1]$, each branch α-limit set is locally expanding and hence coincides with an ω-limit set of the same map f [1]. It follows that each $s\alpha$-limit set of an interval map f is a union of some of its ω-limit sets.

Notably lacking in the list above are purely topological properties. For example, it seems that Hero did not consider the basic question whether all $s\alpha$-limit sets are closed. Outside the realm of one-dimensional dynamics the situation is even more complicated. It has been shown that $s\alpha$-limit sets are always analytic, but not necessarily closed or even Borel [17, 18]. Therefore it seems prudent to study more closely the properties of $s\alpha$-limit sets in one-dimensional dynamics, and especially in the most important one-dimensional space where Hero's work began, the unit interval.

Kolyada, Misiurewicz, and Snoha began this study as a systematic programme in [18]. They investigated special α-limit sets of interval maps and proved that for interval maps with a closed set of periodic points, every special α-limit set has to be closed. This result led to the following conjecture:

Conjecture 1 [18]. For all continuous maps of the unit interval all special α-limit sets are closed.

We disprove the conjecture in theorem 45 by showing a counterexample of a continuous interval map with a special α-limit set which is not closed and give the properties of continuous interval maps that determine if all special α-limit sets are closed in theorem 41. In corollaries

42–44 we identify three classes of continuous interval maps for which all $s\alpha$-limit sets are closed, namely, piecewise monotone maps, zero entropy maps with a closed set of recurrent points and maps which are not Li–Yorke chaotic. On the other hand, we show that for all continuous maps of the unit interval all special α-limit sets are both F_σ and G_δ in theorem 40. We give further topological properties of special α-limit sets of interval maps. If $s\alpha(x)$ is not closed, then it is uncountable and nowhere dense by theorem 39. If $s\alpha(x)$ is closed, then it is the union of a nowhere dense set and finitely many (perhaps zero) closed intervals by theorem 24, and in section 4.2 we prove some amount of transitivity of f on those intervals. Since $s\alpha$-limit sets need not be closed, we propose a new notion of β-limit sets to serve as backward attractors in definition 49. The β-limit set of x is the smallest closed set to which all backward orbit branches of x converge, and it coincides with the closure of the $s\alpha$-limit set.

Kolyada *et al* also made the following conjecture.

Conjecture 2 [18]. The isolated points in a special α-limit set for a continuous interval map are always periodic.

We verify this conjecture in theorem 21. We also show that a countable special α-limit set for an interval map is a union of periodic orbits. These results are opposite to the case of ω-limit sets. The ω-limit sets of a general dynamical system do not posses any periodic isolated points unless $\omega(x)$ is a single periodic orbit [23].

The authors of [18] also investigated the properties of special α-limit sets of *transitive* interval maps and stated the following conjecture:

Conjecture 3 [18]. Let $f : [0, 1] \to [0, 1]$ be a continuous map and $x, y \in [0, 1]$.

- If $x \neq y$ and $s\alpha(x) = s\alpha(y) = [0, 1]$, then f is transitive.
- If $s\alpha(x) = [0, 1]$ then either f is transitive or there is $c \in (0, 1)$ such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are transitive.

We show in theorem 26 that f is transitive if there are three distinct points $x, y, z \in [0, 1]$ with $s\alpha(x) = s\alpha(y) = s\alpha(z) = [0, 1]$. If f has one or two points with special α-limit sets equal to $[0, 1]$, but not more, then $[0, 1]$ is the union of two transitive cycles of intervals by corollary 27.

It is known that if two ω-limit sets of an interval map contain a common open set, then they are equal. The last conjecture in [18] suggested that a similar property holds for special α-limit sets:

Conjecture 4 [18]. Let f be a continuous map $f : [0, 1] \to [0, 1]$ and $x, y \in [0, 1]$. If $\mathrm{Int}(s\alpha(x) \cap s\alpha(y)) \neq \emptyset$ then $s\alpha(x) = s\alpha(y)$.

We correct this conjecture by showing that at most three distinct special α-limit sets of f can contain a given nonempty open set in corollary 29.

One additional motivation for studying $s\alpha$-limit sets is that they provide more information about α-limit sets (see section 2 for the definition), since there is always the containment $sa(x) \subset \alpha(x)$. For transitive interval maps this containment is in fact an equality $s\alpha(x) = \alpha(x)$ for all $x \in [0, 1]$ (this can be deduced from [18, proposition 3.10] or theorem 33). The question then arises whether this is the typical situation, or perhaps typically $s\alpha(x) = \alpha(x)$ at least for 'most' points x. We show in section 4.5 that for the generic interval map $f : [0, 1] \to [0, 1]$ (in the topology of uniform convergence) there is a whole interval of points $x \in [0, 1]$ for which $\alpha(x, f) \neq s\alpha(x, f)$.

To summarize, the key properties of limit sets as they apply to continuous maps of the interval are as follows:

$$x \text{ is recurrent} \iff x \in \omega(x),$$

$$x \text{ is nonwandering} \iff x \in \alpha(x),$$
$$x \text{ is in the attracting centre} \iff x \in s\alpha(x).$$

We conjecture additionally in this paper (see conjecture 53) that

$$x \text{ is in the Birkhoff centre} \iff x \in \overline{s\alpha(x)}.$$

The paper is organised as follows. Sections 1 and 2 are introductory. Section 3 investigates the relation of maximal ω-limit sets to special α-limit sets and provides tools necessary for proving the main results. It also contains a simple example showing that, unlike ω-limit sets, the special α-limit sets of an interval map need not be contained in maximal ones. Section 4 is devoted to the above mentioned results on various properties of special α-limit sets of interval maps. Section 5 studies properties of special α-limit sets which are not closed. The paper closes with open problems and related questions in section 6.

2. Terminology

Let X be a compact metric space and $f : X \to X$ a continuous map. A sequence $\{x_n\}_{n=0}^{\infty}$ is called

- The *forward orbit* of a point x if $f^n(x) = x_n$ for all $n \geqslant 0$,
- A *preimage sequence* of a point x if $f^n(x_n) = x$ for all $n \geqslant 0$,
- A *backward orbit branch* of a point x if $x_0 = x$ and $f(x_{n+1}) = x_n$ for all $n \geqslant 0$.

A point y belongs to the *ω-limit set of a point* x, denoted by $\omega(x)$, if and only if the forward orbit of x has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \to y$. A point y belongs to the *α-limit set of a point* x, denoted by $\alpha(x)$, if and only if some preimage sequence of x has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \to y$. And a point y belongs to the *special α-limit set of a point* x, also written as the *$s\alpha$-limit set* and denoted $s\alpha(x)$, if and only if some backward orbit branch of x has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \to y$. If we wish to emphasize the map, we will write $\omega(x, f)$, $\alpha(x, f)$ and $s\alpha(x, f)$.

To summarize, the ω, α, and $s\alpha$-limit sets of a point x are defined as follows. The set $\omega(x)$ is the set of all accumulation points of its forward orbit and $\alpha(x)$ (resp. $s\alpha(x)$) is the set of all accumulation points of all its preimage sequences (resp. of all its backward orbit branches).

Let $T : X \to X$ and $F : Y \to Y$ be continuous maps of compact metric spaces. If there is a surjective map $\phi : X \to Y$ such that $\phi \circ T = F \circ \phi$ then it is said that ϕ semiconjugates T to F and ϕ is a *semiconjugacy*.

Let $f : X \to X$ be a continuous map. A set $A \subset [0, 1]$ is *invariant* if $f(A) \subset A$. The forward orbit of a point, regarded as a subset of X rather than a sequence, will be denoted by $\mathrm{Orb}(x) = \{f^n(x) : n \geqslant 0\}$. The *forward orbit of a set* is $\mathrm{Orb}(A) = \bigcup\{f^n(A) : n \geqslant 0\}$. We call f *transitive* if for any two nonempty open subsets $U, V \subset X$ there is $n \geqslant 0$ such that $f^n(U) \cap V \neq \emptyset$. We call f *topologically mixing* if for any two nonempty open subsets $U, V \subset X$ there is an integer $N \geqslant 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geqslant N$.

Now let $f : [0, 1] \to [0, 1]$ be an interval map. We say that a point x is *periodic* if $f^n(x) = x$ for some $n \geqslant 1$, x is *recurrent* if $x \in \omega(x)$ and x is *non-wandering* if for every neighbourhood U of x there is $m \geqslant 1$ such that $f^m(U) \cap U \neq \emptyset$. We write $\mathrm{Per}(f)$, $\mathrm{Rec}(f)$, and $\Omega(f)$ for the sets of periodic points, recurrent points, and non-wandering points of f, respectively. We say that x is *preperiodic* if $x \notin \mathrm{Per}(f)$ but $f^n(x) \in \mathrm{Per}(f)$ for some $n \geqslant 1$. We write $\Lambda^1(f) = \bigcup_{x\in[0,1]} \omega(x)$ for the union of all ω-limit sets of f and $\mathrm{SA}(f) = \bigcup_{x\in[0,1]} s\alpha(x)$ for the union of all $s\alpha$-limit sets. Following [29] we define the *attracting centre* of f as $\Lambda^2(f) = \bigcup_{x\in\Lambda^1(f)} \omega(x)$. The *Birkhoff centre* of f is the closure of the set of recurrent points $\overline{\mathrm{Rec}(f)}$ and coincides with $\overline{\mathrm{Per}(f)}$ [8].

If the map f is clear from the context, we may drop it from the notation. The relation of these sets is given by the following summary theorem from the works of Hero and Xiong [15, 29].

Theorem 5 [15, 29]. *For any continuous interval map* $f: [0, 1] \to [0, 1]$, *we have*

$$\mathrm{Per} \subset \mathrm{Rec} \subset \Lambda^2 = \mathrm{SA} \subset \overline{\mathrm{Rec}} \subset \Lambda^1 \subset \Omega.$$

If $K \subset [0, 1]$ is a non-degenerate closed interval such that the sets $K, f(K), \ldots, f^{k-1}(K)$ are pairwise disjoint and $f^k(K) = K$, then we call the set $M = \mathrm{Orb}(K)$ a *cycle of intervals* and the *period* of this cycle is k. We may also call K an *n-periodic interval*. If $f|_M$ is transitive then we call M a *transitive cycle* for f.

3. Maximal ω-limit sets and their relation to special α-limit sets

An important property of the ω-limit sets of an interval map f is that each ω-limit set is contained in a maximal one. These maximal ω-limit sets come in three types: periodic orbits, basic sets, and solenoidal ω-limit sets.

A *solenoidal* ω*-limit set* is a maximal ω-limit set which contains no periodic points. Any solenoidal ω-limit set is uncountable and is contained in a nested sequence $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ of cycles of intervals with periods tending to infinity, also known as a *generating sequence* [4, assertion 4.2]. Here is the theorem relating $s\alpha$-limit sets to solenoidal ω-limit sets; the proof is given in section 3.1.

Theorem 6 (Solenoidal sets). *Let* $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ *be a nested sequence of cycles of intervals for the continuous interval map* f *with periods tending to infinity. Let* $Q = \bigcap \mathrm{Orb}(I_n)$ *and* $S = Q \cap \mathrm{Rec}(f)$.

(*a*) *If* $\alpha(y) \cap Q \neq \emptyset$, *then* $y \in Q$.
(*b*) *If* $y \in Q$, *then* $s\alpha(y) \supset S$ *and* $s\alpha(y) \cap Q = S$.

A *basic set* is an ω-limit set which is infinite, maximal among ω-limit sets, and contains some periodic point. If B is a basic set then with respect to inclusion there is a minimal cycle of intervals M which contains it, and B may be characterized as the set of those points $x \in M$ such that $\overline{\mathrm{Orb}(U)} = M$ for every relative neighbourhood U of x in M, see [4]. Conversely, if M is a cycle of intervals for f, then we will write

$$B(M, f) = \{x \in M : \text{for any relative neighbourhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\mathrm{Orb}(U)} = M\},$$

and if this set is infinite, then it is a basic set [4]. Here is the theorem relating $s\alpha$-limit sets to basic sets; the proof is given in section 3.2.

Theorem 7 (Basic sets). *Let* f *be a continuous interval map and fix* $y \in [0, 1]$.

(*a*) *If* $\alpha(y)$ *contains an infinite subset of a basic set* $B = B(M, f)$, *then* $y \in M$ *and* $s\alpha(y) \supset B$.
(*b*) *If* $s\alpha(y)$ *contains a preperiodic point* x, *then there is a basic set* $B = B(M, f)$ *such that* $x \in B \subset s\alpha(y)$.

The sharpness of the second claim of theorem 7 is illustrated in figure 1. The first map has two basic sets $B([0, 1], f)$ and $B(M, f)$, where M is the invariant middle interval. It is easy to see that set $B([0, 1], f)$ is a Cantor set and it is contained in $s\alpha(1)$. But $s\alpha(1)$ does not contain the basic set $B(M, f)$ although it includes the left endpoint of M, which is preperiodic. The second map shows that we cannot weaken the assumption to $\alpha(y)$. The α-limit set of 1 includes the preperiodic endpoint of M but $s\alpha(1)$ does not contain any basic set.

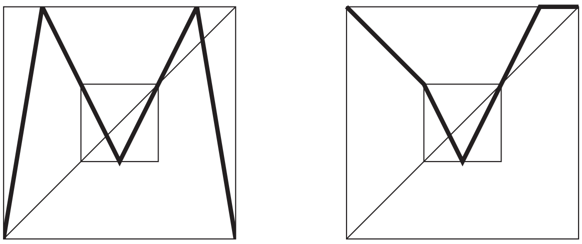

Figure 1. A map where the $s\alpha$-limit set of 1 (respectively, the α-limit set of 1) contains a preperiodic point from a basic set $B(M, f)$, but $s\alpha(1) \not\supset B(M, f)$.

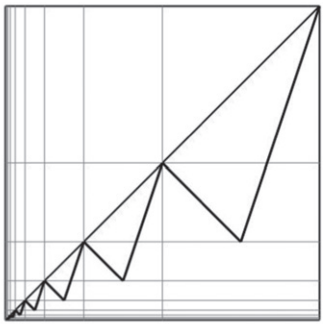

Figure 2. A map with an increasing nested sequence of $s\alpha$-limit sets not contained in any maximal one.

Periodic orbits may also be related to $s\alpha$-limit sets. The following result is one of the main theorems in [18]. Moreover, it holds for all periodic orbits of an interval map, even those which are not maximal ω-limit sets.

Theorem 8 [18, theorem 3.2]. *Let P be a periodic orbit for the continuous interval map f. If $\alpha(y) \cap P \neq \emptyset$, then $s\alpha(y) \supset P$.*

One additional observation is appropriate in this section. Unlike ω-limit sets, the $s\alpha$-limit sets of an interval map need not be contained in maximal ones.

Example 9. Fix two sequences of real numbers $1 = a_1 > b_1 > a_2 > b_2 > \ldots$ both decreasing to 0 and consider the 'connect-the-dots' map $f : [0, 1] \to [0, 1]$ where

$$f(0) = 0, \quad f(a_i) = a_i, \quad f(b_i) = a_{i+2}, \quad (i = 1, 2, \ldots)$$

and f is linear on all the intervals $[a_{i+1}, b_i]$, $[b_i, a_i]$. The graph of such a function f is shown in figure 2. The $s\alpha$-limit sets of this map are

$$s\alpha(x) = \{a_1, \ldots, a_n\} \quad \text{for } x \in (a_{n+1}, a_n] \text{ and } s\alpha(0) = \{0\}.$$

In particular, we get a strictly increasing sequence of $s\alpha$-limit sets and no $s\alpha$-limit set containing them all.

3.1. Solenoidal sets

This section is devoted to the proof of theorem 6.

We start with a broader definition of solenoidal sets, taken from [4]. A *generating sequence* is any nested sequence of cycles of intervals $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ for f with periods tending to infinity. The intersection $Q = \bigcap_n \mathrm{Orb}(I_n)$ is automatically closed and *strongly invariant*, i.e. $f(Q) = Q$, and any closed and strongly invariant subset S of Q (including Q itself) will be called a *solenoidal set*. Two examples described in [4] which we will need later are

(a) The set of all ω-limit points in Q, denoted $S_\omega = S_\omega(Q) = Q \cap \Lambda^1(f)$, and
(b) The set of all recurrent points in Q, denoted $S_{\mathrm{Rec}} = S_{\mathrm{Rec}}(Q) = Q \cap \mathrm{Rec}(f)$.

Blokh showed that Q contains a perfect set S such that $S = \omega(x)$ for all $x \in Q$ [4, theorem 3.1]. Clearly $S = S_{\mathrm{Rec}}$. We refer to S_{Rec} as a *minimal solenoidal set* both because it is the smallest solenoidal set in Q with respect to inclusion, and because the mapping $f|_{S_{\mathrm{Rec}}}$ is minimal, i.e. all forward orbits are dense.

If $\omega(x)$ is a maximal ω-limit set for f and contains no periodic points (what Sharkovskii calls a maximal ω-limit set of genus 1), then it is in fact a solenoidal set [4, assertion 4.2]. Thus it has a generating sequence $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ of cycles of intervals and belongs to their intersection Q. If $Q' = \bigcap_n \mathrm{Orb}(I'_n)$ is formed from another generating sequence for f, then it is well known (and an easy exercise) that Q and Q' are either identical or disjoint. This means that given any solenoidal set S, there is a unique maximal solenoidal set Q which contains it (so Q is uniquely determined, even if the generating sequence is not).

One can use a translation in a zero-dimensional infinite group as a model for the map f acting on a solenoidal set $Q = \bigcap \mathrm{Orb}(I_j)$. Let $D = \{m_j\}_{j=0}^{\infty}$ where m_j is the period of $\mathrm{Orb}(I_j)$ and let $H(D) = \{(r_0, r_1, \ldots) : r_{j+1} = r_j (\mathrm{mod}\, m_j), \text{for all}\, j \geqslant 0\}$ where r_j is an element of the group of residues modulo m_j, for every j. Denote by τ the translation in $H(D)$ by the element $(1, 1, \ldots)$.

Theorem 10 [4, theorem 3.1]. *Let $\{I_j\}_{j=0}^{\infty}$ be generating sequence with periods $D = \{m_j\}_{j=0}^{\infty}$ for a solenoidal set $Q = \cap_{j \geqslant 0} \mathrm{Orb} I_j$. Then there exists a semiconjugacy $\phi : Q \to H(D)$ between $f|_Q$ and τ with the following properties:*

(*a*) *There exists a unique set $S_{\mathrm{Rec}} = Q \cap \mathrm{Rec}(f)$ such that $\omega(x) = S_{\mathrm{Rec}}$ for every $x \in Q$ and $f|_{S_{\mathrm{Rec}}}$ is minimal*
(*b*) *For every $r \in H(D)$, the set $J = \phi^{-1}(r)$ is a connected component of Q and it is either a singleton $J = \{a\}, a \in S_{\mathrm{Rec}}$, or an interval $J = [a, b], \emptyset \neq S_{\mathrm{Rec}} \cap J \subset S_\omega \cap J \subset \{a, b\}$.*

One lemma which we will need several times throughout the paper is the following

Lemma 11. *If A is invariant for f and $\alpha(x) \cap \mathrm{Int}(A) \neq \emptyset$, then $x \in A$. In particular, if $s\alpha(x) \cap \mathrm{Int}(A) \neq \emptyset$, then $x \in A$.*

Proof. Choose $a \in \alpha(x) \cap \mathrm{Int}(A)$ and choose a neighbourhood U of a contained in A. There is $n \in \mathbb{N}$ and a point $x_{-n} \in U$ such that $f^n(x_{-n}) = x$. Since $U \subset A$ and A is invariant, x must belong to A. We get the same conclusion when $s\alpha(x) \cap \mathrm{Int}(A) \neq \emptyset$, because $s\alpha(x) \subset \alpha(x)$. □

Now we are ready to give the proof of theorem 6.

Proof of theorem 6.

(a) Fix $z \in Q \cap \alpha(y)$ and let $S = S_{\mathrm{Rec}} = Q \cap \mathrm{Rec}(f)$ be the minimal solenoidal set in Q. Then by theorem 10 property (a) $S = \omega(z)$ and since $\alpha(y)$ is a closed invariant set it must contain S. In particular, $\alpha(y)$ contains infinitely many points from each cycle of intervals $\mathrm{Orb}(I_n)$, and so by lemma 11 $y \in \mathrm{Orb}(I_n)$, for all n. Therefore $y \in Q$.

(b) Fix $y \in Q$. Since $f(Q) = Q$ we can choose a backward orbit branch for y which never leaves Q. Therefore it has an accumulation point in Q, and so $s\alpha(y) \cap Q \neq \emptyset$. Let $w \in s\alpha(y) \cap Q$. By theorem 10 property (a), $\omega(w)$ is the minimal solenoidal set S. According to [15, lemma 1], if $s\alpha(y)$ contains a point, then it contains the whole ω-limit set of that point as well. Therefore $s\alpha(y) \supset S$ and $s\alpha(y) \cap Q \supset S$. To finish the proof it is enough to show the opposite inclusion $s\alpha(y) \cap Q \subset S$.

We will assume otherwise. Suppose there is a point $z \in s\alpha(y) \cap (Q \backslash S)$. Let $\phi(z) = r$, where ϕ is defined in theorem 10. Since $z \in s\alpha(y) \subset \mathrm{SA}(f) \subset \Lambda^1(f)$, we can assume $z \in \Lambda^1(f) \cap (Q \backslash S) = S_\omega \backslash S$. By theorem 10 property (b), $\phi^{-1}(r)$ has to be an interval and z has to be one of its endpoints, say, the right endpoint, and $\phi^{-1}(r) = [x, z]$, $x \in S$. Since ϕ is a semiconjugacy we have $f^i([x, z]) \subset \phi^{-1}(\tau^i(r))$ for all $i \geqslant 0$. But the intervals $\phi^{-1}(\tau^i(r))$ are pairwise disjoint. This shows that $[x, z]$ is a wandering interval.

Claim. $z \in \mathrm{Int}(\mathrm{Orb}(I_j))$, for every $j \geqslant 0$.

We will assume otherwise. Let K be the connected component of $\mathrm{Orb}(I_N)$, for some $N \geqslant 0$, where z is an endpoint of K. Let v be a point such that $z \in \omega(v)$. By theorem 10 property (a), $S = \omega(z) \subset \omega(v)$, we have $\omega(v) \cap \mathrm{Orb}(I_N)$ infinite and necessarily $\mathrm{Orb}(v) \cap \mathrm{Int}(\mathrm{Orb}(I_N)) \neq \emptyset$. This implies $f^k(v) \in \mathrm{Orb}(I_N)$ for all sufficiently large k. It follows that $\mathrm{Orb}(v)$ accumulates on z from the interior of K and we can find $k > 0$ such that $f^k(v) \in (x, z)$. But $[x, z]$ is a wandering interval, so $\mathrm{Orb}(v)$ cannot accumulate on z which contradicts $z \in \omega(v)$.

Let $\{y_n\}_{n=0}^{\infty}$ be a backward orbit branch of y with a subsequence $\{y_{n_i}\}_{i=0}^{\infty}$ such that $\lim_{i\to\infty} y_{n_i} = z$. Since $z \in \mathrm{Int}(\mathrm{Orb} I_j)$ it follows from lemma 11 that $y_n \in \mathrm{Orb}(I_j)$ for all $j, n \geqslant 0$. Therefore $\{y_n\}_{n=0}^{\infty} \subset Q$. For every $n \geqslant 1$, denote $\phi(y_n) = r_n$. Then by theorem 10 property (b) $\phi^{-1}(r_n)$ are connected, pairwise disjoint sets, each containing an element $s_n \in S$. Since $y_n \in \phi^{-1}(r_n)$, we have $\lim_{i\to\infty} s_{n_i} = z$. But S is a closed set and $z \notin S$, which is impossible. Therefore $s\alpha(y) \cap (Q \backslash S) = \emptyset$ and $s\alpha(y) \cap Q \subset S$. □

Corollary 12. *A $s\alpha$-limit set contains at most one solenoidal set.*

Proof. Let $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ and $\mathrm{Orb}(I'_0) \supset \mathrm{Orb}(I'_1) \supset \ldots$ be nested sequences of cycles of intervals generating two solenoidal sets $Q = \bigcap \mathrm{Orb}(I_n)$ and $Q' = \bigcap_n \mathrm{Orb}(I'_n)$. If $s\alpha(y) \cap Q \neq \emptyset$ and $s\alpha(y) \cap Q' \neq \emptyset$ then, by theorem 6, $y \in Q \cap Q'$. Since two solenoidal sets Q and Q' are either identical or disjoint we have $Q = Q'$. Then the only solenoidal set contained in $s\alpha(y)$ is $S = Q \cap \mathrm{Rec}(f)$. □

3.2. Basic sets

This section is devoted to the proof of theorem 7.

Let f be a continuous map acting on an interval I. We say that an endpoint y of I is *accessible* if there is $x \in \mathrm{Int}(I)$ and $n \in \mathbb{N}$ such that $f^n(x) = y$. If y is not accessible, then it is called *non-accessible*. The following proposition is derived from [21, proposition 2.8].

Proposition 13 [21]. *Let f be a topologically mixing continuous map acting on an interval I. Let $x \in I$ and $\epsilon > 0$ be such that $[x - \epsilon, x + \epsilon] \subset I$ and any endpoints of I in $[x - \epsilon, x + \epsilon]$ are accessible. Then for every non degenerate interval $U \subset I$, there exists an integer N such that $f^n(U) \supset [x - \epsilon, x + \epsilon]$, for all $n \geqslant N$.*

An *m-periodic transitive map of a cycle of intervals* is a transitive map $g : M \to M$, where $M \subset \mathbb{R}$ is a finite union of pairwise disjoint compact intervals $I, g(I), \ldots, g^{m-1}(I)$, and $g^m(I) = I$.

We write End(M) for the endpoints of the connected components of M and refer to these points simply as *endpoints of* M. The set of *exceptional points of* g is defined

$$E := M \setminus \bigcap_{U} \bigcup_{n=1}^{\infty} g^n(U),$$

where U ranges over all relatively open nonempty subsets of M. It is known that E is finite; it can contain some endpoints and at most one non-endpoint from each component of M. If $g^m|_I$ is topologically mixing, then $E = \bigcup_{i=0}^{m-1} E_i$, where E_i is the set of non-accessible endpoints of $g^m|_{g^i(I)}$ by proposition 13. If $g^m|_I$ is transitive but not mixing, then by [21, proposition 2.16] there is an m-periodic orbit $c_0, c_1, \ldots, c_{m-1}$ of points such that $c_i \in \mathrm{Int}(g^i(I))$, $g^{2m}|_{[a_i,c_i]}$ and $g^{2m}|_{[c_i,b_i]}$ are topologically mixing interval maps, where $g^i(I) = [a_i, b_i]$, for $i = 0, \ldots, m-1$. Then $E = \bigcup_{i=0}^{m-1} E_i$, where E_i is the union of the sets of non-accessible endpoints of $g^{2m}|_{[a_i,c_i]}$ and $g^{2m}|_{[c_i,b_i]}$.

By [21, lemma 2.32], every point in E is periodic and therefore $g(E) = E$. By the definition of non-accessible points, $g^{-1}(E) \cap M = E$.

From [18, proposition 3.10] it follows that $s\alpha(x) \supseteq M$ for all $x \in M \setminus E$. On the other hand, if $x \in E$, then its $s\alpha$-limit set is disjoint from the interior of $M \setminus E$ by lemma 11. Therefore we have another characterization of E using $s\alpha$-limit sets,

$$E = \{x \in M : s\alpha(x) \not\supseteq M\}. \tag{1}$$

We use m-periodic transitive maps of cycles of intervals as models for maps acting on basic sets. We recall again Blokh's definition. If M is a cycle of intervals for the map $f : [0,1] \to [0,1]$, then we set

$$B(M, f) = \{x \in M : \text{for any relative neighbourhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\mathrm{Orb}(U)} = M\},$$

and if $B(M,f)$ is infinite, then it is a basic set for f.

Theorem 14 [4, theorem 4.1]. *Let I be an m-periodic interval for f, $M = \mathrm{Orb}(I)$ and $B = B(M,f)$ be a basic set. Then there is a transitive m-periodic map $g : M' \to M'$ and a monotone map $\phi : M \to M'$ such that ϕ semiconjugates $f|_M$ to g and $\phi(B) = M'$. Moreover, for any $x \in M'$, $1 \leqslant \#\{\phi^{-1}(x) \cap B\} \leqslant 2$ and $\mathrm{Int}(\phi^{-1}(x)) \cap B = \emptyset$, and so $\phi^{-1}(x) \cap B \subset \partial\phi^{-1}(x)$. Furthermore, B is a perfect set.*

Lemma 15. *Let B be a basic set, M the smallest cycle of intervals for f which contains B, and $\phi : (M,f) \to (M', g)$ the semiconjugacy to the m-periodic transitive map g given by theorem 14. Let E be the set of exceptional points of the map g acting on M'. Suppose $y \in M$ and $\phi(y) \notin E \cup \mathrm{End}(M')$. Then $s\alpha(y) \supset B$.*

Proof. Let $x \in B$. There is $\epsilon > 0$ such that $\phi|_{(x,x+\epsilon)}$ is not constant and $\phi((x, x+\epsilon)) \cap \mathrm{End}(M') = \emptyset$ or $\phi|_{(x-\epsilon,x)}$ is not constant and $\phi((x-\epsilon, x)) \cap \mathrm{End}(M') = \emptyset$. Otherwise x has a neighbourhood N such that $\phi|_N$ is constant which is in a contradiction with $x \in B$ by theorem 14. We can assume $\phi|_{(x,x+\epsilon)}$ is not constant and $\phi((x, x+\epsilon)) \cap \mathrm{End}(M') = \emptyset$, and denote $V = (x, x+\epsilon)$. Then $U = \phi(V)$ is a non-degenerate interval in M'. Since $\phi(y) \notin E \cup \mathrm{End}(M')$, there is $\delta > 0$ such that $(\phi(y) - \delta, \phi(y) + \delta) \subset M'$ and $(\phi(y) - \delta, \phi(y) + \delta) \cap E = \emptyset$. The set E equals the union of non-accessible endpoints of a topologically mixing map g^m (resp. g^{2m}) acting on the components of M', therefore we can apply the proposition 13 to the map g^m (resp. g^{2m}) acting on the component $I \subset M'$ such that $(\phi(y) - \delta, \phi(y) + \delta) \subset I$. There is an $N > 0$ such that $g^N(U) \supset (\phi(y) - \delta, \phi(y) + \delta)$. But $\phi(f^N(V)) = g^N(U)$, so $\phi(y)$ is in $\phi(f^N(V))$. Since

ϕ is monotone this means either $y \in f^N(V)$ or $\phi(y)$ is an endpoint of the interval $\phi(f^N(V))$. But we have seen that it is not an endpoint. Therefore $y \in f^N(V)$ and we can find $y_1 \in V$ and $N_1 = N$ such that $f^{N_1}(y_1) = y$. Notice that $\phi(y_1) \notin E$ since $g^{N_1}(\phi(y_1)) = \phi(y) \notin E$ and $g^{-N_1}(E) \cap M' = E$; and $\phi(y_1) \notin \mathrm{End}(M')$ since $y_1 \in V$ and $\phi(V) \cap \mathrm{End}(M') = \emptyset$. By the same procedure, we can find $y_2 \in (x, x + \epsilon/2) \cap M$ and $N_2 \in \mathbb{N}$ such that $f^{N_2}(y_2) = y_1$. By repeating this process we construct a sequence $\{y_n\}_{n=1}^{\infty}$ converging to x which is a subsequence of a backward orbit branch of y. Since $x \in B$ was arbitrary, this shows $B \subset s\alpha(y)$. □

Corollary 16. *For every basic set B, there is $y \in B$ such that $s\alpha(y) \supset B$.*

Proof. Let M, ϕ, g, M', E be as in the previous proof. Since the map $\phi|_B$ is at most 2-to-1 and E is a finite set, there are uncountably many points $y \in B$ such that $\phi(y) \notin E \cup \mathrm{End}(M')$. The result follows by lemma 15. □

Before we proceed to the proof of theorem 7 we need to recall the definition of a *prolongation set* and its relation to basic sets. Let M be a cycle of intervals. Let the side T be either the left side $T = L$ or the right side $T = R$ of a point $x \in M$ and $W_T(x)$ be a *one-sided neighbourhood* of x from the T-hand side, i.e. $W_T(x)$ contains for some $\epsilon > 0$ the interval $(x, x + \epsilon)$ (resp. $(x - \epsilon, x)$) when $T = R$ (resp. $T = L$). We do not consider the side $T = R$ (resp. $T = L$) when x is a right endpoint (resp. left endpoint) of a component of M. Now let

$$P_M^T(x) = \bigcap_{W_T(x)} \bigcap_{n \geqslant 0} \overline{\bigcup_{i \geqslant n} f^i(W_T(x) \cap M)},$$

where the intersection is taken over the family of all one-sided neighbourhoods $W_T(x)$ of x. We will write $P^T(x)$ instead of $P^T_{[0,1]}(x)$. The following auxiliary lemmas 17–19 are taken from [4].

Lemma 17 [4, *lemma 2.2*]. *Let $x \in [0, 1]$. Then $P^T(x)$ is a closed invariant set and only one of the following possibilities holds:*

- *There exists a wandering interval $W_T(x)$ with pairwise disjoint forward images and $P^T(x) = \omega(x)$.*
- *There exists a periodic point p such that $P^T(x) = \mathrm{Orb}(p)$.*
- *There exists a solenoidal set Q such that $P^T(x) = Q$.*
- *There exists a cycle of intervals M such that $P^T(x) = M$.*

There is a close relation between prolongation sets and basic sets. If M is a cycle of intervals for f then we define

$$E(M, f) = \{x \in M : \text{there is a side } T \text{ of } x \text{ such that } P_M^T(x) = M\},$$

and, for $x \in E(M, f)$, we call this side T a *source side* of x. By [4, theorem 4.1] if there exists the basic set $B = B(M, f)$ then $E(M, f) = B$. In particular, if $E(M, f)$ is infinite then $E(M, f) = B(M, f)$ (see the discussion on page 48 in [4]).

Lemma 18 [4, *lemma 4.5*]. *Let M be a cycle of intervals. If $E(M,f)$ is a finite set then $E(M,f) = \mathrm{Orb}(p)$ where p is a periodic point. If $E(M,f)$ is infinite then $E(M,f) = B(M,f)$.*

The next property of basic sets follows from step B7 on page 47 in [4]:

Lemma 19 [4]. *Let $B(M,f)$ be a basic set. Then for any $x \in B(M,f)$ with a source side T and any one-sided neighbourhood from the T-hand side $W_T(x)$, we have $W_T(x) \cap B(M,f) \neq \emptyset$.*

Proof of theorem 7.

(a) Let ϕ and g be the maps given in theorem 14 and E be the set of exceptional points of the map g acting on M'. Since $\phi|_B$ is an at most 2-to-1 map, $\phi(\alpha(y) \cap B)$ is an infinite subset of M'. But $E \cup \mathrm{End}(M')$ is a finite set, so we can find a point $z \in \alpha(y) \cap B$ such that $\phi(z) \notin E \cup \mathrm{End}(M')$ and therefore $z \notin \phi^{-1}(E \cup \mathrm{End}(M'))$. The set $\phi^{-1}(E \cup \mathrm{End}(M'))$ is a union of finitely many, possibly degenerate, closed intervals in M. Since $z \in (\alpha(y) \cap B) \backslash \phi^{-1}(E \cup \mathrm{End}(M'))$, there is a pre-image $y' \in M$ of y, $y = f^k(y')$, for some $k \geqslant 0$, and simultaneously $y' \notin \phi^{-1}(E \cup \mathrm{End}(M'))$, which implies $\phi(y') \notin E \cup \mathrm{End}(M')$. Then $y \in M$ by the invariance of M. By lemma 15 applied to y', $s\alpha(y') \supset B$. But the containment $s\alpha(y) \supset s\alpha(y')$ is clear from the definition of $s\alpha$-limit sets, and so $s\alpha(y) \supset B$.

(b) Let $\{y_i\}_{i=0}^{\infty}$ be the backward orbit branch of y accumulating on x. Since x is not a periodic point, it is not contained in $\{y_i\}_{i=0}^{\infty}$ more then one time and we can assume that $\{y_i\}_{i=0}^{\infty}$ accumulates on x from one side T. Consider the prolongation set $P^T(x)$. Clearly $\{y_i\}_{i=0}^{\infty} \subset P^T(x)$. Since $P^T(x)$ is closed and invariant, x and $\mathrm{Orb}(x)$ belong to $P^T(x)$, we see that $P^T(x)$ contains both periodic and non-periodic points. By lemma 17, there is only one possibility $P^T(x) = M$, where M is a cycle of intervals. The other possibilities are ruled out—$\mathrm{Orb}(p)$, where p is a periodic point and $\omega(x)$, where x is a preperiodic point, can not contain a non-periodic point; and a solenoidal set Q can not contain a periodic point. Since $P^T(x) = M$ contains $\{y_i\}_{i=0}^{\infty}$ it must contain a one-sided neighbourhood of x from the T-hand side and therefore $P_M^T(x) = P^T(x) = M$. Let $E(M, f) = \{z \in M :$ there is a side S of z such that $P_M^S(z) = M\}$. Since $x \in E(M, f)$ and x is not periodic, by lemma 18, $E(M, f) = B(M, f)$ and T is a source side of x in $B(M, f)$. Let ϕ and g be the maps given in theorem 14 and E be the set of exceptional points of the map g acting on M'. By lemma 19, $B(M, f)$ accumulates on x from the T-hand side and thus the map ϕ is not constant on some one-sided neighbourhood of x from the T-hand side $W_T(x)$ (otherwise we can find a point $y \in B(M, f) \cap W_T(x)$ such that $y \in \mathrm{Int}(\phi^{-1}(\phi(x)))$ which is in the contradiction with the properties of ϕ from theorem 14). Since $\{y_i\}_{i=0}^{\infty}$ accumulates on x from the T-hand side and ϕ is not constant on $W_T(x)$, we can find $j > 0$ such that $y_j \in W_T(x)$ and $\phi(y_j) \notin E \cup \mathrm{End}(M')$. Then $s\alpha(y_j) \supset B(M, f)$ by lemma 15, and $s\alpha(y) \supset s\alpha(y_j)$ since y_j is a preimage of y. We conclude that $s\alpha(y) \supset B(M, f)$. □

We record here one corollary which we will need several times in the rest of the paper.

Corollary 20. *If $s\alpha(x)$ contains infinitely many points from a transitive cycle M, then $x \in M$ and $s\alpha(x) \supseteq M$.*

Proof. In this case M is itself a basic set, so we may apply theorem 7. □

4. General properties of special α-limit sets for interval maps

4.1. Isolated points are periodic

Unless an ω-limit set is a single periodic orbit, its isolated points are never periodic [23]. The opposite phenomenon holds for the $s\alpha$-limit sets of an interval map.

Theorem 21. *Isolated points in a $s\alpha$-limit set for a continuous interval map are periodic.*

Proof. Let $z \in s\alpha(y)$ such that z is neither periodic nor preperiodic. Then z is a point of an infinite maximal ω-limit set, i.e. a basic set or a solenoidal set. This follows from Blokh's decomposition theorem, that $\Lambda^1(f)$ is the union of periodic orbits, solenoidal sets and basic sets, and from theorem 5, $z \in \mathrm{SA}(f) \subset \Lambda^1(f)$. According to [15, lemma 1], when $s\alpha(y)$ contains a point z, it contains its orbit $\mathrm{Orb}(z)$ as well. If z is in a basic set B then, $\mathrm{Orb}(z) \subset B \cap s\alpha(y)$ is infinite and by theorem 7 (a), $B \subset s\alpha(y)$. Then the point z is not isolated in $s\alpha(y)$ since B is a perfect set. If z is in a solenoidal set $Q = \bigcap_n \mathrm{Orb}(I_n)$ then, by theorem 6, $s\alpha(y) \cap Q = S$. Again, the point z is not isolated in $s\alpha(y)$ since S is a perfect set.

Let $z \in s\alpha(y)$ such that z is a preperiodic point. By theorem 7 (b), there is a basic set B such that $z \in B \subset s\alpha(y)$. Then the point z is not isolated in $s\alpha(y)$ since B is a perfect set. □

In the previous proof, we have shown that if a point $z \in s\alpha(y)$ is not periodic then $s\alpha(y)$ contains either a minimal solenoidal set S or a basic set B. In both cases, $s\alpha(y)$ has to be uncountable. Therefore we have the following corollary.

Corollary 22. *A countable $s\alpha$-limit set for a continuous interval map is a union of periodic orbits.*

4.2. The interior and the nowhere dense part of a special α-limit set

A well-known result by Sharkovsky says that each ω-limit set of an interval map is either a transitive cycle of intervals or a closed nowhere dense set [24]. What can we say in this regard for $s\alpha$-limit sets of interval maps? When $\mathrm{Int}(s\alpha(x))$ is nonempty, Kolyada, Misiurewicz, and Snoha showed that $M = \overline{\mathrm{Int} s\alpha(x)}$ is a cycle of intervals containing x, see [18, proposition 3.6]. We strengthen this result by showing that the non-degenerate components of $s\alpha(x)$ are in fact closed, and the rest of $s\alpha(x)$ is nowhere dense. We also get some amount of transitivity.

The following lemma is simple and we leave the proof to the reader.

Lemma 23. *Let M be a cycle of intervals for f of period k and let K be any of its components. Then*

(*a*) *$f|_M$ is transitive if and only if $f^k|_K$ is transitive.*
(*b*) *If $L \subset K$ is a cycle of intervals for f^k, then $\bigcup_{i<k} f^i(L)$ is a cycle of intervals for f.*
(*c*) *For $x \in K$ and $y \in K\backslash\mathrm{End}(K)$, we have $y \in s\alpha(x, f)$ if and only if $y \in s\alpha(x, f^k|_K)$.*

Theorem 24. *A $s\alpha$-limit set for a continuous interval map f is either nowhere dense, or it is the union of a cycle of intervals M for f and a nowhere dense set. Moreover, M is either a transitive cycle, or it is the union of two transitive cycles.*

Proof. Consider a limit set $s\alpha(x)$ for an interval map $f : [0, 1] \to [0, 1]$. Let M be the union of the non-degenerate components of $\overline{s\alpha(x)}$. If $M = \emptyset$ then $s\alpha(x)$ is nowhere dense. Otherwise M must be a finite or countable union of closed intervals, and since M contains the interior of the closure of the $s\alpha$-limit set we know that $s\alpha(x)\backslash M$ is nowhere dense.

Let K be any component of M. By theorem 5 we have $K \subset \overline{\mathrm{Per}(f)}$, and therefore periodic points are dense in K. Let $n \geqslant 1$ be minimal such that $f^n(K) \cap K \neq \emptyset$. Since $f^n(K)$ is connected and K is a component of the invariant set $\overline{s\alpha(x)}$, we know that $f^n(K) \subseteq K$. Since periodic points are dense in K we must have $f^n(K) = K$. Therefore $\mathrm{Orb}(K)$ is a cycle of intervals, and by lemma 11 we get $x \in \mathrm{Orb}(K)$. Since this holds for every component K of M, we must have $\mathrm{Orb}(K) = M$, i.e. M is a cycle of intervals and $x \in M$.

From now on we take K to be the component of M containing x. Put $g = f^n|_K$. Then $g : K \to K$ is an interval map with a dense set of periodic points. There is a structure theorem for interval maps with a dense set of periodic points [21, theorem 3.9][3] which tells us that K is a union of the transitive cycles and periodic orbits of g,

$$K = \left(\bigcup \{L : L \text{ is a transitive cycle for } g\}\right) \cup \mathrm{Per}(g)$$

By lemma 23, $s\alpha(x, g)$ contains a dense subset of K. By corollary 20 each transitive cycle $L \subseteq K$ for g must contain x. Since transitive cycles have pairwise disjoint interiors, g has at most two transitive cycles. If their union is not K, then K must contain a non-degenerate interval of periodic points of g. But by an easy application of lemma 11, no $s\alpha$-limit set can contain a dense subset of an interval of periodic points. Therefore K is the union of one or two transitive cycles for g. By lemma 23, M is the union of one or two transitive cycles for f.

Finally, if L is one of the (at most two) transitive cycles for f that compose M, then by corollary 20 we have $s\alpha(x) \supseteq L$. Therefore $M \subseteq s\alpha(x)$. □

Remark 25. If $s\alpha(x)$ contains a cycle of intervals M, then $s\alpha(x)$ is in fact a closed set, but we are not yet ready to prove this fact. See theorem 39 below.

4.3. Transitivity and points with $s\alpha(x) = [0, 1]$

Let $f : [0, 1] \to [0, 1]$ be an interval map. We say that the point x has a *full* $s\alpha$-limit set if $s\alpha(x) = [0, 1]$. Kolyada, Misiurewicz, and Snoha proved that when f is transitive, all points $x \in [0, 1]$ with at most three exceptions have a full $s\alpha$-limit set. Conversely, they conjectured that if at least two points $x \in [0, 1]$ have full $s\alpha$-limit sets, then f is transitive. The conjecture was not quite right; the correct result is as follows:

Theorem 26. *A continuous interval map $f : [0, 1] \to [0, 1]$ is transitive if $s\alpha(x_i) = [0, 1]$ for at least three distinct points x_1, x_2, x_3.*

Proof. Suppose that f is not transitive. We will prove that at most two points have a full $s\alpha$-limit set. Suppose there is at least one point x with $s\alpha(x) = [0, 1]$. By theorem 24 there are two transitive cycles L, L' for f such that $[0, 1] = L \cup L'$, and by corollary 20 every point with a full $s\alpha$-limit set belongs to $L \cap L'$. We will show that the cardinality of $L \cap L'$ is at most two.

Let $A_1, A_2, \ldots, A_n$ be the components of L, numbered from left to right in $[0, 1]$. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the cyclic permutation defined by $f(A_i) = A_{\sigma(i)}$. If $n \geqslant 3$ then there must exist i such that $|\sigma(i) - \sigma(i + 1)| \geqslant 2$, so there is some A_j strictly between $A_{\sigma(i)}, A_{\sigma(i+1)}$. Let B be the component of L' between A_i and A_{i+1}, see figure 3. By the intermediate value theorem, $f(B) \supset A_j$. This contradicts the invariance of L'. Therefore L has at most two components. For the same reason L' has at most two components. Moreover, L, L' cannot both have two components; otherwise the middle two of those four components have a point in common, but their images do not, again see figure 3.

There are two cases remaining. L, L' can have one component each, and then $L \cap L'$ has cardinality one. Otherwise, one of the cycles, say L, has two components, while L' has only one, and then $L \cap L'$ has cardinality two. □

In the course of the proof we have also shown the following result:

Corollary 27. *If a continuous map $f : [0, 1] \to [0, 1]$ has one or two points with full $s\alpha$-limit sets, but not more, then $[0, 1]$ is the union of two transitive cycles of intervals.*

[3] In fact, [21, theorem 3.9] tells us that all the transitive cycles for g have period at most 2, and the periodic orbits not contained in transitive cycles also have period at most 2. Some of this extra information can be shown quite easily; it comes up again in our proof of theorem 26.

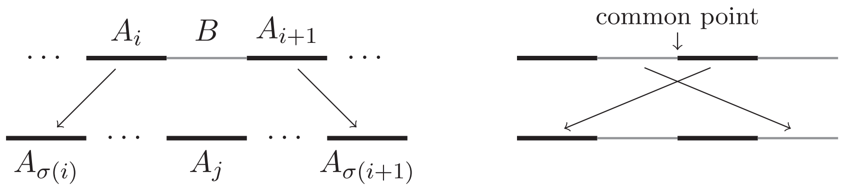

Figure 3. Diagrams for the proof of theorem 26.

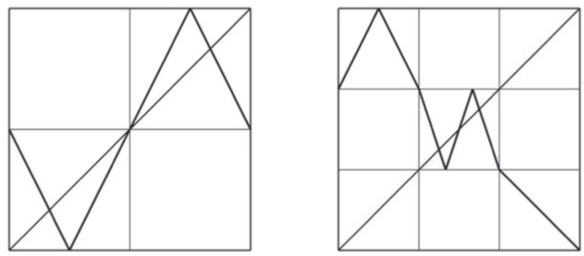

Figure 4. Maps for which $s\alpha(x) = [0, 1]$ for only 1 or 2 points x.

Corollary 27 corrects the second part of conjecture 3 (originally [18, conjecture 4.14]) to allow for two points with a full $s\alpha$-limit set.[4] Both possibilities from the corollary are shown in figure 4. One of the interval maps shown has exactly one point with a full $s\alpha$-limit set, and the other has exactly two.

4.4. Special α-limit sets containing a common open set

Now we study the $s\alpha$-limit sets that contain a given transitive cycle of intervals. We get a sharpening of theorem 7 in the case when $B = M$, i.e. when our basic set is itself a transitive cycle of intervals.

Let $M \subset [0, 1]$ be a transitive cycle of intervals for f. For the reader's convenience, we recall some definitions from section 3.2. We write End(M) for the endpoints of the connected components of M and refer to these points simply as *endpoints of M*. The main role in our analysis is played by the set of *exceptional points of M*

$$E := M \setminus \bigcap_{U} \bigcup_{n=1}^{\infty} f^n(U),$$

where U ranges over all relatively open nonempty subsets of M. It is known that E is finite; it can contain some endpoints and at most one non-endpoint from each component of M. It is also known that E and $M \setminus E$ are both invariant under f, see [21]. Endpoints of M in E are called *non-accessible endpoints*, as explained in section 3.2. Recall from equation (1) that $E = \{x \in M : s\alpha(x) \not\supseteq M\}$.

Theorem 28. *Let M be a transitive cycle of intervals for $f: [0, 1] \to [0, 1]$ and let E be its set of exceptional points.*

[4] Incidentally, when there is exactly one point with a full $s\alpha$-limit set, the conclusion of the conjecture holds as stated in [18]: there is $c \in (0, 1)$ such that such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are both transitive.

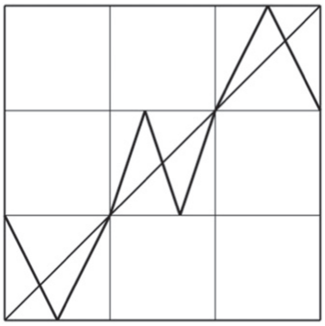

Figure 5. An example where three distinct $s\alpha$-limit sets contain a common open interval.

(a) *Each point* $x \in M \setminus (E \cup \mathrm{End}(M))$ *has the same* $s\alpha$*-limit set.*

(b) *At most three distinct* $s\alpha$*-limit sets of* f *contain* M.

We will see in the course of the proof that if $s\alpha(x') \supseteq M$ is distinct from the $s\alpha$-limit set described in part (a), then x' belongs to a periodic orbit contained in End(M). Since there are at most two such periodic orbits in End(M) we get part (b).

Before giving the proof we discuss some consequences of this theorem.

Corollary 29. *At most three distinct special alpha-limit sets of a continuous map* f *can contain a given nonempty open set.*

Proof. Let U be a nonempty open set in $[0, 1]$. If a $s\alpha$-limit set of f contains U, then by theorem 24, U contains a whole subinterval from some transitive cycle M. Applying corollary 20, we see that any $s\alpha$-limit set which contains U also contains M. By theorem 28 there are at most three such $s\alpha$-limit sets. □

This corollary corrects conjecture 4 (originally [18, conjecture 4.10]), in which it was conjectured that two $s\alpha$-limit sets which contain a common open set must be equal. For comparison, note that if two ω-limit sets of an interval map contain a common open set, then they are in fact equal, since an ω-limit set with nonempty interior is itself a transitive cycle [24]. We also remark that the number three in corollary 29 cannot be improved, as is shown by the following example.

Example 30. Let $f : [0, 1] \to [0, 1]$ be an interval map for which $[0, \frac{1}{3}]$ is a full two-horseshoe, $[\frac{1}{3}, \frac{2}{3}]$ is a full three-horseshoe, and $[\frac{2}{3}, 1]$ is a full two-horseshoe, as shown in figure 5. Then $\frac{1}{2}$ belongs to only one of the three transitive invariant intervals for f and $s\alpha(\frac{1}{2}) = [\frac{1}{3}, \frac{2}{3}]$. But both $\frac{1}{3}$ and $\frac{2}{3}$ are accessible endpoints of adjacent transitive intervals and so $s\alpha(\frac{1}{3}) = [0, \frac{2}{3}]$ and $s\alpha(\frac{2}{3}) = [\frac{1}{3}, 1]$.

In what follows it is necessary to allow for a weaker notion of a cycle of intervals for f. An interval is called *non-degenerate* if it contains more than one point. If U is a non-degenerate interval (not necessarily closed) such that $U, f(U), \ldots, f^{n-1}(U)$ are pairwise disjoint non-degenerate intervals and $f^n(U) \subseteq U$ (not necessarily equal), then we will call Orb(U) a *weak cycle of intervals* of period n. The next lemma records one of the standard ways to produce a weak cycle of intervals. Similar lemmas appear in [4] and several other papers, but since we were unable to find the exact statement we needed, we chose to give our own formulation here.

Lemma 31. *If a subinterval U contains three distinct points from some orbit of f, then* $\mathrm{Orb}(U) = \bigcup_{i=0}^{\infty} f^i(U)$ *is a weak cycle of intervals for f.*

Proof. Let $x, f^n(x), f^m(x)$ be three distinct points in U, $0 < n < m$. Clearly $\mathrm{Orb}(U)$ is invariant. Since the intervals $U, f^n(U)$ both contain $f^n(x)$ we see that $U \cup f^n(U)$ is connected, i.e. it is an interval. Then also $f^n(U) \cup f^{2n}(U)$ is connected, and so on inductively. Therefore the set $A = \bigcup_{j=0}^{\infty} f^{jn}(U)$ is connected. Then $\mathrm{Orb}(U) = \bigcup_{i=0}^{n-1} f^i(A)$ has at most n connected components. Let $B \supseteq A$ be the component of $\mathrm{Orb}(U)$ containing U, and let $k \leqslant n$ be minimal such that $B \cap f^k(B) \neq \emptyset$. Then $f^k(B)$ is a connected subset of $\mathrm{Orb}(U)$, so $f^k(B) \subseteq B$. For $0 \leqslant i < j < k$ if $f^i(B) \cap f^j(B) \neq \emptyset$, then $f^{i+k-j}(B) \cap f^k(B) \neq \emptyset$, so $f^{i+k-j}(B) \cap B \neq \emptyset$, contradicting the choice of k. This shows that $B, f(B), \ldots, f^{k-1}(B)$ are pairwise disjoint. It remains to show that they are all non-degenerate. Clearly all three points $x, f^n(x), f^m(x)$ are in B. From the disjointness of $B, f(B), \ldots, f^{k-1}(B)$ it follows that n, m are multiples of k. And since $x, f^n(x), f^m(x)$ are distinct we get $n = j_1 k$, $m = j_2 k$ with $0 < j_1 < j_2$. If any $f^i(B)$ is a singleton, $i \leqslant k$, then so also is $f^k(B)$. Then using $B \supseteq f^k(B) \supseteq f^{2k}(B) \supseteq \ldots$, we see that all the sets $f^{jk}(B)$, $j \geqslant 1$, are the same singleton. But in that case $f^{j_1 k}(B) = \{f^n(x)\}$, $f^{j_2 k}(B) = \{f^m(x)\}$ are not the same singleton, which is a contradiction. ☐

Lemma 32. *Let N be a weak cycle of intervals for f and M a transitive cycle of intervals. Let E be the set of exceptional points for M. If $N \cap M$ is nonempty, then the following hold:*

(*a*) *Either $N \supseteq M \backslash E$ or else $N \cap M$ is a periodic orbit contained in* $\mathrm{End}(M)$, *and*

(*b*) *The period of N is at most twice the period of M.*

Proof. Let n, m be the periods of N, M, respectively. Let N_i, M_j, be the components of N, M with the temporal ordering, so $f(N_i) \subset N_{i+1 \bmod n}$ for all $i < n$ and $f(M_j) = M_{j+1 \bmod m}$ for all $j < m$.

Suppose first that $N \cap M$ is infinite. Then N contains a non-degenerate interval $U \subset M$. We have $N \supseteq \mathrm{Orb}(U) \supseteq M \backslash E$, where the first containment comes from the invariance of N and the second from the definition of the exceptional set E. Since E contains at most one non-endpoint of each component M_j, it follow that each component M_j meets at most two components N_i. So in this case $n \leqslant 2m$.

For the rest of the proof we suppose that $N \cap M$ is finite. We no longer need transitivity and the sets N, M will play symmetric roles. Clearly each nonempty intersection $N_i \cap M_j$ is at common endpoints. This shows that $N \cap M \subseteq \mathrm{End}(M)$. It also shows that each component N_i contains at most two points from M, and conversely each component M_j contains at most two points from N.

Claim 1. Each component N_i contains the same number of points of M. Conversely, each component M_j contains the same number of points of N. Suppose first that some N_i contains two distinct points a, b from M. Then a, b belong to distinct components of M, and therefore $f(a), f(b)$ also belong to distinct components of M. But they both belong to $N_{i+1 \bmod n}$. Continuing in this way we see that each each component of N contains two points from M. Now suppose instead that each component of N contains at most one point from M. Surely some N_i contains at least one point $a \in M$. Then $N_{i+1 \bmod n}$ contains the point $f(a) \in M$. Continuing in this way we see that each component N_i contains exactly 1 point from M. Moreover, the whole argument still works if we reverse the roles of N and M. This concludes the proof of claim 1.

Claim 2. *The intersection $N \cap M$ is a periodic orbit.* Suppose first that each component of N contains two points from M. We reuse an argument from the proof of theorem 26. Let $A_1, A_2, \ldots, A_m$ be the components of M in the spatial order, i.e. numbered from left to right in $[0, 1]$. Let $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ be the cyclic permutation defined by $f(A_i) \subseteq A_{\sigma(i)}$.

If $m \geqslant 3$ then there must exist i such that $|\sigma(i) - \sigma(i+1)| \geqslant 2$, so there is some A_j strictly between $A_{\sigma(i)}, A_{\sigma(i+1)}$. Let N_k be the component of N which intersects both A_i and A_{i+1}. By the intermediate value theorem, $f(N_k) \supset A_j$. This contradicts the fact that $M \cap N$ is finite. Therefore M has only two components, and N has only one, i.e. $N = N_1$. The two endpoints of N_1 belong to A_1, A_2, respectively, and are therefore interchanged by f. So in this case $M \cap N$ is a periodic orbit of period 2.

The symmetric situation arises if each component of M contains two points from N. Then N has two components and M has only 1, and again $M \cap N$ is a periodic orbit of period 2.

Now suppose that each N_i contains exactly one point from M, and each M_j contains exactly one point from N. Then $m = n$ and we may assume the components are indexed such that $N_i \cap M_j$ is nonempty if and only if $i = j$. Let x_i be the unique point of intersection of N_i and M_i. Since these intersection points are unique we get $f(x_i) = x_{i+1 \bmod n}$ for all i. Thus $N \cap M$ is a periodic orbit of period $m = n$. □

Proof of theorem 28. We have already seen that the set of exceptional points in M may be characterized as $E = \{x \in M : s\alpha(x) \not\supseteq M\}$. Combined with lemma 11 this shows that

$$s\alpha(x) \supseteq M \iff x \in M \backslash E.$$

Our task is to compare the $s\alpha$-limit sets of the various points $x \in M\backslash E$. Since they all contain M it is enough to check whether or not they coincide outside of M. For any point $y \in [0, 1]$ let us write $s\alpha\mathrm{Ba}\sin(y) = \{z : y \in s\alpha(z)\}$.

Step 1: if $x \in M\backslash E$ *and* $y \in s\alpha(x)\backslash M$, *then there is a weak cycle of intervals* N *for* f *such that* $x \in N \subseteq s\alpha\mathrm{Ba}\sin(y)$. To prove this claim, let (x_i) be a backward orbit branch of x accumulating on y. The sequence (x_i) cannot contain y twice, for otherwise y would be a periodic point containing x in its orbit, contradicting the invariance of M. Therefore there is a subsequence (x_{i_j}) which converges to y monotonically from one side. We will suppose $x_{i_j} \searrow y$ from the right. The proof when $x_{i_j} \nearrow y$ from the left is similar.

For $k = 1, 2, \ldots$ let $U_k = (y, y + \frac{1}{k})$. Then U_k contains the points x_{i_j} for large enough j. By lemma 31, the set $V_k = \mathrm{Orb}(U_k)$ is a weak cycle of intervals. Since V_k is invariant and contains points x_i for arbitrarily large natural numbers i, we must have the whole backward orbit branch (x_i) contained in each V_k. Moreover, we have nesting $V_1 \supseteq V_2 \supseteq \ldots$. Letting v_k denote the period (i.e. the number of connected components) of V_k this implies $v_1 \leqslant v_2 \leqslant \ldots$. But each V_k contains the point $x \in M$, so by lemma 32 (b) each $v_k \leqslant 2m$, where m is the period of M. A bounded increasing sequence of natural numbers is eventually constant. So fix k' such that $v_{k'} = v_{k'+1} = \ldots$.

For each k let V_k^0 be the connected component of V_k which contains U_k. Choose some i such that $x_i \in V_{k'}^0$, $x_i > y$. For $k \geqslant k'$, V_k is a weak cycle of intervals contained in $V_{k'}$ and with the same number of components. It follows that V_k^0 is the only component of V_k which meets $V_{k'}^0$. Therefore V_k^0 must contain x_i as well. In particular, setting $\delta = x_i - y$ we find that

$$\forall_{\epsilon>0}, \mathrm{Orb}((y, y+\epsilon)) \supseteq (y, y+\delta).$$

Now let $N = \mathrm{Orb}((y, y+\delta)) = \bigcap_{\epsilon>0} \mathrm{Orb}((y, y+\epsilon))$. It follows easily that $N\backslash\{y\} \subseteq s\alpha\mathrm{Ba}\sin(y)$. For if $z \in N$, $z \neq y$, then taking $\epsilon_1 < \min(\delta, |z-y|)$ we find $z_1 \in (y, y+\epsilon_1)$ and $n_1 \geqslant 1$ such that $f^{n_1}(z_1) = z$. Then taking $\epsilon_2 < z_1 - y$ we find $z_2 \in (y, y+\epsilon_2)$ and $n_2 \geqslant 1$ such that $f^{n_2}(z_2) = z_1$. Continuing inductively, we get a subsequence of a backward orbit branch of z which accumulates on y.

Since $(y, y+\delta)$ contains x_{i_j} for sufficiently large j, lemma 31 also implies that N is a weak cycle of intervals, and since it is forward invariant we have $x \in N$. This concludes step 1.

Step 2: if $x, x' \in M\backslash E$ *and* $y \in s\alpha(x)\backslash M$, *then* $s\alpha(x) \subseteq s\alpha(x')$. To prove this claim, fix an arbitrary point $y \in s\alpha(x)$. We need to show that $y \in s\alpha(x')$ as well. If $y \in M$ then there is nothing to prove, since $M \subset s\alpha(x')$. So suppose $y \notin M$. By step 1 there is a weak cycle of intervals N such that $x \in N \subseteq s\alpha\mathrm{Ba}\sin(y)$. Now we apply lemma 32 (a), noting that $M \cap N$ contains the point $x \notin \mathrm{End}(M)$. Therefore $M\backslash E \subseteq N$. We now have $x' \in M\backslash E \subseteq N \subseteq s\alpha\mathrm{Ba}\sin(y)$, from which it follows that $y \in s\alpha(x')$. This concludes step 2.

Step 3: *each point in* $M\backslash(E \cup \mathrm{End}(M))$ *has the same* $s\alpha$*-limit set*. Suppose $x, x' \in M\backslash(E \cup \mathrm{End}(M))$. Then we may apply step 2 to get both containments $s\alpha(x) \subseteq s\alpha(x')$ and $s\alpha(x') \subseteq s\alpha(x)$. This concludes step 3 and the proof of theorem 28 (a). From now on we will refer to this common $s\alpha$-limit set as S.

Step 4: if *the* $s\alpha$*-limit set of* x' *contains* M *and is distinct from* S*, then* x' *belongs to a periodic orbit contained in* $\mathrm{End}(M)$. The hypothesis $M \subseteq s\alpha(x')$ implies that $x' \in M\backslash E$. Fix $x \in M\backslash(E \cup \mathrm{End}(M))$ and apply step 2 to conclude that $S = s\alpha(x) \subseteq s\alpha(x')$. Our hypothesis is that this containment is strict. So choose $y \in s\alpha(x')\backslash s\alpha(x)$. Clearly $y \notin M$. Applying step 1 to x' we get a weak cycle of intervals N such that $x' \in N \subseteq s\alpha\mathrm{Ba}\sin(y)$. Moreover $x \notin N$ since $y \notin s\alpha(x)$. Now $M \cap N$ contains x' but not x, so we may apply lemma 32 (a) and conclude that $M \cap N$ is a periodic orbit contained in $\mathrm{End}(M)$. This concludes step 4.

Step 5: at *most three distinct* $s\alpha$*-limit sets of* f *contain* M. One of these sets is S from step 3. By step 4, the only other sets to consider are the $s\alpha$-limit sets of those periodic points whose whole orbits are contained in $\mathrm{End}(M)$. Since M is a cycle of intervals it is clear that $\mathrm{End}(M)$ contains at most two periodic orbits. And from the definitions it is clear that all points in a periodic orbit have the same $s\alpha$-limit set. This concludes step 5 and the proof of theorem 28 (b). □

4.5. Comparison of α-limit sets with special α-limit sets

One motivation for studying $s\alpha$-limit sets is that they provide more information about α-limit sets, since there is always the containment $sa(x) \subset \alpha(x)$. For transitive interval maps this containment is in fact an equality $s\alpha(x) = \alpha(x)$ for all $x \in [0, 1]$ (this can be deduced from [18, proposition 3.10] or the theorem below). The question then arises whether this is the typical situation, or perhaps typically $s\alpha(x) = \alpha(x)$ at least for 'most' points x.

If (X, f) is a topological dynamical system and $W \subset X$ is open and wandering (for all $n \geqslant 1$, $f^n(W) \cap W = \emptyset$), then each backward orbit branch visits W at most once, so W is disjoint from all $s\alpha$-limit sets of f. But a wandering set can contain α-limit points. In example 34 below we use this kind of wandering behaviour to show that for the generic interval map $f : [0, 1] \to [0, 1]$ (in the topology of uniform convergence) there is a whole interval of points $x \in [0, 1]$ for which $\alpha(x, f) \neq s\alpha(x, f)$. We show in theorem 33 that if there are no wandering intervals then $\alpha(x, f) = s\alpha(x, f)$ for all $x \in [0, 1]$.

Theorem 33. *Let* $f : [0, 1] \to [0, 1]$ *be a continuous map. If* $\Omega(f) = [0, 1]$ *then* $\alpha(x) = s\alpha(x)$ *for all* $x \in [0, 1]$.

Proof. By [29, theorem 1] the set $\Omega(f)\backslash\overline{\mathrm{Rec}(f)}$ is at most countable and since $\overline{\mathrm{Rec}(f)}$ is a closed set we have $\overline{\mathrm{Per}(f)} = [0, 1]$. Thus f has a dense set of periodic points. By [21, theorem 3.9] (see also the discussion on page 40 in [21]) every non-periodic point of f belongs to the interior of some transitive cycle of intervals L. Fix $x \in [0, 1]$. We will show that $\alpha(x) \subseteq s\alpha(x)$. Let $y \in \alpha(x)$. If y is periodic then by theorem 8 $y \in s\alpha(x)$. Suppose that y

is not periodic and let L be the transitive cycle of intervals for f with $y \in \text{Int}(L)$ where interior is taken relative to [0, 1]. Let $\{x_i\}_{i=1}^{\infty}$ be a preimage sequence of x with some subsequence $x_{i_j} \to y$ as $j \to \infty$. The points from $\{x_i\}_{i=1}^{\infty}$ are all distinct since x is not periodic and thus there are infinitely many preimages of x in L. In particular, one of the preimages is a non-exceptional point of L and $y \in L \subset s\alpha(x)$ by equation (1). □

Let $C^0([0,1])$ be the complete metric space of all maps $f : [0,1] \to [0,1]$ with the usual uniform metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. If some comeager subset of maps in $C^0([0,1])$ all have some property, then we call that property *generic*.

Example 34. Let $\xi : [0,9] \to [0,9]$ be the 'connect-the-dots' map with $\xi(0) = 0$, $\xi(1) = 7$, $\xi(2) = \xi(5) = 4$, $\xi(6) = \xi(8) = 7$, $\xi(9) = 9$, and which is linear (affine) on each of the intervals [0, 1], [1, 2], [2, 5], [5, 6], [6, 8], and [8, 9]. Every map in $C^0([0,1])$ has a fixed point. By perturbation, we can replace that fixed point with a small invariant interval on which we insert a miniature homeomorphic copy of ξ. Call the resulting map f and let A, C, W denote the images under the homeomorphism of [5, 6], [3, 8], and [1, 2], respectively, so that $A \subset f(A) = f(C) = f(W) \subset C$. These containments are stable under further perturbation, that is, for all g sufficiently close to f the three intervals $g(A), g(C), g(W)$ all contain A and are contained in C. It follows that $A \subset g^n(W) \subset C$ for all $n = 1, 2, \ldots$. Then each $x \in A$ has g-preimages of arbitrarily high order in W, so by compactness $\alpha(x,g) \cap W \neq \emptyset$. But since W is disjoint from $\Omega(g)$, it is disjoint from all $s\alpha$-limit sets of g, see theorem 5. This shows that $\alpha(x,g) \neq s\alpha(x,g)$ for all x in the non-degenerate interval A. We have identified an open dense set of maps $g \in C^0([0,1])$ for which the set $\{x | \alpha(x,g) \neq s\alpha(x,g)\}$ contains a whole interval.

5. On special α-limit sets which are not closed

5.1. Points in the closure of a special α-limit set

We start section 5 with two theorems which are important for understanding any non-closed $s\alpha$-limit sets of an interval map. Theorem 35 relates the closure of a $s\alpha$-limit set to the three kinds of maximal ω-limit sets. Theorem 36 determines precisely which points do or do not have a closed $s\alpha$-limit set, and which points from the closure are not present in the limit set. In section 5.2 we apply these results to establish some topological properties of non-closed $s\alpha$-limit sets for interval maps, showing that they are always uncountable, nowhere dense, and of type F_σ and G_δ. In section 5.3 we address the question of which interval maps have all of their $s\alpha$-limit sets closed. Most importantly, this holds in the piecewise monotone case. In section 5.4 we give a concrete example of an interval map with a non-closed $s\alpha$-limit set.

Recall that a *generating sequence* is any nested sequence of cycles of intervals $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ for an interval map f with periods tending to infinity. In light of the discussion at the beginning of section 3.1 we may define a *maximal solenoidal set* as the intersection $Q = \bigcap \text{Orb}(I_n)$ of a generating sequence. Recall that a *solenoidal ω-limit set* is an infinite ω-limit set containing no periodic points, and a solenoidal ω-limit set is always contained in a maximal solenoidal set. Recall also that the *Birkhoff centre* of f is the closure of the set of recurrent points.

Theorem 35. *Let f be a continuous interval map, let $x \in \overline{s\alpha(y)}$, and suppose that any of the following conditions holds:*

(*a*) *x is periodic,*

(*b*) *x belongs to a basic set B, or*

(*c*) *x is a recurrent point in a solenoidal* ω*-limit set.*

Then $x \in s\alpha(y)$.

Theorem 36. *Let* $f\colon [0,1] \to [0,1]$ *be a continuous interval map and* $y \in [0,1]$. *The set* $s\alpha(y)$ *is not closed if and only if y belongs to a maximal solenoidal set Q which contains a nonrecurrent point from the Birkhoff centre of f. In this case*

$$\overline{s\alpha(y)} \backslash s\alpha(y) = Q \cap \left(\overline{\mathrm{Rec}(f)} \backslash \mathrm{Rec}(f)\right).$$

The rest of section 5.1 is devoted to the proofs of these two theorems. For theorem 35 the main idea is that $\overline{s\alpha(y)} \subset \alpha(y)$, so we can apply theorems 6, 7, and 8. However, some extra care is needed for preperiodic points. Recall that x is *preperiodic* for f if $x \notin \mathrm{Per}(f)$ but there exists $n \geqslant 1$ such that $f^n(x) \in \mathrm{Per}(f)$.

Lemma 37. *Let f be a continuous interval map. If* $x \neq f(x) = f^2(x)$ *and* $x \in \overline{s\alpha(y)}$, *then* $x \in s\alpha(y)$.

Proof. We may suppose without loss of generality that $x < f(x)$. A set U is called a *right-hand neighbourhood* (resp. *left-hand neighbourhood*) of x if it contains an interval $(x, x+\epsilon)$ (resp. $(x-\epsilon, x)$) for some $\epsilon > 0$. By hypothesis either $x \in s\alpha(y)$, or every right-hand neighbourhood of x contains points from $s\alpha(y)$, or every left-hand neighbourhood of x contains points from $s\alpha(y)$. In the first case there is nothing to prove. The remaining two cases will be considered separately.

Suppose that every right-hand neighbourhood of x contains points from $s\alpha(y)$. We will construct inductively a sequence of points $y_n \to x$ and times $k_n > 0$ such that $f^{k_0}(y_0) = y$ and $f^{k_n}(y_n) = y_{n-1}$ for $n \geqslant 1$. We also construct points $a_n \in s\alpha(y)$ and open intervals $U_n \ni a_n$ that are compactly contained (i.e. their closure is contained) in $(x, f(x))$. The points a_n will decrease monotonically to x, the intervals U_n will be pairwise disjoint, and the inequalities $\sup U_n < y_{n-1} < \sup U_{n-1}$ will hold for all $n \geqslant 1$.

For the base case, choose any point $a_0 \in s\alpha(y)$ with $x < a_0 < f(x)$. Choose a small open interval $U_0 \ni a_0$ which is compactly contained in $(x, f(x))$. There exists $y_0 \in U_0$ and $k_0 > 0$ such that $f^{k_0}(y_0) = y$. Now we make the induction step. Suppose that $x < y_{n-1} < \sup U_{n-1}$ and choose $a_n \in s\alpha(y)$ with $x < a_n < \min\{y_{n-1}, \inf U_{n-1}, x + \frac{1}{n}\}$. Choose a small open interval $U_n \ni a_n$ compactly contained in $(x, f(x))$ with $\sup U_n < \min\{y_{n-1}, \inf U_{n-1}\}$. By theorem 5 we know that a_n is a non-wandering point, so there exist $b, c \in U_n$ and $k_n > 0$ such that $f^{k_n}(b) = c$. Thus $f^{k_n}([x,b])$ contains both $f(x), c$, but $c < y_{n-1} < f(x)$, so by the intermediate value theorem there is $y_n \in (x,b)$ with $f^{k_n}(y_n) = y_{n-1}$. Clearly $y_n < \sup U_n$, so we are ready to repeat the induction step. This completes the proof in the case when every right-hand neighbourhood of x contains points from $s\alpha(y)$.

Now suppose that every left-hand neighbourhood of x contains points from $s\alpha(y)$. Consider the set

$$W = \bigcap_{\epsilon>0} W_\epsilon, \quad \text{where } W_\epsilon = \bigcup_{t=1}^{\infty} f^t((x-\epsilon, x]).$$

For $\epsilon > 0$ the set W_ϵ is invariant for f. It is connected because the intervals $f^t((x-\epsilon, x])$ all contain the common point $f(x)$. Now choose a point $a \in s\alpha(y) \cap (x-\epsilon, x)$. Since a is non-wandering there are points $b, c \in (x-\epsilon, x)$ such that $f^t(b) = c$ for some $t > 0$. Thus $c \in W_\epsilon$, so by connectedness we get $[x, f(x)] \in W_\epsilon$. Since these properties hold for arbitrary ϵ, we see that W is invariant, connected, and contains $[x, f(x)]$.

We claim that W contains a left-hand neighbourhood of x. Assume to the contrary that $x = \min W$. By continuity there is a point $z > f(x)$ such that $f([f(x), z]) \subset (x, 1]$. Again by continuity there is a point $w < x$ such that $f([w, x]) \subset (x, z)$. Fix $\epsilon > 0$ arbitrary. Put $U = (\max\{w, x - \epsilon\}, x)$. Choose $a \in s\alpha(y) \cap U$. Since a is non-wandering there are points $b, c \in U$ such that $f^s(b) = c$ for some $s > 0$. But $f(b) > x$, so there is some $t \geqslant 1$ such that $f^t(b) > x$ and $f^{t+1}(b) < x$. Since W is invariant, $f^t(b) \notin W$. Therefore $f^t(b) > z$, so it follows that $z \in W_\epsilon$. Since $\epsilon > 0$ was arbitrary, this shows that $z \in W$. Again fix $\epsilon > 0$ arbitrary and let U, a, b, c, s be as they were before. We have $f(b) \in (x, z) \subset W$ and W is invariant, so $f^s(b) = c \in W$, contradicting the assumption that $x = \min W$.

Finally, we construct inductively a monotone increasing sequence of points $y_n \to x$ and a sequence of times $k_n > 0$ such that $f^{k_0}(y_0) = y$ and $f^{k_n}(y_n) = y_{n-1}$ for $n \geqslant 1$. Let $\delta > 0$ be such that W contains the left-hand neighbourhood $(x - \delta, x)$. For the base case we find $a \in s\alpha(y) \cap (x - \delta, x)$. Then there is $y_0 \in (x - \delta, x)$ and $k_0 > 0$ such that $f^{k_0}(y_0) = y$. For the induction step, suppose we are given $y_{n-1} \in (x - \delta, x)$. Choose a positive number $\epsilon < \min\{|x - y_n|, \frac{1}{n}\}$. Since $(0, \delta) \subset W \subset W_\epsilon$, we see from the definition of W that there exist $y_n \in (x - \epsilon, x]$ and $k_n > 0$ such that $f^{k_n}(y_n) = y_{n-1}$. Clearly $y_n \neq x$, since $y_{n-1} \neq f(x)$. Therefore $y_n \in (x - \delta, x)$ and the induction can continue. □

Proposition 38. *Let f be a continuous interval map. If x is preperiodic and $x \in \overline{s\alpha(y)}$, then $x \in s\alpha(y)$.*

Proof. Find n such that $x \neq f^n(x) = f^{2n}(x)$ and apply [18, proposition 2.9], which says $sa(y, f) = \bigcup_{j=0}^{n-1} s\alpha(f^j(y), f^n)$. Since the closure of a finite union is the union of the closures, we find j such that $x \in \overline{s\alpha(f^j(y), f^n)}$. By lemma 1, $x \in s\alpha(f^j(y), f^n)$. Again applying [18, proposition 2.9] we conclude $x \in s\alpha(y, f)$. □

Proof of theorem 35. Let $x \in \overline{s\alpha(y)}$. Since $\alpha(y)$ is a closed set containing $s\alpha(y)$, we have $x \in \alpha(y)$ as well. If x is periodic, then by theorem 8, $x \in s\alpha(y)$. If x is a recurrent point in a solenoidal ω-limit set, then by theorem 6 we again have $x \in s\alpha(y)$. If x belongs to a basic set B, then it must be periodic, preperiodic, or have $\mathrm{Orb}(x)$ infinite. In the periodic case theorem 8 applies. In the preperiodic case proposition 38 shows that $x \in s\alpha(y)$. And when $\mathrm{Orb}(x)$ is infinite, then it must be contained in both B and $\alpha(y)$ since those sets are both invariant. So by theorem 7 we again have $x \in s\alpha(y)$. □

Proof of theorem 36. Suppose first that $s\alpha(y)$ is not closed. Pick any point $x \in \overline{s\alpha(y)} \backslash s\alpha(y)$. By theorem 5 we have $x \in \overline{\mathrm{Rec}(f)}$ and in particular $x \in \Lambda^1(f)$. Let $\omega(z)$ be a maximal ω-limit set of f containing x. By theorem 35, x is not recurrent and $\omega(z)$ is a solenoidal ω-limit set. Then $\omega(z)$ is contained in some maximal solenoidal set Q. By theorem 6 we know that $y \in Q$. But Q also contains x, so we have shown that y belongs to a maximal solenoidal set which contains a nonrecurrent point from the Birkhoff centre. Now if x' is any other point from $\overline{s\alpha(y)} \backslash s\alpha(y)$, then by the same argument, x' is a nonrecurrent point in the Birkhoff centre and belongs to a maximal solenoidal set Q' which also contains y. Since two maximal solenoidal sets are either disjoint or equal, we get $Q = Q'$. This shows that

$$\overline{s\alpha(y)} \backslash s\alpha(y) \subseteq Q \cap \left(\overline{\mathrm{Rec}(f)} \backslash \mathrm{Rec}(f) \right).$$

To prove the converse part of the theorem, suppose that Q is any maximal solenoidal set for f which contains a nonrecurrent point from the Birkhoff centre, and fix $y \in Q$. We will show that $s\alpha(y)$ is not closed, and that

$$\overline{s\alpha(y)}\backslash s\alpha(y) \supseteq Q \cap \left(\overline{\mathrm{Rec}(f)}\backslash \mathrm{Rec}(f)\right).$$

To that end, let x be any point from $Q \cap (\overline{\mathrm{Rec}(f)}\backslash \mathrm{Rec}(f))$. Choose a generating sequence $\mathrm{Orb}(I_0) \supset \mathrm{Orb}(I_1) \supset \ldots$ with $Q = \bigcap \mathrm{Orb}(I_n)$. Without loss of generality we may assume that $x \in I_n$ for each n. Since x is not recurrent we have $x \notin s\alpha(y)$ by theorem 6. To finish the proof it suffices to show that $x \in \overline{s\alpha(y)}$. We do so by finding periodic points arbitrarily close to x which are in the α-limit set of y, and then applying theorem 8.

Let K be the connected component of Q containing x. By theorem 10 we know that the singleton components of Q are recurrent points, and in the non-degenerate components of Q, only the endpoints can appear in ω-limit sets. Since x is in an ω-limit set but is not recurrent, we know that K is a non-degenerate interval with x as one of its endpoints. Without loss of generality we may assume that x is the left endpoint of K and write $K = [x, b]$.

Again using theorem 10 (and the fact that the set S_{Rec} referred to there is perfect) we know that each component of Q has at least one recurrent endpoint, and the endpoint of a component is recurrent if and only if it is the limit of points of other components of Q. Since x is not recurrent, we know that b is. We conclude that Q does not accumulate on x from the left and it does accumulate on b from the right.

Since x is in the Birkhoff centre $x \in \overline{\mathrm{Rec}(f)} = \overline{\mathrm{Per}(f)}$ and K contains no periodic points, we know that the periodic points of f accumulate on x from the left. In particular, x is not the left endpoint of $[0, 1]$. That means we can find non-empty left-hand neighbourhoods of x in $[0, 1]$, i.e. open intervals with right endpoint x.

Since Q does not accumulate on x from the left, we can find a left-hand neighbourhood of x which contains no points from Q, and then there must be some n such that I_n does not contain that left-hand neighbourhood. It follows that $x = \min(Q \cap I_n)$. At this moment we do not know if x is the left endpoint of I_n or not, only that it is the left-most point of Q in I_n.

Let m be the period of the cycle of intervals $\mathrm{Orb}(I_n)$ and let $g = f^m$. Then I_n is an invariant interval for g. Also f and g have the same periodic points (but not necessarily with the same periods). Since $g^i(x) \in I_n$ for all i and $x \in Q$ is neither periodic nor preperiodic, there must be $i \leqslant 2$ such that $g^i(x)$ is not an endpoint of I_n. By continuity there is $\delta > 0$ such that $g^i((x - \delta, x]) \subset I_n$. Then by invariance we get

$$g^j((x-\delta, x]) \subset I_n \quad \text{for all } j \geqslant i. \tag{2}$$

Choose an arbitrary positive real number $\epsilon < \delta$. Choose a periodic point p in $(x - \epsilon, x)$. Let k be the period of p under the map g. Then $g^{ik}(p) = p$ and by (2) $g^{ik}(p) \in I_n$. This shows that $p \in I_n$. (Incidentally, we see that x is not the left endpoint of I_n.)

The set of fixed points of g^k is closed, so let p' be the largest fixed point of g^k in the interval $[p, x]$. Let h be the map $h = g^k = f^{mk}$. We know that $h(x)$ is an element of Q in I_n, and therefore it lies to the right of x. We also know $h(x) \notin [x, b]$ because no component of Q returns to itself. So the order of our points is $p' = h(p') < x < b < h(x)$, and the graph of $h|_{(p',x]}$ lies above the diagonal, since there are no other fixed points there, see figure 6.

Since Q accumulates on b from the right we can find $n' > n$ such that one of the components L of $\mathrm{Orb}(I_{n'}, f)$ lies between b and $h(x)$. By the intermediate value theorem, $h([p', x])$ contains L.

Let U be any right-hand neighbourhood of p', that is $U = (p', z)$ with $z - p' > 0$ as small as we like. We consider the orbit of U. Since the graph of h lies above the diagonal on $(p', x]$ we get a monotone increasing sequence $(h^j(z))_{j=0}^{l}$ with $l \geqslant 1$ minimal such that $h^l(z) \geqslant x$ (if there is no such l, then $h^j(z)$ converges to a fixed point of h in the interval $[z, x]$, which contradicts the choice of p'). Then $h^{l+1}(U) \supset L$. In particular, $\mathrm{Orb}(U, f) \supset \mathrm{Orb}(L, f) \supset \mathrm{Orb}(I_{n'}, f) \supset Q$.

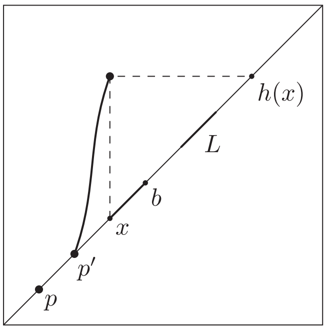

Figure 6. The graph of $h|_{I_n}$. Since $\mathrm{Orb}(L, f) \supset Q$ and each right-hand neighbourhood of p' eventually covers L, we see that $p' \in \alpha(y)$. Note: we do not claim that h is monotone on $(p', x]$, only that the graph stays above the diagonal.

We see that for every right-hand neighbourhood U of p' there is a point $y' \in U$ and a time t such that $f^t(y') = y$. As the neighbourhood U shrinks, the value of t must grow without bound, because p' is periodic. This shows that $p' \in \alpha(y) = \alpha(y, f)$. By theorem 8 it follows that $p' \in s\alpha(y)$.

Since p' was chosen from the ϵ-neighbourhood $(x - \epsilon, x)$ and $\epsilon > 0$ can be arbitrarily small, we conclude that $x \in \overline{s\alpha(y)}$. This completes the proof. □

5.2. *Properties of a non-closed special α-limit set*

As an application of theorems 36 and 35, we get the following results.

Theorem 39. *If $s\alpha(y)$ is not closed, then it is uncountable and nowhere dense.*

Proof. Suppose $s\alpha(y)$ is not closed. By theorem 36 we have $y \in Q$ for some solenoidal set $Q = \bigcap \mathrm{Orb}(I_n)$. By theorem 6 we know that $s\alpha(y)$ contains $Q \cap \mathrm{Rec}(f)$. This set is perfect [4, theorem 3.1], and therefore uncountable.

Let M be a transitive cycle of intervals for f. If $Q \cap M \neq \emptyset$, then by lemma 32, each cycle of intervals $\mathrm{Orb}(I_n)$ has period at most twice the period of M. This contradicts the fact that the periods of the cycles of intervals tend to infinity. Therefore Q does not intersect any transitive cycle of intervals M for f. In particular, y does not belong to any transitive cycle of intervals M. By corollary 20 we conclude that $s\alpha(y)$ does not contain any transitive cycle. By theorem 24 it follows that $s\alpha(y)$ is nowhere dense. □

A subset of a compact metric space X is called F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of open sets. These classes of sets make up the second level of the Borel hierarchy. Closed sets and open sets make up the first level of the Borel hierarchy and they are always both F_σ and G_δ. The next result shows that the $s\alpha$-limit sets of an interval map can never go past the second level of the Borel hierarchy in complexity.

Theorem 40. *Each $s\alpha$-limit set for a continuous interval map f is both F_σ and G_δ.*

Proof. We write $\mathrm{Bas}(f)$ for the union of all basic ω-limit sets of f and $\mathrm{Sol}(f)$ for the union of all solenoidal ω-limit sets of f. We continue to write $\mathrm{Per}(f)$ for the union of all periodic orbits of f.

To prove that $s\alpha(y)$ is of type F_σ we express it as the following union

$$s\alpha(y) = (s\alpha(y) \cap \mathrm{Per}(f)) \cup (s\alpha(y) \cap \mathrm{Bas}(f)) \cup (s\alpha(y) \cap \mathrm{Sol}(f)),$$

and show that each of the three sets in the union is of type F_σ.

The set $\mathrm{Per}(f) = \bigcup_n \{x : f^n(x) = x\}$ is clearly of type F_σ. By theorem 35, $s\alpha(y) \cap \mathrm{Per}(f)$ is a relatively closed subset of $\mathrm{Per}(f)$, and is therefore of type F_σ.

Since an interval map has at most countably many basic sets [4, lemma 5.2], their union $\mathrm{Bas}(f)$ is of type F_σ. By theorem 35, we know that $s\alpha(y) \cap \mathrm{Bas}(f)$ is a relatively closed subset of $\mathrm{Bas}(f)$, and is therefore of type F_σ.

By theorem 6 and corollary 12, we know that $s\alpha(y) \cap \mathrm{Sol}(f)$ is either the empty set, or a single minimal solenoidal set S, and minimal solenoidal sets are closed. Closed sets are trivially of type F_σ.

To prove that $s\alpha(y)$ is of type G_δ it is enough to show that $\overline{s\alpha(y)} \backslash s\alpha(y)$ is at most countable. By theorem 36 we know that this set is of the form

$$\overline{s\alpha(y)} \backslash s\alpha(y) = Q \cap (\overline{\mathrm{Rec}(f)} \backslash \mathrm{Rec}(f))$$

for some maximal solenoidal set Q. But by theorem 10 the only points in Q which can be in the Birkhoff centre but not recurrent are endpoints of non-degenerate components of Q. Since $Q \subset [0, 1]$ has at most countably many non-degenerate components, this completes the proof. □

5.3. Maps which have all special α-limit sets closed

In [18] it was proved that all $s\alpha$-limit sets of f are closed if $\mathrm{Per}(f)$ is closed. This is a very strong condition; it implies in particular that f has zero topological entropy. In this section we give necessary and sufficient criteria to decide if all $s\alpha$-limit sets of f are closed. We show in particular that this is the case for piecewise monotone maps. Note that an interval map $f : [0, 1] \to [0, 1]$ is called *piecewise monotone* if there are finitely many points $0 = c_0 < c_1 < \cdots < c_n = 1$ such that for each $i < n$, the restriction $f|_{[c_i, c_{i+1}]}$ is monotone, i.e. non-increasing or non-decreasing.

Theorem 41. *Let $f : [0, 1] \to [0, 1]$ be a continuous interval map. The following are equivalent:*

(*a*) *For some $y \in [0, 1]$, $s\alpha(y)$ is not closed.*
(*b*) *The attracting centre $\Lambda^2(f)$ is not closed.*
(*c*) *The attracting centre is strictly contained in the Birkhoff centre $\Lambda^2(f) \subsetneq \overline{\mathrm{Rec}(f)}$.*
(*d*) *Some solenoidal ω-limit set of f contains a non-recurrent point in the Birkhoff centre.*

Proof of theorem 41. (a) $\Rightarrow$ (b): suppose there is a point $y \in [0, 1]$ with $s\alpha(y)$ not closed. Choose $x \in \overline{s\alpha(y)} \backslash s\alpha(y)$. By theorem 36 we know that x is a non-recurrent point in a solenoidal set. By theorem 6 it follows that no $s\alpha$-limit set of f contains x. This shows that $x \in \overline{\mathrm{SA}(f)} \backslash \mathrm{SA}(f)$. Thus $\mathrm{SA}(f)$ is not closed. But $\Lambda^2(f) = \mathrm{SA}(f)$ by theorem 5.

(b) $\Rightarrow$ (c): this follows immediately from the containments $\mathrm{Rec}(f) \subset \Lambda^2(f) \subset \overline{\mathrm{Rec}(f)}$ in theorem 5.

(c) $\Rightarrow$ (d): suppose $\mathrm{SA}(f) = \Lambda^2(f) \neq \overline{\mathrm{Rec}(f)}$. Then we can find $x \in \overline{\mathrm{Rec}(f)} \backslash \mathrm{SA}(f)$. By theorem 5 we know that $x \in \Lambda^1(f)$. Because x is in an ω-limit set, it must belong to a periodic orbit, a basic set, or a solenoidal ω-limit set.

Each periodic orbit is contained in the $s\alpha$-limit set of any one of its points. Each basic set is also contained in a $s\alpha$-limit set by corollary 16. Since we supposed that x is not in any $s\alpha$-limit set, we must conclude that x is not in a periodic orbit or a basic set.

Now we know that x belongs to a solenoidal ω-limit set. Since x is not in any $s\alpha$-limit set we may use theorem 5 to conclude that x is not recurrent.

(d) $\Rightarrow$ (a): suppose a solenoidal ω-limit set $\omega(z)$ contains a non-recurrent point x in the Birkhoff centre $x \in \overline{\text{Rec}(f)}$. Since $\omega(z)$ is solenoidal it has a generating sequence, i.e. a nested sequence of cycles of intervals $\text{Orb}(I_0) \supset \text{Orb}(I_1) \supset \ldots$ with $\omega(z) \subseteq Q = \bigcap_n \text{Orb}(I_n)$. By theorem 36 for any $y \in Q$ the set $s\alpha(y)$ is not closed. □

Corollary 42. *If f is a piecewise monotone continuous interval map, then all $s\alpha$-limit sets of f are closed.*

Proof. By [4, lemma PM2] each point in a solenoidal ω-limit set for a piecewise monotone map f is recurrent, so condition (d) of theorem 41 can never be satisfied. □

We remark that in general the conditions of theorem 41 may be difficult to verify. Even condition (d) is difficult, since the non-recurrent points in a solenoidal ω-limit set need not belong to the Birkhoff centre. For an example, see [3]. But for maps with zero topological entropy, the whole picture simplifies considerably. For the definitions of topological entropy and Li–Yorke chaos we refer the reader to any of the standard texts in topological dynamics, e.g. [21].

Corollary 43. *Let $f\colon [0,1] \to [0,1]$ be a continuous interval map with zero topological entropy. Then all $s\alpha$-limit sets of f are closed if and only if* $\text{Rec}(f)$ *is closed.*

Proof. For a continuous interval map f with zero topological entropy, the set of recurrent points is equal to $\Lambda^2(f)$ by [29]. Now apply theorem 41. □

Corollary 44. *Suppose $f\colon [0,1] \to [0,1]$ is not Li–Yorke chaotic. Then all $s\alpha$-limit sets of f are closed.*

Proof. When f is not Li–Yorke chaotic, Steele showed that $\text{Rec}(f)$ is closed [25, corollary 3.4]. Moreover, it is well known that such a map has zero topological entropy, see e.g. [21]. Now apply corollary 43. □

5.4. Example of a non-closed special α-limit set

In 1986 Chu and Xiong constructed a map $f : [0,1] \to [0,1]$ with zero topological entropy such that $\text{Rec}(f)$ is not closed [10]. This example appeared six years before the definition of $s\alpha$-limit sets [15], but by corollary 43 it provides an example of a continuous interval map whose $s\alpha$-limit sets are not all closed.

In this section we give a short direct proof that one of the $s\alpha$-limit sets of Chu and Xiong's map f is not closed. Here are the key properties of the map f from [10].

(a) There is a nested sequence of cycles of intervals $[0,1] = M_0 \supset M_1 \supset M_2 \supset \ldots$ for f, where $M_n = \text{Orb}(J_n)$ has period 2^n.

(b) For each $n \in \mathbb{N}$ the interval J_n is the connected component of M_n which appears farthest to the left in $[0,1]$.

(c) For each $n \in \mathbb{N}$ we can express $J_n = A_n \cup J_{n+1} \cup B_n \cup K_{n+1} \cup C_n$ as a union of five closed non-degenerate intervals with disjoint interiors appearing from left to right in the order $A_n < J_{n+1} < B_n < K_{n+1} < C_n$.

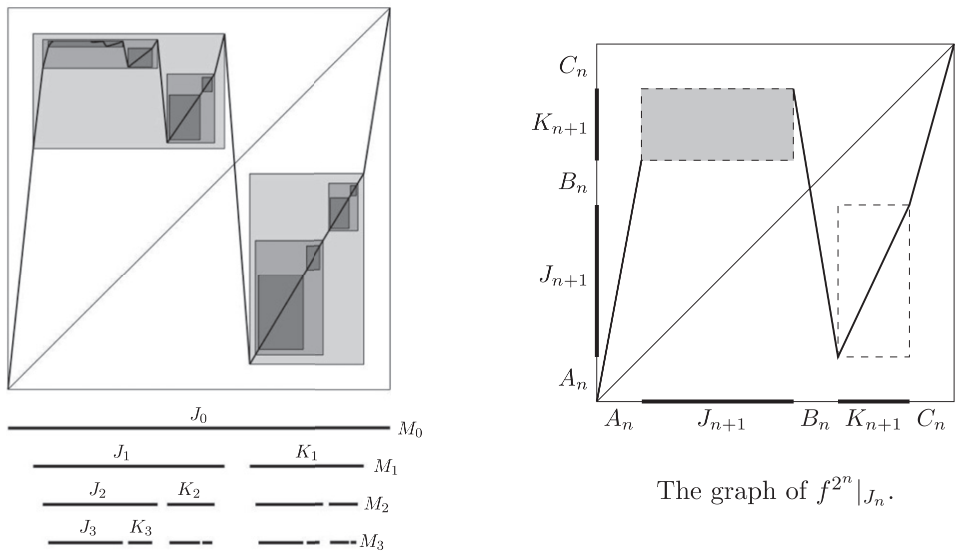

Figure 7. A map with a $s\alpha$-limit set which is not closed.

(d) For each $n \in \mathbb{N}$ the map $f^{2^n} : J_n \to J_n$ has the following properties:

(1) $f^{2^n}|_{A_n} : A_n \to A_n \cup J_{n+1} \cup B_n$ is an increasing linear bijection,
(2) $f^{2^n}|_{J_{n+1}} : J_{n+1} \to K_{n+1}$ is surjective,
(3) $f^{2^n}|_{B_n} : B_n \to K_{n+1} \cup B_n \cup J_{n+1}$ is a decreasing linear bijection,
(4) $f^{2^n}|_{K_{n+1}} : K_{n+1} \to J_{n+1}$ is an increasing linear bijection, and
(5) $f^{2^n}|_{C_n} : C_n \to B_n \cup K_{n+1} \cup C_n$ is an increasing linear bijection.

(e) The nested intersection $J_\infty = \bigcap_{n=0}^{\infty} J_n$ is a non-degenerate interval $J_\infty = [x, y]$.

The graph of the map $f : [0, 1] \to [0, 1]$ is shown in figure 7. Chu and Xiong showed that the left endpoint x of J_∞ is not recurrent, but it is a limit of recurrent points. We give a short direct proof that $s\alpha(x)$ is not closed.

Theorem 45. *Let f be the continuous interval map defined in* [10] *and let x be the left endpoint of J_∞ as defined above. Then $x \in \overline{s\alpha(x)} \setminus s\alpha(x)$, and therefore $s\alpha(x)$ is not closed.*

Proof. Fix $n \in \mathbb{N}$ and let a_n be the left endpoint of the interval A_n. Property (d) tells us that $f^{2^n} : A_n \to A_n \cup J_{n+1} \cup B_n$ is linear, say, with slope λ_n. By property (e) we have $x \in J_{n+1}$. Therefore there is a backward orbit branch $\{x_i\}_{i=0}^{\infty}$ of x such that $x_{k\cdot 2^n} = a_n + \frac{x-a_n}{\lambda_n^k}$ for all $k \in \mathbb{N}$. This shows that $a_n \in s\alpha(x)$. But $n \in \mathbb{N}$ was arbitrary. If we let $n \to \infty$, then $a_n \to x$. This shows that $x \in \overline{s\alpha(x)}$.

Now we will show that $x \notin s\alpha(x)$. Let $\{x_i\}_{i=0}^{\infty}$ be any backward orbit branch of x. Let $Q = \bigcap_{n=0}^{\infty} M_n$. We distinguish two cases. First suppose that $x_i \in Q$ for all i. For any given $i \geqslant 1$ we can choose n with $2^n > i$. Since $x_i \in M_n$, $f^i(x_i) \in J_n$, and M_n is a cycle of intervals of period 2^n, we know that $x_i \notin J_n$. Since J_n is the left-most component of M_n in $[0, 1]$, it follows that $x_i > y > x$ (recall that $J_\infty = \bigcap J_n = [x, y]$ is non-degenerate). Since this holds for all $i \geqslant 1$ we see that the backward orbit branch does not accumulate on x.

Now suppose there is i_0 with $x_{i_0} \notin Q$. For each $i \geqslant i_0$ there is $n(i) \in \mathbb{N}$ such that $x_i \in M_{n(i)} \setminus M_{n(i)+1}$. Since $f(x_{i+1}) = x_i$ and each M_n is invariant, we get $n(i+1) \leqslant n(i)$.

A non-increasing sequence of natural numbers must eventually reach a minimum, say, $n(i_1) = n(i_1 + 1) = \cdots = n$. For $i \geqslant i_1$, $x_i \notin M_{n+1}$, so in particular $x_i \notin J_{n+1}$. But by properties (c) and (e) we know that J_{n+1} is a neighbourhood of x. This shows that this backward orbit branch does not accumulate on x either. □

6. Open problems

Only one problem concerning $s\alpha$-limit sets of interval maps in [18] remains open:

Problem 46 [18]. Characterize all subsets A of [0, 1] for which there exists a continuous map $f : [0, 1] \to [0, 1]$ and a point $x \in [0, 1]$ such that $s\alpha(x, f) = A$.

We have seen that even for interval maps, $s\alpha$-limit sets need not be closed. If we want to work with closed limit sets, then there are several possible solutions. The first one, suggested to us by Snoha, is to answer the following question: *what are some sufficient conditions on a topological dynamical system* (X, f) *so that all of its* $s\alpha$*-limit sets are closed?* In this regard we state one conjecture which we were not able to resolve.

Conjecture 47. If $f : [0, 1] \to [0, 1]$ is continuously differentiable, then all $s\alpha$-limit sets of f are closed.

Another possibility is to ask whether the 'typical' continuous interval map has all $s\alpha$-limit sets closed. Let $C^0([0, 1])$ be the complete metric space of all maps $f : [0, 1] \to [0, 1]$ with the usual uniform metric $d(f, g) = \sup_{x\in[0,1]}|f(x) - g(x)|$. If some comeager subset of maps in $C^0([0, 1])$ all have some property, then we call that property *generic*.

Problem 48. Is the property of having all $s\alpha$-limit sets closed a generic property in $C^0([0, 1])$?

Another possible solution is to work with the closures of $s\alpha$-limit sets. Therefore we propose the following definition.

Definition 49. Let (X, f) be a discrete dynamical system (i.e. a continuous self-map on a compact metric space) and let $x \in X$. The β*-limit set of* x, denoted $\beta(x)$ or $\beta(x, f)$, is the smallest closed set such that $d(x_n, \beta(x)) \to 0$ as $n \to \infty$ for every backward orbit branch $\{x_n\}_{n=0}^{\infty}$ of the point x.

The letter β here means 'backward', since β-limit sets serve as attractors for backward orbit branches. It is clear from the definition that $\beta(x) = \overline{s\alpha(x)}$.

Proposition 50. *If* (X,f) *is a discrete dynamical system and* $x \in X$*, then* $\beta(x)$ *is closed and strongly invariant, i.e.* $f(\beta(x)) = \beta(x)$*. Additionally,* $\beta(x)$ *is nonempty if and only if* $x \in \bigcap_{n=0}^{\infty} f^n(X)$*. In particular,* $\beta(x)$ *is nonempty for every* $x \in X$ *when* f *is surjective.*

Proof. By [18, corollary 2.7] we have $f(s\alpha(x)) = s\alpha(x)$. Taking closures, we get $f(\beta(x)) = \beta(x)$. Additionally, $\beta(x)$ is nonempty if and only if $s\alpha(x)$ is. But by [18, proposition 2.3] we have $s\alpha(x) \neq \emptyset$ if and only if $x \in \bigcap_{n=0}^{\infty} f^n(X)$. □

We can simplify problem 46 by working with β-limit sets, since they are always closed.

Problem 51. Characterize all subsets $A \subseteq [0, 1]$ for which there exists a continuous map $f : [0, 1] \to [0, 1]$ and a point $x \in [0, 1]$ such that $\beta(x, f) = A$.

If X is a compact metric space, then the space $K(X)$ consisting of all nonempty compact subsets of X can be topologized with the Hausdorff metric and it is again compact. The map

$X \ni x \mapsto \omega(x, f) \in K(X)$ associated to a dynamical system (X, f) is usually far from continuous, but its Baire class can be useful, see e.g. [25]. Therefore we propose the following question.

Problem 52. Of what Baire class (if any) is the function $[0, 1] \ni x \mapsto \beta(x, f) \in K([0, 1])$ when f is a surjective continuous interval map?

Hero used $s\alpha$-limit sets to characterize the attracting centre $\Lambda^2(f)$ of a continuous interval map f, and some work has been done to extend his results to trees and graphs [15, 26, 27]. We conjecture that for graph maps, the β-limit sets can be used to characterize the Birkhoff centre $\overline{\mathrm{Rec}(f)}$ as follows:

Conjecture 53. Let $f : X \to X$ be a graph map, and let $x \in X$. The following are equivalent:

(a) $x \in \overline{\mathrm{Rec}(f)}$
(b) $x \in \beta(x)$
(c) There exists $y \in X$ such that $x \in \beta(y)$.

The coexistence of periodic orbits for interval maps was studied by Sharkovsky [22]. A special case of his theorem, proved independently by Li and Yorke, says that if $f : [0, 1] \to [0, 1]$ has a periodic orbit of period three, then it has periodic orbits of all periods [19].

This suggests the problem of studying the coexistence of periodic orbits within special α-limit sets.[5] In this spirit, we offer one conjecture and one open problem.

Conjecture 54. Let $f : [0, 1] \to [0, 1]$ and $x \in [0, 1]$. If $s\alpha(x)$ contains a periodic orbit of period 3, then for every positive integer n, $s\alpha(x)$ contains a periodic orbit of period n or $2n$.

Problem 55. For which subsets $A \subseteq \mathbb{N}$ is there a map $f : [0, 1] \to [0, 1]$ and a point $x \in [0, 1]$ such that A is the set of periods of all periodic orbits of f contained in $s\alpha(x)$*?*

In support of conjecture 54, we show that the conclusion holds for $n = 1$.

Lemma 56. *Let $f : [0, 1] \to [0, 1]$ and $x \in [0, 1]$. If $s\alpha(x)$ contains a periodic orbit of period 3, then it contains a periodic orbit of period 1 or 2 as well.*

Proof. Let $s\alpha(x)$ contain the periodic orbit $\{a, b, c\}$ for the interval map f with $a < b < c$. We may assume without loss of generality that $f(a) = b$, $f(b) = c$, and $f(c) = a$. By continuity we may choose a closed interval $U = [b - \epsilon, b + \epsilon]$ with $\epsilon > 0$ small enough that $f^2(U) < U < f(U)$, that is to say, $\max f^2(U) < \min U < \max U < \min f(U)$. Now find $x_1 \in U$ and $n_1 \geqslant 1$ such that $f^{n_1}(x_1) = x$. By the intermediate value theorem we can find $x_2 \in (\max U, \min f(U))$ such that $f(x_2) = x_1$. Again, by the intermediate value theorem we can find $x_3 \in (x_1, x_2)$ such that $f(x_3) = x_2$. In the next step we find $x_4 \in (x_3, x_2)$ such that $f(x_4) = x_3$. Continuing inductively we find a whole sequence (x_i) such that $f(x_{i+1}) = x_i$, $i \geqslant 1$, arranged in the following order,

$$x_1 < x_3 < x_5 < \cdots < \cdots < x_6 < x_4 < x_2.$$

Since a bounded monotone sequence of real numbers has a limit, we may put $x_\infty^- = \lim_{i\to\infty} x_{2i+1}$ and $x_\infty^+ = \lim_{i\to\infty} x_{2i+2}$, and we have $x_\infty^- \leqslant x_\infty^+$. Then $f(x_\infty^-) = \lim f(x_{2i+1}) = \lim x_{2i} = x_\infty^+$ and similarly $f(x_\infty^+) = x_\infty^-$. This shows that $\{x_\infty^+, x_\infty^-\}$

[5] Or equivalently, within β-limit sets, since by theorem 8 a periodic orbit of an interval map f is contained in $s\alpha(x)$ if and only if it is contained in $\beta(x)$.

is a periodic orbit contained in $s\alpha(x)$. The period is 2 if these points are distinct and 1 if they coincide. □

Example 57. Let $f : [0, 5] \to [0, 5]$ be the 'connect-the-dots' map with $f(0) = 1$, $f(1) = 5$, $f(4) = 2$, $f(5) = 0$, and which is linear (affine) on each of the intervals [0, 1], [1, 4], and [4, 5]. Then $s\alpha(0)$ contains the period-three orbit $\{0, 1, 5\}$ and the period-two orbit $\{2, 4\}$, but not the unique period-one orbit $\{3\}$.

Example 58. Let $f : [0, 8] \to [0, 8]$ be the 'connect-the-dots' map with $f(0) = 4$, $f(4) = 8$, $f(5) = 3$, $f(8) = 0$, and which is linear (affine) on each of the intervals $[0, 4]$, [4, 5], and [5, 8]. Then $s\alpha(0)$ contains the period-three orbit $\{0, 4, 8\}$, the period-four orbit $\{1, 5, 3, 7\}$ and the period-one orbit $\{\frac{14}{3}\}$, but not the unique period-two orbit $\{2, 6\}$.

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Li–Yorke sensitivity does not imply Li–Yorke chaos

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Abstract. We construct an infinite-dimensional compact metric space X, which is a closed subset of $\mathbb{S} \times \mathbb{H}$, where $\mathbb{S}$ is the unit circle and $\mathbb{H}$ is the Hilbert cube, and a skew-product map F acting on X such that (X, F) is Li–Yorke sensitive but possesses at most countable scrambled sets. This disproves the conjecture of Akin and Kolyada that Li–Yorke sensitivity implies Li–Yorke chaos [Akin and Kolyada. Li–Yorke sensitivity. *Nonlinearity* **16**, (2003), 1421–1433].

1. *Introduction*

Li–Yorke sensitivity and Li–Yorke chaos are well-known properties of dynamical systems, where by a dynamical system we mean a phase space X endowed with an evolution map T. We require that the phase space (X, d) is a compact metric space and the evolution map is a continuous surjective mapping $T : X \to X$.

The definition of Li–Yorke sensitivity is a combination of sensitivity and Li–Yorke chaos. Li–Yorke chaos was introduced in 1975 by Li and Yorke in [**8**]. A dynamical system is *Li–Yorke chaotic* if there is an uncountable scrambled set. A set S is scrambled if any two distinct points $x, y \in S$ are proximal (i.e. trajectories of x and y are arbitrarily close for some times) but not asymptotic, which means that

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n(x), T^n(y)) > 0.$$

The initial idea of sensitivity goes back to Lorenz [**9**], but it was firstly used in topological dynamics by Auslander and Yorke in [**2**] and popularized later by Devaney in [**5**]. A map T is *sensitive* if there is $\epsilon > 0$ such that, for each $x \in X$ and each $\delta > 0$, there is $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) > \epsilon$. By Huang and Ye in [**6**], T is sensitive if and only if there is $\epsilon > 0$ with the property that any neighbourhood of any $x \in X$ contains a point y such that trajectories of x and y are separated by ϵ for infinitely many times, that is,

$$\limsup_{n\to\infty} d(T^n(x), T^n(y)) > \epsilon.$$

Inspired by the above results, Akin and Kolyada introduced Li–Yorke sensitivity in [**1**]. A map T is *Li–Yorke sensitive* if there is $\epsilon > 0$ with the property that any neighbourhood of any $x \in X$ contains a point y proximal to x, such that trajectories of x and y are separated by ϵ for infinitely many times. Thus,

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n(x), T^n(y)) > \epsilon.$$

Akin and Kolyada proved, among other things, that weak mixing systems are Li–Yorke sensitive and stated five conjectures concerning Li–Yorke sensitivity. Three of them were disproved in [**3**] and [**4**], one was confirmed recently in [**10**]. Only one problem remained open until now, as follows.

Question 1. Are all Li–Yorke sensitive systems Li–Yorke chaotic?

This question was also included in the list of important open problems in contemporary chaos theory in topological dynamics in [**7**].

We show that the answer is negative. We construct an infinite-dimensional compact metric space X, which is a closed subset of $\mathbb{S} \times \mathbb{H}$, where $\mathbb{S}$ is the unit circle and $\mathbb{H}$ is the Hilbert cube, and a skew-product map F, which is a combination of a rotation on $\mathbb{S}$ and a contraction on $\mathbb{H}$, such that (X, F) is Li–Yorke sensitive but possesses at most countable scrambled sets. The mapping F can be continuously extended to get a connected dynamical system with the same properties, see Remark 1.

We recall here some notation used throughout the paper. A pair of points (x, y) in X^2 is *asymptotic* if $\lim_{n\to\infty} d(T^n(x), T^n(y)) = 0$. A pair of points (x, y) in X^2 is *proximal* if $\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0$; if (x, y) is not proximal then it is called *distal*. A pair of points (x, y) in X^2 is *scrambled* if it is proximal but not asymptotic. A pair of points (x, y) in X^2 is *scrambled with modulus* ϵ if it is proximal and $\limsup_{n\to\infty} d(T^n(x), T^n(y)) \geq \epsilon$. A system (X, T) is *minimal* if every point $x \in X$ has a dense orbit $\{T^n(x)\}_{n=0}^{\infty}$. A system is *transitive* if, for every pair of open, non-empty subsets $U, V \subset X$, there is a positive integer $n \in \mathbb{N}$ such that $U \cap T^n(V) \neq \emptyset$. A system (X, T) is *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive.

2. *Main result*

Here we state the main result and outline of its proof. Technical details of the proof can be found in the form of lemmas and claims in the last section.

THEOREM 1. *There is a Li–Yorke sensitive dynamical system which is not chaotic in the Li–Yorke sense.*

Proof. Let X_0 be the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ equipped with the metric $d_0(x, y) = \min\{|x - y|, 1 - |x - y|\}$ and, for $i \geq 1$, $X_i = \mathbb{N} \cup \{\infty\}$ equipped with the metric $d_i(x, y) = |(1/x) - (1/y)|$, where $1/\infty = 0$. Then $\prod_{i=0}^{\infty} X_i$ with the product topology is a compact space. The product topology is equivalent to the metric topology induced by the metric $D(x, y) = \sum_{i=0}^{\infty} (d_i(x_i, y_i)/2^i)$. Let $Y = \{x \in \prod_{i=0}^{\infty} X_i : \{x_i\}_{i=1}^{\infty}$ is non-decreasing$\}$. Y is a closed subset of $\prod_{i=0}^{\infty} X_i$ and therefore it is a compact metric space. Notice that, for $i \geq 1$, X_i can be embedded into the unit interval $[0, 1]$ equipped with the natural topology, so Y can be identify with a closed subset of $\mathbb{S} \times \mathbb{H}$, where $\mathbb{H}$ is the Hilbert cube.

Let $F : Y \to Y$ be a mapping defined for a point $x = (x_0, x_1, x_2, \ldots)$ in Y by $F(x) = (f_0(x), f_1(x), f_2(x), \ldots)$, where

$$f_0(x) = \left(x_0 + \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{1}{x_i}\right) \bmod 1, \tag{1}$$

$$f_i(x) = x_i + 1 \quad \text{for } i \geq 1, \tag{2}$$

where $\infty + 1 = \infty$. F is a continuous mapping, since f_i is continuous, for every $i \geq 0$. First, we will show that (F, Y) is Li–Yorke sensitive. It is enough to show that, for a given $x = (x_0, x_1, x_2, \ldots) \in Y$ and $U \in Nb_x$, there is $y = (y_0, y_1, y_2, \ldots) \in U$ such that

$$\liminf_{n\to\infty} D(F^n(x), F^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} D(F^n(x), F^n(y)) \geq \tfrac{1}{2}.$$

Case I. $\{x_i\}_{i=1}^{\infty}$ is a non-decreasing sequence containing at least one ∞. Since $\{x_i\}_{i=1}^{\infty}$ is non-decreasing, there is $M \in \mathbb{N}$ such that x_i is finite, for $i < M$, and $x_i = \infty$, for $i \geq M$. The neighbourhood U is defined by $U = V \cap Y$, where V is a neighbourhood of x in $\prod_{i=0}^{\infty} X_i$. Let $V = V_0 \times V_1 \times V_2 \times \ldots$, where V_i is a neighbourhood of x_i such that $V_i = X_i$ for all but finitely many $i \geq 0$. Let $K \in \mathbb{N}$, sufficiently large to satisfy $K \in V_i$, for $i \geq M$, and simultaneously $K \geq x_{M-1}$. We define the point y as follows:

$$y_i = x_i \quad \text{for } 0 \leq i < M, \tag{3}$$

$$y_i = K \quad \text{for } i \geq M. \tag{4}$$

It is easy to see that y belongs to U. By Claim 1 in the next section, (x, y) is scrambled with modulus $\frac{1}{2}$.

Case II. $\{x_i\}_{i=1}^{\infty}$ is a non-decreasing sequence of finite numbers. The neighbourhood U is defined by $U = V \cap Y$, where V is a neighbourhood of x in $\prod_{i=0}^{\infty} X_i$. Without loss of generality, suppose $V = V_0 \times V_1 \times V_2 \times \cdots$, where $V_0 = (x_0 - \delta, x_0 + \delta)$, for some $\delta > 0$, and, for $i > 0$, V_i is a neighbourhood of x_i such that $V_i = X_i$ for all but finitely many i. Let $M \in \mathbb{N}$ such that $2^{-M} < \delta$ and simultaneously $V_i = X_i$, for $i \geq M$. We define the point y as follows:

$$y_0 = \left(x_0 + 1 - \sum_{i=1}^{\infty} \frac{1}{2^{i+M}} \delta_i\right) \bmod 1, \tag{5}$$

$$y_i = x_i \quad \text{for } 0 < i < M, \tag{6}$$

$$y_i = \infty \quad \text{for } i \geq M, \tag{7}$$

where

$$\delta_i = \begin{cases} 0 & \text{if } x_{i+M} = x_M, \\ \left(\displaystyle\sum_{j=0}^{x_{i+M}-x_M-1} \frac{1}{x_M + j}\right) \bmod 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $\sum_{i=1}^{\infty} (1/2^{i+M})\delta_i \leq 2^{-M}$ and y belongs to U. By Claim 2, (x, y) is scrambled with modulus $\frac{1}{2}$.

Notice that, in both cases, one point of the pair (x, y) has ∞ coordinates while the other has all coordinates finite. By Claim 3, if x_i and y_i are finite, for all $i \geq 1$, then $\lim_{n\to\infty} D(F^n(x), F^n(y))$ exists and (x, y) is not a scrambled pair. Therefore in each scrambled set $S \subset Y$, there is at most one $z \in S$ such that z_i is finite, for $i \geq 1$. We finish our proof by finding an injection between $S \setminus \{z\}$ and $\mathbb{N}$.

Let $l_x = \min\{i : x_i = \infty\}$. Then the mapping $\iota : S\backslash\{z\} \to \mathbb{N}$ defined by $\iota(x) = l_x$ is injective. We proceed by assuming the opposite. Let $x \neq y$ in $S\backslash\{z\}$ such that $l = l_x = l_y$. Since $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are non-decreasing,

$$x_i < \infty \wedge y_i < \infty \quad \text{for } 0 < i < l, \quad x_i = y_i = \infty \quad \text{for } i \geq l. \tag{8}$$

By Claim 4, $\lim_{n\to\infty} D(F^n(x), F^n(y))$ exists, which contradicts (x, y) being a scrambled pair. □

Remark 1. The mapping F can be continuously extended to get a *connected* dynamical system with the same properties. Let X_0 be the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ equipped with the metric $d_0(x, y) = \min\{|x - y|, 1 - |x - y|\}$ and, for $i \geq 1$, X_i be the unit interval [0, 1] equipped with the natural topology. Then $\prod_{i=0}^{\infty} X_i$ equipped with the product topology is $\mathbb{S} \times \mathbb{H}$, where $\mathbb{H}$ is the Hilbert cube. The product topology is equivalent to the metric topology induced by the metric $D(x, y) = \sum_{i=0}^{\infty} (d_i(x_i, y_i)/2^i)$. Let $Z = \{x \in \mathbb{S} \times \mathbb{H} : \{x_i\}_{i=1}^{\infty}$ is non-increasing$\}$. Z is a closed and pathwise connected subset of $\mathbb{S} \times \mathbb{H}$.

Let $x = (x_0, x_1, x_2, \ldots)$ be a point in Z. We will express every $x_i \in X_i\backslash\{0\} = (0, 1]$, for $i \geq 1$, as $x_i = 1/k_i + t_i \cdot |(1/(k_i - 1)) - (1/k_i)|$, where $t_i \in (0, 1]$ and $k_i \in \mathbb{N}\backslash\{1\}$. Let $G : Z \to Z$ be a mapping defined by $G(x) = (g_0(x)g_1(x), g_2(x), \ldots)$, where

$$g_0(x) = \left(x_0 + \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot x_i\right) \bmod 1$$

and, for $i \geq 1$,

$$g_i(x) = \begin{cases} 0 & \text{if } x_i = 0, \\ \dfrac{1}{k_i + 1} + t_i \cdot \left|\dfrac{1}{k_i} - \dfrac{1}{k_i + 1}\right| & \text{otherwise.} \end{cases}$$

Then G is a continuous extension of F and (G, Z) is a Li–Yorke sensitive but not Li–Yorke chaotic system.

Remark 2. The mapping F is not minimal (it is even not transitive). In the case of minimal maps, we still have the following open question.

Question 2. Are all Li–Yorke sensitive minimal systems Li–Yorke chaotic?

Remark 3. Y is an infinite-dimensional space. We can examine the relation between Li–Yorke sensitivity and Li–Yorke chaos for low-dimensional dynamical systems. It is known that in the case of graph mappings (in particular, interval mappings) Li–Yorke sensitivity implies Li–Yorke chaos, since, for graph mappings, the existence of a single scrambled pair implies the existence of an uncountable scrambled set. But this is not true for other classes of dynamical systems—shifts, maps on dendrites, triangular maps of the square.

Question 3. Are all Li–Yorke sensitive shifts/maps on dendrites/triangular maps of the square Li–Yorke chaotic?

3. *Proofs*

LEMMA 1. *Let $p \in \mathbb{N}$, $m \in \mathbb{N}$ and $\epsilon > 0$. There are sequences $\{v_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ such that*

$$\text{for every } n \in \mathbb{N}, \quad \left(\frac{1}{2^p} \sum_{j=0}^{v_n - 1} \frac{1}{m + j}\right) \bmod 1 < \epsilon, \tag{9}$$

and

$$\text{for every } n \in \mathbb{N}, \quad \left|\left(\frac{1}{2^p}\sum_{j=0}^{u_n-1}\frac{1}{m+j}\right) \bmod 1 - \frac{1}{2}\right| < \epsilon. \tag{10}$$

Lemma 1 follows by the simple fact that the harmonic series is divergent while its increment tends to 0. Therefore the nth partial sum of harmonic series modulo 1 is ϵ-close to any number from $[0, 1)$ for infinitely many n.

LEMMA 2. *For any $i \in \mathbb{N}$, let $\{\delta_i^n\}_{n=1}^{\infty}$ be a sequence of positive numbers not greater than* 1, *such that* $\lim_{n\to\infty} \delta_i^n = 0$. *Then*

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{1}{2^i}\delta_i^n = 0.$$

Proof. For every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $\epsilon > 2^{-k+1}$. Since $\lim_{n\to\infty} \delta_i^n = 0$, for every $i \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that, for $n \geq N$ and $i \leq k$, $\delta_i^n < 2^{-k}$. We can estimate, for $n \geq N$,

$$\sum_{i=1}^{\infty}\frac{1}{2^i}\delta_i^n < \sum_{i=1}^{k}\frac{1}{2^i}\delta_i^n + \sum_{i=k+1}^{\infty}\frac{1}{2^i} < (1-2^{-k})\cdot 2^{-k} + 2^{-k} < \epsilon. \qquad \square$$

LEMMA 3. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}$. Then*

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\left(\frac{1}{k+j} - \frac{1}{k+r+j}\right) = \sum_{j=0}^{r-1}\frac{1}{k+j}.$$

Proof. The sums telescope. For sufficiently large n,

$$\sum_{j=0}^{n-1}\left(\frac{1}{k+j} - \frac{1}{k+r+j}\right) = \sum_{j=0}^{r-1}\frac{1}{k+j} - \sum_{j=n-r}^{n-1}\frac{1}{k+r+j},$$

where the second term on the right-hand side tends to 0 for $n \to \infty$. $\square$

CLAIM 1. *x and y defined in (3) and (4) are a scrambled pair with modulus $\frac{1}{2}$.*

Proof. Denote the ith coordinate of $F^n(x)$ by x_i^n. The first members of the sequences $\{x_0^n\}_{n=1}^{\infty}$ and $\{y_0^n\}_{n=1}^{\infty}$ are

$$x_0 \mapsto \left(x_0 + \sum_{i=1}^{M-1}\frac{1}{2^i}\frac{1}{x_i}\right) \bmod 1 \mapsto \left(x_0 + \sum_{i=1}^{M-1}\frac{1}{2^i}\frac{1}{x_i} + \sum_{i=1}^{M-1}\frac{1}{2^i}\frac{1}{x_i+1}\right) \bmod 1 \ldots,$$

$$y_0 = x_0 \mapsto \left(x_0 + \sum_{i=1}^{M-1}\frac{1}{2^i}\frac{1}{x_i} + \sum_{i=M}^{\infty}\frac{1}{2^i}\frac{1}{K}\right) \bmod 1$$

$$\mapsto \left(x_0 + \sum_{i=1}^{M-1}\frac{1}{2^i}\frac{1}{x_i} + \sum_{i=M}^{\infty}\frac{1}{2^i}\frac{1}{K} + \sum_{i=1}^{M-1}\frac{1}{2^i}\frac{1}{x_i+1} + \sum_{i=M}^{\infty}\frac{1}{2^i}\frac{1}{K+1}\right) \bmod 1 \ldots.$$

The following equations are with modulus 1 whenever necessary. Since $d_0(x_0^n, y_0^n) \leq |x_0^n - y_0^n|$, where

$$|x_0^n - y_0^n| = \sum_{i=M}^{\infty} \sum_{j=0}^{n-1} \frac{1}{2^i} \frac{1}{K+j} = 2^{-M+1} \cdot \sum_{j=0}^{n-1} \frac{1}{K+j} \tag{11}$$

and

$$d_i(x_i^n, y_i^n) = 0 \quad \text{for } 0 < i < M, \quad d_i(x_i^n, y_i^n) = \frac{1}{K+n} \quad \text{for } i \geq M,$$

we can estimate

$$D(F^n(x), F^n(y)) \leq 2^{-M+1} \cdot \sum_{j=0}^{n-1} \frac{1}{K+j} + \sum_{i=M}^{\infty} \frac{1}{2^i} \frac{1}{K+n}.$$

Let $\epsilon > 0$. By (9) in Lemma 1, there is $\{v_n\}_{n=1}^{\infty}$ such that

$$\left(2^{-M+1} \cdot \sum_{j=0}^{v_n-1} \frac{1}{K+j}\right) \bmod 1 < \epsilon \quad \text{for } n \geq 1.$$

By Lemma 2 and since $\lim_{n\to\infty} (1/(K+n)) = 0$, we have $\sum_{i=M}^{\infty} (1/2^i)(1/(K+v_n)) < \epsilon$, for sufficiently large v. Therefore $\lim_{v\to\infty} D(F^{v_n}(x), F^{v_n}(y)) < 2\epsilon$ and $\liminf_{n\to\infty} D(F^n(x), F^n(y)) = 0$. Similarly, by (10) in Lemma 1, there is $\{u_n\}_{n=1}^{\infty}$ such that $\left|\left(2^{-M+1} \sum_{j=0}^{u_n-1} (1/(K+j))\right) \bmod 1 - \frac{1}{2}\right| < \epsilon$, for sufficiently large n. Therefore, by (11), $d_0(x_0^{u_n}, y_0^{u_n}) > \frac{1}{2} - \epsilon$ and $\limsup_{n\to\infty} D(F^n(x), F^n(y)) \geq \frac{1}{2}$. □

CLAIM 2. *x and y defined in (5)–(7) are a scrambled pair with modulus $\frac{1}{2}$.*

Proof. Denote the ith coordinate of $F^n(x)$ by x_i^n. The first members of the sequences $\{x_0^n\}_{n=1}^{\infty}$ and $\{y_0^n\}_{n=1}^{\infty}$ are

$$x_0 \mapsto \left(x_0 + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{x_i}\right) \bmod 1 \mapsto \left(x_0 + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{x_i} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{x_i+1}\right) \bmod 1 \ldots,$$

$$\begin{aligned} y_0 &= \left(x_0 + 1 - \sum_{i=1}^{\infty} \frac{1}{2^{i+M}} \delta_i\right) \bmod 1 \\ &\mapsto \left(x_0 + 1 - \sum_{i=1}^{\infty} \frac{1}{2^{i+M}} \delta_i + \sum_{i=1}^{M-1} \frac{1}{2^i} \frac{1}{x_i}\right) \bmod 1 \\ &\mapsto \left(x_0 + 1 - \sum_{i=1}^{\infty} \frac{1}{2^{i+M}} \delta_i + \sum_{i=1}^{M-1} \frac{1}{2^i} \frac{1}{x_i} + \sum_{i=1}^{M-1} \frac{1}{2^i} \frac{1}{x_i+1}\right) \bmod 1 \ldots. \end{aligned}$$

Notice that, for sufficiently large n,

$$\delta_i + \sum_{j=0}^{n-1} \frac{1}{x_{i+M}+j} = \sum_{j=0}^{n-1} \frac{1}{x_M+j} + \gamma_i^n, \tag{12}$$

where

$$\gamma_i^n = \begin{cases} 0 & \text{if } x_{i+M} = x_M, \\ \left(\sum_{j=x_M}^{x_{i+M}-1} \frac{1}{n+j} \right) \bmod 1 & \text{otherwise.} \end{cases}$$

The following equations are with modulus 1 whenever necessary. Since

$$\begin{aligned} d_0(x_0^n, y_0^n) &\leq |x_0^n - y_0^n| = \sum_{i=1}^{\infty} \frac{1}{2^{i+M}} \delta_i + \sum_{i=M}^{\infty} \sum_{j=0}^{n-1} \frac{1}{2^i} \frac{1}{x_i + j} \\ &= \sum_{i=1}^{\infty} \frac{1}{2^{i+M}} \left(\delta_i + \sum_{j=0}^{n-1} \frac{1}{x_{i+M} + j} \right) + \sum_{j=0}^{n-1} \frac{1}{2^M} \frac{1}{x_M + j} \\ &\overset{(12)}{=} \sum_{i=0}^{\infty} \frac{1}{2^{i+M}} \left(\sum_{j=0}^{n-1} \frac{1}{x_M + j} + \gamma_i^n \right) + \sum_{j=0}^{n-1} \frac{1}{2^M} \frac{1}{x_M + j} \\ &= 2^{-M+1} \cdot \sum_{j=0}^{n-1} \frac{1}{x_M + j} + \sum_{i=0}^{\infty} \frac{1}{2^{i+M}} \gamma_i^n \end{aligned}$$

and

$$d_i(x_i^n, y_i^n) = 0 \quad \text{for } 0 < i < M, \quad d_i(x_i^n, y_i^n) = \frac{1}{x_i + n} \quad \text{for } i \geq M,$$

we can estimate

$$D(F^n(x), F^n(y)) \leq 2^{-M+1} \cdot \sum_{j=0}^{n-1} \frac{1}{x_M + j} + \sum_{i=0}^{\infty} \frac{1}{2^{i+M}} \gamma_i^n + \sum_{i=M}^{\infty} \frac{1}{2^i} \frac{1}{x_i + n}.$$

Let $\epsilon > 0$. By (9) in Lemma 1, there is $\{v_n\}_{n=1}^{\infty}$ such that

$$\left(2^{-M+1} \cdot \sum_{j=0}^{v_n - 1} \frac{1}{x_M + j} \right) \bmod 1 < \epsilon \quad \text{for } n \geq 1.$$

By Lemma 2 and since $\lim_{n\to\infty} \gamma_i^n = 0$ and $\lim_{n\to\infty} (1/(x_i + n)) = 0$, for $i \geq 1$, we have

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+M}} \gamma_i^{v_n} < \epsilon \quad \text{and} \quad \sum_{i=M}^{\infty} \frac{1}{2^i} \frac{1}{x_i + v_n} < \epsilon,$$

for sufficiently large n. Therefore $\lim_{v\to\infty} D(F^{v_n}(x), F^{v_n}(y)) < 3\epsilon$ and $\liminf_{n\to\infty} D(F^n(x), F^n(y)) = 0$. Similarly, by (10) in Lemma 1, there is $\{u_n\}_{n=1}^{\infty}$ such that

$$\left| \left(2^{-M+1} \sum_{j=0}^{u_n - 1} \frac{1}{x_M + j} \right) \bmod 1 + \sum_{i=0}^{\infty} \frac{1}{2^{i+M}} \gamma_i^{u_n} - \frac{1}{2} \right| < 2\epsilon,$$

for sufficiently large n. Therefore $d_0(x_0^{u_n}, y_0^{u_n}) > \frac{1}{2} - 2\epsilon$ and $\limsup_{n\to\infty} D(F^n(x), F^n(y)) \geq \frac{1}{2}$. □

CLAIM 3. *If x_i and y_i are finite, for all $i \geq 1$, then $\lim_{n\to\infty} D(F^n(x), F^n(y))$ exists and (x, y) is not a scrambled pair.*

Proof. Let $r_i = |x_i - y_i|$. The following equations are with modulus 1 whenever necessary. Observe that, by Lemma 3,

$$\lim_{n\to\infty} \sum_{\substack{i=1\\ x_i\le y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{n-1} \left(\frac{1}{x_i+j} - \frac{1}{y_i+j}\right) = \lim_{n\to\infty} \sum_{\substack{i=1\\ x_i\le y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{n-1} \left(\frac{1}{x_i+j} - \frac{1}{x_i+r_i+j}\right)$$
$$= \sum_{\substack{i=1\\ x_i\le y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{r_i-1} \frac{1}{x_i+j} \tag{13}$$

and, similarly,

$$\lim_{n\to\infty} \sum_{\substack{i=1\\ x_i> y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{n-1} \left(\frac{1}{y_i+j} - \frac{1}{x_i+j}\right) = \sum_{\substack{i=1\\ x_i> y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{r_i-1} \frac{1}{y_i+j}. \tag{14}$$

Since

$$|x_0^n - y_0^n| = \left| x_0 - y_0 + \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{j=0}^{n-1} \left(\frac{1}{x_i+j} - \frac{1}{y_i+j}\right)\right|$$
$$= \left| x_0 - y_0 + \sum_{\substack{i=1\\ x_i\le y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{n-1} \left(\frac{1}{x_i+j} - \frac{1}{y_i+j}\right)\right.$$
$$\left. - \sum_{\substack{i=1\\ x_i> y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{n-1} \left(\frac{1}{y_i+j} - \frac{1}{x_i+j}\right)\right| \tag{15}$$

and

$$d_i(x_i^n, y_i^n) = \left|\frac{1}{x_i+n} - \frac{1}{y_i+n}\right| \quad \text{for } i \ge 1,$$

it follows, by Lemma 2 and by $\lim_{n\to\infty} |(1/(x_i+n)) - (1/(y_i+n))| = 0$, that

$$\lim_{n\to\infty} D(F^n(x), F^n(y)) = \lim_{n\to\infty} d_0(x_0^n, y_0^n) + \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{1}{2^i} \left|\frac{1}{x_i+n} - \frac{1}{y_i+n}\right|$$
$$= \min\{\lim_{n\to\infty} |x_0^n - y_0^n|, 1 - \lim_{n\to\infty} |x_0^n - y_0^n|\},$$

where, by equations (13)–(15),

$$\lim_{n\to\infty} |x_0^n - y_0^n| = \left| x_0 - y_0 + \sum_{\substack{i=1\\ x_i\le y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{r_i-1} \frac{1}{x_i+j} - \sum_{\substack{i=1\\ x_i> y_i}}^{\infty} \frac{1}{2^i} \sum_{j=0}^{r_i-1} \frac{1}{y_i+j}\right|.$$

Therefore $\lim_{n\to\infty} D(F^n(x), F^n(y))$ exists, which contradicts (x, y) being a scrambled pair. □

CLAIM 4. *For x and y defined in (8), $\lim_{n\to\infty} D(F^n(x), F^n(y))$ exists and (x, y) is not a scrambled pair.*

Proof. Let $r_i = |x_i - y_i|$, for $0 < i < l$. By a calculation similar to that in Claim 3,

$$\begin{aligned}\lim_{n\to\infty} D(F^n(x), F^n(y)) &= \lim_{n\to\infty} d_0(x_0^n, y_0^n) + \lim_{n\to\infty} \sum_{i=1}^{l-1} \frac{1}{2^i}\left|\frac{1}{x_i+n} - \frac{1}{y_i+n}\right| \\ &= \min\{\lim_{n\to\infty} |x_0^n - y_0^n|, 1 - \lim_{n\to\infty} |x_0^n - y_0^n|\},\end{aligned}$$

where

$$\lim_{n\to\infty} |x_0^n - y_0^n| = \left|x_0 - y_0 + \sum_{\substack{i=1\\ x_i \le y_i}}^{l-1} \frac{1}{2^i} \sum_{j=0}^{r_i-1} \frac{1}{x_i+j} - \sum_{\substack{i=1\\ x_i > y_i}}^{l-1} \frac{1}{2^i} \sum_{j=0}^{r_i-1} \frac{1}{y_i+j}\right|.$$

Therefore $\lim_{n\to\infty} D(F^n(x), F^n(y))$ exists, which contradicts (x, y) being a scrambled pair. □

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Distributionally scrambled invariant sets in a compact metric space☆

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ABSTRACT

The paper solves a question posed by Oprocha on the existence of invariant distributionally chaotic scrambled sets. We show, among other things, that a continuous map f acting on compact metric space (X, d) with a weak specification property, fixed point, and infinitely many mutually distinct periods has a dense Mycielski (i.e., c dense set of type F_σ) invariant distributionally scrambled set. As a consequence, we describe a class of maps with a distributionally scrambled invariant set of full Lebesgue measure in the case when X is a k-dimensional cube.

1. Introduction and main results

In 2005, it was proved by Du that an interval map f has positive topological entropy if and only if some of its iterates have an invariant Li–Yorke scrambled set [1]. Thus we get a new characterization of a class of interval maps with positive topological entropy, and this shows that invariant scrambled sets are of great interest. In 2010, Balibrea et al. considered invariant Li–Yorke chaos in [2] and constructed a dense Mycielski invariant ϵ-scrambled set. Another result about points with trajectories forming scrambled sets can be found in [3]. One of the most important extensions of the concept of Li–Yorke chaos is distributional chaos, introduced in [4]. Recently, Oprocha proved that if f is a continuous turbulent map of unit interval $f : I \to I$ then there exists a distributionally scrambled set D such that $f(D) \subset D$ and the distributional chaos is uniform [5]. His method cannot be extended to the mapping of a general compact metric space, so he stated following open question.

Question. *Does every map f with the specification property and a fixed point contain a distributionally scrambled set such that $f(D) \subset D$?*

The aim of this article is to prove that the answer is positive (but with some additional assumption about the set of periodic points) and also to study the topological size of such sets.

1.1. Terminology

Let (X, d) be a compact metric space. Let us denote $\mathcal{C}(X)$ the set of continuous self-maps acting on X. Let $f \in \mathcal{C}(X)$. Point x is *periodic* if $f^n(x) = x$ with period $n > 1$ but $f^i(x) \neq x$ for all $0 < i < n$. Point x is said to be *fixed* if $f(x) = x$. We define the *forward orbit* of x, denoted by $Orb_f^+(x)$, as the set $\{f^n(x) : n \geq 0\}$.

By I we denote the compact unit interval [0, 1]. By the k-dimensional unit cube we mean the set I^k, where $k \geq 1$. The Lebesgue measure on I^k will be denoted by λ. By a *perfect set* we mean a compact set without isolated points. By *Cantor set*

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we mean a nonempty, perfect, and totally disconnected set. We say that a set $D \subset X$ is *invariant* if $f(D) \subset D$. The *Mycielski set* is defined as a countable union of Cantor sets.

A function f mapping a compact metric space (X, d) into itself has the *strong specification property* (briefly *SSP*) if, for any $\delta > 0$, there is a positive integer $K(\delta)$ such that, for any integer $s \geq 2$, any set $\{y_1, y_2, \ldots, y_s\}$ of s points of X, and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 \cdots < j_s \leq k_s$ of $2s$ integers with $j_{m+1} - k_m \geq K(\delta)$ for $m = 0, \ldots, s-1$, there is a point x in X such that

$$f^n(x) = x, \tag{1}$$

where $n = K(\delta) + k_s$ and for each positive integer $m \leq s$ and all integers i with $j_m \leq i \leq k_m$

$$d(f^i(x), f^i(y_m)) < \delta. \tag{2}$$

A function f has *weak specification property* (briefly *WSP*) if f fulfills the above-mentioned conditions only for the special case $s = 2$. Because we do not need the full force of this notation to obtain our result, we can omit the periodicity condition (1) and denote this version of specification property by *WSP**. Actually, there are many various kinds of specification property (for example the generalized specification property in [6]).

For a pair (x, y) of points in X, define the *lower distribution function* generated by f, x, and y as

$$\Phi_{x,y}(\delta) = \liminf_{n\to\infty} \frac{1}{n} \#\{0 < j < n;\, d(f^j(x), f^j(y)) < \delta\},$$

and the *upper distributional function* as

$$\Phi^*_{x,y}(\delta) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 < j < n;\, d(f^j(x), f^j(y)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A.

A pair of points $(x, y) \in X^2$ is called *distributionally chaotic of type 1* (briefly *DC*1) if

$$\Phi^*_{x,y} \equiv 1 \quad \text{and} \quad \Phi_{x,y}(\delta) = 0, \quad \text{for some } 0 < \delta \leq \operatorname{diam} X.$$

A set containing at least two points is called a *distributionally scrambled set of type 1* (briefly a D1-scrambled set) if any pair of its distinct points is distributionally chaotic of type 1. We say that a continuous map from the unit cube I^k into itself exhibits *invariant distributional chaos of type 1 almost everywhere* (briefly invariant DC1 a.e.) if there exists a distributionally scrambled set $D \subset I^k$ of type 1 such that $\lambda(D) = 1$ and $f(D) \subset D$.

1.2. Distributionally scrambled invariant sets in a compact metric space

In this section we will show how an invariant distributionally scrambled set can be constructed. We will use the specification property because there is a close relation between distributional chaos and the specification property (see for example [6]).

Theorem 1. *Let (X, d) be a compact metric space, $\#X > 1$, and let $f : X \to X$ be a continuous mapping with WSP* which has a fixed point and infinitely many periodic points with mutually different periods. Then there is a point $x \in X$ such that $(f^i(x), f^j(x))$ is a DC1 pair for all $i \neq j$, i.e., the forward orbit of x is a D1-scrambled set.*

Remark. The assumption about fixed point is natural: if x belongs to an invariant scrambled set, then $(x, f(x))$ is proximal. By the compactness of X there is an increasing sequence k_i and a point $p \in X$ such that $\lim_{i\to\infty} f^{k_i}(x) = p$ and $d(f^{k_i}(x), f^{k_i}(f(x))) = 0$, which by continuity of f implies that $f(p) = p$.

In the previous theorem we obtained a countable D1-scrambled invariant set. Another interesting question is how large can this set be.

Theorem 2. *Let (X, d) be a compact metric space, $\#X > 1$, and let $f : X \to X$ be a continuous mapping with WSP* which has a fixed point and infinitely many periodic points with mutually different periods. Then there is a dense invariant D1-scrambled Mycielski set.*

Remark. Let I^k be the k-dimensional unit cube and $D \subset I^k$ be a dense union of perfect sets. Then D is homeomorphic to a set of full Lebesgue measure. An appropriate homeomorphism is obtained by application of the Oxtoby–Ulam theorem [7]. A similar method was used by Oprocha and Štefánková in [8].

Theorem 3 (*Oxtoby, Ulam*)**.** *Let $B \subset I^k$ and suppose that there exists a sequence $\{S_n\}_{n=1}^{\infty}$ of perfect sets S_n such that $\cup_{n=1}^{\infty} S_n$ is dense in I^k. Then there exists a homeomorphism $h : I^k \to I^k$ such that $h|_{\partial I^k} = id$ and $\lambda(h(B)) = 1$.*

Corollary. *Every map $f \in \mathcal{C}(I^k)$ with WSP*, fixed point, and infinitely many periodic points with mutually different periods is conjugate to some map $g \in \mathcal{C}(I^k)$ which exhibits invariant DC1 a.e.*

Proof. The map g is obtained by application of Theorems 2 and 3. It was proved in [9] that the image of a D1-scrambled set via conjugacy remains D1-scrambled and also that the image of an invariant set remains invariant. □

2. Proofs

Proof of Theorem 1. Denote by p the fixed point and by $\{q_i\}_{i=1}^{\infty}$ the set of periodic points with different periods. Let $\{a_j\}_{j=1}^{\infty}$ be the sequence

$$p, q_1, p, q_1, q_2, p, q_1, q_2, q_3, p, q_1, q_2, q_3, q_4, p, \dots,$$

and $\delta_i = \frac{\delta}{2^i}$ for any $i \geq 1$, where $\delta > 0$ is arbitrary. Let K_i be $K(\delta_i)$, for $i \geq 1$, from the definition of *WSP**. We can construct an increasing sequence of integers $0 = j_1 < k_1 < j_2 < k_2 < j_3 < k_3 < \cdots$ such that $j_{i+1} - k_i > K_i$, for any $i \geq 1$, and

$$\lim_{i\to\infty} \frac{k_i}{j_i} = \infty.$$

We now go through the recursive procedure.

Stage 1. Let $v \in X$ be an arbitrary point. Apply the definition of *WSP** with $\delta = \delta_1, y_1 = v, y_2 = a_1$, and integers j_1, k_1, j_2, k_2 to obtain a point x that satisfies

$$d(f^i(x), f^i(v)) < \delta_1, \quad \text{for } j_1 \leq i \leq k_1,$$
$$d(f^i(x), f^i(a_1)) < \delta_1, \quad \text{for } j_2 \leq i \leq k_2. \tag{3}$$

Denote this point x_1. Since f and its iterates are by compactness of X uniformly continuous, there is $\delta_{j(1)} \leq \delta_2$ such that every point x in the closed ball $B(x_1, \delta_{j(1)})$ with center in x_1 and radius $\delta_{j(1)}$ also satisfies (3).

Stage n, for $n \geq 2$. Apply the definition of *WSP** with $\delta = \delta_{j(n-1)}, y_1 = x_{n-1}, y_2 = a_n$, and integers $j_1, k_n, j_{n+1}, k_{n+1}$ to obtain a point x that satisfies

$$d(f^i(x), f^i(x_{n-1})) < \delta_{j(n-1)}, \quad \text{for } j_1 \leq i \leq k_n,$$
$$d(f^i(x), f^i(a_n)) < \delta_{j(n-1)}, \quad \text{for } j_{n+1} \leq i \leq k_{n+1}. \tag{4}$$

Denote this point by x_n, and find a number $\delta_{j(n)} \leq \delta_{n+1}$ such that every point x in the closed ball $B(x_n, \delta_{j(n)})$ satisfies (4) and also $B(x_n, \delta_{j(n)}) \subset B(x_{n-1}, \delta_{j(n-1)})$. By this recursive procedure we get a nested sequence $\{B(x_n, \delta_{j(n)})\}$ of nonempty closed sets. There is therefore at least one point $x \in \bigcap_{n=1}^{\infty} B(x_n, \delta_{j(n)})$, and by (4), for every $n \geq 1$,

$$d(f^i(x), f^i(a_n)) < \delta_n, \quad \text{for } j_{n+1} \leq i \leq k_{n+1}. \tag{5}$$

We claim that this x is the wanted point. Let $z = f^l(x), y = f^m(x)$, suppose that $l > m$, and denote $\Delta = l - m$. Because $a_{s_n} = p$, where $s_n = \sum_{j=1}^{n} j$, for every $n \geq 1$, we get

$$d(f^i(x), p) < \delta_{s_n}, \quad \text{for } j_{s_n+1} \leq i \leq k_{s_n+1},$$

and hence

$$d(f^i(y), f^i(z)) < 2\delta_{s_n}, \quad \text{for } j_{s_n+1} - m \leq i \leq k_{s_n+1} - m - \Delta.$$

It follows that

$$\#\{0 \leq i \leq k_{s_n+1};\, d(f^i(y), f^i(z)) < 2\delta_{s_n}\} \geq k_{s_n+1} - j_{s_n+1} - \Delta.$$

Since $\lim_{n\to\infty} \delta_{s_n} = 0$, it is easy to see that $\Phi^*_{y,z} \equiv 1$ for all $y \neq z$. Let us proceed with the lower distributional function. We can find periodic point q_r such that $|q_r| > \Delta$, where $|q_r|$ denotes the period of q_r. Let

$$\epsilon_r = \min_{\substack{i,j\in\{0,1,\dots,|q_r|-1\}\\ i\neq j}} d(f^i(q_r), f^j(q_r)).$$

Because $a_{s_n} = q_r$, where $s_n = r + \sum_{j=1}^{n+r-1} j$, for every $n \geq 1$, we get

$$d(f^i(x), f^i(q_r)) < \delta_{s_n}, \quad \text{for } j_{s_n+1} \leq i \leq k_{s_n+1},$$

and hence

$$d(f^i(y), f^i(z)) > \epsilon_r - 2\delta_{s_n}, \quad \text{for } j_{s_n+1} - m \leq i \leq k_{s_n+1} - m - \Delta.$$

It follows that

$$\#\{0 \leq i \leq k_{s_n+1};\, d(f^i(y), f^i(z)) < \epsilon_r - 2\delta_{s_n}\} \leq j_{s_n+1} + \Delta.$$

Since $\Phi_{y,z}$ is left-continuous, we get $\Phi_{y,z}(\epsilon_r) = \lim_{n\to\infty} \Phi_{y,z}(\epsilon_r - 2\delta_{s_n}) = 0$. □

Before proving Theorem 2, we need the following lemma.

Lemma 1. *There is a Cantor set* $B \subset \{0, 1\}^{\mathbb{N}}$ *such that, for any distinct* $\alpha = \{\alpha(i)\}_{i=1}^{\infty}$ *and* $\beta = \{\beta(i)\}_{i=1}^{\infty}$ *in B, the set*

$$\{j \in \mathbb{N}; \alpha(j) \neq \beta(j)\} \quad \text{is infinite.} \tag{6}$$

Proof. By Lemma 5.4 in [4] there is an uncountable Borel set $B \subset \{0, 1\}^{\mathbb{N}}$ with the desired property. The result follows from the Alexandrov–Hausdorff theorem [10]. □

Proof of Theorem 2. Let $\alpha \in B$, where B is the set from Lemma 1. Let $\{a_j^{(\alpha)}\}_{j=1}^{\infty}$ be the sequence

$$p, q_1, b_1^{(\alpha)}, p, q_1, q_2, b_2^{(\alpha)}, p, q_1, q_2, q_3, b_3^{(\alpha)}, p, q_1, q_2, q_3, q_4, b_4^{(\alpha)}, p, \dots, \tag{7}$$

with the same notation as in Theorem 1, where $\{b_j^{(\alpha)}\}_{j=1}^{\infty}$ is the sequence defined in the following way:

$$b_j^{(\alpha)} = p \quad \text{if } \alpha(j) = 0,$$
$$b_j^{(\alpha)} = q_1 \quad \text{if } \alpha(j) = 1.$$

By the recursive procedure from Theorem 1 we get a point $x^{(\alpha)}$ such that, for every $n \geq 1$,

$$d(f^i(x^{(\alpha)}), f^i(a_n^{(\alpha)})) < \delta_n, \quad \text{for } j_{n+1} \leq i \leq k_{n+1}.$$

Denote $C = \{x^{(\alpha)}; \alpha \in B\}$ and $\hat{D} = \cup_{i=0}^{\infty} f^i(C)$. We claim that $\hat{D}$ is a D1-scrambled set. If $y, z \in \hat{D}$ then there are $l, m \in \mathbb{N}_0$ and $\alpha, \beta \in B$ such that

$$y = f^l(x^{(\alpha)}) \quad \text{and} \quad z = f^m(x^{(\beta)}).$$

Case 1. If $\alpha = \beta$, $l \neq m$ then, as in the proof of Theorem 1, (y, z) is a DC1 pair.

Case 2. If $\alpha \neq \beta$, then suppose that $l \geq m$ and denote $\Delta = l - m$. By (7), there is a sequence of integers $\{s_n\}_{n=1}^{\infty}$ such that, for every $n \geq 1$, $a_{s_n} = p$, and therefore

$$d(f^i(y), p) < \delta_{s_n} \wedge d(f^i(z), p) < \delta_{s_n},$$

for all $j_{s_n+1} - m \leq i \leq k_{s_n+1} - m - \Delta$. Thus it follows that $\Phi_{y,z}^* \equiv 1$.

Because $\alpha \neq \beta$ and $\alpha, \beta \in B$, there is a subsequence $\{b_{j_k}\}_{k=1}^{\infty} \subset \{b_j\}_{j=1}^{\infty}$ such that, for every $k \geq 1$,

$$b_{j_k}^{(\alpha)} = p \wedge b_{j_k}^{(\beta)} = q_1 \quad \text{or} \quad b_{j_k}^{(\alpha)} = q_1 \wedge b_{j_k}^{(\beta)} = p.$$

By (7), there is a sequence of integers $\{s_n\}_{n=1}^{\infty}$ such that, for every $n \geq 1$, $a_{s_n} = b_{j_k}$, and therefore

$$d(f^i(y), f^i(b_{j_k}^{(\alpha)})) < \delta_{s_n} \wedge d(f^i(z), f^i(b_{j_k}^{(\beta)})) < \delta_{s_n},$$

for all $j_{s_n+1} - m \leq i \leq k_{s_n+1} - m - \Delta$. Then $\Phi_{y,z}(\epsilon) = 0$, where $\epsilon = \min_{i \in \{0,1,\dots,|q_1|-1\}} d(p, f^i(q_1))$.

Let $h : B \to C$ be a bijection such that, for all $\alpha \in B$,

$$h(\alpha) = x^{(\alpha)}.$$

To prove that h is a homeomorphism, it is sufficient to show that h is continuous. Let $\{\alpha_m\}_{m=1}^{\infty}$ be a converging sequence in B, i.e. $\lim_{m\to\infty} \alpha_m = \alpha$; then, for an arbitrary $i > 0$, there is m_0 such that, for all $m > m_0$, the first i members of sequences α_m and α are equal. Therefore also the first i members of $\{a_j^{(\alpha_m)}\}_{j=1}^{\infty}$ and $\{a_j^{(\alpha)}\}_{j=1}^{\infty}$ are equal, and both x^{α} and x^{α_m} belong to the same $\{B(x_i, \delta_{j(i)})\}$ from the recursive procedure. This exactly means that $\lim_{m\to\infty} x^{\alpha_m} = x^{\alpha}$; hence h is a homeomorphism and C is a Cantor set. Since C is D1-scrambled, the mapping $f^i|_C : C \to f^i(C)$ is one-to-one and $f^i|_C$ is a homeomorphism for every $i \geq 1$. Thus $\hat{D}$ is a union of Cantor sets, but it is not dense in X. From Theorem 1, for every $\alpha \in B$, $x^{(\alpha)} \in B(v, \delta)$, where $v \in X$ and $\delta > 0$ are arbitrary, and hence $C \subset B(v, \delta)$. Let $\{G_i\}_{i=1}^{\infty}$ denote a countable base of the topology of X consisting of open balls. Let $B(v, \delta) = G_1$ to obtain a Cantor D1-scrambled set C_1. If $\cup_{i=0}^{\infty} f^i(C_1) \neq X$, let $v \in X$ and $\delta > 0$ be the center and radius of the first open ball G_k such that $G_k \cap (\cup_{i=0}^{\infty} f^i(C_1)) = \emptyset$ to obtain a Cantor set $C_2 \subset G_k$. We must change the construction of the sequence $\{a_j\}_{j=1}^{\infty}$ to guarantee that the union

$$\bigcup_{\substack{i=0 \\ k\in\{1,2\}}}^{\infty} f^i(C_k)$$

remains D1-scrambled set as follows:

$$x^{(\alpha,1)} \in C_1 \Leftrightarrow \{a_j^{(\alpha,1)}\}_{j=1}^{\infty} = p, q_1, b_1^{(\alpha)}, \mathbf{q_1}, p, q_1, q_2, b_2^{(\alpha)}, \mathbf{q_1}, p, q_1, q_2, q_3, b_3^{(\alpha)}, \mathbf{q_1}, p, q_1, q_2, q_3, q_4, b_4^{(\alpha)}, \mathbf{q_1}, p, \dots,$$
$$x^{(\alpha,2)} \in C_2 \Leftrightarrow \{a_j^{(\alpha,2)}\}_{j=1}^{\infty} = p, q_1, b_1^{(\alpha)}, \mathbf{q_2}, p, q_1, q_2, b_2^{(\alpha)}, \mathbf{q_2}, p, q_1, q_2, q_3, b_3^{(\alpha)}, \mathbf{q_2}, p, q_1, q_2, q_3, q_4, b_4^{(\alpha)}, \mathbf{q_2}, p, \dots.$$

By induction we obtain $\cup_{i,k=0}^{\infty} f^i(C_k)$, which is invariant, dense, and D1-scrambled. □

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Scrambled and distributionally scrambled n-tuples

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This article investigates the relation between the distributional chaos and the existence of a scrambled triple. We show that for a continuous mapping f acting on a compact metric space (X, d), the possession of an infinite extremal distributionally scrambled set is not sufficient for the existence of a scrambled triple. We also construct an invariant Mycielski set with an uncountable extremal distributionally scrambled set without any scrambled triple.

Keywords: distributional chaos; scrambled tuples; morse minimal set

MSC (2010) Classification: Primary 37D45; 37B10

1. Introduction

The first definition of chaotic pairs appeared in the paper [5]. One of the most important extensions of the concept of Li–Yorke chaos is distributional chaos introduced in [6]. This extended definition is much stronger – there are many mappings which are chaotic in the sense of Li–Yorke but not distributionally chaotic. Another way how to extend the Li–Yorke chaos is looking on dynamics of tuples instead of dynamics of pairs. In 2005, [9] introduced the notation of n-chaos, for $n \geq 2$. The classical Li–Yorke chaos is just 2-chaos in this sense. Since Xiong [8] and Smítal [7] constructed some interval maps with zero topological entropy which are Li–Yorke chaotic, the Li–Yorke chaos is not a sufficient condition for positive topological entropy. But interval maps with zero topological entropy never contain scrambled triples [3] and hence existence of a scrambled triple implies positive topological entropy. Consequently we can find a dynamical system which is Li–Yorke chaotic but contains no scrambled triple. The natural question was if there is a dynamical system which is distributionally chaotic but contains no distributionally scrambled triple. Example in [4] contains no distributionally scrambled triple but still there were some scrambled triples (in the sense of Li–Yorke) and therefore another open problem appeared – is there a distributionally chaotic dynamical system without any scrambled triple? In this paper, we construct a dynamical system which possesses an infinitely countable extremal distributionally scrambled set but without any scrambled triple. We show the existence of a non-compact dynamical system which possesses an uncountable extremal distributionally scrambled set and has no scrambled triple. But the following question remains open: *In a compact dynamical system, does the existence of uncountable distributionally scrambled set imply the existence of a scrambled triple?*

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2. Terminology

Let (X, d) be a non-empty compact metric space. Let us denote by (X,f) the *topological dynamical system*, where f is a continuous self-map acting on X. We define the *forward orbit* of x, denoted by $\mathrm{Orb}_f^+(x)$ as the set $\{f^n(x) : n \geq 0\}$. A non-empty closed invariant subset $Y \subset X$ defines naturally a subsystem (Y,f) of (X,f). For $n \geq 2$, we denote by $(X^n,f^{(n)})$ the product system $(X \times X \times \cdots \times X, f \times f \times \cdots \times f)$ and put $\Delta^{(n)} = \{(x_1, x_2, \ldots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}$. By a *perfect set* we mean a non-empty compact set without isolated points. A *Cantor set* is a non-empty, perfect and totally disconnected set. A set $D \subset X$ is *invariant* if $f(D) \subset D$. A *Mycielski set* is defined as a countable union of Cantor sets.

Definition 1. *A tuple* $(x_1, x_2, \ldots, x_n) \in X^n$ *is called* n-scrambled *if*

$$\liminf_{k\to\infty} \max_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) = 0 \tag{1}$$

and

$$\limsup_{k\to\infty} \min_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) > 0. \tag{2}$$

A subset S *of* X *is called* n-scrambled *if every* n-*tuple* $(x_1, x_2, \ldots, x_n) \in S^n \backslash \Delta^{(n)}$ *is* n-*scrambled. The system* (X,f) *is called* n-chaotic *if there exists an uncountable* n-*scrambled set.*

Definition 2. *For an* n-*tuple* $(x_1, x_2, \ldots, x_n)$ *of points in* X, *define the* lower distribution function *generated by f as*

$$\Phi_{(x_1,x_2,\ldots,x_n)}(\delta) = \liminf_{m\to\infty} \frac{1}{m} \# \left\{ 0 < k < m;\ \min_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) < \delta \right\},$$

and the upper distributional function as

$$\Phi^*_{(x_1,x_2,\ldots,x_n)}(\delta) = \limsup_{m\to\infty} \frac{1}{m} \# \left\{ 0 < k < m;\ \max_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) < \delta \right\},$$

where #A denotes the cardinality of set A.

A tuple $(x_1, x_2, \ldots, x_n) \in X^n$ *is called* distributionally n-scrambled *if*

$$\Phi^*_{(x_1,x_2,\ldots,x_n)} \equiv 1 \text{ and } \Phi_{(x_1,x_2,\ldots,x_n)}(\delta) = 0, \text{ for some } 0 < \delta \leq \operatorname{diam} X.$$

A subset S *of* X *is called* distributionally n-scrambled *if every* n-*tuple* $(x_1, x_2, \ldots, x_n) \in S^n \backslash \Delta^{(n)}$ *is distributionally* n-*scrambled. The system* (X,f) *is called* distributionally n-chaotic *if there exists an uncountable distributionally* n-*scrambled set.*

Definition 3. *A subset* S *of* X *is called* extremal distributionally n-scrambled *if every* n-*tuple* $(x_1, x_2, \ldots, x_n) \in S^n \backslash \Delta^{(n)}$ *is distributionally* n-*scrambled with* $\Phi_{(x_1,x_2,\ldots,x_n)}(\delta) = 0$, *for any* $\delta <$ diam$X$.

Let $A = \{0, 1, \ldots, n-1\}, n \geq 2$ be a finite alphabet and Σ_n a set of all infinite sequences on A, that is, for $u \in \Sigma_n$, $u = u_1 u_2 u_3 \ldots$, where $u_i \in A$ for all $i \geq 1$. We define a metric on Σ_n by

$$d(u, v) = \sum_{i=1}^{\infty} \frac{\delta(u_i, v_i)}{2^i},$$

where

$$\delta(u_i, v_i) = \begin{cases} 0, & u_i = v_i \\ 1, & u_i \neq v_i. \end{cases}$$

The *shift* transformation is a continuous map $\sigma : \Sigma_n \to \Sigma_n$ given by $\sigma(u)_i = u_{i+1}$. The dynamical system (Σ_n, σ) is called the one-sided shift on n symbols. Any closed subset $X \subset \Sigma_n$ invariant for σ is called a subshift of (Σ_n, σ).

Any finite string B of some $u \in \Sigma_n$ is called a word (or a block) and the length of B is denoted by $|B|$. Let $B = b_1 \ldots b_n$ and $G = g_1 \cdots g_n$ be words. Denote by $BG = b_1 \cdots b_n g_1 \cdots g_n$ and for the case of Σ_2 denote by $\bar{B}$ the binary complement of B.

The *Morse block* M_i is defined inductively such that $M_0 = 0$, and $M_i = M_{i-1}\bar{M}_{i-1}$, for all $i > 0$. The *Morse sequence* $m \in \Sigma_2$ is the limit of the Morse blocks, i.e. $m = \lim_{i\to\infty} M_i$. This sequence m generates the infinite Morse minimal set $M = \mathrm{cl}\{m, \sigma(m), \sigma^2(m), \ldots\}$ and it is known that, for all words $B \subset m$, the sequence m contains no block BBb, where b is the first element of block B (cf. [2]). Denote this property by $\mathcal{P}$

$$m \text{ has property } \mathcal{P} \Leftrightarrow \nexists BBb \subset m. \tag{3}$$

3. Scrambled and distributionally scrambled *n*-tuples

We will show that the existence of an infinitely countable extremal distributionally scrambled set does not imply the existence of a scrambled triple. Then we construct a non-compact dynamical system with an uncountable extremal distributionally scrambled set without any scrambled triple.

LEMMA 1. *Let* $\{a_i\}_{i=1}^{\infty}$ *be a strictly increasing sequence of positive integers such that, for every* $n \geq 1$, a_n *and* n *have the same parity. Then the point* $x = M_{a_1}M_{a_2}M_{a_3}\cdots$ *is contained in the Morse minimal set.*

Proof. Since a_n and a_{n+1} have different parity, for every $n \geq 1$, the Morse block $M_{a_{n+1}}$ ends with M_{a_n}

$$M_{a_{n+1}} = M_{a_n}\bar{M}_{a_n}\bar{M}_{a_n}M_{a_n}\cdots\bar{M}_{a_n}. \tag{4}$$

We construct a sequence $\{m_k\}_{k=1}^{\infty}$, such that every $m_k \in \{0,1\}^{\mathbb{N}}$ starts with blocks $M_{a_1}M_{a_2}M_{a_3}\cdots M_{a_k}$, in the following way:

$$m_1 = m = M_{a_1}\bar{M}_{a_1}\bar{M}_{a_1}M_{a_1}\bar{M}_{a_1}M_{a_1}M_{a_1}\bar{M}_{a_1}\cdots,$$

We rewrite m using (4)

$$m = M_{a_2}\bar{M}_{a_2}\underbrace{\bar{M}_{a_1}M_{a_1}M_{a_1}\bar{M}_{a_1}\cdots M_{a_1}}_{\bar{M}_{a_2}}M_{a_2}\bar{M}_{a_2}M_{a_2}M_{a_2}\bar{M}_{a_2}\cdots$$

and set $m_2 = \sigma^{3\cdot 2^{a_2}-2^{a_1}}(m)$, since $|M_{a_n}| = 2^{a_n}$. We can proceed similarly for an arbitrary k,

$$m = M_{a_k}\bar{M}_{a_k}\underbrace{\bar{M}_{a_{k-1}}M_{a_{k-1}}M_{a_{k-1}}\bar{M}_{a_{k-1}}\cdots\underbrace{\bar{M}_{a_{k-2}}M_{a_{k-2}}M_{a_{k-2}}\bar{M}_{a_{k-2}}\cdots\bar{M}_{a_{k-2}}M_{a_{k-2}}}_{\bar{M}_{a_{k-1}}}M_{a_{k-1}}}_{\bar{M}_{a_k}}M_{a_k}\cdots,$$

and set $m_k = \sigma^{r_k}(m)$, where $r_k = 3\cdot 2^{a_k} - \sum_{i=1}^{k-1}2^{a_i}$. Since $\sigma^{r_k}(m)$ starts with blocks $M_{a_1}M_{a_2}M_{a_3}\cdots M_{a_k}$, where $r_k = 3\cdot 2^{a_n} - \sum_{i=1}^{k-1}2^{a_i}$, for all $n > 1$, the point $x = \lim_{k\to\infty}\sigma^{r_k}(m)$ is contained in the Morse minimal set. □

THEOREM 1. *There exists a dynamical system X with an infinite extremal distributionally scrambled set but without any scrambled triple.*

Proof. For $k \in \mathbb{N}$, let W_k be the set of positive integers of the form $2^n\cdot(2k-1), n \geq 1$. Obviously, $W_1, W_2, W_3, \ldots$ is a decomposition of the set $2\mathbb{N}$ of even integers.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then

$$\lim_{n\to\infty}(a_n - a_{n-1}) = \infty \text{ implies } \lim_{n\to\infty}\frac{\sum_{i=1}^{n-1}2^{a_i}}{2^{a_n}} = 0. \tag{5}$$

The formula follows easily since, for every large n,

$$\frac{\sum_{i=1}^{n-1}2^{a_i}}{2^{a_n}} \leq \frac{2\cdot 2^{a_{n-1}}-1}{2^{a_n}} < \frac{2}{2^{a_n-a_{n-1}}}.$$

We construct the point x^i as a sequence of blocks

$$M_{a_1}M^i_{a_2}M_{a_3}M^i_{a_4}M_{a_5}M^i_{a_6}M_{a_7}M^i_{a_8}\cdots,$$

where

$$M^i_{a_j} = \begin{cases} M_{a_j}, & \text{if } j \notin W_i \\ \bar{M}_{a_j}, & \text{if } j \in W_i. \end{cases}$$

Let i be a fixed positive integer. Then the first complementary block $\bar{M}_{a_j}$ appears in the construction of x^i for $j = 2\cdot(2i-1)$,

$$x^i = M_{a_1}M_{a_2}M_{a_3}M_{a_4}M_{a_5}\cdots M_{a_{2\cdot(2i-1)-1}}\bar{M}_{a_{2\cdot(2i-1)}}\cdots.$$

and the sequence $\{x^i\}_{i=1}^{\infty}$ converges. Since we can choose the sequence $\{a_n\}_{n=1}^{\infty}$ so that, for every $n \geq 1$, a_n and n have the same parity,

$$\lim_{i\to\infty} x^i = x, \tag{6}$$

where x is the point constructed in Lemma 1.

Let $D = \{x^i\}_{i=1}^{\infty}$. We claim that D is a distributionally 2-scrambled set and $X = \mathrm{cl}(\bigcup_{i=0}^{\infty}\sigma^i(D))$ is the wanted dynamical system.

I. D is an extremal distributionally 2-scrambled set

Let $(x^i, x^j) \in D^2$ be a pair of distinct points. For simplicity denote $s_k = \sum_{n=1}^{k} 2^{a_n}$, where $|M_{a_n}| = 2^{a_n}$. Let l be a fixed positive integer and $\epsilon = 1/2^l$. Since $(x^i)_n = (x^j)_n$ if $s_{2k} < n \leq s_{2k+1}$, for any $k > 0$, we have $d(\sigma^n(x^i), \sigma^n(x^j)) < \epsilon$ for all $s_{2k} < n < s_{2k+1} - l$. By (5),

$$\lim_{k\to\infty} \frac{2^{a_{2k+1}} - l}{2^{a_{2k+1}} + s_{2k}} = 1,$$

so it is easy to see that $\Phi^*_{x^i,x^j}(\epsilon) = 1$, for arbitrary small ϵ, and hence $\Phi^*_{x^i,x^j} \equiv 1$.

On the other hand, there is a sequence $\{l_k\}_{k=1}^{\infty} \subset W_i \cup W_j$ such that $(x^i)_n = \overline{(x^j)}_n$ if $s_{l_k-1} < n \leq s_{l_k}$, for any integer k. Since $(x^i)_m = \overline{(x^j)}_m$, for $m = 1, 2, \ldots r$, implies $d(x^i, x^j) \geq \sum_{m=1}^{r}(1/2^m)$ and $(\sigma^n(x^i))_m = \overline{(\sigma^n(x^j))}_m$, for all $s_{l_k-1} < n < s_{l_k} - r$ and $m = 1, 2, \ldots r$, it follows $d(\sigma^n(x^i), \sigma^n(x^j)) \geq \sum_{m=1}^{r}(1/2^m)$ for all $s_{l_k-1} < n < s_{l_k} - r$. Because

$$\lim_{k\to\infty} \frac{s_{l_k-1}}{s_{l_k-1} + 2^{a_{l_k}} - r} = 0,$$

it is easy to see that $\Phi_{x^i,x^j}(\sum_{m=1}^{r}(1/2^m)) = 0$, for arbitrary large r, and hence $\Phi_{x^i,x^j}(\delta) = 0$, for any $0 < \delta < 1$.

II. $\bigcup_{i=0}^{\infty}\sigma^i(D)$ has no scrambled triples

Let $(x^i, x^j, x^k) \in D^3 \setminus \Delta^{(3)}$. Since

$$M^i_{2n} = M^j_{2n} \text{ or } M^i_{2n} = M^k_{2n} \text{ or } M^j_{2n} = M^k_{2n},$$

and M_{2n-1} is the common block for all x^i, x^j, x^k and every $n > 1$, we can assume

$$\lim_{n\to\infty} \min\{d(\sigma^n(x^i), \sigma^n(x^j)), d(\sigma^n(x^i), \sigma^n(x^k)), d(\sigma^n(x^k), \sigma^n(x^j))\} = 0$$

and consequently, condition (2) is not satisfied and $\sigma^n(D)$ has no scrambled triples, for any $n \geq 0$. By the same argument, (x^i, x^j, x) is never a scrambled triple, where $x = \lim_{i\to\infty} x^i$. It follows that any potential scrambled triple in X must contain some pair $\sigma^p(u), \sigma^q(v)$, where $p < q$ and $u, v \in D$. To prove that for such tuple condition (1) is not fulfilled, it is sufficient to show that

$$\liminf_{k\to\infty} d(\sigma^k(\sigma^p(u)), \sigma^k(\sigma^q(v))) > 0, \tag{7}$$

where $p < q$ and $u, v \in D$. Assume the contrary – let $\liminf_{k\to\infty} d(\sigma^k(\sigma^p(u)), \sigma^k(\sigma^q(v))) = 0$ and denote $r = q - p > 0$. Then we can find an infinite subsequence $\{k_n\}_{n=1}^{\infty}$ such that both $\sigma^{k_n}(\sigma^p(u))$ and $\sigma^{k_n}(\sigma^q(v))$ begin with the same block G_n of length $14r$ and obviously these blocks can be found also in sequence u and, shifted by r, in v. For sufficiently large n, G_n is in the sequence u either contained in some Morse block or is on the edge of two following Morse blocks, but at least the first $7r$ digits or the last $7r$ digits of block G_n are contained in a single Morse block. Denote this block $M^{(u)}_{a_j}$, where $M^{(u)}_{a_j}$ is either M_{a_j} or $\bar{M}_{a_j}$), depending on u and j, and these $7r$ consecutive digits of G_n by $G = g^u_1 g^u_2 \cdots g^u_{7r}$. This G appears in v shifted by r, so we can conclude that the first $6r$ digits $g^v_1 g^v_2 \cdots g^v_{6r}$ of G in v are in $M^{(v)}_{a_j}$. The block $M^{(v)}_{a_j}$ is either the same Morse block as $M^{(u)}_{a_j}$ or its binary complement. In the first case, $g^u_1 = g^v_1 = g^u_{r+1} = g^v_{r+1} = g^u_{2r+1} = \cdots = g^u_{5r+1}$ and similarly for $g^u_2, \ldots, g^u_r$ and therefore we obtained a block BBBBBB which is impossible since $M^{(u)}_{a_j}$ is a Morse block and has property $\mathcal{P}$ in (3). In the second case, $g^u_1 = g^v_1 = \bar{g}^u_{r+1} = \bar{g}^v_{r+1} = \bar{\bar{g}}^u_{2r+1} = g^u_{2r+1} = \cdots = g^u_{4r+1}$ and similarly for $g^u_2, \ldots, g^u_{2r}$ and therefore we obtained a block BBB which is a contradiction with $M^{(u)}_{a_j}$ is a Morse block.

III. If $y \in X \backslash \bigcup_{i=0}^{\infty} \sigma^i(D)$, then there exists a non-negative integer n such that $\sigma^n(y)$ is contained in the Morse minimal set M.

Suppose the contrary. By [2], the points of Morse minimal sets are characterized by the property $\mathcal{P}$ in (3) and therefore we can find two blocks $B_1B_1b_1$ and $B_2B_2b_2$ which appear in different places of y, where b_1 denotes the first element of B_1 and b_2 denotes the first element of B_2. Let the last element of $B_1B_1b_1$ be on the k_1-th position in y, the last element of $B_2B_2b_2$ be on the k_2-th position in y and $k_2 > k_1 + 2 \cdot |B_2|$. Since y is contained in the closure of $\bigcup_{i=0}^{\infty} \sigma^i(D)$, there are sequences $\{x^{m_i}\}_{i=1}^{\infty}$ and $\{n_i\}_{i=1}^{\infty}$ such that

$$y = \lim_{i \to \infty} \sigma^{n_i}(x^{m_i}),$$

and suppose all sequences $\sigma^{n_i}(x^{m_i})$ have the same first k_2 symbols. Let j be an integer such that $2^{a_j} > k_2$. We can observe that n_i is bounded by $\sum_{l=1}^{j-1} 2^{a_l}$ for all $i > 0$ – for larger n_i, either $B_1B_1b_1$ or $B_2B_2b_2$ would be part of a single Morse block. Hence there is a non-negative integer N and a subsequence $\{x^{m_{i_k}}\}_{k=1}^{\infty}$ such that $y = \lim_{i\to\infty} \sigma^{n_i}(x^{m_i}) = \lim_{i\to\infty} \sigma^N(x^{m_{i_k}}) = \sigma^N(x)$, where the last identity follows by (6). By Lemma 1, $y = \sigma^N(x) \in M$ and this is a contradiction with assumptions.

IV. X has no scrambled triples.

By [1] the Morse minimal set is an almost distal system, i.e. all pairs in this set are either distal or asymptotic. Hence by II and (7) the only potential scrambled triples are (x^i, x^j, y), where $y \in X \backslash \bigcup_{i=0}^{\infty} \sigma^i(D)$ and $x^i, x^j \in D$. By the previous step, it is sufficient to show that (x^i, x^j, y) is not a scrambled triple, where $y \in M$. Since $y \in M$ and $x \in M$ by Lemma 1, the pair (x, y) is either distal or asymptotic:

(a) (x, y) is a distal pair. Sequences x, x^i, x^j are exactly the same except or blocks $M^i_{a_l}$ and $M^j_{a_l}$ where $l \in W_i \cup W_j$ and it holds $M^i_{a_l} = \bar{M}^j_{a_l}$, for $l \in W_i \cup W_j$. Therefore (x^i, x^j, y) is not proximal.

(b) (x, y) is an asymptotic pair. The triple (x^i, x^j, x) is not scrambled by II, therefore (x^i, x^j, y) is not scrambled.

□

To prove Theorem 2, we need the next lemma:

LEMMA 2. *There is a Cantor set $B \subset \{0,1\}^{\mathbb{N}}$ such that, for any distinct $\alpha = \{\alpha(i)\}_{i=1}^{\infty}$ and $\beta = \{\beta(i)\}_{i=1}^{\infty}$ in B, the set*

$$\{j \in \mathbb{N}; \alpha(j) \neq \beta(j)\} \text{ is infinite.} \tag{8}$$

Proof. Let $h : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be a mapping such that

$$h(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\ldots) = \alpha_1\alpha_1\alpha_2\alpha_1\alpha_2\alpha_3\alpha_1\alpha_2\alpha_3\alpha_4\cdots.$$

Since h is clearly a continuous bijection onto some subset $B \subset \{0,1\}^{\mathbb{N}}$, it is sufficient to show that any pair of distinct points $\alpha = \{\alpha(i)\}_{i=1}^{\infty}$ and $\beta = \{\beta(i)\}_{i=1}^{\infty}$ in B satisfies (8). But if there is i for which $\alpha(i) \neq \beta(i)$, then, by construction of h, this non-equality must be repeated on infinitely many places. □

Theorem 2. *There exists an invariant Mycielski set* $X \subset \Sigma_2$ *with an uncountable extremal distributionally* 2*-scrambled set but without any scrambled triple.*

Proof. We will denote by M_i^0 the Morse block M_i and by M_i^1 the binary complement of M_i. Let $\alpha = \{\alpha(i)\}_{i=1}^{\infty}$ be a point of B where B is the set from Lemma 2. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers with $\lim_{n\to\infty} a_n - a_{n-1} = \infty$. We construct a point x^{α} as a sequence of blocks

$$M_{a_1}^{\alpha_1}M_{a_2}^{0}M_{a_3}^{\alpha_2}M_{a_4}^{0}M_{a_5}^{\alpha_3}M_{a_6}^{0}M_{a_7}^{\alpha_4}\cdots.$$

Let $D = \{x^{\alpha}; \alpha \in B\}$. We claim that D is an extremal distributionally 2-scrambled set and $X = \bigcup_{i=0}^{\infty}\sigma^i(D)$ is the wanted Mycielski set. Since part of the proof is similar to steps I and II of the proof of Theorem 1, we show only a shortened explanation:

I. D is an extremal distributionally 2*-scrambled set*

Let $(u,v) \in D^2$ be a pair of distinct points. For simplicity denote $s_i = \sum_{j=1}^{i} 2^{a_j}$. Let l be a fixed integer and $\epsilon = 1/2^l$. Since $u_i = v_i$ if $s_{2k-1} < i \le s_{2k}$, for any $k > 0$, we have $d(\sigma^i(u), \sigma^i(v)) < \epsilon$ for all $s_{2k-1} < i < s_{2k} - l$. By (5),

$$\lim_{k\to\infty} \frac{2^{a_{2k}} - l}{2^{a_{2k}} + s_{2k-1}} = 1,$$

so it is easy to see that $\Phi_{u,v}^{*}(\epsilon) = 1$, for arbitrary small ϵ, and hence $\Phi_{u,v}^{*} \equiv 1$.

On the other hand, there is a sequence $\{l_k\}_{k=1}^{\infty}$ such that $u_i = \bar{v}_i$ if $s_{l_k-1} < i \le s_{l_k}$, for any integer k. Since $u_m = \bar{v}_m$, for $m = 1, 2, \ldots r$, implies $d(u,v) \ge \sum_{m=1}^{r}(1/2^r)$ and it follows $d(\sigma^i(u), \sigma^i(v)) \ge \sum_{m=1}^{r}(1/2^r)$ for all $s_{l_k-1} < i < s_{l_k} - r$ and $m = 1, 2, \ldots r$. Because

$$\lim_{k\to\infty} \frac{s_{l_k-1}}{s_{l_k-1} + 2^{a_{l_k}} - r} = 0,$$

it is easy to see $\Phi_{u,v}(\sum_{m=1}^{r}(1/2^m)) = 0$, for arbitrary large r, and hence $\Phi_{u,v}(\delta) = 0$, for any $0 < \delta < 1$.

II. X has no scrambled triples

Let $(x^\alpha, x^\beta, x^\gamma) \in D^3 \setminus \Delta^{(3)}$. Since $\alpha_i, \beta_i, \gamma_i \in \{0, 1\}$, for any integer i,

$$M_{2i-1}^{\alpha_i} = M_{2i-1}^{\beta_i} \text{ or } M_{2i-1}^{\alpha_i} = M_{2i-1}^{\gamma_i} \text{ or } M_{2i-1}^{\beta_i} = M_{2i-1}^{\gamma_i},$$

and M_{2i}^0 is the common block for all $x^\alpha, x^\beta, x^\gamma$, we can assume

$$\lim_{k\to\infty} \min\{d(\sigma^k(x^\alpha), \sigma^k(x^\beta)), d(\sigma^k(x^\alpha), \sigma^k(x^\gamma)), d(\sigma^k(x^\gamma), \sigma^k(x^\beta))\} = 0$$

and consequently, condition (2) is not satisfied and D has no scrambled triples. For the same reason $\sigma^i(D)$ has no scrambled triples, for any $i > 0$. It follows that any potential scrambled triple in X must contain some pair $\sigma^p(u), \sigma^q(v)$, where $p < q$ and $u, v \in D$. The fact that for such tuple condition (1) is not fulfilled, can be proven in a similar way to the second step of the proof of the previous theorem.

III. X is a Mycielski set

Let $h : B \to D$ be a bijection such that, for all $\alpha \in B$,

$$h(\alpha) = x^{(\alpha)}.$$

To prove that h is homeomorphism, it is sufficient to show that h is continuous. Let $\{\alpha_m\}_{m=1}^\infty$ be a converging sequence in B, i.e. $\lim_{m\to\infty} \alpha_m = \alpha$. Then for an arbitrary $i > 0$ there is an m_0 such that, for all $m > m_0$, the first i members of the sequences α_m and α are equal. Therefore also the first $2^{a_1} + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_{(2i-1)}}$ members of x^{α_m} and x^α are equal and this exactly means $\lim_{m\to\infty} x^{\alpha_m} = x^\alpha$, hence h is homeomorphism and D is a Cantor set. Since D is a distributionally 2-scrambled, the mapping $\sigma^i|_D : D \to \sigma^i(D)$ is one-to-one and $\sigma^i|_D$ is homeomorphism for every $i \geq 1$. Thus X is the union of Cantor sets. □

Remark. There are some scrambled triples (x^α, x^β, x) in the closure of X, where $x \in \mathrm{cl}(X) \setminus X$ and depends on the parity of $\{a_n\}_{n=1}^\infty$. If a_n and n have the same parity, then there exist α and β such that (x^α, x^β, x) is scrambled, where $x = \bar{M}_{a_1} \bar{M}_{a2} \bar{M}_{a_3} \bar{M}_{a_4} \ldots$.

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Distributional chaos and factors

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We show the existence of a dynamical system without any distributionally scrambled pair which is semiconjugated to a distributionally chaotic factor.

Keywords: distributional chaos; semi-conjugacy; chaotic factor

MSC (2010) Classification: 37D45; 37B40

1. Introduction

Semi-conjugacy is used as a common tool for proving topological chaos or positive topological entropy. The usual technique is to find a semi-conjugacy π with a chaotic system and transfer the chaos to the extension. By continuity of π, the topological entropy of the extension is not smaller than the entropy of factor system. Unfortunately, semi-conjugacy may not automatically guarantee the distributional chaos, which was introduced in [3]. Authors in [1,2,4] developed several techniques for proving distributional chaos via semi-conjugacy, usually using a symbolic space as the factor space. Example in [2] shows the existence of distributionally chaotic factor which is semi-conjugated to the system with no three-points distributionally scrambled sets. The aim of this study is to improve this result and find a distributionally chaotic factor which has an extension without any distributionally scrambled pair.

2. Terminology

Let (X, d) be a non-empty compact metric space. Let us denote by (X, f) the *topological dynamical system*, where f is a continuous self-map acting on X. We define the *forward orbit* of x, denoted by $Orb_f^+(x)$ as the set $\{f^n(x) : n \geq 0\}$. Let (X, F) and (Y, f) be dynamical systems on compact metric spaces. A continuous map $\pi : X \longrightarrow Y$ is called a semi-conjugacy between f and F if π is surjective and $\pi \circ f = F \circ \pi$. In this case, we can say that (Y, f) is a factor of the system (X, F), equivalently (X, F) is an extension of the system (Y, f).

Definition 1. *A pair* $(x_1, x_2) \in X^2$ *with* $x_1 \neq x_2$ *is called* scrambled *if*

$$\liminf_{k\longrightarrow\infty} \mathrm{d}\big(f^k(x_1), f^k(x_2)\big) = 0 \tag{1}$$

and

$$\limsup_{k\longrightarrow\infty} \mathrm{d}\big(f^k(x_1), f^k(x_2)\big) > 0. \tag{2}$$

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A subset S of X is called scrambled *if every pair of distinct points in S scrambled. The system* (X, f) *is called* chaotic *if there exists an uncountable scrambled set.*

DEFINITION 2. *For a pair* $(\mathrm{x}_1, \mathrm{x}_2)$ *of points in* X, *define the lower distribution function generated by f as*

$$\Phi_{(x_1,x_2)}(\delta) = \liminf_{m \to \infty} \frac{1}{m} \#\{0 < k < m; \mathrm{d}\big(f^k(x_1), f^k(x_2)\big) < \delta\},$$

and the upper distributional function as

$$\Phi^*_{(x_1,x_2)}(\delta) = \limsup_{m \to \infty} \frac{1}{m} \#\{0 < k < m; \mathrm{d}\big(f^k(x_1), f^k(x_2)\big) < \delta\},$$

where #A denotes the cardinality of the set A.

A pair $(x_1, x_2) \in X^2$ *is called* distributionally scrambled of type 1 *if*

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \quad \text{for every} \quad 0 < \delta \le \text{diam} \ X$$

and

$$\Phi_{(x_1,x_2)}(\varepsilon) = 0, \quad \text{for some} \quad 0 < \varepsilon \le \text{diam} \ X,$$

distributionally scrambled of type 2 *if*

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \quad \text{for every} \quad 0 < \delta \le \text{diam} \ X$$

and

$$\Phi_{(x_1,x_2)}(\delta) < \Phi^*_{(x_1,x_2)}(\delta), \quad \text{for every} \quad \delta \in (a, b), \quad \text{where} \quad 0 \le a < b \le \text{diam} \ X,$$

distributionally scrambled of type 3 *if*

$$\Phi_{(x_1,x_2)}(\delta) < \Phi^*_{(x_1,x_2)}(\delta), \quad \text{for every} \quad \delta \in (a, b), \quad \text{where} \quad 0 \le a < b \le \text{diam} \ X.$$

The dynamical system (X, f) is distributionally chaotic of type i (DCi for short), where $i = 1, 2, 3$, if there is an uncountable set $S \subset X$ such that any pair of distinct points from S is distributionally scrambled of type i.

3. Distributional chaos and factors

We will show the existence of a system without any distributionally scrambled pair which is semi-conjugated to a distributionally chaotic factor. This system is three-dimensional union of countably many homocentric cylinders with unit height and converging radius. First, we state the following technical lemma about rotation on circle. Let I be the unit closed interval $\langle 0, 1 \rangle$ and $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ be the unit circle. Let $u \in \mathbb{S}$ and $v \in \mathbb{S}$ be determined by normed angles $\phi_u \in I$ and $\phi_v \in I$. These points rotate along the circle by different angles $r_u \in I$, respectively, $r_v \in I$, i.e.

$$\begin{aligned} \phi_u &\mapsto (\phi_u + r_u) \mod 1 \\ \phi_v &\mapsto (\phi_v + r_v) \mod 1. \end{aligned} \tag{3}$$

We denote the relative angle of rotation by $\Delta r = |r_u - r_v|$ and assume that the metric on $\mathbb{S}$ is $\rho(\alpha, \beta) = \min\{|\alpha - \beta|, 1 - |\alpha - \beta|\}$.

LEMMA 1. *For every number $\delta > 0$ and every integer $p > 2/\Delta r$, the following estimation holds:*

$$\frac{1}{p}\#\{0 < i < p; \rho((\phi_u + ir_u) \mod 1, (\phi_v + ir_v) \mod 1) < \delta\} < 3\delta.$$

Proof. Because $\rho((\phi_u + ir_u) \mod 1, (\phi_v + ir_v) \mod 1) = \rho(\phi_u, (\phi_v + i\Delta r) \mod 1)$, it is sufficient to show

$$\frac{1}{p}\#\{0 < i < p; \rho(\phi_u, (\phi_v + i\Delta r) \mod 1) < \delta\} < 3\delta.$$

The expression $[p \cdot \Delta r]$ determines the number of turns that the point v makes along the circle by rotation through the angle Δr after p iterations, where $[x]$ denotes the integer part of x. The maximal number of iterations i during one turn, for which $\rho(\phi_u, (\phi_v + i\Delta r) \mod 1) < \delta.$, is $2\delta/\Delta r$. It follows

$$\frac{1}{p}\#\{0 < i < p; \rho(\phi_u, (\phi_v + i\Delta r) \mod 1) < \delta\} < \frac{1}{p}\left([p \cdot \Delta r]\frac{2\delta}{\Delta r} + \frac{2\delta}{\Delta r}\right) < 2\delta + \frac{2\delta}{p\Delta r}.$$

Because $p > 2/\Delta r$, we can estimate the second term by δ, i.e. $(2\delta/p\Delta r) < \delta$. □

THEOREM 1. *There exists a DC*1 *dynamical system* (Y, f) *which is semi-conjugated to an extension* (X, F) *which possess no distributionally scrambled pair* (*of type* 1 *or* 2).

Proof. The space X is defined

$$X = \left(\left\{\left[\left(2 - \frac{1}{k}\right)\cos 2\pi\phi, \left(2 - \frac{1}{k}\right)\sin 2\pi\phi\right] : k \in \mathbb{N}, \phi \in I\right\} \bigcup \{[2\cos 2\pi\phi, 2\sin 2\pi\phi] : \phi \in I\}\right) \times I,$$

where I is the unit interval. Each point $u = [r_u \cos 2\pi\phi_u, r_u \sin 2\pi\phi_u, z_u]$ in X is determined by its angle $\phi_u \in I$, radius

$$r_u \in \left\{2 - \frac{1}{k} : k \in \mathbb{N}\right\} \bigcup \{2\}$$

and height $z_u \in I$.

The space is endowed with max-metric

$$\mathrm{d}(u,v) = \max\{|r_u - r_v|, |z_u - z_v|, \rho(\phi_u, \phi_v)\}, \tag{4}$$

where $\rho(\phi_u, \phi_v) = \min\{|\phi_u - \phi_v|, 1 - |\phi_u - \phi_v|\}$. We define the mapping $F : X \to X$ as identity on the limit cylinder $X_0 = \left\{\left[2\cos 2\pi\phi, 2\sin 2\pi\phi\right] : \phi \in I\right\} \times I$,

$$[2\cos 2\pi\phi, 2\sin 2\pi\phi, z] \mapsto [2\cos 2\pi\phi, 2\sin 2\pi\phi, z],$$

and as a composition of rotation and continuous mapping g_k on inner cylinders,

$$\left[\left(2-\frac{1}{k}\right)\cos 2\pi\phi, \left(2-\frac{1}{k}\right)\sin 2\pi\phi, z\right] \mapsto \left[\left(2-\frac{1}{k+1}\right)\cos 2\pi(\phi + \Psi(k,z)),\right.$$
$$\left.\left(2-\frac{1}{k+1}\right)\sin 2\pi(\phi + \Psi(k,z)), g_k(z)\right].$$

Let $||h|| = \sup\{h(x) : x \in I\}$ be the uniform norm. To define $g_k : I \to I$ and $\Psi : \mathbb{N} \times I \to I$, let $\{r_i\}_{i=1}^{\infty} = \mathbb{Q}|_{(0,1)}$ be a sequence of all rationals in $(0,1)$, and $m_1 < m_2 < m_3 < \ldots$ an increasing sequence of integers which we specify later. Then

$$g_k = \begin{cases} h_l & \text{if } m_{3l+1} \le k < m_{3l+2} \\ Id & \text{if } m_{3l+2} \le k < m_{3l+3} \quad k, l \in \mathbb{N}_0 \\ h_l^{-1} & \text{if } m_{3l+3} \le k < m_{3l+4} \end{cases}, \tag{5}$$

where $h_l : I \to I$ is a continuous strictly increasing mapping with three fixed points $0, 1, r_l$ and

$$\lim_{l\to\infty} ||h_l - Id|| = 0; \quad h_l(x) < x \text{ for } x \in (0, r_l); \quad h_l(x) > x \text{ for } x \in (r_l, 1).$$

The sequence $\{m_i\}_{i=1}^{\infty}$ is defined in the following way:

$$m_{3l+2} - m_{3l+1} = m_{3l+4} - m_{3l+3} = n_l,$$

where n_l is integer satisfying

$$h_l^{n_l}\left(\left\langle 0, r_l - \frac{1}{l}\right\rangle\right) \subset \langle 0, \frac{1}{l}) \quad \wedge \quad h_l^{n_l}\left(\left\langle r_l + \frac{1}{l}, 1\right\rangle\right) \subset (1 - \frac{1}{l}, 1\rangle. \tag{6}$$

Notice that $h_l^{n_l}$ is a continuous bijection, hence we can find ε_l for which

$$|x - y| > \frac{1}{l} \Rightarrow |h_l^{n_l}(x) - h_l^{n_l}(y)| > \varepsilon_l.$$

Simultaneously $\{m_i\}_{i=1}^{\infty}$ can be chosen such that

$$m_{3l+3} - m_{3l+2} > \frac{2l}{\varepsilon_l}, \tag{7}$$

$$\begin{aligned} &\lim_{l\to\infty} \frac{m_{3l+1}}{m_{3l+2}} = \lim_{l\to\infty} \frac{m_{3l+3}}{m_{3l+4}} = 1, \\ &\lim_{l\to\infty} \frac{m_{3l+2}}{m_{3l+3}} = 0. \end{aligned} \tag{8}$$

For example, the sequence defined inductively by $m_1 = 0$ and

$$m_{3l+2} = m_{3l+1} + n_l,$$
$$m_{3l+3} = \left(m_{3l+2} + \tfrac{2l}{\varepsilon_l}\right) \cdot 2^l \cdot n_l \cdot n_{l+1},$$
$$m_{3l+4} = m_{3(l+1)+1} = m_{3l+3} + n_l,$$

satisfies these requirements. The angle of rotation $\Psi : \mathbb{N} \times I \to I$ is defined as

$$\Psi(k,z) = \begin{cases} z & \text{if } \ 1 \leq k < m_4 \\ z/l & \text{if } \ m_{3l+1} \leq k < m_{3l+4} \quad l \in \mathbb{N}. \end{cases}$$

The factor space Y is simply X with fixed $\phi = 0$ i.e. for each point $x \in X$,

$$\pi(x) = \pi([r_x \cos 2\pi\phi_x, r_x \sin 2\pi\phi_x, z_x]) = [r_x, 0, z_x].$$

To simplify the notation, we skip the second zero coordinate and treat Y as a two-dimensional space. The space Y is union of converging sequence of unit fibres and the limit fibre,

$$Y = \left\{2 - \frac{1}{k} : k \in \mathbb{N}\right\} \times I \cup \{2\} \times I.$$

Then the system (X, F) is semi-conjugated with skew-product map $f : Y \to Y$, which is identity on the limit fibre,

$$[2, z] \mapsto [2, z],$$

and which is g_k on inner fibres,

$$\left[2 - \frac{1}{k}, z\right] \mapsto \left[2 - \frac{1}{k+1}, g_k(z)\right], \quad k \in \mathbb{N}.$$

(i) The factor system (*Y,f*) is DC1.We show that set $S = \{1\} \times I$ is a distributionally scrambled set, i.e. for any pair $(u, v) \in S^2$ with $u \neq v$,

$$\Phi^*_{(u,v)} \equiv 1 \ \text{ and } \ \Phi_{(u,v)}(\varepsilon) = 0, \ \text{ for some } \ 0 < \varepsilon < 1. \qquad (9)$$

Suppose $u^2 > v^2$, where x^2 denotes the second coordinate of a point x. Since $\{r_i\}_{i=1}^{\infty}$ is dense in I, we can find a subsequence $\{r_{s_k}\}_{k=1}^{\infty}$ such that

$$u^2 < r_{s_k} - \frac{1}{s_k}$$

and

$$v^2 < r_{s_k} - \frac{1}{s_k}$$

for every k. By (6), $\mathrm{d}(f^i(u), f^i(v)) < \frac{1}{s_k}$, for $m_{3s_k+2} \leq i < m_{3s_k+3}$, and therefore, by

(8), $\Phi^*_{(u,v)} \equiv 1$.

We can find another subsequence $\{r_{q_k}\}_{k=1}^{\infty}$ such that

$$u^2 > r_{q_k} + \frac{1}{q_k}$$

and

$$v^2 < r_{q_k} - \frac{1}{q_k},$$

for every k. By (6), $\mathrm{d}(f^i(u), f^i([1,1])) < \frac{1}{q_k}$ and simultaneously $\mathrm{d}(f^i(v), f^i([1,0])) < \frac{1}{q_k}$, for $m_{3q_k+2} \leq i < m_{3q_k+3}$. Since f preserves the distance between the endpoints of any fibre, $\mathrm{d}(f^i([1,1]), f^i([1,0])) = 1$, for $i \geq 0$, we can conclude, by (8), $\Phi_{(u,v)}(\varepsilon) = 0$, for any $\varepsilon < 1$.

(ii) (X, F) *has no distributionally scrambled pair* (*of type* 1 *or* 2).

We claim $\Phi^*_{(u,v)} < 1$ for any pair of distinct points in X. Let X_0 be the limit cylinder $X_0 = \{[2\cos 2\pi\phi, 2\sin 2\pi\phi] : \phi \in I\} \times I$ and $\bar{X} = X \setminus X_0$. Consider five possible cases:

(a) $(u, v) \in \bar{X}$ with $z_u = z_v = z$, $k_u = k_v = k$, $\phi_u \neq \phi_v$.
The angle of rotation is the same for both u and v, $\Psi(k_u, z_u) = \Psi(k_v, z_v) = \Psi(k, z)$, hence, by (4),

$$\mathrm{d}(F(u), F(v)) = \rho(\phi_u + \Psi(k, z), \phi_v + \Psi(k, z)) = \rho(\phi_u, \phi_v) = \mathrm{d}(u, v).$$

F is isometric in this case and $\Phi^*_{(u,v)} \neq 1$.

(b) $(u, v) \in \bar{X}$ with $z_u = z_v = z$, $k_u \neq k_v$.
Consider the image $(F^N(u), F^N(v))$ instead of (u, v), where N is the first integer for which $z_{F^N(u)} \neq z_{F^N(v)}$ and reduce this case to d).

(c) $(u, v) \in \bar{X}$ with $z_u \neq z_v$, $k_u = k_v = k$.
Without loss of generality suppose $k = 1$ (otherwise consider the pre images $(F^{-k}(u), F^{-k}(v))$) and let L be an integer such that $|z_u - z_v| > 1/L$. It is sufficient to show that there is $0 < \delta < 1/3$, for which

$$\frac{1}{m_{3l+3} - m_{3l+2}} \#\left\{m_{3l+2} < i < m_{3l+3}; \mathrm{d}(F^i(u), F^i(v)) < \delta\right\} < 3\delta, \quad \text{for any} \quad L \leq l.$$

Since d is max-metric, it is sufficient to prove

$$\frac{1}{m_{3l+3} - m_{3l+2}} \#\left\{m_{3l+2} < i < m_{3l+3}; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\right\} < 3\delta.$$

By the definition of ε_L,

$$|x - y| > \frac{1}{L} \Rightarrow |h_L^{n_L}(x) - h_L^{n_L}(y)| > \varepsilon_L.$$

Since $\left|h_L^{n_L}(z_u) - h_L^{n_L}(z_v)\right| > \varepsilon_L$, and $\left|h_L^{n_L}(z_u) - h_L^{n_L}(z_u)\right|$ is the minimal distance between trajectories of u and v between times m_{3L+1} and m_{3L+4} (see the definition of g_k in (5)), it follows

$$\min_{3L+1 < k \leq 3L+4} |g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_u) - g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_v)| > \varepsilon_L. \tag{10}$$

Denote the relative angle of rotation of points with height z_u and z_v in the k-the cylinder by

$$\Delta\Psi_k(z_u, z_v) = |\Psi(k, z_u) - \Psi(k, z_v)| = \frac{|z_u - z_v|}{L}$$

for $m_{3L+1} \leq k < m_{3L+4}$. By (10),

$$\Delta\Psi_k(g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_u), g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_v)) > \frac{\varepsilon_L}{L}, \text{ for } m_{3L+1} \leq k < m_{3L+4}.$$

Since

$$m_{3L+3} - m_{3L+2} > \frac{2L}{\varepsilon_L}$$

we can use Lemma 1 and conclude, for any $\delta > 0$,

$$\frac{1}{m_{3L+3} - m_{3L+2}} \#\{m_{3L+2} < i < m_{3L+3}; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\} < 3\delta.$$

We obtain the result for any $l > L$ using the same argument, since for every $l > L$, $|z_u - z_v| > \frac{1}{l}$.

(d) $(u, v) \in \bar{X}$ with $z_u \neq z_v$, $k_u \neq k_v$.

Without loss of generality suppose $k_u = 1$ and $k_v = p$. If $|z_u - z_v| > 1/L$, then by case b)

$$\#\{m_{4L+2} + p < i < m_{4L+3} - p; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\} < 3\delta \cdot (m_{3L+3} - m_{3L+2})$$

and hence

$$\frac{1}{m_{3L+3} - m_{3L+2}} \#\{m_{3L+2} < i < m_{3L+3}; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\}$$
$$< 3\delta + \frac{2p}{m_{3L+3} - m_{3L+2}} < 1,$$

for sufficiently large L.

(e) $u \in \bar{X}$ and $v \in X_0$

Since $v \in X_0$ is fixed and $\phi_v = \phi_{F(v)}$, we can find another point in $\bar{X}$, $w = \left[\left(2 - \frac{1}{k_u}\right)\cos 2\pi\phi_v, \left(2 - \frac{1}{k_u}\right)\sin 2\pi\phi_v, 0\right]$, which is also fixed under rotation. Therefore

$$\rho(\phi_{F(u)}, \phi_{F(v)}) = \rho(\phi_{F(u)}, \phi_{F(w)})$$

and we can apply case a) or c) to investigate the pair (u, w) instead of (u, v).

□

Remark. Notice that the distributional functions for projected pair (u, v) in factor space Y are greater or equal than the corresponding distributional function for the pair $(\pi^{-1}(u), \pi^{-1}(v))$ in X, because X is equipped with max-metric. Nevertheless the upper distributional function remains positive even in the extension, $\Phi^*_{(\pi^{-1}(u),\pi^{-1}(v))}(\delta) > 0$, for any pair in $I \times \{1\}$ and $\delta > 0$. By (9), $\Phi_{(\pi^{-1}(u),\pi^{-1}(v))}(\delta) \leq \Phi_{(u,v)}\delta = 0$, hence

$$\Phi_{(\pi^{-1}(u),\pi^{-1}(v))} < \Phi^*_{(\pi^{-1}(u),\pi^{-1}(v))}$$

and the system (X, F) is distributionally chaotic of type 3. This fact implies an open question: *Is there a DC3 system which is semi-conjugated to an extension without any distributionally scrambled pairs of type 3?*

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On the Weakest Version of Distributional Chaos

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The aim of the paper is to correct and improve some results concerning distributional chaos of type 3. We show that in a general compact metric space, distributional chaos of type 3, denoted DC3, even when assuming the existence of an uncountable scrambled set, is a very weak form of chaos. In particular, (i) the chaos can be unstable (it can be destroyed by conjugacy), and (ii) such an unstable system may contain no Li–Yorke pair. However, the definition can be strengthened to get DC$2\frac{1}{2}$ which is a topological invariant and implies Li–Yorke chaos, similarly as types DC1 and DC2; but unlike them, strict DC$2\frac{1}{2}$ systems must have zero topological entropy.

Keywords: Distributional chaos; Li–Yorke chaos; distal chaotic system.

1. Introduction

The study of chaotic pairs in dynamics goes back to Li and Yorke [1975], who studied pairs of points with the property that their orbits neither approach each other asymptotically, nor do they eventually separate from each other by any fixed positive distance. Schweizer and Smítal [1994] introduced the related concept of a distributionally chaotic pair, which means, roughly speaking, that the statistical distribution of distances between the orbits does not converge. Distributional chaos was later divided into three types, DC1, DC2, and DC3, see [Balibrea *et al.*, 2005].

The relations between the three versions of distributional chaos, and the relation between distributional chaos and Li–Yorke chaos are investigated by many authors, see e.g. [Balibrea *et al.*, 2005; Wang *et al.*, 2008, 2014; Oprocha, 2009]. One can easily see from the definitions that DC1 implies DC2 and DC2 implies DC3. On the other hand, there are examples which show that DC1 is stronger than DC2 and DC2 is stronger than DC3 (see [Balibrea *et al.*, 2005; Wang *et al.*, 2008]). It is also obvious that either DC1 or DC2 implies Li–Yorke chaos. Moreover, there are Li–Yorke chaotic continuous maps of the interval with zero topological entropy; by [Schweizer & Smítal, 1994], such maps cannot be distributionally chaotic. This shows that Li–Yorke chaos need not imply any of the three versions of distributional chaos.

We focus on the properties of the weakest form of distributional chaos, DC3. Unlike its stronger relatives, DC3 chaos does not imply Li–Yorke chaos, and DC3 chaos is not an invariant of topological conjugacy. In a weak sense, these two results were already stated in [Balibrea *et al.*, 2005, Theorems 1 and 2]. However, it should be noticed that the distributional chaos in [Balibrea *et al.*, 2005] was defined as the existence of a single distributionally scrambled pair, but nowadays it is generally assumed that distributional chaos means the existence of an uncountable distributionally scrambled set. Moreover, the proof of [Balibrea *et al.*, 2005, Theorem 2] is unfortunately in error — the authors constructed a conjugacy which destroys a DC3 pair, but they overlooked many other DC3 pairs which persist.

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Our first goal, then, is to give stronger statements and correct proofs of these two theorems.

Our second goal is to strengthen the definition of the DC3 pair in such a way that it is preserved under conjugacy and implies Li–Yorke chaos — we denote the new definition by DC2$\frac{1}{2}$. The only difference between DC2$\frac{1}{2}$ and DC3 is the following: a pair (x, y) is DC3 iff the lower and upper distribution functions satisfy $\Phi(\delta) < \Phi^*(\delta)$, for every δ in some interval I. We say a pair (x, y) is DC2$\frac{1}{2}$ iff $\Phi(0) < \Phi^*(0)$, where the distribution functions at 0 are defined as limits of their values as $\delta \to 0^+$. We also provide an example which shows that DC2$\frac{1}{2}$ is essentially weaker than DC2. This example possesses no DC2 pair; hence, by results in [Downarowicz & Lacroix, 2014], its topological entropy must be zero.

Another strengthened distributional chaos, denoted by DC1$\frac{1}{2}$, was introduced in [Downarowicz & Lacroix, 2014]. DC1$\frac{1}{2}$ chaos is stronger than DC2 and is implied by positive topological entropy.

The paper is organized as follows: the first and second sections are introductory. In the third section we show the error in [Balibrea *et al.*, 2005] and simultaneously prove that even the presence of an uncountable DC3 scrambled set does not imply Li–Yorke chaos. The fourth section proves (with two new examples) that both the existence of a DC3 pair and the existence of an uncountable DC3 scrambled set are not conjugacy invariants. The fifth section introduces our new definition of DC2$\frac{1}{2}$.

2. Terminology

Let (X, d) be a nonempty compact metric space. A pair (X, f), where f is a continuous self-map acting on X, is called a *topological dynamical system*. The *orbit* of a point $x \in X$ is the set $\{f^n(x) : n \geq 0\}$. Let (X, f) and (Y, g) be dynamical systems on compact metric spaces. A continuous map $\pi : X \to Y$ is a *conjugacy* between f and g if π is one-to-one and onto and $\pi \circ f = g \circ \pi$.

Definition 2.1. A pair of two different points $(x_1, x_2) \in X^2$ is *scrambled* or *Li–Yorke* if

$$\liminf_{k\to\infty} d(f^k(x_1), f^k(x_2)) = 0 \tag{1}$$

and

$$\limsup_{k\to\infty} d(f^k(x_1), f^k(x_2)) > 0. \tag{2}$$

A subset S of X is *scrambled* if every pair of distinct points in S is scrambled. The system (X, f) is *Li–Yorke chaotic* if there exists an uncountable scrambled set. We call a pair $(x_1, x_2) \in X^2$ *proximal* if (1) holds (otherwise we say $(x_1, x_2) \in X^2$ is *distal*). If (2) does not hold, i.e.

$$\limsup_{k\to\infty} d(f^k(x_1), f^k(x_2)) = 0,$$

we say that $(x_1, x_2) \in X^2$ is *asymptotic*. A pair of points is scrambled simply if it is proximal but not asymptotic. A dynamical system (X, f) is *distal* if every pair of distinct points in X is distal.

Definition 2.2. For a pair (x_1, x_2) of points in X, define the *lower distribution function* generated by f as

$$\Phi_{(x_1,x_2)}(\delta) = \liminf_{m\to\infty} \frac{1}{m}\#\{0 \leq k \leq m; d(f^k(x_1), f^k(x_2)) < \delta\}$$

and the *upper distribution function* as

$$\Phi^*_{(x_1,x_2)}(\delta) = \limsup_{m\to\infty} \frac{1}{m}\#\{0 \leq k \leq m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A.

A pair $(x_1, x_2) \in X^2$ is called *distributionally scrambled of type 1* (or a DC1 pair) if

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \quad \text{for every } 0 < \delta \leq \operatorname{diam} X$$

and

$$\Phi_{(x_1,x_2)}(\epsilon) = 0, \quad \text{for some } 0 < \epsilon \leq \operatorname{diam} X,$$

distributionally scrambled of type 2 (or a DC2 pair) if

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \quad \text{for every } 0 < \delta \leq \operatorname{diam} X$$

and

$$\Phi_{(x_1,x_2)}(\epsilon) < 1, \quad \text{for some } 0 < \epsilon \leq \operatorname{diam} X$$

and *distributionally scrambled of type 3* (or a DC3 pair) if

$$\Phi_{(x_1,x_2)}(\delta) < \Phi^*_{(x_1,x_2)}(\delta), \quad \text{for every } \delta \text{ in some interval } (a, b), \text{ where } 0 \leq a < b \leq \operatorname{diam} X.$$

A subset S of X is *distributionally scrambled of type i* (or a DC*i* set), where $i = 1, 2, 3$, if every pair of distinct points in S is a DC*i* pair. The dynamical system (X, f) is *distributionally chaotic of type i* (a DCi system), where $i = 1, 2, 3$, if there is an uncountable distributionally scrambled set $S \subset X$ of type i.

Let X, Y be compact metric spaces and $X \times Y$ be equipped with the product topology. Then a continuous map $F : X \times Y \to X \times Y$ is a *skew-product mapping* if it has the form $F((x, y)) = (f(x), g_x(y))$. Then $f : X \to X$ is called the *base map* and the maps $g_x : Y \to Y$ are called *fiber maps*.

3. Distal DC3 System

The main goal of this section is to prove that DC3 chaos does not imply Li–Yorke chaos. We will prove this statement in the strongest possible sense — we present a system with an uncountable DC3 set but without any Li–Yorke pairs.

Theorem 1. *There exists a distal dynamical system which is DC3 chaotic. Thus, DC3 chaos does not imply Li–Yorke chaos.*

Proof. Our proof analyzes a distal system first introduced in [Balibrea *et al.*, 2005], in which the authors mistakenly claimed (without any proof) that there are no DC3 pairs.

Definition of the System: Consider the skew product mapping

$$F : \Omega \times \mathbb{S}^1 \to \Omega \times \mathbb{S}^1, \quad (\omega, \theta) \mapsto (\tau(\omega), \theta + p(\omega)),$$

where $\Omega = \{0, 1\}^{\mathbb{N}}$ is the Cantor space, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle, $\tau : \Omega \to \Omega$ is the binary adding machine $\omega \mapsto \omega + 10\,000\cdots$ with "carrying" to the right (for details, see for example [Downarowicz, 2005]), and $p(\omega)$ is a rotation angle as defined below.

The function $p : \Omega \to \mathbb{S}^1$ is given by the following algorithm. First, fix an arbitrary parameter α with $\frac{3}{4} < \alpha < 1$. Then choose an increasing sequence of natural numbers $\{n_i\}_{i=1}^{\infty}$ such that[1]

$$\lim_{i\to\infty} \frac{(2^{n_1} - 2)}{2^{n_1}} \cdot \frac{(2^{n_2} - 4)}{2^{n_2}} \cdot \frac{(2^{n_3} - 8)}{2^{n_3}} \cdot \cdots \cdot \frac{(2^{n_i} - 2^i)}{2^{n_i}} = \alpha. \qquad (3)$$

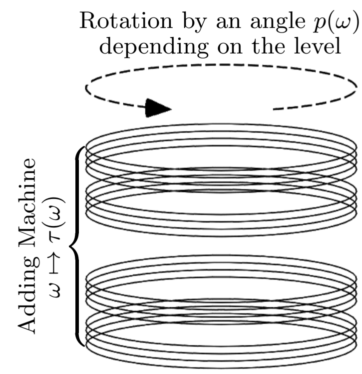

Fig. 1. A distal system with an uncountable DC3 scrambled set.

We will also use the notation $m_i := n_1 + n_2 + \cdots + n_i$, and $m_0 := 0$. If[2] $\omega = 1^\infty$, then set $p(\omega) = 0$. Otherwise, decompose ω into the infinite concatenation $\omega = \omega^{(1)}\omega^{(2)}\omega^{(3)}\cdots$ where each $\omega^{(i)}$ is a block of length n_i. To be clear, if $\omega = \omega_1\omega_2\cdots$, then $\omega^{(i)} = \omega_{m_{i-1}+1}\omega_{m_{i-1}+2}\cdots\omega_{m_i} \in \{0, 1\}^{n_i}$. Then define

$$k = k(\omega) = \min\{i; \omega^{(i)} \neq 1^{n_i}\}. \qquad (4)$$

By $|\omega^{(k)}|$ we denote the *evaluation* of the block $\omega^{(k)}$, where the evaluation operator is defined by

$$|x_1x_2x_3\cdots x_q| = x_1 + 2x_2 + 2^2x_3 + \cdots + 2^{q-1}x_q, \quad q \in \mathbb{N}, x_1, \ldots, x_q \in \{0, 1\}. \qquad (5)$$

Observe that $0 \leq |\omega^{(k)}| < 2^{n_k} - 1$, since we know that $\omega^{(k)}$ consists of n_k symbols, but $\omega^{(k)} \neq 1^{n_k}$ by the definition of k. Finally, set

$$p(\omega) = \begin{cases} 0, & \text{if } 2^{k-1} \leq |\omega^{(k)}| < 2^{n_k} - 2^{k-1} - 1 \\ \dfrac{1}{2^k}, & \text{otherwise.} \end{cases} \qquad (6)$$

Definition of the Metric: To discuss DC3 chaos we must specify a metric. If two points in $\mathbb{S}^1$ have representatives $\theta, \theta' \in [0, 1)$, then the distance between them will be $\min(|\theta - \theta'|, 1 - |\theta - \theta'|)$, so that points on opposite sides of the circle are separated by a distance $\frac{1}{2}$. We wish that Ω will have diameter $\frac{1}{4}$, so we define the distance between two points $\omega, \omega' \in \Omega$ to be $2^{-1-\inf\{i; \omega_i \neq \omega'_i\}}$. Finally, we

[1] We note that it is possible to choose such a sequence n_i. It suffices (analysis exercise) to define recursively n_{i+1} as the smallest integer greater than n_i such that the $(i+1)$th partial product in (3) is still larger than α.

[2] As is usual in symbolic dynamics, we use exponents to indicate repetition of a symbol. Thus, 1^∞ denotes the infinite sequence $111\cdots$, while 1^l denotes the block $11\cdots1$ formed by concatenating l copies of the symbol 1.

equip the product space $\Omega \times \mathbb{S}^1$ with the maximum metric

$$d((\omega,\theta),(\omega',\theta')) = \max(\mathrm{dist}_\Omega(\omega,\omega'), \mathrm{dist}_{\mathbb{S}^1}(\theta,\theta')).$$

It is clear that F determines a distal dynamical system — for points from distinct fibers we recall that the adding machine is distal, and for distinct points from the same fiber we notice that the rotations preserve the distance in the circle.

Lemma 1 (Rotation along Orbits). *Define $\sigma : \Omega \to \Omega$ by $\sigma(\omega_1\omega_2\omega_3\cdots) = 0^{n_1-1}\omega_1 0^{n_2-1}\omega_2 0^{n_3-1}\omega_3\cdots$. For each $\omega \in \Omega$ and $i \in \mathbb{N}$ we have the following estimate regarding net rotations along the orbit of $\sigma(\omega)$. Let $\beta_i := \left(2^{n_i-2} - 2^{i-1} - 1\right) \cdot \prod_{l=1}^{i-1}\left(2^{n_l} - 2^l\right)$. Then*

$$\#\left\{ n \in \{1,\ldots,2^{m_i-2}\}; \sum_{j=0}^{n-1} p(\tau^j\sigma\omega) = \frac{1}{2}\Big((1-\omega_1)+\cdots+(1-\omega_i)\Big) \bmod 1 \right\} \geq \beta_i.$$

Moreover,

$$\alpha_i := \frac{\beta_i}{2^{m_i-2}} \to \alpha \text{ as } i \to \infty.$$

The proof of Lemma 1 is technical and is deferred to the Appendix. The interpretation of Lemma 1 is that during the first 2^{m_i-2} steps along the orbit, the position of the point $F^n(\sigma\omega, 0)$ in the circle is "usually" equal to $\frac{1}{2}((1-\omega_1)+\cdots+(1-\omega_i)) \bmod 1$, i.e. with frequency at least α_i.

The Scrambled Set: Equip Ω with the tail equivalence relation $\omega_1\omega_2\cdots \sim \omega_1'\omega_2'\cdots$ if and only if $\exists n \in \mathbb{N} : \omega_n\omega_{n+1}\cdots = \omega_n'\omega_{n+1}'\cdots$. We wish to choose an uncountable set $\Lambda \subset \Omega$ such that no two points of Λ are tail equivalent. One quick solution is to invoke the axiom of choice and let Λ consist of one representative point from each equivalence class of $\sim$. Alternatively, we may follow a more constructive approach and take $\Lambda = \lambda(\{0,1\}^{\mathbb{N}})$, where $\lambda(\omega_1\omega_2\omega_3\cdots) = \omega_1\ \omega_1\omega_2\ \omega_1\omega_2\omega_3\ \omega_1\omega_2\omega_3\omega_4\ \cdots$, so that $\lambda(\omega) \not\sim \lambda(\omega')$ for $\omega \neq \omega'$. We claim that the uncountable set

$$S = \sigma(\Lambda) \times \{0\} \subset \Omega \times \mathbb{S}^1$$

is pairwise DC3 scrambled for the map F.

Verification of DC3 Chaos: Let $s = (\sigma\omega, 0)$, $s' = (\sigma\omega', 0)$ be distinct points in S. The distance between $F^n(s)$, $F^n(s')$ is small whenever the points are on the same side of the circle, and large whenever the points are on opposite sides of the circle. We state this notion precisely in the following implications:

$$\sum_{j=0}^{n-1} p(\tau^j\sigma\omega) = \sum_{j=0}^{n-1} p(\tau^j\sigma\omega') \mod 1 \Rightarrow d(F^n(s),F^n(s')) \leq \frac{1}{4}, \tag{7}$$

$$\sum_{j=0}^{n-1} p(\tau^j\sigma\omega) - \sum_{j=0}^{n-1} p(\tau^j\sigma\omega') = \frac{1}{2} \mod 1 \Rightarrow d(F^n(s),F^n(s')) = \frac{1}{2}. \tag{8}$$

Now partition the natural numbers into the sets

$$A = \{i \in \mathbb{N}; \omega_1 + \cdots + \omega_i = \omega_1' + \cdots + \omega_i' \bmod 2\},$$

$$B = \{i \in \mathbb{N}; \omega_1 + \cdots + \omega_i \neq \omega_1' + \cdots + \omega_i' \bmod 2\}.$$

Since $\omega \not\sim \omega'$, it follows that both sets A, B are infinite.

Suppose that $i \in A$. Lemma 1 gives us the "usual" positions of the points $F^n(s), F^n(s')$ in the circle, but does not give any sense of synchrony. Thus, Lemma 1 in conjunction with (7) gives

$$\frac{1}{2^{m_i-2}} \cdot \#\{n \in \{1,\ldots,2^{m_i-2}\}; d(F^n(s),F^n(s')) < \delta\} \geq 2\alpha_i - 1, \quad \delta > \frac{1}{4}.$$

Taking the limit as $i \to \infty$, $i \in A$, we obtain an estimate for the upper distribution function

$$\Phi^*_{(s,s')}(\delta) \geq 2\alpha - 1, \quad \delta > \frac{1}{4}.$$

Now suppose that $i \in B$. It follows from (8) and Lemma 1 that

$$\frac{1}{2^{m_i-2}} \cdot \#\{n \in \{1,\ldots,2^{m_i-2}\}; d(F^n(s),F^n(s')) < \delta\} \leq 1 - (2\alpha_i - 1), \quad \delta < \frac{1}{2}.$$

Taking the limit as $i \to \infty$, $i \in B$, we obtain an estimate for the lower distribution function

$$\Phi_{(s,s')}(\delta) \leq 2 - 2\alpha, \quad \delta < \frac{1}{2}.$$

Since $\alpha > \frac{3}{4}$, it follows that

$$\Phi_{(s,s')}(\delta) \le 2 - 2\alpha < \frac{1}{2} < 2\alpha - 1$$

$$\le \Phi^*_{(s,s')}(\delta), \quad \frac{1}{4} < \delta < \frac{1}{2}.$$

Thus s, s' are a DC3 pair. This completes the proof of Theorem 1. ■

4. Conjugacy Problem

We show that DC3 is not preserved by conjugacy, i.e. distributional chaos of type 3 can be destroyed (or created) by using a conjugating homeomorphism. There are two ways we can understand this statement. In Sec. 4.1, we interpret distributional chaos as the existence of a chaotic pair, and we construct a system with a DC3 pair which is conjugate to a dynamical system without one. This constitutes the first correct proof of [Balibrea *et al.*, 2005, Theorem 2]. In Sec. 4.2, we interpret distributional chaos as the existence of an uncountable chaotic set and we present a DC3 system which is conjugate to a dynamical system with only DC3 pairs (the maximum cardinality of any distributionally scrambled set in this system is 2).

4.1. *DC3 pairs*

Throughout the whole subsection, we use the cylindrical coordinate system for $\mathbb{R}^3$. This means that the point $(r\cos(2\pi\phi), r\sin(2\pi\phi), z)$, $r \ge 0$, $\phi \in [0,1]$, will be denoted more compactly as (r, ϕ, z). Let d be the max-metric on $\mathbb{R}^3$ given by $d(a,b) = \max\{|r_a - r_b|, |z_a - z_b|, \rho(\phi_a, \phi_b)\}$, where ρ will be the circle metric given by $\rho(\phi_a, \phi_b) = \min\{|\phi_a - \phi_b|, 1 - |\phi_a - \phi_b|\}$.

Lemma 2. *Suppose we are given two angles* ϕ_x, $\phi_y \in [0,1]$ *(mod 1), a natural number* k *and a number* $\delta \in (0, 0.5)$. *Let* $\pi_k = \pi_k(\phi_x, \phi_y, \delta) := \#\{0 \le i \le k-1; \rho(\phi_x + \frac{i}{k}, \phi_y) < \delta\}$. *Then*

$$2\delta k - 1 \le \pi_k \le 2\delta k + 1.$$

Proof. The reader may picture k points equally spread around the circle $[0,1]$ (mod 1) and an arc $(\phi_y - \delta, \phi_y + \delta)$ of radius $\delta \in (0, 1/2)$. Without loss of generality we will take $\phi_x = 0$. Then $\pi_k = \#\{0 \le i \le k-1; \rho(\frac{i}{k}, \phi_y) < \delta\}$ is equal to $\#\{i \in \mathbb{Z}; |\frac{i}{k} - \phi_y| < \delta\} = \#\{i \in \mathbb{Z}; |i - k\phi_y| < \delta k\}$. By these equalities we moved the problem from the circle to the real line and so the question is: How many integers lie in the open interval $(k\phi_y - \delta k, k\phi_y + \delta k)$ with length $2\delta k$? At least $2\delta k - 1$ and at most $2\delta k + 1$. ■

Now we construct two conjugate dynamical systems (X, f) and (Y, g). We also denote the jth iterate of a point x by $f^j(x) = (r_x^j, \phi_x^j, z_x^j)$, and when no confusion can result, we use the same notation for the iterates of x by g.

Definition of the Conjugate Systems (X, f) and (Y, g): The space X consists of two concentric columns of "rings"; in each column the rings are accumulating on a bottom-most ring. The definition is

$$X = \left\{(r, \phi, z) : r \in \{0.01, 0.02\}, \phi \in [0,1], \; z \in \left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup \{0\}\right\}.$$

We need the radius difference $|r_1 - r_2|$ to be smaller than 1/2. For concreteness we choose $r_1 = 0.01$ and $r_2 = 0.02$, but it is not necessary to work with these two exact numbers.

The map $f : X \to X$ carries each ring down to the next lower ring with some rotation and fixes the bottom ring. The definition is

$$f\left(r, \phi, \frac{1}{n}\right) = \left(r, \phi + \varphi_n^{(r)}, \frac{1}{n+1}\right) \quad \text{and} \qquad f(r, \phi, 0) = (r, \phi, 0), \tag{9}$$

where $\phi + \varphi_n^{(r)}$ is computed modulo 1 and the rotation angle $\varphi_n^{(r)}$ is given by

$$\varphi_n^{(0.01)} = \frac{1}{k}, \quad \text{for } l_{k-1} < n \le l_k, \; k \in \mathbb{N},$$

$$\varphi_n^{(0.02)} = \begin{cases} \frac{1}{k}, & \text{for } l_{k-1} < n \le l_k, \quad \text{odd } k, \\ \frac{2}{k}, & \text{for } l_{k-1} < n \le l_k, \quad \text{even } k, \end{cases}$$

$$\text{where } l_k = \sum_{i=1}^{k} i^i, \quad \text{for } k \in \mathbb{N} \text{ and } l_0 = 0. \tag{10}$$

We may think of $\varphi_n^{(r)}$ as a sequence being divided into blocks, where the kth block consists of the angles $\frac{1}{k}$ or $\frac{2}{k}$ repeated k^k times. We need the block lengths l_k to be multiples of k and with the property that $\frac{l_k}{l_{k+1}-l_k} \to 0$ for $k \to \infty$. It is easy to see that for our blocks it is true and so for $k \to \infty$

$$\frac{l_k}{(k+1)^{k+1}} \to 0 \quad \text{and} \quad \frac{k^k}{l_k} \to 1. \qquad (11)$$

We will see (in the proof of Theorem 2) that every DC3 pair in (X, f) contains one point from each cylinder, and neither of those two points is fixed. The chaos is detected by distances of angles, which are always smaller than 1/2. Since we are using a maximum metric, we "kill" the chaos in the conjugate system by separating the cylinders by a distance more than 1/2. We lift the inner cylinder (by homeomorphism) by, for concreteness, 1.3 units, and we keep the map on each cylinder the same as before. We can imagine this space like an extended telescope.

We construct a conjugate dynamical system (Y, g) by the following definitions

$$Y = \Pi(X) \quad \text{and} \quad g = \Pi \circ f \circ \Pi^{-1}, \quad \text{where}$$

$$\Pi((r, \phi, z)) = \begin{cases} (r, \phi, z + 1.3), & \text{if } r = 0.01, \\ (r, \phi, z), & \text{if } r = 0.02. \end{cases} \qquad (12)$$

Theorem 2. *The existence of DC3 pairs is not preserved by topological conjugacy.*

Proof. For the proof we will use the systems (X, f) and (Y, g) constructed above. We will first show that in (Y, g) there do not exist any DC3 pairs and after that we will show that in (X, f) there are DC3 pairs.

(A) We claim that there are no DC3 pairs in (Y, g).

We can see all possible different cases for pairs of distinct points $(x, y) \in Y \times Y$ in Fig. 2, where $x = (r_x, \phi_x, z_x)$ and $y = (r_y, \phi_y, z_y)$.

(A1) $r_x \neq r_y$.

When x and y belong to different cylinders, the distance between their images converges to 1.3 because we are using the max metric.

(A2) $r_x = r_y = r$, $z_x = z_y = z$ $(\phi_x \neq \phi_y)$.

Since x and y are at the same height in the same cylinder, they rotate by the same angle, so the distance between them stays constant. That means $d(g^j(x), g^j(y)) = \rho(\phi_x, \phi_y)$ for all j, and so (x, y) is not DC3.

(A3) $z_x \neq z_y$ and $z_x, z_y \neq 1.3$, $r_x = r_y = r = 0.01$.

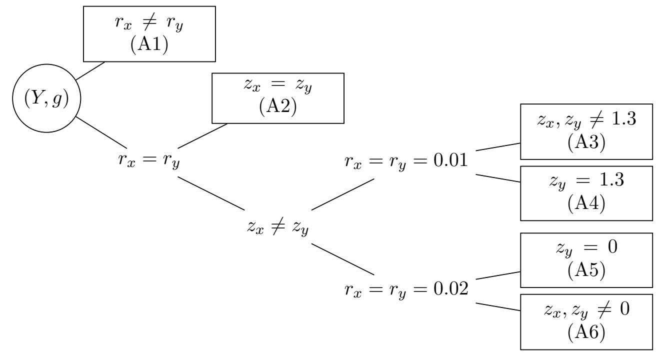

Fig. 2. Possible pairs in the system (Y, g).

Let $z_x = \frac{1}{c_x} + 1.3$, $z_y = \frac{1}{c_y} + 1.3$, and without loss of generality we suppose $c_y = c_x + c$, where $c \in \mathbb{N}$. Then $\lim_{j\to\infty} |z_x^j - z_y^j| = 0$ and $|r_x^j - r_y^j| = 0$ for any $j \in \mathbb{N}$. That means we can make calculations using the circle metric ρ in place of the maximum metric d. We can write $\phi_x^j = \phi_x + \sum_{n=c_x}^{c_x+j-1} \varphi_n^{(r)}$, and similarly for ϕ_y^j. By expanding these sums we obtain

$$\lim_{j\to\infty} \rho(\phi_x^j, \phi_y^j) =$$

$$= \lim_{j\to\infty} \rho \left[\left(\phi_x + \sum_{n=c_x}^{c_x+c-1} \varphi_n^{(r)} + \sum_{n=c_x+c}^{c_x+j-1} \varphi_n^{(r)} \right), \left(\sum_{n=c_x+c}^{c_x+j-1} \varphi_n^{(r)} + \sum_{n=c_x+j}^{c_x+j+c-1} \varphi_n^{(r)} + \phi_y \right) \right].$$

We may cancel the innermost sums. By (10), we find $\lim_{j\to\infty} \sum_{n=c_x+j}^{c_x+j+c-1} \varphi_n^{(r)} = 0$. Therefore the limit $\lim_{j\to\infty} d(g^j(x), g^j(y))$ exists and so (x, y) is not DC3.

(A4) $z_x \neq z_y$ and $z_y = 1.3$, $r_x = r_y = 0.01$.

As in (A3), we make calculations using ρ in place of d. We will calculate Φ and Φ^* explicitly to show that they are equal. Notice that $\rho(\phi_x^j, \phi_y^j) \in [0, 0.5]$ for all j, so $\Phi^*_{(x,y)}(\delta) = \Phi_{(x,y)}(\delta) = 1$ for $\delta > 0.5$. From the definition of the map g, the point $y = (0.01, \phi_y, 1.3)$ is a fixed point, and for simplicity let us take $z_x = 1 + 1.3$ (if it is not, we replace x by some higher preimage from the uppermost circle). As the point x approaches the bottom ring, it rotates by finer and finer steps. Thus, we hope to obtain the distribution functions of a smoothly rotating point and a fixed point. Our careful choice of rotation angles and block sizes in (10) fulfills this hope.

We write $\phi_x^j = \phi_x + \sum_{n=1}^{j} \varphi_n^{(r)}$. We may think about ϕ_x^j in blocks. In the kth block (between l_{k-1} and l_k) we can see the sequence of these k angles $\{\phi_x^{l_{k-1}} + 0, \phi_x^{l_{k-1}} + \frac{1}{k}, \phi_x^{l_{k-1}} + \frac{2}{k}, \ldots, \phi_x^{l_{k-1}} + \frac{k-1}{k}\}$ exactly k^{k-1} times. Then by Lemma 2 we obtain

$$2\delta k - 1 \le \pi_k \le 2\delta k + 1, \quad \text{where}$$

$$\pi_k := \#\{l_{k-1} \le j \le l_{k-1} + k - 1; \rho(\phi_x^j, \phi_y) < \delta\} \tag{13}$$

and so in the whole kth block

$$2\delta k^k - k^{k-1} \le B_k \le 2\delta k^k + k^{k-1}, \quad \text{where}$$

$$B_k := \#\{l_{k-1} \le j < l_k; \rho(\phi_x^j, \phi_y) < \delta\} = k^{k-1}\pi_k. \tag{14}$$

To show equality $\Phi^*_{(x,y)}(\delta) = \Phi_{(x,y)}(\delta)$, it is not enough to work along subsequences — we need estimates for all n. For each n, there exist unique k_n, α_n and β_n such that $\alpha_n < k_n^{(k_n-1)}$, $\beta_n < k_n$ and

$$\begin{aligned} n &= 1^1 + 2^2 + \cdots + (k_n - 1)^{(k_n-1)} + \alpha_n k_n + \beta_n \\ &= l_{k_n-1} + \alpha_n k_n + \beta_n. \end{aligned} \tag{15}$$

Let us mark $p_n := \#\{0 \le j \le n; \rho(\phi_x^j, \phi_y^j) < \delta\}$. Then by (14) and (15), for each n there exists a unique $\gamma_n \le \pi_{k_n}$ such that

$$\begin{aligned} p_n &= \sum_{j=1}^{k_n-1} B_j + \alpha_n \pi_{k_n} + \gamma_n \\ &= \sum_{j=1}^{k_n-1} j^{j-1}\pi_j + \alpha_n \pi_{k_n} + \gamma_n. \end{aligned} \tag{16}$$

By (13) and (16)

$$\begin{aligned} p_n &\le \sum_{j=1}^{k_n-1} (j^{j-1}(2\delta j + 1)) + \alpha_n(2\delta k_n + 1) + \gamma_n \\ &= l_{k_n-1} 2\delta + \sum_{j=1}^{k_n-1} j^{j-1} + \alpha_n(2\delta k_n + 1) + \gamma_n \\ &= 2\delta(l_{k_n-1} + \alpha_n k_n) + \gamma_n + \sum_{j=1}^{k_n-1} j^{j-1} + \alpha_n. \end{aligned} \tag{17}$$

And similarly

$$p_n \ge l_{k_n-1} 2\delta - \sum_{j=1}^{k_n-1} j^{j-1} + \alpha_n(2\delta k_n - 1) + \gamma_n. \tag{18}$$

By (11), (15) and (17) we get

$$\Phi^*_{(x,y)}(\delta) = \limsup_{n\to\infty} \left[\frac{2\delta(l_{k_n-1} + \alpha_n k_n) + \gamma_n}{l_{k_n-1} + \alpha_n k_n + \beta_n} + \frac{\sum_{j=1}^{k_n-1} j^{j-1} + \alpha_n}{l_{k_n-1} + \alpha_n k_n + \beta_n} \right] = 2\delta \tag{19}$$

and by (11), (15) and (18) we get

$$\Phi_{(x,y)}(\delta) \ge \liminf_{n\to\infty} \left[\frac{2\delta(l_{k_n-1} + \alpha_n k_n) + \gamma_n}{l_{k_n-1} + \alpha_n k_n + \beta_n} - \frac{\sum_{j=1}^{k_n-1} j^{j-1} + \alpha_n}{l_{k_n-1} + \alpha_n k_n + \beta_n} \right] = 2\delta. \tag{20}$$

From (19) and (20), we obtain

$$\begin{aligned} \Phi_{(x,y)}(\delta) &= \Phi^*_{(x,y)}(\delta) = 2\delta, \quad \text{for } \delta < \frac{1}{2}, \\ \Phi_{(x,y)}(\delta) &= \Phi^*_{(x,y)}(\delta) = 1, \quad \text{for } \delta \ge \frac{1}{2}. \end{aligned} \tag{21}$$

This shows that (x, y) is not DC3.

(A5) $z_x \ne z_y$ and $z_y = 0$, $r_x = r_y = 0.02$.

The only difference from (A4) is that when we think in blocks indexed by k, now we have to distinguish between even and odd k. For odd k, the blocks are exactly the same as in (A4) — see (13)–(16). But for even k, we add the angle $2/k$ instead of $1/k$ in every iteration. For odd k, we define $\pi_k := \#\{l_{k-1} \le j \le l_{k-1} + k - 1; \rho(\phi_x^j, \phi_y) < \delta\}$ and observe by Lemma 2 that $2\delta k - 1 \le \pi_k \le 2\delta k + 1$. For even k, we define $\pi_k := \#\{l_{k-1} \le j \le l_{k-1} + \frac{k}{2} - 1; \rho(\phi_x^j, \phi_y) < \delta\}$ and observe in the same way that $2\delta\frac{k}{2} - 1 \le \pi_k \le 2\delta\frac{k}{2} + 1$.

You can see that for even k we get full rotation in $k/2$ steps instead of k steps. Then for the whole block

$$B_k = \begin{cases} k^{k-1}\pi_k, & \text{for odd } k, \\ 2k^{k-1}\pi_k, & \text{for even } k, \end{cases} \tag{22}$$

where $B_k := \#\{l_{k-1} \leq j < l_k; \rho(\phi_x^j, \phi_y) < \delta\}$. Hence

$$2\delta k^k - 2k^{k-1} \leq B_k \leq 2\delta k^k + 2k^{k-1}, \quad \text{for every } k \in \mathbb{N}. \tag{23}$$

Then by the same computation as in (A4) we obtain

$$\begin{aligned} \Phi_{(x,y)}(\delta) &= \Phi^*_{(x,y)}(\delta) = 2\delta, \quad \text{for } \delta < \frac{1}{2}, \\ \Phi_{(x,y)}(\delta) &= \Phi^*_{(x,y)}(\delta) = 1, \quad \text{for } \delta \geq \frac{1}{2}. \end{aligned} \tag{24}$$

This shows that (x, y) is not DC3.

(A6) $z_x \neq z_y$ and $z_x, z_y \neq 0$, $r_x = r_y = r = 0.02$.

The computing is the same as in (A3) so $\lim_{j\to\infty} d(g^j(x), g^j(y))$ converges and (x, y) is not DC3.

(B) We claim that there is a DC3 pair in (X, f).

We will show that $x = (0.01, 0, 1)$ and $y = (0.02, 0, 1)$ is DC3. For all j, $|r_x^j - r_y^j| = 0.01$, $|z_x^j - z_y^j| = 0$ and $\rho(\phi_x^j, \phi_y^j) \in [0, 0.5]$. We will compute $\Phi^*_{(x,y)}(\delta)$ and $\Phi_{(x,y)}(\delta)$ for $\delta \in [0.01, 0.5]$.

After every l_k steps (in the end of every kth block), $\phi_x^{l_k} = \phi_y^{l_k} = 0$. Let us denote

$$A_k := \#\{0 \leq j \leq l_k; d(f^j(x), f^j(y)) < \delta\} \leq l_k + 1. \tag{25}$$

After l_{k-1} steps, if $k - 1$ is even, then for the next k^k steps $\phi_x^j = \phi_y^j$ so $d(f^j(x), f^j(y)) = 0.01$ and

$$\#\{l_{k-1} < j \leq l_{k-1} + k^k; d(f^j(x), f^j(y)) < \delta\} = k^k. \tag{26}$$

Then by (11), (25) and (26) we get

$$\begin{aligned} \Phi^*_{(x,y)}(\delta) &\geq \limsup_{k\to\infty} \frac{A_{k-1} + k^k}{l_{k-1} + k^k} \\ &= \limsup_{k\to\infty} \frac{A_{k-1} + k^k}{l_k} = 1. \end{aligned} \tag{27}$$

If $k - 1$ is odd, then for the next k^k steps, the inner point is rotating with speed $1/k$ while the outer point is rotating with speed $2/k$. From the point of view of calculating distances, the situation is exactly the same as if the inner point made no rotation and the outer point rotated with speed $1/k$. Applying Lemma 2 [the same as we did in (14)] we obtain

$$\begin{aligned} \#\{l_{k-1} < j \leq l_k; d(f^j(x), f^j(y)) < \delta\} &\leq \#\{l_{k-1} < j \leq l_k; \rho(\phi_x^j, \phi_y^j) < \delta\} \\ &\leq 2\delta k^k + k^{k-1} \end{aligned} \tag{28}$$

and by (11), (25) and (28) we get

$$\Phi_{(x,y)}(\delta) \leq \liminf_{k\to\infty} \frac{A_{k-1} + 2\delta k^k + k^{k-1}}{l_k} = 2\delta. \tag{29}$$

By (27) and (29) we get

$$\Phi_{(x,y)} \leq 2\delta < 1 \leq \Phi^*_{(x,y)}(\delta), \quad \text{for } \delta \in (0.01, 0.5). \tag{30}$$

That shows that (x, y) is a DC3 pair.

We have shown that (X, f) contains a DC3 pair, while the conjugate system (Y, g) does not. ■

4.2. *DC3 — Uncountable scrambled set*

Theorem 3. *The existence of an uncountable DC3 set is not preserved by topological conjugacy.*

Proof. Let I be the unit interval and C be the middle-third Cantor set inside I. Let $\tilde{C}$ be C translated by 1, i.e. $\tilde{C} = C + 1$ and $\mathbb{S}_r$ be the circle with radius $r \in \tilde{C}$ and with center at the origin. The desired counterexample is constructed on the union in $\mathbb{R}^3$ of uncountably many concentric cylinders above $\{\mathbb{S}_r : r \in \tilde{C}\}$ (height of points in these cylinders ranges only in $\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\}$), with a skew-product mapping. The base map of this skew-product mapping decreases the height of a point in the limit to 0 and each fiber map is a rigid rotation.

Define

$$X = \left\{(r\cos(2\pi\varphi), r\sin(2\pi\varphi), z) : r \in \tilde{C}, \varphi \in I, z \in \left\{\frac{1}{k} : k \in \mathbb{N}\right\} \cup \{0\}\right\}.$$

We equip our space with the max-metric $d(x, y) = \max\{|r_x - r_y|, \rho(\varphi_x, \varphi_y), |z_x - z_y|\}$, where ρ is the metric on the unit circle $\rho(\alpha, \beta) = \min\{|\alpha - \beta|, 1 - |\alpha - \beta|\}$.

The transformation $f : X \to X$ is the identity for points with zero third coordinate,

$$(r\cos(2\pi\varphi), r\sin(2\pi\varphi), 0) \mapsto (r\cos(2\pi\varphi), r\sin(2\pi\varphi), 0)$$

and for the other points f is defined as follows

$$\left(r\cos(2\pi\varphi), r\sin(2\pi\varphi), \frac{1}{k}\right) \mapsto \left(r\cos(2\pi(\varphi + p(k))), r\sin(2\pi(\varphi + p(k))), \frac{1}{k+1}\right).$$

To define the function $p : \mathbb{N} \to I$, let $0 < n_0 < n_1 < \cdots$ be an increasing sequence of integers such that

$$\lim_{k\to\infty} \frac{s_k}{n_k} = 0 \quad \text{and simultaneously} \quad \lim_{k\to\infty} \frac{2^k}{n_k} = 0, \tag{31}$$

where $s_m = \sum_{i=0}^{m-1} n_i$ and $s_0 = 0$. Set

$$p(k) = \begin{cases} \dfrac{1}{2^{m+1}} & \text{if } s_m < k \le s_m + 2^m \\ 0 & \text{if } s_m + 2^m < k \le s_m + n_m. \end{cases} \tag{32}$$

Let (Y, F) be a dynamical system conjugated with (X, f) via the following homeomorphism

$$\Pi((x, y, z)) = \begin{cases} (x, y, z) & \text{if } x \ge 0, \\ (2x, y, z) & \text{if } x < 0, \end{cases} \tag{33}$$

where $Y = \Pi(X)$ and $F = \Pi \circ f \circ \Pi^{-1}$.

(I) (Y, F) is DC3.

We claim that the set $S \subset Y$,

$$S = \{(r, 0, 1) : r \in \tilde{C}\},$$

is distributionally scrambled of type 3. Denote $f^i(x) = (r\cos(2\pi\eta^i), r\sin(2\pi\eta^i), z^i)$, for $i \ge 0$. Then for $x \in S$,

$$z^i = \frac{1}{1+i} \quad \text{and} \quad \eta^i = \sum_{j=1}^{i} p(j) \bmod 1.$$

Let

$$U_k = \#\{0 \le i \le s_k + n_k; \eta^i = 0\},$$
$$L_k = \#\left\{0 \le i \le s_k + n_k; \eta^i = \frac{1}{2}\right\}.$$

By (32) and the definition of η^i, we can see that for $k \ge 0$

$$L_{2k} \ge n_{2k} - 2^{2k} \quad \text{and} \quad U_{2k+1} \ge n_{2k+1} - 2^{2k+1}.$$

Let x_1, x_2 be two distinct points in S such that $x_1 = (r_1, 0, 1)$ and $x_2 = (r_2, 0, 1)$. Then $d(x_1, x_2) = |r_1 - r_2| = \epsilon$. By the definition of F,

$$d(F^i(x_1), F^i(x_2)) = d(f^i(x_1), f^i(x_2)) = \epsilon, \quad \text{if } \eta^i = 0$$

and

$$d(F^i(x_1), F^i(x_2)) = 2 \cdot d(f^i(x_1), f^i(x_2)) = 2\epsilon, \quad \text{if } \eta^i = \frac{1}{2}.$$

It follows for any $\delta \in (\epsilon, 2\epsilon]$,

$$\Phi^*_{(x_1,x_2)}(\delta) \ge \lim_{k\to\infty} \frac{1}{s_{2k+1} + n_{2k+1}} U_{2k+1} \ge \lim_{k\to\infty} \frac{n_{2k+1} - 2^{2k+1}}{n_{2k+1} + s_{2k+1}}, \tag{34}$$

$$\Phi_{(x_1,x_2)}(\delta) \le 1 - \lim_{k\to\infty} \frac{1}{s_{2k} + n_{2k}} L_{2k} \le 1 - \lim_{k\to\infty} \frac{n_{2k} - 2^{2k}}{s_{2k} + n_{2k}}. \tag{35}$$

By (31), (34) and (35), we get for any $\delta \in (\epsilon, 2\epsilon]$,

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \quad \Phi_{(x_1,x_2)}(\delta) = 0.$$

(II) (X, f) is not DC3.

It is sufficient to show that every DC3 pair in X contains a fixed point and therefore the maximum cardinality of any distributionally scrambled set in X is 2. Let

$$\text{Fix} = \{(r\cos 2\pi\varphi, r\sin 2\pi\varphi, 0) : r \in \tilde{C}, \varphi \in I\}$$

be the set of all fixed points. We prove that for every $(x, y) \in (X\backslash\text{Fix})^2$ the limit $\lim_{n\to\infty} d(f^n(x), f^n(y))$ always exists and hence $\Phi^*_{(x,y)}(\delta) = \Phi_{(x,y)}(\delta)$, for any $\delta > 0$.

Let $x = (r_x\cos 2\pi\varphi_x, r_x\sin 2\pi\varphi_x, \frac{1}{k_x})$ and $y = (r_y\cos 2\pi\varphi_y, r_y\sin 2\pi\varphi_y, \frac{1}{k_y})$. We assume without loss of generality that $k_x \le k_y$, and after replacing x and y by their $(k_x - 1)$th preimages, we may assume that $k_x = 1$ and we may write $l = k_y$. By the definition of f, the distance $|r_{f^n(x)} - r_{f^n(y)}| = |r_x - r_y|$ is constant and the distance $|\frac{1}{k_{f^n(x)}} - \frac{1}{k_{f^n(y)}}|$ decreases to zero. Moreover, using the fact that $\lim_{n\to\infty} \sum_{k=n+1}^{n+l} p(k) = 0$, we find that the difference between the angles (with all calculations modulo 1) converges to a constant:

$$\begin{aligned}\lim_{n\to\infty}\varphi_{f^n(x)}-\varphi_{f^n(y)} &= \lim_{n\to\infty}\left(\varphi_x+\sum_{k=1}^{n}p(k)\right)-\left(\varphi_y+\sum_{k=l}^{n+l}p(k)\right)\\ &= \varphi_x-\varphi_y+\sum_{k=1}^{l-1}p(k)-\lim_{n\to\infty}\sum_{k=n+1}^{n+l}p(k)\\ &= \varphi_x-\varphi_y+\sum_{k=1}^{l-1}p(k).\end{aligned}$$

It follows that $\rho(\varphi_{f^n(x)},\varphi_{f^n(y)})$ converges to a constant. The existence of the limit $\lim_{n\to\infty} d(f^n(x), f^n(y))$ follows from the separate convergence in all three coordinates. ∎

5. Distributional Chaos of Type $2\frac{1}{2}$

Nevertheless, the notion of DC3 can be strengthened in such a way that it is preserved under conjugacy and implies Li–Yorke chaos:

Definition 5.1. A pair $(x_1, x_2) \in X^2$ is called *distributionally scrambled of type* $2\frac{1}{2}$ if there are positive numbers c and s such that, for any $0 < \delta \le s$,

$$\Phi_{(x_1,x_2)}(\delta) < c < \Phi^*_{(x_1,x_2)}(\delta).$$

We can define both distribution functions at 0 as limits $\Phi_{(x_1,x_2)}(0) = \lim_{\delta\to 0^+}\Phi_{(x_1,x_2)}(\delta)$ and $\Phi^*_{(x_1,x_2)}(0) = \lim_{\delta\to 0^+}\Phi^*_{(x_1,x_2)}(\delta)$. Then (x_1, x_2) being DC$2\frac{1}{2}$ is equivalent to

$$\Phi_{(x_1,x_2)}(0) < \Phi^*_{(x_1,x_2)}(0).$$

We define also DC$2\frac{1}{2}$ sets and DC$2\frac{1}{2}$ systems in the same way as for the other versions of distributional chaos (see Sec. 2).

By simple observation we can see that if $(x_1, x_2) \in X^2$ is DC$2\frac{1}{2}$, then it is Li–Yorke scrambled. Since $\Phi^*_{(x_1,x_2)}(\delta) > c$ for arbitrary small δ, (x_1, x_2) must be proximal (for distal pairs $\Phi^*_{(x_1,x_2)}(0) = 0$). Similarly, (x_1, x_2) is not asymptotic (for asymptotic pairs $\Phi_{(x_1,x_2)}(\delta) = 1$ for every $\delta > 0$).

Remark 5.1. We call a dynamical system *strictly* DC$2\frac{1}{2}$ if it possesses an uncountable DC$2\frac{1}{2}$ set but no DC2 pairs. By results in [Downarowicz & Lacroix, 2014], positive topological entropy implies existence of an uncountable DC2 set, hence strictly DC$2\frac{1}{2}$ systems must have zero topological entropy.

Theorem 4. *Let f and g be topologically conjugate continuous maps of a compact metric space (X, d). Then f is* DC$2\frac{1}{2}$ *if and only if g is* DC$2\frac{1}{2}$.

Proof. Suppose (u, v) is a DC$2\frac{1}{2}$ pair with respect to f, i.e. there are positive numbers c and s such that, for any $0 < \delta \le s$, $\Phi_{(u,v)}(\delta) < c < \Phi^*_{(u,v)}(\delta)$. Let h be a homeomorphism conjugating f and g such that $h \circ f = g \circ h$. By uniform continuity of h and h^{-1}, for any $\epsilon > 0$ there is $\delta > 0$ (and we may take $\delta < s$) such that for any $u, v \in X$,

$$d(u,v) < \delta \quad \text{implies } d(h(u),h(v)) < \epsilon \tag{36}$$

and

$$d(h(u),h(v)) < \delta \quad \text{implies } d(u,v) < \epsilon. \tag{37}$$

Since $f^n = h^{-1}\circ g^n \circ h$, it follows by (36),

$$d(f^n(u), f^n(v)) < \delta \quad \text{implies} \quad d(g^n\circ h(u), g^n\circ h(v)) < \epsilon$$

and consequently $c < \Phi^*_{(u,v)}(\delta) \le \Psi^*_{(h(u),h(v))}(\epsilon)$, where Φ^* and Ψ^* are the upper distribution functions of f and g respectively. Similarly by (37), $\Psi_{(h(u),h(v))}(\delta) \le \Phi_{(u,v)}(\epsilon)$, where Φ and Ψ are the lower distribution functions of f and g respectively. Hence for given s we can find $s' > 0$ such that $\Psi_{(h(u),h(v))}(s') \le \Phi_{(u,v)}(s) < c$. By monotonicity of the distribution function $\Psi_{(h(u),h(v))}$, for any $0 < \epsilon \le s'$,

$$\Psi_{(h(u),h(v))}(\epsilon) < c < \Psi^*_{(h(u),h(v))}(\epsilon).$$ ∎

The following example illustrates that DC$2\frac{1}{2}$ is essentially weaker than DC2.

Example 5.1. Let X be the union of a converging sequence of unit fibers and the limit fiber,

$$X = \left\{\frac{1}{k} : k\in\mathbb{N}\right\}\times I \cup \{0\}\times I,$$

where $I = [0, 1]$, and let f be a skew-product map $f : X \to X$ which is the identity on the limit fiber,

$$(0,z)\mapsto(0,z)$$

and which is f_k on the other fibers,

$$\left(\frac{1}{k}, z\right)\mapsto\left(\frac{1}{k+1}, f_k(z)\right),\quad k\in\mathbb{N}.$$

To define $f_k : I \to I$, let $0 = m_0 < m_1 < m_2 < m_3 < \cdots$ be an increasing sequence of integers satisfying

$$\lim_{l\to\infty}\frac{m_l}{m_{l+1}-m_l}=0,\quad \lim_{l\to\infty}\frac{l}{m_l}=0$$

and the difference $(m_l - m_{l-1})$ is divisible by $7l$, for any $l \in \mathbb{N}$. Let $h_l(x) = l^{-1/l}x$ and $\overline{h_l}(x) = \min\{1, l^{1/l}x\}$. The sequence $\{f_k\}_{k=1}^{\infty}$ is defined in two ways. If l is odd, then f_k for $m_{l-1} < k \le m_l$ is given by

$$f_{m_{l-1}+1}\cdots f_{m_l} = [(h_l)^l(Id)^{4l}(\overline{h_l})^l(Id)^l]^{\frac{1}{7}(m_l-m_{l-1})},$$

where $(h)^l$ means $\underbrace{h,\ldots,h}_{l\text{-times}}$. If l is even, then we change the order and f_k for $m_{l-1} < k \le m_l$ is given by

$$f_{m_{l-1}+1}\cdots f_{m_l} = [(h_l)^l(Id)^{l}(\overline{h_l})^l(Id)^{4l}]^{\frac{1}{7}(m_l-m_{l-1})}.$$

We can express our construction of the sequence of maps f_k more precisely by writing that for odd l,

$$f_k = \begin{cases} h_l, & m_{l-1}+i7l < k \le m_{l-1}+i7l+l \\ Id, & m_{l-1}+i7l+l < k \le m_{l-1}+i7l+5l \\ & \quad\text{where } k\in\mathbb{N}, \\ \overline{h_l}, & m_{l-1}+i7l+5l < k \le m_{l-1}+i7l+6l \\ & \quad\text{where } i \in \left\{0,1,\ldots,\dfrac{m_l-m_{l-1}}{7l}-1\right\}, \\ Id, & m_{l-1}+i7l+6l < k \le m_{l-1}+i7l+7l \end{cases}$$

and for even l,

$$f_k = \begin{cases} h_l, & m_{l-1}+i7l < k \le m_{l-1}+i7l+l \\ Id, & m_{l-1}+i7l+l < k \le m_{l-1}+i7l+2l \\ & \quad\text{where } k\in\mathbb{N}, \\ \overline{h_l}, & m_{l-1}+i7l+2l < k \le m_{l-1}+i7l+3l \\ & \quad\text{where } i \in \left\{0,1,\ldots,\dfrac{m_l-m_{l-1}}{7l}-1\right\}. \\ Id, & m_{l-1}+i7l+3l < k \le m_{l-1}+i7l+7l. \end{cases}$$

The sequence $\{f_k\}_{k=1}^{\infty}$ uniformly converges to the identity and therefore f is continuous. Let $(u,v) \in \{1\}\times I$ and $\alpha = d(u,v)$. After l applications of h_l the distance is contracted from α to $\frac{\alpha}{l}$ and it remains the same until l applications of $\overline{h_l}$ recover the distance to the original α. The identity mappings between h_l and $\overline{h_l}$ ensure that the distance is contracted to $\frac{\alpha}{l}$ for four sevenths of the times i and then it is recovered to α for one seventh of the times i, when $m_{l-1} < i \le m_l$ and l is odd. Hence $\frac{6}{7} \ge \Phi^*_{(u,v)}(\delta) \ge \frac{4}{7}$, for any $0<\delta<\alpha$. When l is even, the distance is contracted to $\frac{\alpha}{l}$ for one seventh of the times i and then it is recovered to α for four sevenths of the times i, for $m_{l-1} < i \le m_{l+1}$. Therefore $\frac{3}{7} \ge \Phi_{(u,v)}(\delta) \ge \frac{1}{7}$, for any $0<\delta<\alpha$ and the set $\{1\}\times I$ is DC2$\frac{1}{2}$ but not DC2.

With the same reasoning we conclude that pairs in any fiber $\{\frac{1}{k}\}\times I$ are either DC2$\frac{1}{2}$ but not DC2 scrambled or they have eventually equal trajectories (because of the constant part of the function $\overline{h_l}$). If $u \in \{\frac{1}{k}\}\times I$ and $v \in \{\frac{1}{k+n}\}\times I$, then we apply functions to u with a delay of n times. Blocks of identities are arbitrarily long and n is fixed, hence the delay does not change the limits — $\frac{6}{7} \ge \Phi^*_{(u,v)}(\delta) \ge \frac{4}{7}$ and $\frac{3}{7} \ge \Phi_{(u,v)}(\delta) \ge \frac{1}{7}$, for any $0<\delta<\beta$, where $\beta = d(f^n(u),v)$. In this case (u,v) is also a DC2$\frac{1}{2}$ but not DC2 pair. If $\beta = 0$, then v belongs to the orbit of u and (u,v) is asymptotic.

Notice that points in $\{0\}\times I$ are fixed and hence there are no scrambled pairs in $\{0\}\times I$. Suppose $u \notin \{0\}\times I$ and $v \in \{0\}\times I$. Then $\frac{6}{7} \ge \Phi^*_{(u,v)}(\delta)$, for sufficiently small δ, since the height of u is contracted for at least $1/7$ of the time and (u,v) is not DC2 (if $v=(0,0)$, (u,v) is either an asymptotic pair or $\frac{6}{7} \ge \Phi^*_{(u,v)}(\delta)$, since u is recovered for at least $1/7$ of the time). Therefore X has no DC2 pairs.

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Appendix A

We give the proof of Lemma 1.

Proof. First we confirm the convergence $\alpha_i \to \alpha$ as $i \to \infty$. If we multiply the numerator and denominator of the expression for α_i by 4 and rearrange terms, we find

$$\alpha_i = \frac{\prod_{l=1}^{i}(2^{n_l} - 2^l)}{2^{m_i}} - \frac{2^i + 4}{2^{n_i}} \cdot \frac{\prod_{l=1}^{i-1}(2^{n_l} - 2^l)}{2^{m_{i-1}}}, \quad i \in \mathbb{N}.$$

Since n_i is strictly increasing in i, it follows that $n_i - i$ is nondecreasing. Moreover, $n_i - i \to \infty$, for otherwise, $n_i - i$ would be eventually constant, which contradicts (3). Now pass to the limit.

Now we derive the estimate for the rotation along the orbit. We introduce the following functions (cf. (4) and (6) for the definitions of k and p)

$$\varphi_l(n, \sigma\omega) = \sum_{j \in J} p(\tau^j \sigma\omega), \quad \text{where}$$
$$J = \{j; 0 \le j \le n-1 \text{ and } k(\tau^j \sigma\omega) = l\}. \tag{A.1}$$

Thus, φ_l counts the contribution to the net rotation due to block l. We have

$$\sum_{j=0}^{n-1} p(\tau^j \sigma\omega) = \sum_{l=1}^{\infty} \varphi_l(n, \sigma\omega), \quad n \in \mathbb{N}.$$

Notice that $\tau^{2^{m_i-2}}$ represents addition by $000\cdots010\cdots$, where 1 appears in the next to last coordinate of block i. Thus, if $n \le 2^{m_i-2}$, then for all $j < n$ there is still a zero in the ith block of $\tau^j\sigma\omega$ so that $k(\tau^j\sigma\omega) \le i$. Therefore we only need to add together contributions from blocks 1 through i,

$$\sum_{j=0}^{n-1} p(\tau^j \sigma\omega) = \sum_{l=1}^{i} \varphi_l(n, \sigma\omega), \quad n \in \{1, \ldots, 2^{m_i-2}\}.$$

Comparing definitions (6) and (A.1), we find that we can evaluate $\varphi_l(n, \sigma\omega)$ by looking up the word appearing in the lth block of $\tau^n\sigma\omega$ on one of the tables in Fig. 3. We will say that the word in block l is a *good word* if $\varphi_l(n, \sigma\omega) = \frac{1}{2}(1 - \omega_l) \bmod 1$.

Use this table if $\omega_l = 0$

Block l from $\tau^n\sigma\omega$	Evaluation	$\varphi_l(n,\sigma\omega)$ mod 1	
$00\cdots0 - 000\cdots00$	0	0	Bad
$\vdots$	$\vdots$	$\vdots$	
$11\cdots1 - 000\cdots00$	$2^{l-1} - 1$	$\frac{2^{l-1}-1}{2^l}$	
$00\cdots0 - 100\cdots00$	2^l	$\frac{1}{2}$	Good
$\vdots$	$\vdots$	$\vdots$	
$11\cdots1 - 011\cdots11$	$2^{n_l} - 2^{l-1} - 1$	$\frac{1}{2}$	
$00\cdots0 - 111\cdots11$	$2^{n_l} - 2^{l-1}$	$\frac{2^{l-1}+1}{2^l}$	Bad
$\vdots$	$\vdots$	$\vdots$	
$11\cdots1 - 111\cdots11$	$2^{n_l} - 1$	1	

Use this table if $\omega_l = 1$

Block l from $\tau^n\sigma\omega$	Evaluation	$\varphi_l(n,\sigma\omega)$ mod 1	
$00\cdots0 - 000\cdots01$	2^{n_l-1}	0	Good
$\vdots$	$\vdots$	$\vdots$	
$11\cdots1 - 011\cdots11$	$2^{n_l} - 2^{l-1} - 1$	0	
$00\cdots0 - 111\cdots11$	$2^{n_l} - 2^{l-1}$	$\frac{1}{2^l}$	Bad
$\vdots$	$\vdots$	$\vdots$	
$11\cdots1 - 000\cdots00$	$2^{l-1} - 1$	$\frac{2^l-1}{2^l}$	
$00\cdots0 - 100\cdots00$	2^{l-1}	1	Good
$\vdots$	$\vdots$	$\vdots$	
$11\cdots1 - 111\cdots10$	$2^{n_l-1} - 1$	1	

Fig. 3. Table of values for $\varphi_l(n, \sigma\omega)$. Each word has length n_l. The first $l - 1$ digits have been typographically separated from the remaining digits by a hyphen.

If $\tau^n\sigma\omega$ contains good words in each of the blocks 1 through i, then we obtain the desired sum

$$\sum_{j=0}^{n-1} p(\tau^j\sigma\omega) = \frac{1}{2}((1-\omega_1) + \cdots + (1-\omega_i)) \bmod 1.$$

How many ways are there to choose good words in all blocks?

Take the set $\{\tau^n(\sigma\omega)\}_{n=1}^{2^{m_i}-2}$ and truncate each member of this set to the first i blocks. We obtain in this way the set of all $x \in \{0,1\}^{m_i}$ in the interval

$$\overbrace{0\cdots 0\omega_1}^{\text{Block 1}}\overbrace{0\cdots 0\omega_2}^{\text{Block 2}}\cdots\overbrace{0\cdots 00\omega_i}^{\text{Block i}}$$
$$< x \le \overbrace{0\cdots 0\omega_1}^{\text{Block 1}}\overbrace{0\cdots 0\omega_2}^{\text{Block 2}}\cdots\overbrace{0\cdots 01\omega_i}^{\text{Block i}},$$

where the order relation corresponds to the temporal ordering of our orbit segment; we may express this explicitly in terms of the evaluation operator (5) with the rule that $x < y$ whenever $|x| < |y|$. Unfortunately, this interval does not contain arbitrary combinations of words in the first i blocks. It does, however, contain the subinterval

$$\overbrace{00\cdots 0}^{\text{Block 1}}\overbrace{00\cdots 0}^{\text{Block 2}}\cdots\overbrace{10\cdots 00\omega_i}^{\text{Block i}}$$
$$\le x \le \overbrace{11\cdots 1}^{\text{Block 1}}\overbrace{11\cdots 1}^{\text{Block 2}}\cdots\overbrace{11\cdots 10\omega_i}^{\text{Block i}}.$$

This interval contains words from the set $\{x \in \{0,1\}^{n_i}; 10\ldots 00\omega_i \le x \le 11\ldots 10\omega_i\}$ in block i combined with arbitrary words in blocks 1 through $i-1$. Counting the number of good words available in each block and multiplying completes the proof of the lemma. ■

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Iteration Problem for Distributional Chaos

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We disprove the conjecture that the existence of a DC3-scrambled pair is preserved under iteration and show that a slightly strengthened definition of distributional chaos of type 3, denoted by DC$2\frac{1}{2}$, is iteration invariant, i.e. that f^n is DC$2\frac{1}{2}$ if and only if f is. Unlike DC3, DC$2\frac{1}{2}$ is also conjugacy invariant and implies Li–Yorke chaos. The definition of DC$2\frac{1}{2}$ is the following: a pair $\langle x, y\rangle$ is DC$2\frac{1}{2}$-scrambled iff $\Phi_{\langle x,y\rangle}(0) < \Phi^*_{\langle x,y\rangle}(0)$, where $\Phi_{\langle x,y\rangle}(\delta)$ (resp., $\Phi^*_{\langle x,y\rangle}(\delta)$) is lower (resp., upper) asymptotic density of the set of times k when $d(f^k(x), f^k(y)) < \delta$, and both densities are defined at 0 as limits of their values for $\delta \to 0^+$. DC$2\frac{1}{2}$ shares similar properties with DC1 and DC2 but it is essentially weaker than DC2.

Keywords: Distributional chaos; Li–Yorke chaos; iteration invariant; zero topological entropy.

1. Introduction

The study of chaotic pairs in dynamical systems started with Li and Yorke [1975], who considered pairs of points with the property that their orbits are neither asymptotic nor separated by any fixed positive constant. Schweizer and Smítal [1994] introduced a related concept of a distributionally chaotic pair as two points for which the statistical distribution of distances between their orbits does not converge. In case of dynamics on a compact interval, the existence of at least one distributionally chaotic pair is equivalent to the positivity of topological entropy (and some other notions of chaos).

Later, distributional chaos was divided into three types, DC1, DC2, and DC3, see [Balibrea *et al.*, 2005]. Relations between them and the relation between distributional chaos, Li–Yorke chaos and positivity of topological entropy were investigated by many authors, see e.g. [Balibrea *et al.*, 2005; Downarowicz, 2014; Oprocha, 2009; Wang *et al.*, 2008b; Wang *et al.*, 2014]. One can easily see from the definitions that DC1 implies DC2 and DC2 implies DC3. On the other hand, there are examples which show that DC1 is stronger than DC2 and DC2 is stronger than DC3. It is also obvious that DC2 (thus also DC1) implies Li–Yorke chaos. Moreover, there are Li–Yorke chaotic continuous maps of the interval with zero topological entropy; by [Schweizer & Smítal, 1994], such maps cannot be distributionally chaotic. This shows that Li–Yorke chaos need not imply any of the three versions of distributional chaos. Recently, Downarowicz in [Downarowicz, 2014] proved that positive topological entropy implies DC2.

It is proved in [Balibrea *et al.*, 2005] that DC3 does not imply chaos in the sense of Li and Yorke and it is not invariant with respect to topological conjugacy. Hence the definition of DC3 was strengthened in such a way that it is preserved under conjugacy and implies Li–Yorke chaos, but is still weaker than DC2 — the new definition was denoted by DC$2\frac{1}{2}$ (see [Doleželová-Hantáková *et al.*, 2016]). A pair $\langle x, y\rangle$ is DC3-scrambled iff $\Phi(\delta) < \Phi^*(\delta)$, for every $\delta > 0$ in some interval I. We say that a pair $\langle x, y\rangle$ is DC$2\frac{1}{2}$-scrambled iff

$\Phi(0) < \Phi^*(0)$, where the distribution functions at 0 are defined as limits of their values for $\delta \to 0^+$. Since both Φ and Φ^* are nondecreasing, $\Phi(0) < \Phi^*(0)$ implies $\Phi(\delta) < \Phi^*(\delta)$, for every $\delta > 0$ in some interval I with left endpoint 0, which shows that DC$2\frac{1}{2}$ implies DC3. An example which proved that DC$2\frac{1}{2}$ is essentially weaker than DC2 can be found in [Doleželová-Hantáková *et al.*, 2016]. This example possesses no DC2-scrambled pair; hence, by results in [Downarowicz, 2014], its topological entropy must be zero. We will show in this paper that DC$2\frac{1}{2}$ is (like DC1 and DC2) iteration invariant.

Another strengthened distributional chaos, denoted by DC$1\frac{1}{2}$, was proposed by authors in [Downarowicz & Lacroix, 2014]. DC$1\frac{1}{2}$ chaos is stronger than DC2 but is still implied by positive topological entropy.

Li in [Li, 2011] and Wang *et al.* in [Wang *et al.*, 2008a] show that DC1 and DC2 are iteration invariants and posed an open question whether DC3 is also preserved under iteration. Dvořaková proved in [Dvořáková, 2012] one implication — if a function f is DC3, then f^n is DC3, for every $n \in \mathbb{N}$, and conjectured that the opposite implication also holds. We disprove this conjecture by finding a dynamical system which has a DC3-scrambled pair with respect to f^2 but no DC3-scrambled pairs with respect to f.

However, it should be noticed that the distributional chaos considered in [Balibrea *et al.*, 2005; Li, 2011; Wang *et al.*, 2008a; Dvořáková, 2012] was defined as the existence of at least one distributionally scrambled pair. In the considerations by most other authors, it is usually assumed that distributional chaos means the existence of an uncountable distributionally scrambled set. Thus, a natural question arises — does the existence of an uncountable scrambled set behave under iterates of a function in the same manner as the existence of at least one scrambled pair? The answer to this question strongly depends on the type of distributional chaos. Since we will deal with both conceptions of distributional chaos, we adapt the following notation. The existence of at least one DCi-scrambled pair is denoted by DCi_p and the existence of an uncountable DCi set is denoted by DCi_u.

The paper is organized as follows: the first two sections are introductory. The third section investigates distributional chaos of type $2\frac{1}{2}$ and proves that both DC$2\frac{1}{2}_p$ and DC$2\frac{1}{2}_u$ are iteration invariants. In the fourth section we show that DC3_p is not iteration invariant, by creating a counter-example. The fifth section discusses whether the existence of an infinite or an uncountable distributionally scrambled set is preserved under iteration.

2. Terminology

Let (X, d) be a nonempty compact metric space. A pair (X, f), where f is a continuous self-map acting on X, is called a *topological dynamical system*. A property P is *iteration invariant* if, for any dynamical system (X, f) and $n \in \mathbb{N}$, (X, f) has P if and only if (X, f^n) has P. We define the *forward orbit* of x, denoted by $\mathrm{Orb}_f^+(x)$ as the set $\{f^n(x) : n \geq 0\}$. We say that a pair $\langle x, y\rangle$ is *asymptotic* if $\lim_{n\to\infty} d(f^i(x), f^i(y)) = 0$ or *eventually equal* if there is $j \in \mathbb{N}$ such that $f^j(x) = f^j(y)$. We call a pair $\langle x, y\rangle \in X^2$ *proximal* if $\liminf_{n\to\infty} d(f^i(x), f^i(y)) = 0$ (otherwise we say that $\langle x, y\rangle$ is *distal*). A pair of points is *Li–Yorke scrambled* simply if it is proximal but not asymptotic.

For a pair $\langle x_1, x_2\rangle$ of points in X, we define the *lower distribution function* generated by f as

$$\Phi_{\langle x_1,x_2\rangle}(\delta) = \liminf_{m\to\infty} \frac{1}{m}\#\{0 \leq k \leq m; \; d(f^k(x_1), f^k(x_2)) < \delta\}$$

and the *upper distribution function* as

$$\Phi^*_{\langle x_1,x_2\rangle}(\delta) = \limsup_{m\to\infty} \frac{1}{m}\#\{0 \leq k \leq m; \; d(f^k(x_1), f^k(x_2)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A.

A pair $\langle x_1, x_2\rangle \in X^2$ is called *distributionally scrambled of type 1* (*briefly DC1*) if

$$\Phi^*_{\langle x_1,x_2\rangle}(\delta) = 1, \quad \text{for every } 0 < \delta \leq \operatorname{diam} X,$$

$$\Phi_{\langle x_1,x_2\rangle}(\epsilon) = 0, \quad \text{for some } 0 < \epsilon \leq \operatorname{diam} X,$$

distributionally scrambled of type $1\frac{1}{2}$ (*briefly DC*$1\frac{1}{2}$) if

$$\Phi^*_{\langle x_1,x_2\rangle}(\delta) = 1, \quad \text{for every } 0 < \delta \leq \operatorname{diam} X, \text{ and, for every } c > 0, \text{ there is } \epsilon_c > 0 \text{ such that } \Phi_{\langle x_1,x_2\rangle}(\epsilon_c) \leq c,$$

distributionally scrambled of type 2 (*briefly DC2*) if

$$\Phi^*_{\langle x_1,x_2\rangle}(\delta) = 1, \quad \text{for every } 0 < \delta \leq \operatorname{diam} X,$$

$$\Phi_{\langle x_1,x_2\rangle}(\epsilon) < 1, \quad \text{for some } 0 < \epsilon,$$

distributionally scrambled of type $2\frac{1}{2}$ (*briefly DC$2\frac{1}{2}$*) if there exist numbers $c, \epsilon > 0$ such that

$$\Phi_{\langle x_1,x_2\rangle}(\delta) < c < \Phi^*_{\langle x_1,x_2\rangle}(\delta), \quad \text{for every } 0 < \delta \leq \epsilon,$$

distributionally scrambled of type 3 (*briefly DC3*) if

$$\Phi_{\langle x_1,x_2\rangle}(\delta) < \Phi^*_{\langle x_1,x_2\rangle}(\delta), \quad \text{for every } \delta \in (a,b),$$
$$\text{where } 0 \leq a < b \leq \operatorname{diam} X.$$

A subset S of X is *distributionally scrambled of type i* (or a DCi set), where $i = 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$, if every pair of distinct points in S is a DCi-scrambled pair. The dynamical system (X, f) is DCi_u, where $i = 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$, if there is an uncountable distributionally scrambled set $S \subset X$ of type i. The dynamical system (X, f) is DCi_p, where $i = 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$, if there is at least one distributionally scrambled pair of type i in X. The dynamical system is strictly DCi_u (resp., strictly DCi_p) if it is DCi_u (resp., DCi_p) and possesses no distributionally scrambled pairs of types smaller than i.

We can define both distribution functions at 0 as the limit $\Phi_{\langle x_1,x_2\rangle}(0) = \lim_{\delta\to 0^+} \Phi_{\langle x_1,x_2\rangle}(\delta)$ and $\Phi^*_{\langle x_1,x_2\rangle}(0) = \lim_{\delta\to 0^+} \Phi^*_{\langle x_1,x_2\rangle}(\delta)$. Then (x_1, x_2) being DC1-scrambled is equivalent to

$$\Phi^*_{\langle x_1,x_2\rangle}(0) = 1, \quad \Phi_{\langle x_1,x_2\rangle}(\epsilon) = 0,$$
$$\text{for some } 0 < \epsilon \leq \operatorname{diam} X;$$

DC$1\frac{1}{2}$-scrambled is equivalent to

$$\Phi^*_{\langle x_1,x_2\rangle}(0) = 1, \quad \Phi_{\langle x_1,x_2\rangle}(0) = 0;$$

DC2-scrambled is equivalent to

$$\Phi^*_{\langle x_1,x_2\rangle}(0) = 1, \quad \Phi_{\langle x_1,x_2\rangle}(0) < 1;$$

DC$2\frac{1}{2}$-scrambled is equivalent to

$$\Phi_{\langle x_1,x_2\rangle}(0) < \Phi^*_{\langle x_1,x_2\rangle}(0).$$

3. Iteration Problem for DC$2\frac{1}{2}$

For completeness, we first state all existing results about distributional chaos of type $2\frac{1}{2}$ from [Doleželová-Hantáková *et al.*, 2016]. First of all, if $\langle x, y\rangle \in X^2$ is DC$2\frac{1}{2}$, then it is Li–Yorke scrambled. Indeed, $\Phi^*_{\langle x,y\rangle}(0) > 0$ implies $\langle x, y\rangle$ being proximal (for distal pairs, $\Phi^*_{\langle x,y\rangle}(0) = 0$). Similarly, $\Phi_{\langle x,y\rangle}(0) < 1$ implies $\langle x, y\rangle$ being not asymptotic (for asymptotic pairs, $\Phi_{\langle x,y\rangle}(0) = 1$).

DC$2\frac{1}{2}{}_p$ is strictly stronger than DC3_p (any distal DC3_p system must not have DC$2\frac{1}{2}$ pairs) and strictly weaker than DC2_p (see example of strictly DC$2\frac{1}{2}{}_p$ system in [Doleželová-Hantáková *et al.*, 2016]). By results in [Downarowicz, 2014], positive topological entropy implies existence of an uncountable DC2 set, hence strictly DC$2\frac{1}{2}{}_p$ systems and strictly DC$2\frac{1}{2}{}_u$ systems must have zero topological entropy.

Both DC$2\frac{1}{2}{}_p$ and DC$2\frac{1}{2}{}_u$ are conjugacy invariants — let f and g be topologically conjugate continuous maps of a compact metric space. Then f is DC$2\frac{1}{2}{}_p$ (resp., DC$2\frac{1}{2}{}_u$) if and only if g is DC$2\frac{1}{2}{}_p$ (resp., DC$2\frac{1}{2}{}_u$). This claim is an easy consequence of the following Lemma 1.

Lemma 1. *Let f and g be topologically conjugate continuous maps of a compact metric space (X, d). Then the upper and the lower distribution functions of any pair $\langle x, y\rangle$ in X calculated with respect to g have the same value at zero as calculated with respect to f.*

Proof. Let h be a homeomorphism conjugating f and g such that $h \circ f = g \circ h$. By uniform continuity of h and h^{-1}, for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x, y \in X$,

$$d(x, y) < \delta \quad \text{implies } d(h(x), h(y)) < \epsilon \tag{1}$$

and

$$d(h(x), h(y)) < \delta \quad \text{implies } d(x, y) < \epsilon. \tag{2}$$

Since $f^n = h^{-1} \circ g^n \circ h$, it follows by (1),

$$d(f^n(x), f^n(y)) < \delta$$
$$\text{implies } d(g^n \circ h(x), g^n \circ h(y)) < \epsilon$$

and consequently $\Phi_{\langle x,y\rangle}(\delta) \leq \Psi_{\langle h(x),h(y)\rangle}(\epsilon)$ and $\Phi^*_{\langle x,y\rangle}(\delta) \leq \Psi^*_{\langle h(x),h(y)\rangle}(\epsilon)$, where Φ and Φ^* (resp., Ψ and Ψ^*) are the lower and the upper distribution functions of f (resp., of g). Similarly by (2), $\Psi_{\langle h(x),h(y)\rangle}(\delta) \leq \Phi_{\langle x,y\rangle}(\epsilon)$ and $\Psi^*_{\langle h(x),h(y)\rangle}(\delta) \leq \Phi^*_{\langle x,y\rangle}(\epsilon)$. It follows $\Phi_{\langle x,y\rangle}(0) = \Psi_{\langle h(x),h(y)\rangle}(0)$ and $\Psi^*_{\langle h(x),h(y)\rangle}(0) = \Phi^*_{\langle x,y\rangle}(0)$. ■

Now we will show that DC$2\frac{1}{2}{}_p$ and DC$2\frac{1}{2}{}_u$ are also iteration invariants:

Theorem 1. *For any integer $N > 1$, the function f^N is DC$2\frac{1}{2}{}_p$ (resp., DC$2\frac{1}{2}{}_u$) if and only if f is DC$2\frac{1}{2}{}_p$ (resp., DC$2\frac{1}{2}{}_u$).*

Proof. For a given function f of a compact metric space (X, d), an integer N and two points x, y in X, denote the distribution functions with respect to f by Φ

$$\Phi(\delta) = \liminf_{k\to\infty} \frac{1}{k}\#\{0 \le i < k; d(f^i(x), f^i(y)) < \delta\},$$

$$\Phi^*(\delta) = \limsup_{k\to\infty} \frac{1}{k}\#\{0 \le i < k; d(f^i(x), f^i(y)) < \delta\}$$

and with respect to f^N by Ψ

$$\Psi(\delta) = \liminf_{k\to\infty} \frac{1}{k}\#\{0 \le i < k; d(f^{iN}(x), f^{iN}(y)) < \delta\},$$

$$\Psi^*(\delta) = \limsup_{k\to\infty} \frac{1}{k}\#\{0 \le i < k; d(f^{iN}(x), f^{iN}(y)) < \delta\}.$$

Notice that we can calculate $\Phi(\delta)$ (resp., $\Phi^*(\delta)$) using only subsequence of multiples of N. For every element n_i of the subsequence of times with minimal (resp., maximal) asymptotic density, there is an integer k such that $kN \le n_i < kN + N$, therefore errors in computation using subsequence of multiples of N tend to 0 with $i \to \infty$.

We derive another metric D on X,

$$D(x, y) = \max_{j=0,1,\dots,N}\{d(f^j(x), f^j(y))\}.$$

By a simple observation,

$$D(x, y) < \delta \quad \text{implies } d(f^j(x), f^j(y)) < \delta, \quad \text{for } j = 0, 1, \dots, N-1 \tag{3}$$

and

$$d(f^N(x), f^N(y)) \ge \delta \quad \text{implies } D(f^j(x), f^j(y)) \ge \delta, \quad \text{for } j = 0, 1, \dots, N-1. \tag{4}$$

It follows by (3), $\Psi_D(\delta) \le \Phi(\delta)$ and $\Psi^*_D(\delta) \le \Phi^*(\delta)$, and similarly by (4), $1 - \Psi(\delta) \le 1 - \Phi_D(\delta)$ and $1 - \Psi^*(\delta) \le 1 - \Phi^*_D(\delta)$, where the distribution functions with subscript D were computed for the metric D and all distribution functions with respect to f were computed along the subsequence of multiples of N. By Lemma 1, the distribution functions do not depend on the metric at zero. Therefore $\Psi(0) = \Psi_D(0) \le \Phi(0)$ and $\Psi(0) \ge \Phi_D(0) = \Phi(0)$. The equality $\Psi^*(0) = \Phi^*(0)$ is obtained in the same manner. Consequently, $\langle x, y\rangle$ is DC2$\frac{1}{2}$ with respect to f if and only if it is DC2$\frac{1}{2}$ with respect to f^N. ■

4. Iteration Problem for DC3

Theorem 2. *DC3$_p$ is not iteration invariant.*

Proof of this theorem consists of finding a dynamical system which has a DC3-scrambled pair with respect to f^2 but no DC3-scrambled pairs with respect to f. The main obstacle in the creation of such system is that by [Li, 2011], DC2$_p$ is an iteration invariant, hence the desired system has to be strictly DC3$_p$. There are only few such examples in the literature (see [Balibrea *et al.*, 2005; Li, 2011; Oprocha, 2009]).

In this section we will gradually modify a very simple dynamical system from Sec. 4.1 to get a DC3$_p$ system in Sec. 4.2 and then prove our theorem in Sec. 4.3.

Since we are considering only DC3$_p$ in this section, we pose the following open question for DC3$_u$.

Question 4.1. *Is there a dynamical system which has an uncountable DC3 set with respect to f^2 but no DC3-scrambled pairs with respect to f?*

4.1. *Oscillator*

Our first goal is to construct an oscillatory dynamical system, where points regularly move from the right endpoint of an interval to the left endpoint (and back). Let I be the unit interval and $g_m : I \to I$ be the mapping defined as

$$g_m(x) = \begin{cases} 0 & 0 \le x < \dfrac{1}{m} \\ x - \dfrac{1}{m} & \dfrac{1}{m} \le x \le 1 \end{cases} \tag{5}$$

and $\hat{g}_m : I \to I$ defined as

$$\hat{g}_m(x) = \begin{cases} x + \dfrac{1}{m} & 0 \le x < 1 - \dfrac{1}{m} \\ 1 & 1 - \dfrac{1}{m} \le x \le 1. \end{cases} \tag{6}$$

Dynamical system O_1 consists of a compact metric space $I \times (\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\})$ endowed with the max-metric $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ and a function F such that, for $x \in I$,

$$F(\langle x, 0\rangle) = \langle x, 0\rangle$$
$$F\left(\left\langle x, \frac{1}{k}\right\rangle\right) = \left\langle f_k(x), \frac{1}{k+1}\right\rangle,$$

with

$$f_k = \begin{cases} g_m & s_m + 2im < k \le s_m + 2im + m \\ \hat{g}_m & s_m + (2i+1)m < k \le s_m + (2i+2)m \\ & i \in \{0, 1, \ldots, n_m - 1\}, \end{cases} \tag{7}$$

where $s_1 = 0$, $s_m = n_1 \cdot 2 \cdot 1 + n_2 \cdot 2 \cdot 2 + \cdots + n_{m-1} \cdot 2 \cdot (m-1)$, for $m > 1$, and $\{n_m\}_{m=1}^{\infty}$ is an increasing sequence of positive integers which will be specified later. The first terms of the sequence $\{f_k\}_{k=1}^{\infty}$ are

$$(g_1, \hat{g}_1)^{n_1}(g_2, g_2, \hat{g}_2, \hat{g}_2)^{n_2}(g_3, g_3, g_3, \hat{g}_3, \hat{g}_3, \hat{g}_3)^{n_3} \cdots$$

where $(h)^l$ means $\underbrace{h, \ldots, h}_{l\text{-times}}$. Notice that the point $\langle 1, 1\rangle$ moves from left to the right applying m-times g_m and then from left to right applying m-times $\hat{g}_m$ and repeat this movement n_m-times in each time interval (s_m, s_{m+1}). Other points in O_1 are either fixed or lie on the orbit of $\langle 1, 1\rangle$, or are eventually mapped on the orbit of $\langle 1, 1\rangle$. We will show that O_1 is not DC3 — since points on the orbit of $\langle 1, 1\rangle$ are asymptotic to $\langle 1, 1\rangle$, it is enough to show that $\langle x, y\rangle$ is not DC3-scrambled, where $x = \langle 1, 1\rangle$ and $y = \langle z, 0\rangle$, $z \in I$. The pair $\langle x, y\rangle$ is not DC3-scrambled, in particular, if $\Phi_{\langle x,y\rangle}(\delta) = \Phi^*(\delta)_{\langle x,y\rangle}$, for all $\delta > 0$. Because the second coordinate of x decreases with time to zero and we are considering the max-metric, it is sufficient to prove that the limit

$$\lim_{n\to\infty} \frac{1}{n} \#\{i \le n : |f^i(1) - z| < \delta\}$$

exists, where $f^i = f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1$.

Denote $J_\delta = (z - \delta, z + \delta) \cap I$ and the length of J_δ by $|J_\delta|$. The number of times i such that $|f^i(1) - z| < \delta$, for $s_m < i \le s_{m+1}$, is the same as the number of times when a point oscillating between the endpoints of I with velocity $\frac{1}{m}$ for $2m$ times hits the subinterval $J_\delta \subset I$. Denote the number of hitting times by P_m. We estimate P_m by

$$|J_\delta| \cdot 2m - 2 \le P_m \le |J_\delta| \cdot 2m + 2. \tag{8}$$

For every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that

$$n = s_m + 2m\alpha + \beta,$$

where $0 \le \alpha < n_m$ and $0 \le \beta < 2m$. Since

$$\begin{aligned} &\#\{i \le n : |f^i(1) - z| < \delta\} \\ &\quad = P_1 n_1 + P_2 n_2 + \cdots + P_{m-1} n_{m-1} \\ &\qquad + P_m \alpha + \gamma, \quad 0 \le \gamma \le \beta, \end{aligned}$$

we can estimate $\#\{i \le n : |f^i(1) - z| < \delta\}$ according to (8) from below by

$$\begin{aligned} &(|J_\delta| \cdot 2 \cdot 1 - 2)n_1 + (|J_\delta| \cdot 2 \cdot 2 - 2)n_2 \\ &\quad + \cdots + (|J_\delta| \cdot 2 \cdot m - 2)\alpha \end{aligned}$$

and from above by

$$\begin{aligned} &(|J_\delta| \cdot 2 \cdot 1 + 2)n_1 + (|J_\delta| \cdot 2 \cdot 2 + 2)n_2 \\ &\quad + \cdots + (|J_\delta| \cdot 2 \cdot m + 2)\alpha + 2m. \end{aligned}$$

If the sequence $\{n_i\}_{i=1}^{\infty}$ grows rapidly such that $\lim_{i\to\infty} \frac{s_i}{n_i} = 0$, then

$$\begin{aligned} |J_\delta| &= \lim_{m\to\infty} \frac{1}{s_m + 2m\alpha + \beta}(|J_\delta| \cdot (s_m + 2m\alpha) \\ &\qquad - (2n_1 + 2n_2 + 2n_{m-1} \cdots + 2\alpha)) \\ &\le \lim_{n\to\infty} \frac{1}{n} \#\{i \le n : |f^i(1) - z| < \delta\} \\ &\le \lim_{m\to\infty} \frac{1}{s_m + 2m\alpha + \beta}(|J_\delta| \cdot (s_m + 2m\alpha) \\ &\qquad + (2n_1 + 2n_2 + \cdots + 2n_{m-1} + 2\alpha + 2m)) \\ &= |J_\delta|, \end{aligned}$$

implying $\Phi_{\langle x,y\rangle}(\delta) = \Phi^*(\delta)_{\langle x,y\rangle} = |J_\delta|$.

4.2. *Distributionally chaotic oscillators*

We extend the dynamical system from the previous section by adding one more oscillator with distance 1 to the right side of O_1. Let K be the interval $[2, 3]$ and $h_m : K \to K$ be a mapping defined as

$$h_m(x) = \begin{cases} 2 & 2 \le x < 2 + \dfrac{1}{m} \\ x - \dfrac{1}{m} & 2 + \dfrac{1}{m} \le x \le 3 \end{cases} \tag{9}$$

and $\hat{h}_m : K \to K$ defined as

$$\hat{h}_m(x) = \begin{cases} x + \dfrac{1}{m} & 2 \le x < 3 - \dfrac{1}{m} \\ 3 & 3 - \dfrac{1}{m} \le x \le 3. \end{cases} \tag{10}$$

The dynamical system O_2 consists of the compact metric space $K \times (\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\})$ and the function $\hat{F}$ defined by

$$\hat{F}(\langle x, 0\rangle) = \langle x, 0\rangle, \quad x \in K$$
$$\hat{F}\left(\left\langle x, \frac{1}{k}\right\rangle\right) = \left\langle \hat{f}_k(x), \frac{1}{k+1}\right\rangle, \quad x \in K,\ k \in \mathbb{N}. \tag{11}$$

The function $\hat{f}_k$ is defined for $k \in \{s_m, s_m + 1, \ldots, s_{m+1}\}$ differently for even and odd m. For odd m,

$$\hat{f}_k = \begin{cases} \mathrm{Id} & s_m < k \le s_m + 2m \\ h_m & s_m + m < k \le s_m + 2m \\ \hat{h}_m & s_m + 2im < k \le s_m + 2im + m \\ h_m & s_m + (2i+1)m < k \le s_m + (2i+2)m \\ & \qquad i \in \{1, \ldots, n_m - 1\}, \end{cases} \tag{12}$$

for even m,

$$\hat{f}_k = \begin{cases} \mathrm{Id} & s_m < k \le s_m + 2m \\ \hat{h}_m & s_m + m < k \le s_m + 2m \\ h_m & s_m + 2im < k \le s_m + 2im + m \\ \hat{h}_m & s_m + (2i+1)m < k \le s_m + (2i+2)m \\ & \qquad i \in \{1, \ldots, n_m - 1\}, \end{cases} \tag{13}$$

where s_m and n_m are taken the same as in the previous construction. O_2 is made similarly as O_1, we are just using $\hat{f}_k$ instead of f_k. The first terms of sequence $\{\hat{f}_k\}_{k=1}^{\infty}$ are

$$\mathrm{Id}, h_1, (h_1, \hat{h}_1)^{n_1 - 1}\mathrm{Id}^2, \hat{h}_2^2, (h_2, h_2, \hat{h}_2, \hat{h}_2)^{n_2-1}\mathrm{Id}^3,$$
$$\hat{h}_3^3, (h_3, h_3, h_3, \hat{h}_3, \hat{h}_3, \hat{h}_3)^{n_3-1} \ldots .$$

For a better understanding of the dynamical systems O_1 and O_2 see Fig. 1. Adding m identity mappings at the beginning of each time interval (s_m, s_{m+1}) causes change in the movement of $y = \langle 3, 1\rangle$ — for m odd, y starts to oscillate from the right endpoint but for m even, y starts to oscillate from the left endpoint. Nevertheless, these identity mappings do not affect the calculation of distribution functions of y and some fixed point in K — we get the same results as in (8). We conclude that there are no DC3-scrambled pairs either in O_1 or in O_2.

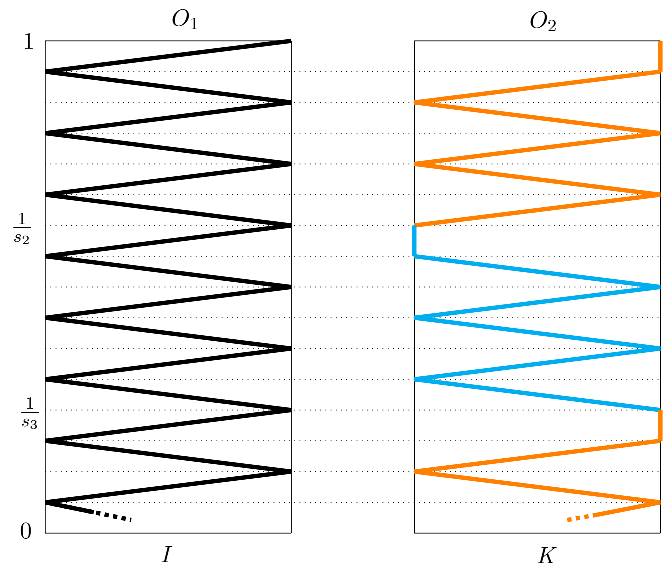

Fig. 1. Movement of points $x = \langle 1, 1\rangle$ in O_1 and $y = \langle 3, 1\rangle$ in O_2.

Consider the union of the dynamical systems $O_1 \cup O_2$ defined naturally as the space $(I \cup K) \times (\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\})$ with a function G such that G restricted to $I \times (\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\})$ is equal to F and G restricted to $K \times (\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\})$ is equal to $\hat{F}$.

Now we investigate the behavior of pairs in $O_1 \cup O_2$. We have already seen that there are no DC3-scrambled pairs inside O_1 or O_2. By Remark 1, any fixed point in K (resp., in I) paired with $x = \langle 1, 1\rangle$ (resp., $y = \langle 3, 1\rangle$) also cannot be DC3-scrambled. All other possible pairs consist of points asymptotic to $x = \langle 1, 1\rangle$ or $y = \langle 3, 1\rangle$, so it is sufficient to examine only $\Phi_{\langle x,y\rangle}$ and $\Phi^*_{\langle x,y\rangle}$.

In the time interval $(s_m + 2m, s_{m+1})$, where m is even, the orbits of the points x and y are *synchronic* (see the cyan part of the trajectory of y in Fig. 1) — they maintain the same distance. If we denote the first coordinate of $G^i(x)$ by x_i and the first coordinate of $G^i(y)$ by y_i, then

$$y_i = 2 + x_i, \quad \text{for } s_m + 2m < i \le s_m + 2mn_m, \quad m \text{ is even},$$

therefore $\#\{s_m + 2m < i \le s_m + 2mn_m, d(F^i(x), F^i(y)) < \delta\}$ is either 0, for $\delta \le 2$, or $2mn_m - 2m$, for $\delta > 2$.

Since $\lim_{m\to\infty} \frac{2mn_m}{s_m+2mn_m} = 1$, we obtain

$$\Phi^e_{\langle x,y\rangle} = \lim_{\substack{m\to\infty\\ m \text{ is even}}} \frac{1}{s_m + 2mn_m} \{0 < i \le s_m + 2mn_m, d(G^i(x), G^i(y)) < \delta\} = \begin{cases} 0 & \delta \le 2 \\ 1 & \delta > 2. \end{cases} \tag{14}$$

In time interval $(s_m + 2m, s_{m+1})$, where m is odd, the orbits of the points x and y are *asynchronic* (see orange parts of trajectory of y in Fig. 1) — the image of x is on the left endpoint of its interval when the image of y is on the right endpoint of its interval (and vice versa), therefore

$$y_i = 3 - x_i, \quad \text{for } s_m + 2m < i \le s_m + 2mn_m, \quad m \text{ is odd.}$$

From the perspective of the image of x, the orbit of y is approaching the orbit of x to the distance 1 and then is leaving for distance 3 with doubled speed $\frac{2}{m}$. This type of movement (one point is fixed and one point is oscillating) was investigated in the previous section — see calculation between (7) and (8), hence

$$\Phi^o_{\langle x,y\rangle} = \lim_{\substack{m\to\infty\\ m \text{ is odd}}} \frac{1}{s_m + 2mn_m} \{0 < i \le s_m + 2mn_m, d(G^i(x), G^i(y)) < \delta\} = \begin{cases} 0 & \delta \le 1 \\ \dfrac{\delta - 1}{2} & 1 < \delta \le 3 \\ 1 & \delta > 3. \end{cases} \tag{15}$$

Finally, we can conclude

$$\Phi_{\langle x,y\rangle} = \min\{\Phi^o_{\langle x,y\rangle}, \Phi^e_{\langle x,y\rangle}\},$$

$$\Phi^*_{\langle x,y\rangle} = \max\{\Phi^o_{\langle x,y\rangle}, \Phi^e_{\langle x,y\rangle}\}.$$

By (14) and (15), $\Phi_{\langle x,y\rangle}(\delta) < \Phi^*_{\langle x,y\rangle}(\delta)$, for $\delta \in (1, 3)$, hence $\langle x, y\rangle$ — and all pairs consisting of points asymptotic to x and y — is DC3-scrambled.

4.3. *Iteration problem*

Dvořáková in [Dvořáková, 2012] proved that if $\langle x, y\rangle$ is a DC3-scrambled pair with respect to G then there is a $j \in \{0, 1\}$ such that $\langle G^j(x), G^j(y)\rangle$ is a DC3-scrambled pair with respect to G^2. We keep the notation from the previous sections and will define a new function H such that $H^2(z) = G^2(z)$, for $z \in \mathrm{Orb}^+_G(x) \cup \mathrm{Orb}^+_G(y)$. Hence $\langle G^j(x), G^j(y)\rangle$ remains DC3-scrambled with respect to H^2 but there will be no DC3-scrambled pairs with respect to H, which will complete the proof of Theorem 2.

We add one more oscillator with distance 1 to the left side of O_1. Let J be the interval $[-2, -1]$ and $l_m : J \to J$ be the mapping defined as

$$l_m(x) = \begin{cases} -2 & -2 \le x < -2 + \dfrac{1}{m} \\ x - \dfrac{1}{m} & -2 + \dfrac{1}{m} \le x \le -1 \end{cases} \tag{16}$$

and $\hat{l}_m : J \to J$ defined as

$$\hat{l}_m(x) = \begin{cases} x + \dfrac{1}{m} & -2 \le x < -1 - \dfrac{1}{m} \\ -1 & -1 - \dfrac{1}{m} \le x \le -1. \end{cases} \tag{17}$$

The definition of $\tilde{f}_k$ is symmetrical to $\hat{f}_k$, we use l_m (resp., $\hat{l}_m$) instead of $\hat{h}_m$ (resp., h_m). For odd m,

$$\tilde{f}_k = \begin{cases} \mathrm{Id} & s_m < k \le s_m + 2m \\ \hat{l}_m & s_m + m < k \le s_m + 2m \\ l_m & s_m + 2im < k \le s_m + 2im + m \\ \hat{l}_m & s_m + (2i+1)m < k \le s_m + (2i+2)m \\ & i \in \{1, \ldots, n_m - 1\}, \end{cases} \tag{18}$$

for even m,

$$\tilde{f}_k = \begin{cases} \mathrm{Id} & s_m < k \le s_m + 2m \\ l_m & s_m + m < k \le s_m + 2m \\ \hat{l}_m & s_m + 2im < k \le s_m + 2im + m \\ l_m & s_m + (2i+1)m < k \le s_m + (2i+2)m \\ & i \in \{1, \ldots, n_m - 1\}. \end{cases} \tag{19}$$

The dynamical system O consists of the compact metric space $(I \cup K \cup J) \times (\{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\})$ and the function H defined by

$$H(\langle x, 0\rangle) = \langle x, 0\rangle \qquad x \in I \cup K \cup J$$

$$H\left(\left\langle x, \frac{1}{k}\right\rangle\right) = \left\langle f_k(x), \frac{1}{k+1}\right\rangle \quad x \in I,\ k \in \mathbb{N}$$

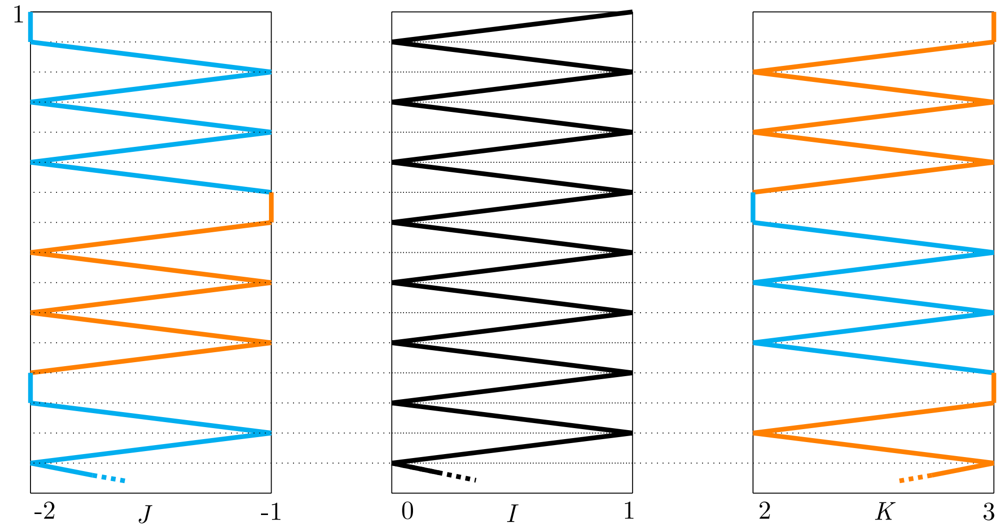

Fig. 2. Movement of points $x = \langle 1, 1\rangle$, $y = \langle 3, 1\rangle$, $z = \langle -2, 1\rangle$. Cyan parts of trajectories of y and z indicate when they are synchronic with x and orange parts indicate when they are asynchronic.

$$H\left(\left\langle x, \frac{1}{k}\right\rangle\right) = \left\langle 1 - \hat{f}_k(x), \frac{1}{k+1}\right\rangle \quad x \in K,\ k \in \mathbb{N}$$
$$H\left(\left\langle x, \frac{1}{k}\right\rangle\right) = \left\langle 1 - \tilde{f}_k(x), \frac{1}{k+1}\right\rangle \quad x \in J,\ k \in \mathbb{N}. \tag{20}$$

The idea of dynamical system O is represented in Fig. 2. By (20), the oscillator above K is mapped onto the oscillator above J and vice versa. Moreover, these oscillators are symmetric to each other about the axis $S = \frac{1}{2}$. It is easy to see that

$$H^2(x) = G^2(x),$$
$$\text{for } x \in (I \cup K) \times \left(\left\{\frac{1}{k} : k \in \mathbb{N}\right\} \cup \{0\}\right),$$

therefore existence of the DC3-scrambled pair for G^2 implies the same for H^2.

There are two types of points in O — fixed in $I \cup K \cup J$ and oscillating. Fixed points cannot be part of any DC3-scrambled pair by arguments given in previous sections. Oscillating points are either $x = \langle 1, 1\rangle$, $y = \langle 3, 1\rangle$ and its mirror image $z = \langle -2, 1\rangle$, or points which are asymptotic to them. Therefore it is sufficient to investigate distribution functions among x, y and z.

Denote the upper and lower distribution functions with respect to H by Ψ and Ψ^*. The orbits of the pair $\langle y, z\rangle$ are *asynchronic* for the whole time — the distance between orbits of y and z ranges from 3 to 5. By a similar argument as in (15), we have

$$\begin{aligned}\Psi^*_{(y,z)}(\delta) &= \Psi_{(y,z)}(\delta) \\ &= \lim_{m\to\infty} \frac{1}{s_m} \{0 < i \le s_m, d(H^i(x), H^i(y)) < \delta\} \\ &= \begin{cases} 0 & \delta \le 1 \\ \dfrac{\delta - 3}{2} & 3 < \delta \le 5 \\ 1 & \delta > 5. \end{cases}\end{aligned} \tag{21}$$

We proceed with the calculation of the distribution function of $\langle x, y\rangle$. In the time interval $(s_m + 2m, s_{m+1})$, where m is even, the image of y is for half times above K and the orbits of $\langle x, y\rangle$ are *synchronic* in those times — see Fig. 2. The other half times is the image of y above J and the orbits of $\langle x, y\rangle$ are *asynchronic*. Therefore we can use $\Phi^e_{\langle x,y\rangle}$ (as a result of synchronic movement) and $\Phi^o_{\langle x,y\rangle}$ (as a result of asynchronic movement) from (14) and (15) to calculate the distribution function $\Psi^e_{\langle x,y\rangle}$ as the arithmetic average of $\Phi^e_{\langle x,y\rangle}$ and $\Phi^o_{\langle x,y\rangle}$,

$$\begin{aligned}\Psi^e_{\langle x,y\rangle}(\delta) &= \lim_{\substack{m\to\infty \\ m \text{ is even}}} \frac{1}{s_m + 2mn_m} \\ &\quad \{0 < i \le s_m + 2mn_m, d(H^i(x), H^i(y)) < \delta\} \\ &= \frac{\Phi^e_{\langle x,y\rangle}(\delta) + \Phi^o_{\langle x,y\rangle}(\delta)}{2}.\end{aligned} \tag{22}$$

Similarly, in the time interval $(s_m + 2m, s_{m+1})$, where m is odd, the image of the point y is for half times above K and the orbits of $\langle x, y\rangle$ are *asynchronic*. The other half times is the image of y above J and the orbits of $\langle x, y\rangle$ are *synchronic*. Hence

$$\Psi^o_{\langle x,y\rangle}(\delta) = \lim_{\substack{m\to\infty\\ m \text{ is odd}}} \frac{1}{s_m + 2mn_m} \{0 < i \le s_m + 2mn_m, d(H^i(x), H^i(y)) < \delta\} = \frac{\Phi^e_{\langle x,y\rangle}(\delta) + \Phi^o_{\langle x,y\rangle}(\delta)}{2}, \tag{23}$$

which shows $\Psi^e_{\langle x,y\rangle} = \Psi^o_{\langle x,y\rangle}$. We conclude that

$$\Psi_{\langle x,y\rangle} = \Psi^*_{\langle x,y\rangle} = \frac{\Phi^e_{\langle x,y\rangle} + \Phi^o_{\langle x,y\rangle}}{2}.$$

Since z is a mirror image of y, $\langle x, z\rangle$ has the same distribution functions, i.e.

$$\Psi_{\langle x,z\rangle} = \Psi^*_{\langle x,z\rangle} = \frac{\Phi^e_{\langle x,y\rangle} + \Phi^o_{\langle x,y\rangle}}{2},$$

therefore there are no DC3-scrambled pairs in the entire system O with respect to H.

5. Chaotic Sets for Iterated Function

We now turn our attention to the distributional chaos defined using an uncountable distributionally scrambled set.

DC1$_u$, DC2$_u$ and DC2$\frac{1}{2}_u$ are iteration invariants, since $\langle x, y\rangle$ is a DCi-scrambled pair with respect to f if and only if it is DCi-scrambled with respect to f^n, for $i = 1, 2, 2\frac{1}{2}$. But the situation is more complicated for DC3$_u$ — by [Dvořáková, 2012], if $\langle x, y\rangle$ is a DC3-scrambled pair with respect to f, then there is a $j \in \{0, 1, \ldots, n-1\}$ such that $\langle f^j(x), f^j(y)\rangle$ is a DC3-scrambled pair with respect to f^n. This j can be different for different pairs in the DC3 set, hence the uncountable DC3 set can be split into DC3-scrambled pairs or into DC3 sets with smaller cardinalities.

Let S be a DC3 set with respect to f. We can generate an undirected graph G in the following way — the set of vertices of G is labeled by all points in S and we add an edge between vertices x and y if $\langle x, y\rangle$ is a DC3-scrambled pair. Then for a fixed $x \in S$ there is exactly one edge leading to every $y \in S\backslash\{x\}$ — hence G is a complete graph. Next, on this graph, we assign colors $c_0, \ldots, c_{n-1}$ in such a way that the edge between x and y has color c_j if $\langle f^j(x), f^j(y)\rangle$ is a DC3-scrambled pair with respect to f^n. By [Dvořáková, 2012], there is always at least one such j (in case of multiple choices for j we pick one arbitrarily). Since the graph G is colored by n colors, we can use Ramsey theory to find a complete monochromatic subgraph which will represent a chaotic set with respect to f^n.

Let us recall a classic result from [Ramsey, 1930], reformulated for our purposes:

Theorem 3. *Let G be a complete graph with infinite set of vertices and let each edge in this graph be colored by exactly one of colors $c_0, \ldots, c_{n-1}$. Then G contains an infinite subgraph H such that edges between every two distinct vertices in H have the same color c_i, for some $i \in \{0, 1, \ldots, n-1\}$.*

An immediate consequence of Infinite Ramsey theorem is the following corollary:

Corollary 5.1. *Let S be an infinite DC3 set with respect to f. Then there exists an infinite subset $R \subset S$ such that $f^j(R)$ is a DC3 set with respect to f^n, for some $j \in \{0, 1, \ldots, n-1\}$.*

Unfortunately the existence of an *uncountable* monochromatic subgraph is not ensured — Sierpinski coloring in [Sierpinski, 1933] serves as an example. Thus we pose an open question:

Question 5.1. *Does the existence of an uncountable DC3 set with respect to f imply the same with respect to f^n?*

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