Slezská univerzita v Opavě

Matematický ústav v Opavě

První kohomologické grupy minimálních toků

Habilitační práce

Matematika — Matematická analýza

Opava 2022 Matúš Dirbák

Silesian university in Opava

Mathematical institute in Opava

First cohomology groups of minimal flows

Habilitation thesis

Mathematics — Mathematical analysis

Opava 2022 Matúš Dirbák

Introduction

This habilitation thesis is based on our work

(D) M. Dirbák, *First cohomology groups of minimal flows*, Dissertationes Math. 562 (2021), 206 pp.,

which is also a part of the thesis as an attachment. Our interest in the thesis is in group extensions of minimal flows with compact abelian groups in the fibres. We study their structure from categorical and algebraic points of view, and describe relations of their dynamics to the one-dimensional algebraic-topological invariants. We determine the first cohomology groups of flows with simply connected acting groups and those of topologically free flows possessing a free cycle. As an application we show that minimal extensions of these flows not only do exist, but they have a rich algebraic structure.

Summary

In what follows we formulate the problems, describe the necessary tools and summarize the main results obtained in (D). Chapter 1 in (D) explains motivation for our study of group extensions and proposes the main problems of our research. Chapter 2 is preliminary, its purpose is to collect the necessary auxiliary material from the category theory, theory of abelian groups, topology, theory of compact abelian groups and Pontryagin duality, and topological dynamics. Chapters 3–6 form the core of (D) and contain our main results. In the following pages we will try to summarize them in a brief (but close-to-complete) way. Despite our effort to make this summary as self-contained as possible, on occasions the reader might find it helpful to consult List of symbols or Index contained at the end of (D).

Notation and basic definitions

Minimal flows. By a flow $\mathcal{F}$ we shall understand a representation of a topological group Γ in the group of homeomorphisms of a topological space X, which is continuous as a map $\Gamma \times X \to X$. We write $\mathcal{F} \colon \Gamma \curvearrowright \mathrm{X}$ and use T_γ $(\gamma \in \Gamma)$ to denote the acting transformations of $\mathcal{F}$. We refer to Γ as the acting group of $\mathcal{F}$ and to X as the phase space of $\mathcal{F}$. A flow $\mathcal{F}$ is called minimal if all its orbits $\mathcal{O}_{\mathcal{F}}(x)$ $(x \in X)$ are dense in the phase space X.

Given $z \in X$, consider the associated transition map

$$\mathcal{F}_z \colon \Gamma \ni \gamma \mapsto T_\gamma(z) \in X.$$

If $\mathcal{F}_z$ is injective then z is called a free point of $\mathcal{F}$. One of the classes of flows $\mathcal{F}$ that we work with is formed by the minimal flows with the following properties:

- the acting group Γ of $\mathcal{F}$ is locally compact, non-compact, second countable and amenable,
- the phase space X of $\mathcal{F}$ is compact and second countable,

- the flow $\mathcal{F}$ has a free point.

Since every abelian locally compact group is amenable, this includes the cases $\Gamma = \mathbb{Z}$ and $\Gamma = \mathbb{R}$.

Another class of minimal flows $\mathcal{F}$ that appear in our work is formed by those with the following properties:

- the acting group Γ of $\mathcal{F}$ is a simply connected Lie group,
- the phase space X of $\mathcal{F}$ is a (not necessarily locally connected) compact metrizable space.

Notice that by minimality of $\mathcal{F}$, X is a connected space. In this setting we shall frequently make use of the first cohomotopy group $\pi^1(X)$ of X (which is isomorphic to the first Čech cohomology group of X). In Section 1.7 we notice that for every torsion-free abelian group A with $\operatorname{rank}(A) \leq \mathfrak{c}$ there is a compact connected space X with $\pi^1(X)$ isomorphic to A, which admits a minimal continuous flow $\mathcal{F}\colon \mathbb{R} \curvearrowright X$. Moreover, if the group A is, in addition, countable then such a space X can be found in the class of metrizable continua.

Finally, we are interested also in minimal flows $\mathcal{F}$ with the following properties:

- the acting group Γ of $\mathcal{F}$ is a connected Lie group,
- the phase space X of $\mathcal{F}$ is a compact (connected) manifold.

In this context we shall often make us of the fundamental groups $\pi_1(\Gamma)$ and $\pi_1(X)$ of Γ and X, respectively. The first homology groups of these spaces, $H_1(\Gamma)$ and $H_1(X)$, are interpreted exclusively as abelianisations of their fundamental groups. We shall often remove torsion parts of the first homology groups, thus obtaining the first weak homology groups $H_1^w(\Gamma)$ and $H_1^w(X)$ of Γ and X, respectively. Given $z \in X$, we consider the morphism

$$\mathcal{F}_z^\sharp\colon H_1^w(\Gamma) \to H_1^w(X)$$

induced by the transition map $\mathcal{F}_z\colon \Gamma \to X$ and set $H_1^w(\mathcal{F}) = \operatorname{im}(\mathcal{F}_z^\sharp)$. (As follows from Lemma 2.7, the subgroup $H_1^w(\mathcal{F})$ of $H_1^w(X)$ does not depend on the choice of z.) Set $n = \operatorname{rank}(H_1^w(\mathcal{F}))$, $n + m = \operatorname{rank}(H_1^w(X))$ and denote by $d_1, \dots, d_n$ the elementary divisors of $H_1^w(\mathcal{F})$ in $H_1^w(X)$. Then, up to isomorphism, the inclusion $H_1^w(\mathcal{F}) \subseteq H_1^w(X)$ takes the form

$$(d_1\mathbb{Z} \oplus \dots \oplus d_n\mathbb{Z}) \oplus 0 \subseteq \mathbb{Z}^n \oplus \mathbb{Z}^m$$

and we have an isomorphism of groups

$$H_1^w(X)/H_1^w(\mathcal{F}) \cong \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_n} \oplus \mathbb{Z}^m.$$

We say that the flow $\mathcal{F}$

- is topologically free if $\mathcal{F}_z^\sharp\colon H_1^w(\Gamma) \to H_1^w(X)$ is a monomorphism (equivalently, if the restriction $\mathcal{F}_z^\sharp\colon H_1^w(\Gamma) \to H_1^w(\mathcal{F})$ is an isomorphism),
- has a free cycle if $\operatorname{rank}(H_1^w(\mathcal{F})) < \operatorname{rank}(H_1^w(X))$ (equivalently, if $m > 0$).

In Subsections 2.5.3–2.5.5 we discuss these two properties and give examples of minimal flows which are topologically free and/or have a free cycle.

Cocycles, coboundaries and cohomology groups. Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow with acting homeomorphisms T_γ ($\gamma \in \Gamma$). We are interested in group extensions of $\mathcal{F}$ with values in compact abelian groups. To simplify notation, write CAGp for the category of compact abelian groups. (The group operation on $G \in \mathsf{CAGp}$ will be denoted either multiplicatively or additively, depending on what is more convenient or appropriate.) Given $G \in \mathsf{CAGp}$, a group extension of $\mathcal{F}$ with values in G is a skew product over $\mathcal{F}$ with the fibre G, whose acting homeomorphisms act by rotations on the fibres. Formally, let $\mathcal{C}\colon \Gamma \times X \to G$ be a continuous map and for every $\gamma \in \Gamma$ consider the homeomorphism

$$\widetilde{T}_\gamma\colon X \times G \ni (x,g) \mapsto (T_\gamma x, \mathcal{C}(\gamma,x)g) \in X \times G.$$

Then the family $\widetilde{T}_\gamma$ ($\gamma \in \Gamma$) constitutes a Γ-flow on $X \times G$ if and only if the map $\mathcal{C}$ satisfies the cocycle identity

$$\mathcal{C}(\alpha, T_\beta x)\mathcal{C}(\beta,x) \equiv \mathcal{C}(\alpha\beta, x).$$

If this is the case then we call $\mathcal{C}$ a cocycle over $\mathcal{F}$ with values in G. We also denote the flow with acting homeomorphisms $\widetilde{T}_\gamma$ ($\gamma \in \Gamma$) by $\mathcal{F}_\mathcal{C}$ and call it a group extension of $\mathcal{F}$. When convenient, we shall identify $\mathcal{F}_\mathcal{C}$ with $\mathcal{C}$ and attribute to $\mathcal{C}$ the dynamical properties of $\mathcal{F}_\mathcal{C}$. In particular, we say that $\mathcal{C}$ is minimal if $\mathcal{F}_\mathcal{C}$ is minimal.

Given $G \in \mathsf{CAGp}$, denote by $\mathbf{Z}_\mathcal{F}(G)$ the set of all cocycles over $\mathcal{F}$ with values in G. Clearly, $\mathbf{Z}_\mathcal{F}(G)$ is an abelian group with operations defined pointwise. If both Γ and X are locally compact then we equip $\mathbf{Z}_\mathcal{F}(G)$ with the topology of uniform convergence on compact sets (briefly, u.c.s. convergence) or, equivalently, with the compact-open topology, thus turning it into a complete topological abelian group. If all Γ, X and G are additionally second countable then $\mathbf{Z}_\mathcal{F}(G)$ is a Polish abelian group. Often in our work we deal with a flow $\mathcal{F}$ whose acting group Γ is connected. By minimality of $\mathcal{F}$ it follows that the phase space X of $\mathcal{F}$ is also connected. In this case every cocycle $\mathcal{C} \in \mathbf{Z}_\mathcal{F}(G)$ takes its values in the identity component G_0 of G. For this reason, we often restrict ourselves to cocycles with values in connected groups G.

A cocycle $\mathcal{C} \in \mathbf{Z}_\mathcal{F}(G)$ is called a coboundary over $\mathcal{F}$ if it is of the form

$$\mathcal{C}(\gamma,x) = \xi(T_\gamma x)\xi(x)^{-1}$$

for an appropriate continuous map $\xi\colon X \to G$. Such a map ξ is called a transfer function of $\mathcal{C}$. By minimality of $\mathcal{F}$, the transfer function of a coboundary is unique up to an additive constant. In fact, if $z \in X$ and e is the identity of G then there is a unique transfer function ξ of $\mathcal{C}$ with $\xi(z) = e$. The coboundaries over $\mathcal{F}$ with values in G form a subgroup $\mathbf{B}_\mathcal{F}(G)$ of $\mathbf{Z}_\mathcal{F}(G)$. The corresponding quotient group

$$\mathbf{H}_\mathcal{F}(G) = \mathbf{Z}_\mathcal{F}(G)/\,\mathbf{B}_\mathcal{F}(G)$$

is called the (first) cohomology group of $\mathcal{F}$ (underlying to G). The congruence relation on $\mathbf{Z}_\mathcal{F}(G)$ determined by $\mathbf{B}_\mathcal{F}(G)$ is called the cohomology relation or the equivalence of extensions and we shall denote it by $\simeq$. We denote by π_G the quotient morphism

$$\pi_G\colon\ \mathbf{Z}_\mathcal{F}(G) \to \mathbf{H}_\mathcal{F}(G)$$

and by μ_G the inclusion morphism

$$\mu_G\colon\ \mathbf{B}_\mathcal{F}(G) \to \mathbf{Z}_\mathcal{F}(G).$$

In order to simplify notation, we may occasionally use the same symbol $\mathcal{C}$ for an extension from $\mathbf{Z}_{\mathcal{F}}(G)$ as well as for its cohomology class $\pi_G(\mathcal{C}) \in \mathbf{H}_{\mathcal{F}}(G)$. Of special importance for us will be extensions with values in the 1-dimensional torus $\mathbb{T}^1$. For the sake of simplicity of notation we write $\mathbf{Z}_{\mathcal{F}}$, $\mathbf{B}_{\mathcal{F}}$, $\mathbf{H}_{\mathcal{F}}$, μ and π instead of $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^1)$, $\mathbf{B}_{\mathcal{F}}(\mathbb{T}^1)$, $\mathbf{H}_{\mathcal{F}}(\mathbb{T}^1)$, $\mu_{\mathbb{T}^1}$ and $\pi_{\mathbb{T}^1}$, respectively.

We will be particularly interested in minimal extensions $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$. The set of such extensions together with the trivial extension (that is, with the identity $\mathcal{C} = e$ of $\mathbf{Z}_{\mathcal{F}}(G)$) will be denoted by $\mathbf{MinZ}_{\mathcal{F}}(G)$. The set $\mathbf{MinZ}_{\mathcal{F}}(G)$ is, in general, not a subgroup of $\mathbf{Z}_{\mathcal{F}}(G)$. It is a groupoid in the sense that the product in $\mathbf{Z}_{\mathcal{F}}(G)$ is a partial operation on $\mathbf{MinZ}_{\mathcal{F}}(G)$. Notice that if the group G is non-trivial then the trivial extension $\mathcal{C} = e$ is not minimal. Despite this fact we prefer including it into the groupoid $\mathbf{MinZ}_{\mathcal{F}}(G)$, for it will simplify formulations of some of our subsequent results. We shall also work with the groupoid of cohomology classes of extensions from $\mathbf{MinZ}_{\mathcal{F}}(G)$ and we denote it by $\mathbf{MinH}_{\mathcal{F}}(G)$. Thus, formally, $\mathbf{MinH}_{\mathcal{F}}(G) = \pi_G(\mathbf{MinZ}_{\mathcal{F}}(G))$. We also write $\mathbf{MinZ}_{\mathcal{F}}$ and $\mathbf{MinH}_{\mathcal{F}}$ instead of $\mathbf{MinZ}_{\mathcal{F}}(\mathbb{T}^1)$ and $\mathbf{MinH}_{\mathcal{F}}(\mathbb{T}^1)$, respectively.

The group extensions over a given minimal flow $\mathcal{F}$ form a category $\mathsf{CAGpZ}_{\mathcal{F}}$. Objects of this category are extensions $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ with $G \in \mathsf{CAGp}$. If $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ and $\mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(H)$ are objects of $\mathsf{CAGpZ}_{\mathcal{F}}$ then the set of morphisms $\mathcal{C} \to \mathcal{D}$ is defined as

$$\mathrm{Hom}(\mathcal{C}, \mathcal{D}) = \{q \in \mathrm{Hom}(G, H) \colon q\mathcal{C} = \mathcal{D}\}.$$

The composition of morphisms in $\mathsf{CAGpZ}_{\mathcal{F}}$ is their composition in CAGp and the identity in $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ with $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ is id_G. As we show in Proposition 2.20, $\mathsf{CAGpZ}_{\mathcal{F}}$ is a co-complete category in the sense that every inverse system in $\mathsf{CAGpZ}_{\mathcal{F}}$ has a limit.

Results of Chapter 3. Fundamental tools

Extensions as group morphisms. Let $\mathcal{F} \colon \Gamma \curvearrowright \mathrm{X}$ be a minimal flow, $G \in \mathsf{CAGp}$ and G^* be the Pontryagin dual of G. With every extension $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ we may associate a morphism of groups $\mathcal{C}^* \colon G^* \to \mathbf{Z}_{\mathcal{F}}$ acting by the rule $\mathcal{C}^*(\chi) = \chi\mathcal{C}$ for every $\chi \in G^*$. The assignment $\mathcal{C} \mapsto \mathcal{C}^*$ defines a morphism of groups

$$\Phi_G \colon\ \mathbf{Z}_{\mathcal{F}}(G) \to \mathrm{Hom}(G^*, \mathbf{Z}_{\mathcal{F}}).$$

As we show in Theorem 3.1, Φ_G is in fact an isomorphism. Moreover, if Γ and X are locally compact, $\mathbf{Z}_{\mathcal{F}}(G)$ and $\mathbf{Z}_{\mathcal{F}}$ carry the topology of u.c.s. convergence and $\mathrm{Hom}(G^*, \mathbf{Z}_{\mathcal{F}})$ is equipped with the topology of pointwise convergence, then Φ_G is a topological isomorphism. Further, in Corollary 3.8 we notice that Φ_G restricts to an isomorphism of groups

$$\Phi_G \colon\ \mathbf{B}_{\mathcal{F}}(G) \to \mathrm{Hom}(G^*, \mathbf{B}_{\mathcal{F}})$$

as well as to the isomorphism of groupoids

$$\Phi_G \colon\ \mathbf{MinZ}_{\mathcal{F}}(G) \to \mathrm{Mon}(G^*, \mathbf{MinZ}_{\mathcal{F}}),$$

where $\mathrm{Mon}(G^*, \mathbf{MinZ}_{\mathcal{F}})$ is the groupoid of monomorphisms $G^* \to \mathbf{Z}_{\mathcal{F}}$ which take their values in $\mathbf{MinZ}_{\mathcal{F}}$. We infer from these isomorphisms that in order to determine the groups $\mathbf{Z}_{\mathcal{F}}(G)$, $\mathbf{B}_{\mathcal{F}}(G)$ and the groupoid $\mathbf{MinZ}_{\mathcal{F}}(G)$ for every $G \in \mathsf{CAGp}$, it is in principle sufficient to determine $\mathbf{Z}_{\mathcal{F}}$, $\mathbf{B}_{\mathcal{F}}$ and $\mathbf{MinZ}_{\mathcal{F}}$, respectively.

Some useful consequences of Theorem 3.1 are collected in Corollaries 3.3—3.5. In particular, a contravariant functor is constructed, which allows us to view the category $\mathsf{CAGpZ}_{\mathcal{F}}$ as an opposite to a full subcategory of the category $\mathsf{Hom}(\mathsf{DAGp}, \mathbf{Z}_{\mathcal{F}})$, where DAGp denotes the category of discrete abelian groups.

Functorial approach to sections. Given a minimal flow $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$, a group $G \in \mathsf{CAGp}$ with the identity e, $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ and $x \in X$, denote by $F(\mathcal{C})$ the vertical x-section of the orbit-closure of (x, e) under the action of $\mathcal{F}_{\mathcal{C}}$. Then $F(\mathcal{C})$ is a closed subgroup of G, which does not depend on x. In Theorem 3.7 we consider the assignment $\mathcal{C} \mapsto F(\mathcal{C})$ as a covariant functor $F\colon \mathsf{CAGpZ}_{\mathcal{F}} \to \mathsf{CAGp}$. We treat it axiomatically and verify its uniqueness. In particular, we show that F detects coboundaries:

$$\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(G) \text{ if and only if } F(\mathcal{C}) = e,$$

as well as minimal extensions:

$$\mathcal{F}_{\mathcal{C}} \text{ is minimal if and only if } F(\mathcal{C}) = G.$$

Furthermore, F respects the cohomology relation:

$$\text{if } \mathcal{C}, \mathcal{D} \in \mathbf{Z}_{\mathcal{F}}(G) \text{ are cohomologous then } F(\mathcal{C}) = F(\mathcal{D}),$$

and it is continuous in the sense that it preserves limits of inverse systems. As a useful application of these properties of F we notice in Corollary 3.9 that

$$\mathbf{MinH}_{\mathcal{F}} \setminus 1 = \mathbf{H}_{\mathcal{F}} \setminus \operatorname{tor}(\mathbf{H}_{\mathcal{F}}),$$

that is, cohomology classes of the minimal extensions $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}$ are precisely the elements of the group $\mathbf{H}_{\mathcal{F}}$ of infinite order.

We wish to mention that this abstract, functorial approach to sections turns out to be very useful to us. Very often we manage with the defining conditions of the functor F (and with its other properties listed in Theorem 3.7) and thus do not have to invoke its explicit construction. This makes many proofs throughout our whole work much more transparent.

A summation formula for F. Let $H_1, H_2 \in \mathsf{CAGp}$. We say that H_1, H_2 are group-disjoint if $H_1 \times H_2$ is the only closed subgroup of $H_1 \times H_2$ with full projections onto both H_1 and H_2. For some results related to the notion of group-disjointness we refer the reader to Subsection 2.4.17.

Now let $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$ be a minimal flow, $G \in \mathsf{CAGp}$ and $\mathcal{C}_1, \dots, \mathcal{C}_n \in \mathbf{Z}_{\mathcal{F}}(G)$ be extensions, whose sections $F(\mathcal{C}_1), \dots, F(\mathcal{C}_n)$ are pairwise group-disjoint. Then, by Proposition 3.19,

$$F(\mathcal{C}_1, \dots, \mathcal{C}_n) = F(\mathcal{C}_1) \times \cdots \times F(\mathcal{C}_n) \subseteq G^n \quad \text{and} \quad F\left(\sum_{i=1}^{n} \mathcal{C}_i\right) = \sum_{i=1}^{n} F(\mathcal{C}_i) \subseteq G.$$

The assumption of group-disjointness is essential for the validity of these equalities and can not be omitted in general. The first equality generalizes to sequences of extensions

$\mathcal{C}_n \in \mathbf{Z}_{\mathcal{F}}(G)$ $(n \in \mathbb{N})$ in the sense that

$$F((\mathcal{C}_n)_{n\in\mathbb{N}}) = \prod_{n\in\mathbb{N}} F(\mathcal{C}_n),$$

provided that the sections $F(\mathcal{C}_n)$ $(n \in \mathbb{N})$ are pairwise group-disjoint. However, the second equality does not generalize to sequences in general. Nevertheless, in Theorem 3.23 we show that if Γ and X are locally compact second countable, $G \in \mathsf{CAGp}$ is second countable, $\mathcal{C}_n \longrightarrow e$ u.c.s. and the sections $F(\mathcal{C}_n)$ $(n \in \mathbb{N})$ are pairwise group-disjoint, then there is an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that

$$F\left(\sum_{n=1}^{\infty} \mathcal{C}_{k_n}\right) \supseteq \overline{\sum_{n=1}^{\infty} F(\mathcal{C}_{k_n})}, \tag{1}$$

where the right-hand side of this inclusion denotes the closed subgroup of G generated by the groups $F(\mathcal{C}_{k_n})$:

$$\overline{\sum_{n=1}^{\infty} F(\mathcal{C}_{k_n})} = \overline{\bigcup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n} F(\mathcal{C}_{k_i})\right)}.$$

Proposition 3.28 then shows that passing to a subsequence is in general necessary for the validity of (1). Finally, in Theorem 3.29 we show that the inclusion converse to that from (1) does not hold, not even when passing to a subsequence is allowed.

The ext-topology. Given a group $G \in \mathsf{CAGp}$, we denote by 2^G the set of all non-empty closed subsets of G. As we notice in Subsection 2.4.16, with the usual product of sets

$$HK = \{hk \colon h \in H, k \in K\}$$

and with the Vietoris topology, 2^G is a compact abelian semigroup. The closed subgroups of G form a closed subsemigroup of 2^G.

Now let $\mathcal{F} \colon \Gamma \curvearrowright \mathrm{X}$ be a minimal flow and $G \in \mathsf{CAGp}$. The subgroup $\mathbf{B}_{\mathcal{F}}(G)$ of $\mathbf{Z}_{\mathcal{F}}(G)$ is not closed in general if we equip $\mathbf{Z}_{\mathcal{F}}(G)$ with the topology of u.c.s. convergence. For this reason, the corresponding quotient topology on $\mathbf{H}_{\mathcal{F}}(G)$ is not Hausdorff and so $\mathbf{H}_{\mathcal{F}}(G)$ with this topology is not a topological group in the usual sense. However, as we show in Section 3.4, it is possible to turn $\mathbf{H}_{\mathcal{F}}(G)$ into a (Hausdorff) topological group in a natural way. As a matter of fact, in Theorem 3.34 we show that there is the coarsest of all the topologies on $\mathbf{H}_{\mathcal{F}}(G)$, with respect to which $\mathbf{H}_{\mathcal{F}}(G)$ is a (Hausdorff) topological group and the mapping $F \colon \mathbf{H}_{\mathcal{F}}(G) \to 2^G$ is continuous. We call this topology the *ext-topology* and denote it by τ_{ext}. In fact, τ_{ext} is induced by the translation-invariant 2^G-valued pseudo-metric δ on $\mathbf{Z}_{\mathcal{F}}(G)$, given by

$$\delta(\mathcal{C}, \mathcal{D}) = F(\mathcal{C}\mathcal{D}^{-1}),$$

see Proposition 3.22 and Remark 3.33. This means that a net $(\mathcal{C}_i)$ in $\mathbf{H}_{\mathcal{F}}(G)$ converges to $\mathcal{C} \in \mathbf{H}_{\mathcal{F}}(G)$ with respect to τ_{ext} if and only if the net $F(\mathcal{C}_i\mathcal{C}^{-1})$ converges to the identity e in 2^G.

Lifts of extensions. In Section 3.5 we consider the problem whether extensions lift into extensions across covering morphisms. In Theorem 3.38 we show that if the acting group Γ of a minimal flow $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$ is simply connected then this is the case. More precisely, if $p \in \mathrm{Hom}(G', G)$ is a covering morphism between abelian topological groups and $C \in \mathbf{Z}_{\mathcal{F}}(G)$ then there is a unique $\mathcal{C}' \in \mathbf{Z}_{\mathcal{F}}(G')$ with $p\mathcal{C}' = \mathcal{C}$. In subsequent parts of our work this theorem is frequently applied to the covering morphisms

$$p\colon \mathbb{R} \to \mathbb{S}^1, p(x) = \exp(i2\pi x), \quad \text{and} \quad \kappa_d\colon \mathbb{S}^1 \to \mathbb{S}^1, \kappa_d(z) = z^d \ \ (d \in \mathbb{N}).$$

Example 3.40 demonstrates that the assumption of simple connectedness of Γ can not be omitted from the statement of Theorem 3.38.

As a first application of Theorem 3.38 we show in Corollary 3.41 that for a flow $\mathcal{F}$ with a simply connected acting Lie group Γ and a compact second countable phase space X, the group $\mathbf{Z}_{\mathcal{F}}(G)$ is the additive topological group of a real separable Fréchet space for every connected second countable group $G \in \mathsf{CAGp}$. Moreover, if G is additionally of finite topological dimension then $\mathbf{Z}_{\mathcal{F}}(G)$ is the additive topological group of a real separable Banach space.

Secondly, we use Theorem 3.38 to show in Theorem 3.43 that for a minimal flow $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$ with Γ simply connected and X compact there is a short exact sequence of abelian groups

$$0 \longrightarrow \pi^1(X) \longrightarrow \mathbf{H}_{\mathcal{F}}(\mathbb{R}) \longrightarrow \mathbf{H}_{\mathcal{F}} \longrightarrow 0.$$

We also show that $\mathbf{H}_{\mathcal{F}}$ splits into a direct sum

$$\mathbf{H}_{\mathcal{F}} = \mathrm{tor}(\mathbf{H}_{\mathcal{F}}) \oplus \mathcal{D},$$

where $\mathcal{D}$ is a divisible torsion-free subgroup of the groupoid $\mathbf{MinH}_{\mathcal{F}}$. Finally, if $\pi^1(X) \neq 0$ and $\mathrm{rank}(\pi^1(X)) < \mathfrak{c}$ then $\mathrm{rank}(\mathcal{D}) \geq \mathfrak{c}$ and $\mathbf{MinH}_{\mathcal{F}}$ thus contains a subgroup isomorphic to $\mathbb{R}$. This result suggests the possibility that the groupoids $\mathbf{MinH}_{\mathcal{F}}(G)$ ($G \in \mathsf{CAGp}$) might have large algebraic structures. We consider this problem in Chapter 6.

Results of Chapter 4. Topological-algebraic aspects

Free group extensions. One of the most important notions introduced and studied in our work is that of the free group extension of a given minimal flow $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$, which is defined as the free object for the category $\mathsf{CAGpZ}_{\mathcal{F}}$. It consists of a group $\gimel_{\mathcal{F}} \in \mathsf{CAGp}$ and an extension $\daleth_{\mathcal{F}} \in \mathbf{Z}_{\mathcal{F}}(\gimel_{\mathcal{F}})$ such that for every $G \in \mathsf{CAGp}$ and every $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ there is a unique $h_{\mathcal{C}} \in \mathrm{Hom}(\gimel_{\mathcal{F}}, G)$ with $h_{\mathcal{C}} \daleth_{\mathcal{F}} = \mathcal{C}$. In other words, for every object $\mathcal{C}$ of $\mathsf{CAGpZ}_{\mathcal{F}}$, the set $\mathrm{Hom}(\daleth_{\mathcal{F}}, \mathcal{C})$ is a singleton and each extension from $\mathsf{CAGpZ}_{\mathcal{F}}$ is thus a composition of $\daleth_{\mathcal{F}}$ with a unique morphism from CAGp. This means that the pair $(\gimel_{\mathcal{F}}, \daleth_{\mathcal{F}})$ carries full information about all extensions from $\mathsf{CAGpZ}_{\mathcal{F}}$. The existence of such a universal object is proved in Theorem 4.5, where it is shown that $\gimel_{\mathcal{F}}$ can be identified with the Pontryagin dual $(\mathbf{Z}_{\mathcal{F}})_d^*$ of the discretization $(\mathbf{Z}_{\mathcal{F}})_d$ of the group $\mathbf{Z}_{\mathcal{F}}$ and, under this identification, $\daleth_{\mathcal{F}}$ acts by the rule

$$\daleth_{\mathcal{F}}(\gamma, x)\colon\ \mathbf{Z}_{\mathcal{F}} \ni \mathcal{C} \mapsto \mathcal{C}(\gamma, x) \in \mathbb{T}^1$$

for all $\gamma \in \Gamma$ and $x \in X$. The group $\gimel_{\mathcal{F}}$ is, typically, very large. We illustrate it in Example 4.6 by showing that $\gimel_{\mathcal{F}}$ is topologically isomorphic to the Bohr compactification

$b\mathbb{R}$ of $\mathbb{R}$, provided that the acting group Γ of $\mathcal{F}$ is a simply connected Lie group and the phase space X of $\mathcal{F}$ is compact second countable.

In Theorem 4.7 we collect some basic information about the free group extension $(\gimel_{\mathcal{F}}, \daleth_{\mathcal{F}})$ of $\mathcal{F}$. We show that under the standard identification $(\gimel_{\mathcal{F}})^* \cong (\mathbf{Z}_{\mathcal{F}})_d$ we have

$$F(\daleth_{\mathcal{F}})^{\perp} = \mathbf{B}_{\mathcal{F}} \quad \text{and} \quad (\daleth_{\mathcal{F}})^* = \mathrm{Id}_{\mathbf{Z}_{\mathcal{F}}}.$$

Further, we identify the quotient group $\gimel_{\mathcal{F}} / F(\daleth_{\mathcal{F}})$ as the free compact abelian group over the space X. Finally, we describe an isomorphism between the groups $\mathbf{Z}_{\mathcal{F}}(G)$ and $\mathrm{Hom}(\gimel_{\mathcal{F}}, G)$ for every $G \in \mathsf{CAGp}$, and an isomorphism between the categories $\mathsf{CAGpZ}_{\mathcal{F}}$ and $\mathsf{Hom}(\gimel_{\mathcal{F}}, \mathsf{CAGp})$.

Divisibility and torsion-freeness. For a given minimal flow $\mathcal{F} \colon \Gamma \curvearrowright \mathrm{X}$, in Section 4.3 we study certain topological-algebraic properties of the group $\gimel_{\mathcal{F}}$. In Theorem 4.10 we show that the connectedness of $\gimel_{\mathcal{F}}$ is equivalent to the torsion-freeness of $\mathbf{Z}_{\mathcal{F}}$ or, equivalently, of $\mathbf{Z}_{\mathcal{F}}(G)$ for every $G \in \mathsf{CAGp}$. Moreover, this occurs if and only if X is connected and Γ has no non-trivial finite abelian quotient groups.

Further, as shown in Theorem 4.12, the torsion-freeness of $\gimel_{\mathcal{F}}$ is equivalent to the divisibility of $\mathbf{Z}_{\mathcal{F}}$ or, equivalently, of $\mathbf{Z}_{\mathcal{F}}(G)$ for every connected group $G \in \mathsf{CAGp}$. This can be equivalently expressed by saying that objects of $\mathsf{CAGpZ}_{\mathcal{F}}$ lift across the finite-to-one epimorphisms from CAGp or, equivalently, across all epimorphisms from CAGp. Moreover, this occurs if and only if the short exact sequences in CAGp give rise to short exact sequences of associated groups of cocycles.

Finally, in Theorem 4.14 we show that $\gimel_{\mathcal{F}}$ is both torsion-free and connected if and only if $\mathbf{Z}_{\mathcal{F}}$ is both divisible and torsion-free, and this in turn occurs if and only if $\mathbf{Z}_{\mathcal{F}}(G)$ is divisible (and torsion-free) for every $G \in \mathsf{CAGp}$. Moreover, this is the case if and only if the objects of $\mathsf{CAGpZ}_{\mathcal{F}}$ lift uniquely across the epimorphisms from CAGp with finite (or totally disconnected) kernels.

In Proposition 4.16 we apply these results to show that if $\mathbf{Z}_{\mathcal{F}}$ is both torsion-free and divisible then certain morphisms in CAGp induce isomorphisms (or topological isomorphisms) of associated groups of cocycles. These include epimorphisms with totally disconnected kernels and, in particular, epimorphisms coming from projective resolutions and maximal toral quotient sequences. Finally, we conclude by arriving at an isomorphism $\mathbf{Z}_{\mathcal{F}} \cong \mathbb{R}$ and a topological isomorphism $\gimel_{\mathcal{F}} \cong b\mathbb{R}$ under assumptions of local compactness and second countability of Γ and X.

Non-existence of free minimal extensions. Let $\mathcal{F} \colon \Gamma \curvearrowright \mathrm{X}$ be a minimal flow. Being inspired by our construction of the free group extension of $\mathcal{F}$, we search for an analogous free object for the subcategory of $\mathsf{CAGpZ}_{\mathcal{F}}$ formed by the minimal extensions. To be concrete, we are searching for a group $\gimel^{\mathrm{m}}_{\mathcal{F}} \in \mathsf{CAGp}$ and a minimal extension $\daleth^{\mathrm{m}}_{\mathcal{F}} \in \mathbf{Z}_{\mathcal{F}}(\gimel^{\mathrm{m}}_{\mathcal{F}})$ such that for every $G \in \mathsf{CAGp}$ and every minimal extension $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ there is $q \in \mathrm{Hom}(\gimel^{\mathrm{m}}_{\mathcal{F}}, G)$ with $q\,\daleth^{\mathrm{m}}_{\mathcal{F}} = \mathcal{C}$. In Theorem 4.23 we show that such an object typically does not exist. More precisely, if both $\mathbf{B}_{\mathcal{F}}$ and $\mathbf{MinZ}_{\mathcal{F}}$ are non-trivial then such a pair $(\gimel^{\mathrm{m}}_{\mathcal{F}}, \daleth^{\mathrm{m}}_{\mathcal{F}})$ does not exist. However, in Theorem 4.27 we show that for every minimal extension $\mathcal{C}$ from $\mathsf{CAGpZ}_{\mathcal{F}}$ we can construct a minimal extension $\beth$ in $\mathsf{CAGpZ}_{\mathcal{F}}$ such that $\mathcal{C}$ is an epimorphic image of $\beth$ and every minimal extension $\beth'$ from $\mathsf{CAGpZ}_{\mathcal{F}}$ with $\mathrm{Hom}(\beth', \beth) \neq \emptyset$ is isomorphic to $\beth$. Thus every minimal extension from $\mathsf{CAGpZ}_{\mathcal{F}}$ is an epimorphic image of a "maximal" minimal extension. Example 4.29 then demonstrates that two such extensions $\beth$ need not be isomorphic in $\mathsf{CAGpZ}_{\mathcal{F}}$ and in Example 4.30 we describe a situation when their sections $F(\beth)$ are not isomorphic in CAGp.

Cohomology classes as group morphisms. Let $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$ be a minimal flow and $G \in \mathsf{CAGp}$. As we explained before, there is an isomorphism

$$\Phi_G\colon\ \mathbf{Z}_{\mathcal{F}}(G) \to \mathrm{Hom}(G^*, \mathbf{Z}_{\mathcal{F}}),$$

which restricts to an isomorphism

$$\Phi_G\colon\ \mathbf{B}_{\mathcal{F}}(G) \to \mathrm{Hom}(G^*, \mathbf{B}_{\mathcal{F}}).$$

These two isomorphisms give rise to a monomorphism

$$\Psi_G\colon\ \mathbf{H}_{\mathcal{F}}(G) \to \mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}}).$$

In fact, as we show in Theorem 4.32, Ψ_G is a topological isomorphism onto its image, provided that $\mathbf{H}_{\mathcal{F}}(G)$ and $\mathbf{H}_{\mathcal{F}}$ carry the ext-topology and $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$ is equipped with the topology of pointwise convergence. Since $\mathbf{H}_{\mathcal{F}}$ is always discrete, this implies, in particular, that $\mathbf{H}_{\mathcal{F}}(G)$ is always totally disconnected. The image $\mathrm{im}(\Psi_G)$ of Ψ_G may or may not be closed in $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$, depending on a particular situation. We shall discuss this fact more thoroughly later.

In Corollary 4.34 we use Ψ_G to construct a topological isomorphism onto its image

$$\widetilde{\Psi}_G\colon\ \mathbf{H}_{\mathcal{F}}(G) \to \mathrm{Hom}(F(\daleth_{\mathcal{F}}), G),$$

where $\mathbf{H}_{\mathcal{F}}(G)$ carries the ext-topology and $\mathrm{Hom}(F(\daleth_{\mathcal{F}}), G)$ is equipped with the topology of uniform convergence.

Let us emphasize that unlike Φ_G, Ψ_G is in general not an isomorphism. In Theorem 4.36 we find exact sequences which determine the extent, to which Ψ_G fails to be epic. We then infer that Ψ_G is an isomorphism provided that the group $\mathrm{Ext}(G^*, \pi^1(X))$ of extensions of G^* by $\pi^1(X)$ vanishes. Moreover, if $\mathbf{Z}_{\mathcal{F}}$ is divisible and G is connected then (under the identification $\mathrm{im}(\Psi_G) \cong \mathbf{H}_{\mathcal{F}}(G)$) there is a direct sum

$$\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}}) = \mathbf{H}_{\mathcal{F}}(G) \oplus \mathrm{Ext}(G^*, \pi^1(X)).$$

In Theorem 4.39 we continue our study of Ψ_G under the assumption of divisibility of $\mathbf{Z}_{\mathcal{F}}$ and connectedness of G. We begin by noticing that there is a direct sum

$$\mathbf{Z}_{\mathcal{F}}(G) = \mathbf{Z}^{\mathrm{td}}_{\mathcal{F}}(G) \oplus \mathbf{Z}^{\mathrm{cn}}_{\mathcal{F}}(G),$$

where $\mathbf{Z}^{\mathrm{td}}_{\mathcal{F}}(G)$ is the group of all extensions $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ with a totally disconnected section $F(\mathcal{C})$ and $F(\mathcal{D})$ is connected for every $\mathcal{D} \in \mathbf{Z}^{\mathrm{cn}}_{\mathcal{F}}(G)$. This direct sum then yields a (topological) direct sum of the corresponding subgropus of $\mathbf{H}_{\mathcal{F}}(G)$, namely

$$\mathbf{H}_{\mathcal{F}}(G) = \mathbf{H}^{\mathrm{td}}_{\mathcal{F}}(G) \oplus \mathbf{H}^{\mathrm{cn}}_{\mathcal{F}}(G),$$

We also show that Ψ_G maps $\mathbf{H}^{\mathrm{cn}}_{\mathcal{F}}(G)$ isomorphically onto $\mathrm{Hom}(G^*, \mathbf{H}^{\mathrm{cn}}_{\mathcal{F}})$ and that there is a direct sum

$$\mathrm{Hom}(G^*, \mathbf{H}^{\mathrm{td}}_{\mathcal{F}}) = \mathbf{H}^{\mathrm{td}}_{\mathcal{F}}(G) \oplus \mathrm{Ext}(G^*, \pi^1(X)).$$

Thus we may say that the non-surjectivity of Ψ_G is caused by the extensions with totally disconnected sections.

Torsions as non-minimal extensions. As we mentioned earlier, the torsion elements in $\mathbf{H}_\mathcal{F}$ correspond to non-minimal extensions from $\mathbf{Z}_\mathcal{F}$ and minimal extensions from $\mathbf{Z}_\mathcal{F}$ thus correspond to elements of $\mathbf{H}_\mathcal{F}$ of infinite order. Being motivated by these facts, we study the torsion subgroup $\mathrm{tor}(\mathbf{H}_\mathcal{F})$ of $\mathbf{H}_\mathcal{F}$ for a given minimal flow $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$.

In Theorem 4.41 we show that if Γ has no non-trivial finite abelian quotient groups, X is a compact connected space and the group $\mathbf{Z}_\mathcal{F}$ is divisible (which includes the case when Γ is simply connected) then for every $k \in \mathbb{N}$ there are isomorphisms

$$\mathrm{tor}_k(\mathbf{H}_\mathcal{F}) \cong \pi^1(X)/k\pi^1(X)$$

and

$$\mathrm{tor}(\mathbf{H}_\mathcal{F}) \cong (\mathbb{Q}/\mathbb{Z}) \otimes \pi^1(X).$$

Moreover, $\mathrm{tor}(\mathbf{H}_\mathcal{F})$ is a direct summand in $\mathbf{H}_\mathcal{F}$ and

$$\mathbf{H}_\mathcal{F} = \mathcal{D} \oplus \mathrm{tor}(\mathbf{H}_\mathcal{F})$$

for some divisible subgroup $\mathcal{D}$ of the groupoid $\mathbf{MinH}_\mathcal{F}$. This allows us to prove in Corollary 4.43 that the identity component $F(\daleth_\mathcal{F})_0$ of $F(\daleth_\mathcal{F})$ is a topological direct summand in $F(\daleth_\mathcal{F})$ and the group of components $F(\daleth_\mathcal{F})/F(\daleth_\mathcal{F})_0$ is topologically isomorphic to the Pontryagin dual of the discrete group $(\mathbb{Q}/\mathbb{Z}) \otimes \pi^1(X)$.

In Theorem 4.44 we study the torsion subgroup of $\mathbf{H}_\mathcal{F}$ in the case when Γ is a connected Lie group and X is a compact manifold. Set $n = \mathrm{rank}(H_1^w(\mathcal{F}))$, $n + m = \mathrm{rank}(H_1^w(X))$, denote by $d_1, \ldots, d_n$ the elementary divisors of $H_1^w(\mathcal{F})$ in $H_1^w(X)$ and, for $k \geq 2$ and $i = 1, \ldots, n$, let $\delta_i = \gcd(d_i, k)$. Then there are isomorphisms of groups

$$\mathrm{tor}_k(\mathbf{H}_\mathcal{F}) \cong \mathbb{Z}_{\delta_1} \oplus \cdots \oplus \mathbb{Z}_{\delta_n} \oplus (\mathbb{Z}_k)^m$$

and

$$\mathrm{tor}(\mathbf{H}_\mathcal{F}) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n} \oplus (\mathbb{Q}/\mathbb{Z})^m.$$

If, in addition, the flow $\mathcal{F}$ is topologically free then, by Theorem 4.47, $\mathrm{tor}(\mathbf{H}_\mathcal{F})$ is a direct summand in $\mathbf{H}_\mathcal{F}$ and we have

$$\mathbf{H}_\mathcal{F} = \mathcal{D} \oplus \mathrm{tor}(\mathbf{H}_\mathcal{F})$$

for an appropriate divisible subgroup $\mathcal{D}$ of the groupoid $\mathbf{MinH}_\mathcal{F}$. We also have a topological isomorphism

$$F(\daleth_\mathcal{F}) \cong \mathcal{D}^* \times ((\mathbb{Q}/\mathbb{Z})^*)^m \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n},$$

where both $\mathcal{D}$ and $\mathbb{Q}/\mathbb{Z}$ carry the discrete topology. Finally, as we show in Theorem 4.46, under the assumption of topological freeness of $\mathcal{F}$, we have direct sums

$$\mathbf{Z}_\mathcal{F} \cong \mathrm{Div}(\mathbf{Z}_\mathcal{F}) \oplus \mathbb{Z}^n$$

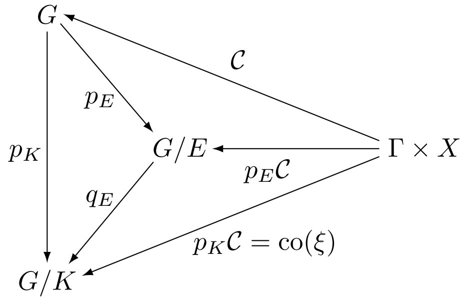
and

$$\mathbf{H}_\mathcal{F} \cong \mathrm{Div}(\mathbf{H}_\mathcal{F}) \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n},$$

where $\mathrm{Div}(\mathbf{Z}_\mathcal{F})$ and $\mathrm{Div}(\mathbf{H}_\mathcal{F})$ denote the divisible subgroups of $\mathbf{Z}_\mathcal{F}$ and $\mathbf{H}_\mathcal{F}$, respectively.

Results of Chapter 5. Algebraic-topological aspects

Lifts of transfer functions. Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow with a connected phase space X and suppose that the acting group Γ of $\mathcal{F}$ has no non-trivial finite abelian quotient groups. (This is the case, for instance, if Γ is connected.) Assume that $G \in \mathsf{CAGp}$ is a group with the identity e, $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ is an extension and the section $F(\mathcal{C})$ of $\mathcal{C}$ is contained in a closed totally disconnected subgroup K of G. If $p_K\colon G \to G/K$ denotes the quotient morphism then $p_K\mathcal{C}$ is a coboundary. Fix $z \in X$ and let ξ be the transfer function of $p_K\mathcal{C}$ with $\xi(z) = K$. Given a closed subgroup E of K, let $p_E\colon G \to G/E$ be the quotient morphism and $q_E\colon G/E \to G/K$ be the unique topological morphism with $q_E p_E = p_K$. In Proposition 5.3 we show that the section $F(\mathcal{C})$ of $\mathcal{C}$



- coincides with the smallest of all the closed subgroups E of K, for which ξ lifts across q_E to a continuous map $\eta\colon X \to G/E$,
- coincides with the smallest of all the closed subgroups E of K, for which the induced morphism $\xi^*\colon (G/K)^* \to C_z(X, \mathbb{T}^1)$ extends through $(q_E)^*\colon (G/K)^* \to (G/E)^*$ to a morphism $\sigma\colon (G/E)^* \to C_z(X, \mathbb{T}^1)$ (here $C_z(X, \mathbb{T}^1)$ denotes the group of all continuous maps $f\colon X \to \mathbb{T}^1$ with $f(z) = 1$).

This (auxiliary) result is an important step towards our proofs of the main results of Chapter 5.

Sections and monodromy. Let $G \in \mathsf{CAGp}$, K be a closed totally disconnected subgroup of G and $p_K\colon G \to G/K$ be the quotient morphism. Denote by $p_K^\sharp$ the morphism $\pi_1(G) \to \pi_1(G/K)$ induced by p_K with the identities of the groups $G, G/K$ serving as base points for their fundamental groups. Let us recall the monodromy action of $\pi_1(G/K)$ on K. Given a loop $f \in \pi_1(G/K)$ and $k \in K$, let $\widetilde{f}$ be the (continuous) lift of f across p_K starting at k. Then we let the endpoint of $\widetilde{f}$ be the result of the monodromy action of f on k.

Given a subgroup Q of $\pi_1(G/K)$, let $\mathcal{E}_K(Q)$ be the orbit of the identity e of G under the monodromy action of Q on K. Thus, $\mathcal{E}_K(Q)$ consists of the endpoints of lifts of loops in Q starting at e. Since $\mathcal{E}_K(Q) = \mathcal{E}_K(Q + p_K^\sharp \pi_1(G))$, we shall sometimes restrict ourselves to the subgroups Q of $\pi_1(G/K)$ containing $p_K^\sharp \pi_1(G)$. Conversely, given a subgroup E of K, let $\mathcal{Q}_K(E)$ consist of those elements of $\pi_1(G/K)$ which lift across p_K to paths both starting and ending in E. Notice that $\mathcal{Q}_K(E) = \mathcal{Q}_K(E \cap G_a)$, where G_a denotes the identity arc-component of G.

In Lemma 5.8 we collect some important properties of $\mathcal{E}_K$ and $\mathcal{Q}_K$. We show that for $Q \supseteq p_K^\sharp \pi_1(G)$ and E as above,

- $\mathcal{E}_K(Q)$ is a subgroup of K contained in G_a,
- $\mathcal{Q}_K(E)$ is a subgroup of $\pi_1(G/K)$ containing $p_K^\sharp \pi_1(G)$,
- $\mathcal{E}_K(\mathcal{Q}_K(E)) = E \cap G_a$ and $\mathcal{Q}_K(\mathcal{E}_K(Q)) = Q$,
- $\mathcal{E}_K(Q) \cong Q/p_K^\sharp \pi_1(G)$.

(In Examples 5.10 and 5.12 we describe $\mathcal{E}_K$ in the situation when G is a torus or a solenoid, respectively.) Let us mention that for a given Q, the subgroup $\mathcal{E}_K(Q)$ of K may not be closed in K. We denote its closure by $\overline{\mathcal{E}_K}(Q)$. Thus, $\overline{\mathcal{E}_K}(Q)$ is the orbit closure of the identity e of G under the monodromy action of Q on K.

Keeping the assumptions and notation as above, let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ be an extension with $p_K\mathcal{C} \in \mathbf{B}_{\mathcal{F}}(G/K)$. Fix $z \in X$ and let ξ be the transfer function of $p_K\mathcal{C}$ with $\xi(z) = K$. In Theorem 5.14 we express the section $F(\mathcal{C})$ of $\mathcal{C}$ in terms of the monodromy action. To be concrete, we show that if X is a connected manifold and Γ has no non-trivial finite abelian quotient groups then

$$F(\mathcal{C}) = \overline{\mathcal{E}_K}\left(\xi^\sharp \pi_1(X) + p_K^\sharp \pi_1(G)\right) = \overline{\mathcal{E}_K}\left(\xi^\sharp \pi_1(X)\right). \tag{2}$$

It follows, in particular, that $F(\mathcal{C})$ depends only on the homotopy class of ξ. Further, if the group K is finite then

- we have
$$F(\mathcal{C}) = \mathcal{E}_K\left(\xi^\sharp \pi_1(X) + p_K^\sharp \pi_1(G)\right) = \mathcal{E}_K\left(\xi^\sharp \pi_1(X)\right),$$
- $F(\mathcal{C}) \subseteq G_a$ and there is an isomorphism
$$F(\mathcal{C}) \cong \left(\xi^\sharp \pi_1(X) + p_K^\sharp \pi_1(G)\right)/p_K^\sharp \pi_1(G),$$
- $F(\mathcal{C}) = K$ occurs if and only if $K \subseteq G_a$ and
$$\xi^\sharp \pi_1(X) + p_K^\sharp \pi_1(G) = \pi_1(G/K).$$

Let us also mention that for an infinite group K, the closure operator can not be removed from (2) and the inclusion $F(\mathcal{C}) \subseteq G_a$ may fail to hold. Examples demonstrating these assertions are constructed in Proposition 5.18 and Remark 5.19.

Existence of prescribed sections, part 1. The monodromy formula (2) enables us to determine which closed totally disconnected subgroups of a given fibre group are values of the functor F. To be concrete, let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow with Γ a connected Lie group and X a compact connected manifold. Set $n = \operatorname{rank}(H_1^w(\mathcal{F}))$, $n + m = \operatorname{rank}(H_1^w(X))$ and let $d_1, \dots, d_n$ be the elementary divisors of $H_1^w(\mathcal{F})$ in $H_1^w(X)$. Assume that $G \in \mathsf{CAGp}$ is a connected group with the identity arc-component G_a and K is a closed totally disconnected subgroup of G. As we show in Theorem 5.22, under these assumptions the following conditions are equivalent:

- there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ with $F(\mathcal{C}) = K$,

- there are $k_1, \dots, k_{n+m} \in K \cap G_a$ which generate a dense subgroup of K and such that $k_i^{d_i} = e$ for $i = 1, \dots, n$.

(If follows, in particular, that if $\mathcal{F}$ has no free cycle (that is, if $m = 0$) then all totally disconnected values of F are finite.)

In Corollary 5.24 we use the result of Theorem 5.22 to show that if K is finite then the following conditions are equivalent:

- there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ with $F(\mathcal{C}) = K$,
- K is contained in G_a and it is a quotient group of $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n} \oplus \mathbb{Z}^m$.

By using the preceding results together with the summation formula (1), we can prove analogous existence results for more general (not necessarily totally disconnected) closed subgroups K of G. We illustrate this in situations when G is a torus or a solenoid. Firstly, in Theorem 5.25 we show that for every positive integer k the following conditions are equivalent:

- for every closed subgroup K of $\mathbb{T}^k$ there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^k)$ with $F(\mathcal{C}) = K$,
- for every finite subgroup K of $\mathbb{T}^k$ there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^k)$ with $F(\mathcal{C}) = K$,
- $m = \operatorname{rank}(H_1^w(X)) - \operatorname{rank}(H_1^w(\mathcal{F})) \geq k$.

In particular, if $m \geq k$ then $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^k)$ contains a minimal extension.

Secondly, for a given solenoid S and its non-trivial proper closed subgroup K, we study in Theorem 5.30 the existence of an extension $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(S)$ with $F(\mathcal{C}) = K$. We begin by showing that if $\mathcal{F}$ does not have a free cycle (that is, if $m = 0$) then such an extension $\mathcal{C}$ does not exist. Then we show that if $\mathcal{F}$ has a free cycle then an extension $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(S)$ with $F(\mathcal{C}) = K$ exists if and only if the annihilator $K^{\perp}$ of K in S^* is infinite cyclic. Finally, if $\mathcal{F}$ has a free cycle then $\mathbf{Z}_{\mathcal{F}}(S)$ contains a minimal extension.

Sections and first cohomotopy groups. The monodromy formula (2) applies to extensions of minimal flows $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$ on manifolds X. Since many important examples of minimal flows are supported on more general, not necessarily locally connected spaces, we are motivated to search for a result analogous to (2) in the setting of more general spaces X. So let the phase space X of $\mathcal{F}$ be compact connected and let the acting group Γ of $\mathcal{F}$ have no non-trivial finite abelian quotient groups. Let $G \in \mathsf{CAGp}$ be connected, K be a finite subgroup of G and $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(G)$ be an extension with $F(\mathcal{C}) \subseteq K$. Given $z \in X$, let ξ be the transfer function of $p_K\mathcal{C}$ with $\xi(z) = K$. Consider the morphism $\xi^\flat\colon \pi^1(G/K) \to \pi^1(X)$ induced by ξ. Since $\pi^1(G/K)$ is isomorphic to $(G/K)^* \cong K^{\perp}$, we may view $\xi^\flat$ as a morphism $K^{\perp} \to \pi^1(X)$. In Theorem 5.32 we show that if d is the largest torsion coefficient of K and η is the cardinality of K then

$$F(\mathcal{C})^{\perp} = \frac{1}{d}\left(\xi^\flat\right)^{-1}\left(d\pi^1(X)\right) = \frac{1}{\eta}\left(\xi^\flat\right)^{-1}\left(\eta\pi^1(X)\right). \tag{3}$$

To be even more precise, let us notice that the first equality in (3) states that

$$F(\mathcal{C})^{\perp} = \left\{\chi \in G^* : \xi^\flat(\chi^d) \in d\pi^1(X)\right\};$$

the second equality is understood in the same way. Moreover, we also show that the following conditions are equivalent:

- $F(\mathcal{C}) = K$,
- $dG^* \cap (\xi^\flat)^{-1}\left(d\pi^1(X)\right) = dK^\perp$,
- $\xi^\flat\left(dG^* \setminus dK^\perp\right) \cap d\pi^1(X) = \emptyset$.

Theorem 5.32 enables us to find an algebraic criterion for the existence of an extension $\mathcal{C} \in \mathbf{Z}_\mathcal{F}(G)$ with $F(\mathcal{C}) = K$. This is done in Corollary 5.36. We show that if the acting group Γ of $\mathcal{F}$ is simply connected then the existence of $\mathcal{C} \in \mathbf{Z}_\mathcal{F}(G)$ with $F(\mathcal{C}) = K$ is equivalent to the existence of a morphism $\varrho \in \operatorname{Hom}\left(K^\perp, \pi^1(X)\right)$ with

$$dG^* \cap \varrho^{-1}\left(d\pi^1(X)\right) = dK^\perp.$$

In Theorem 5.40 we generalize the results from Theorem 5.32 to the situation when K is a closed (possibly infinite) totally disconnected subgroup of G. So let the other assumptions of Theorem 5.32 be fulfilled and assume that K is a closed totally disconnected subgroup of G. Consider a generating net of torsions $(d_j)_{j\in J}$ for K as defined in Section 5.5. Then

$$\begin{aligned} F(\mathcal{C})^\perp &= \bigcup_{j\in J} \frac{1}{d_j}\left(\xi^\flat\right)^{-1}\left(d_j\pi^1(X)\right) = \sum_{j\in J} \frac{1}{d_j}\left(\xi^\flat\right)^{-1}\left(d_j\pi^1(X)\right) \\ &= \varinjlim \frac{1}{d_j}\left(\xi^\flat\right)^{-1}\left(d_j\pi^1(X)\right). \end{aligned}$$

As we mention in Remark 5.41, the equality $F(\mathcal{C}) = K$ holds if and only if

$$\frac{1}{d_j}\left(\xi^\flat\right)^{-1}\left(d_j\pi^1(X)\right) \subseteq K^\perp$$

for every $j \in J$.

Existence of prescribed sections, part 2. Now we use results of Theorems 5.32 and 5.40 together with the summation formula (1) to find conditions, under which a given closed subgroup K of $G \in \mathsf{CAGp}$ is a value of the functor F. We assume that $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$ is a minimal flow with a simply connected acting group Γ and a compact connected phase space X.

First, in Thoerem 5.42 we consider the case when X is a manifold. Given a connected group $G \in \mathsf{CAGp}$ and a non-trivial closed totally disconnected subgroup K of G with a generating net of torsions $(d_j)_{j\in J}$, we show that the following conditions are equivalent:

- there is $\mathcal{C} \in \mathbf{Z}_\mathcal{F}(G)$ with $F(\mathcal{C}) = K$,
- the group $K^\perp$ splits into a direct sum $K^\perp = F \oplus Q$, where F is a free abelian group with $\operatorname{rank}(F) \le \operatorname{rank}(\pi^1(X))$ and Q is a d_j-pure subgroup of G^* for every $j \in J$,
- the group G/K has a topological direct summand T, which is a torus with topological dimension $\dim(T) \le \operatorname{rank}(\pi^1(X))$ and whose preimage under the quotient morphism $p_K\colon G \to G/K$ is connected.

Notice that by our assumptions on X we have isomorphism $\pi^1(X) \cong H_1^w(X)$, from which it follows that $\operatorname{rank}(\pi^1(X))$ is finite.

Further, in Theorem 5.44 we consider the situation when X is an arbitrary compact (connected) space and G is a torus $\mathbb{T}^n$. We show that for a given integer $d \ge 2$ the following conditions are equivalent:

- for every finite subgroup K of $\mathbb{T}^n$, whose largest torsion coefficient divides d, there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^n)$ with $F(\mathcal{C}) = K$,
- $\pi^1(X)/d\pi^1(X)$ has rank at least n when considered as a $\mathbb{Z}_d$-module.

Moreover, in Proposition 5.45 we give a sufficient condition for the existence of a minimal extension $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(\mathbb{T}^n)$. We show that such an extension exists if the space X is second countable and for infinitely many prime numbers p, the group $\pi^1(X)$ is not p-divisible (that is, $\pi^1(X) \neq p\pi^1(X)$). In fact, under these assumptions such an extension $\mathcal{C}$ exists in the closure of $\mathbf{Z}^{\mathrm{td}}_{\mathcal{F}}(\mathbb{T}^n)$, so that it is expressible as a limit of a sequence of extensions $\mathcal{C}_k$ $(k \in \mathbb{N})$, each of which has a totally disconnected section $F(\mathcal{C}_k)$.

We continue our study of the existence of finite sections in Proposition 5.47, this time concentrating on the case when $G = \mathbb{T}^1$. Let $d \geq 2$ be an integer with the prime decomposition $d = p_1^{k_1} \dots p_n^{k_n}$. Then the following conditions are equivalent:

- there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}$ with $F(\mathcal{C}) = \mathbb{Z}_d$,
- for every $i = 1, \dots, n$, the group $\pi^1(X)$ is not p_i-divisible.

Then we concentrate on the situation when G is a solenoid S. In Theorem 5.48 we show that for a non-trivial closed totally disconnected subgroup K of S with a generating net of torsions $(d_j)_{j \in J}$, the following conditions are equivalent:

- there is $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(S)$ with $F(\mathcal{C}) = K$,
- $\pi^1(X)$ contains a subgroup isomorphic to $K^{\perp}$, which is d_j-pure in $\pi^1(X)$ for every $j \in J$.

In particular, if $d \geq 2$ is an integer and $K \cong \mathbb{Z}_d$ then the existence of an extension $\mathcal{C} \in \mathbf{Z}_{\mathcal{F}}(S)$ with $F(\mathcal{C}) = K$ is equivalent to the existence of a d-pure subgroup of $\pi^1(X)$ isomorphic to $K^{\perp}$, see Corollary 5.49.

Finally, we study the existence of minimal extensions in $\mathbf{Z}_{\mathcal{F}}(G)$ for connected finite-dimensional groups $G \in \mathsf{CAGp}$. In Proposition 5.51 we show that if $\dim(G) = n \in \mathbb{N}$ then $\mathbf{Z}_{\mathcal{F}}(G)$ contains a minimal extension if and only if $\mathbf{Z}_{\mathcal{F}}(\mathbb{T}^n)$ contains a minimal extension. In Corollary 5.52 we use this equivalence to verify the equivalence of the following three statements:

- for every solenoid S, $\mathbf{Z}_{\mathcal{F}}(S)$ contains a minimal extension,
- there is a solenoid S such that $\mathbf{Z}_{\mathcal{F}}(S)$ contains a minimal extension,
- $\mathbf{Z}_{\mathcal{F}}$ contains a minimal extension.

Results of Chapter 6. First cohomology groups

Groups of minimal extensions. Let $\mathcal{F}\colon \Gamma \curvearrowright X$ be a minimal flow and let $G \in \mathsf{CAGp}$. As we mentioned earlier, $\mathbf{MinZ}_{\mathcal{F}}(G)$ is a subgroupoid but not a subgroup of $\mathbf{Z}_{\mathcal{F}}(G)$. It is therefore natural to ask whether or not $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of a given abelian group A and, in particular, of $\mathbf{Z}_{\mathcal{F}}(G)$. Of course, the group A can not be arbitrary and must reflect certain features of $\mathbf{Z}_{\mathcal{F}}(G)$, including cardinality and group-theoretical properties.

We begin our study of the problem formulated above in Theorem 6.1, where we show that for an arbitrary minimal flow $\mathcal{F}$ and an infinite cardinal number $\mathfrak{k}$ the following conditions are equivalent:

- for every torsion-free abelian group A with $\operatorname{card}(A) \leq \mathfrak{k}$ and every non-trivial connected group $H \in \mathsf{CAGp}$ with weight $\mathrm{w}(H) \leq \mathfrak{k}$, $\mathbf{MinZ}_{\mathcal{F}}(H)$ contains an isomorphic copy of A,
- for every torsion-free abelian group A with $\operatorname{card}(A) \leq \mathfrak{k}$, $\mathbf{MinZ}_{\mathcal{F}}$ contains an isomorphic copy of A,
- for every non-trivial connected group $G \in \mathsf{CAGp}$ with weight $\mathrm{w}(G) \leq \mathfrak{k}$, $\mathbf{MinZ}_{\mathcal{F}}(G) \setminus 1 \neq \emptyset$.

Let us mention that the assumption of torsion-freeness of A is necessary, for the group $\mathbf{Z}_{\mathcal{F}}(G)$ is torsion-free as soon as $G \in \mathsf{CAGp}$ is connected.

Next we turn to describing situations in which the three conditions stated above are fulfilled. First, in Theorem 6.3 we show that if Γ is locally compact second countable amenable, X is compact second countable and $\mathcal{F}$ has a free point then the following statements hold:

- for every non-trivial connected group $G \in \mathsf{CAGp}$ with $\mathrm{w}(G) \leq \mathfrak{c}$, $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of $\mathbb{R}$,
- for every non-trivial connected second countable group $G \in \mathsf{CAGp}$, the groupoid $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of the quotient group $\mathbf{Z}_{\mathcal{F}}(G)/\operatorname{tor}(\mathbf{Z}_{\mathcal{F}}(G))$.

In connection with the first statement recall that the group $\mathbb{R}$ is universal among torsion-free abelian groups A with $\operatorname{card}(A) \leq \mathfrak{c}$ in the sense that each such group A is isomorphic to a subgroup of $\mathbb{R}$. In connection with the second statement notice that $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains no torsion elements and so factoring out the torsion subgroup $\operatorname{tor}(\mathbf{Z}_{\mathcal{F}}(G))$ of $\mathbf{Z}_{\mathcal{F}}(G)$ is necessary in general.

Further, in Theorem 6.5 we consider the situation when Γ is a simply connected Lie group and X is a compact space with $1 \leq \operatorname{rank}(\pi^1(X)) < \mathfrak{c}$. We show that for every non-trivial connected group $G \in \mathsf{CAGp}$ with $\mathrm{w}(G) \leq \mathfrak{c}$, $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of $\mathbb{R}$. If, in addition, X is second countable then for every non-trivial connected second countable group $G \in \mathsf{CAGp}$, $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of $\mathbf{Z}_{\mathcal{F}}(G)$.

Finally, in Theorem 6.6 we turn our attention to the case when Γ is a connected Lie group, X is a compact (connected) manifold and the flow $\mathcal{F}$ has a free cycle. We show that the following statements hold:

- for every non-trivial connected group $G \in \mathsf{CAGp}$ with $\mathrm{w}(G) \leq \mathfrak{c}$, $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of $\mathbb{R}$,
- for every non-trivial connected second countable group $G \in \mathsf{CAGp}$, $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of $\mathbf{Z}_{\mathcal{F}}(G)$.

First cohomology groups. The results obtained thus far enable us to determine first cohomology groups of certain minimal flows $\mathcal{F}\colon \Gamma \curvearrowright \mathrm{X}$. We begin by studying the groups $\mathbf{H}_{\mathcal{F}} = \mathbf{H}_{\mathcal{F}}(\mathbb{T}^1)$ in Section 6.2.

In Theorem 6.12 we consider the case when Γ is a simply connected Lie group and X is a second countable compact (connected) space with a non-trivial first cohomotopy group $\pi^1(X)$. We show that the inclusion of groups $\mathbf{B}_{\mathcal{F}} \subseteq \mathbf{Z}_{\mathcal{F}}$ takes the form

$$0 \oplus \mathbb{R} \oplus \pi^1(X) \subseteq \mathbb{R} \oplus \mathbb{R} \oplus \big(\mathbb{Q} \otimes \pi^1(X)\big)$$

and infer that there is an isomorphism of groups

$$\mathbf{H}_{\mathcal{F}} \cong \mathbb{R} \oplus \big((\mathbb{Q}/\mathbb{Z}) \otimes \pi^1(X)\big).$$

We also show that the inclusion of topological groups $F(\daleth_{\mathcal{F}}) \subseteq \gimel_{\mathcal{F}}$ takes the form

$$b\mathbb{R} \times 1 \times \pi^1(X)^{\perp} \subseteq b\mathbb{R} \times b\mathbb{R} \times \big(\mathbb{Q} \otimes \pi^1(X)\big)^*,$$

where $\pi^1(X)^{\perp}$ denotes the annihilator of $\pi^1(X)$ in $(\mathbb{Q} \otimes \pi^1(X))^*$. If the group $\pi^1(X)$ is additionally finitely generated with rank k then the inclusion $\mathbf{B}_{\mathcal{F}} \subseteq \mathbf{Z}_{\mathcal{F}}$ becomes

$$0 \oplus \mathbb{R} \oplus \mathbb{Z}^k \subseteq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{Q}^k,$$

there is an isomorphism

$$\mathbf{H}_{\mathcal{F}} \cong \mathbb{R} \oplus (\mathbb{Q}/\mathbb{Z})^k$$

and the inclusion $F(\daleth_{\mathcal{F}}) \subseteq \gimel_{\mathcal{F}}$ takes the form

$$b\mathbb{R} \times 1 \times \big(\mathbb{Z}^{\perp}\big)^k \subseteq b\mathbb{R} \times b\mathbb{R} \times (\mathbb{Q}^*)^k.$$

In Theorem 6.14 we consider the case when Γ is a connected Lie group, X is a compact (connected) manifold, the flow $\mathcal{F}$ is topologically free and has a free cycle. As usual, we set $n = \operatorname{rank}(H_1^w(\mathcal{F}))$, $n + m = \operatorname{rank}(H_1^w(X))$ and denote by $d_1, \dots, d_n$ the elementary divisors of $H_1^w(\mathcal{F})$ in $H_1^w(X)$. We show that the inclusion $\mathbf{B}_{\mathcal{F}} \subseteq \mathbf{Z}_{\mathcal{F}}$ takes the form

$$0 \oplus \mathbb{R} \oplus \mathbb{Z}^m \oplus \left(\bigoplus_{i=1}^n d_i\mathbb{Z}\right) \subseteq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{Q}^m \oplus \mathbb{Z}^n.$$

Consequently, we have an isomorphism

$$\mathbf{H}_{\mathcal{F}} \cong \mathbb{R} \oplus (\mathbb{Q}/\mathbb{Z})^m \oplus \left(\bigoplus_{i=1}^n \mathbb{Z}_{d_i}\right).$$

Also, the inclusion of topological groups $F(\daleth_{\mathcal{F}}) \subseteq \gimel_{\mathcal{F}}$ becomes

$$b\mathbb{R} \times 1 \times \big(\mathbb{Z}^{\perp}\big)^m \times \left(\prod_{i=1}^n \mathbb{Z}_{d_i}\right) \subseteq b\mathbb{R} \times b\mathbb{R} \times (\mathbb{Q}^*)^m \times \mathbb{T}^n.$$

In Corollary 6.16 we use results of Theorem 6.14 to express the short exact sequence

$$E_{\mathcal{F}}\colon 0 \longrightarrow \mathbf{B}_{\mathcal{F}} \longrightarrow \mathbf{Z}_{\mathcal{F}} \longrightarrow \mathbf{H}_{\mathcal{F}} \longrightarrow 0$$

as a direct sum of elementary short exact sequences and to determine the 2-cocycle $\varphi_{\mathcal{F}} \in Z^2(\mathbf{H}_{\mathcal{F}}, \mathbf{B}_{\mathcal{F}})$ which leads to $E_{\mathcal{F}}$. We observe that $E_{\mathcal{F}}$ does not split (that is, $\mathbf{B}_{\mathcal{F}}$ is not a direct summand in $\mathbf{Z}_{\mathcal{F}}$) and so $\varphi_{\mathcal{F}}$ is not a 2-coboundary.

In Section 6.3 we study the first cohomology groups $\mathbf{H}_{\mathcal{F}}(G)$ of $\mathcal{F}$, where $G \in \mathsf{CAGp}$ is an arbitrary connected second countable group. We begin by noticing that G^*, being a countable torsion-free abelian group, splits into a direct sum

$$G^* = \mathfrak{f}(G^*) \oplus \mathfrak{t}(G^*),$$

where $\mathfrak{f}(G^*)$ is a free abelian group and the group $\mathfrak{t}(G^*)$ is torsion-less (that is, all morphisms $\mathfrak{t}(G^*) \to \mathbb{Z}$ vanish). We set

$$\mathfrak{r} = \operatorname{rank}(G^*), \quad \mathfrak{f} = \operatorname{rank}(\mathfrak{f}(G^*)) \quad \text{and} \quad \mathfrak{t} = \operatorname{rank}(\mathfrak{t}(G^*)).$$

In Theorem 6.19 we show that under the assumptions and notation from Theorem 6.14 (that is, for a topologically free flow $\mathcal{F}$ having a free cycle), the inclusion of groups $\mathbf{B}_{\mathcal{F}}(G) \subseteq \mathbf{Z}_{\mathcal{F}}(G)$ takes the form

$$\begin{array}{ccccccccccc} & 0 & \oplus & \mathbb{R}^{\mathfrak{r}} & \oplus & \mathbb{Z}^{m\mathfrak{f}} & \oplus & 0 & \oplus & (d_1\mathbb{Z} \oplus \cdots \oplus d_n\mathbb{Z})^{\mathfrak{f}} & \subseteq \\ \subseteq & \mathbb{R}^{\mathfrak{r}} & \oplus & \mathbb{R}^{\mathfrak{r}} & \oplus & \mathbb{Q}^{m\mathfrak{f}} & \oplus & \mathbb{Q}^{m\mathfrak{t}} & \oplus & (\mathbb{Z}^n)^{\mathfrak{f}}, & \end{array}$$

and we have an isomorphism of groups

$$\mathbf{H}_{\mathcal{F}}(G) \cong \mathbb{R}^{\mathfrak{r}} \oplus (\mathbb{Q}/\mathbb{Z})^{m\mathfrak{f}} \oplus \mathbb{Q}^{m\mathfrak{t}} \oplus (\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n})^{\mathfrak{f}}.$$

We also express the topological morphism

$$\Psi_G \colon \mathbf{H}_{\mathcal{F}}(G) \to \operatorname{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$$

as an inclusion of concrete topological groups. As a corollary we observe that Ψ_G is an isomorphism if $\mathfrak{t}(G^*) = 0$, that is, if G is a torus. (In connection with the two isomorphisms above notice that if the group G is non-trivial then $1 \leq \mathfrak{r} \leq \aleph_0$ and so $\mathbb{R}^{\mathfrak{r}}$ is isomorphic to $\mathbb{R}$.)

In Theorem 6.23 we work under the assumptions and notation from Theorem 6.12. We thus assume that Γ is a simply connected Lie group and X is a second countable compact (connected) space with $\pi^1(X) \neq 0$. Given a connected group $G \in \mathsf{CAGp}$ with $\mathfrak{r} = \operatorname{rank}(G^*)$, we show that the inclusion $\mathbf{B}_{\mathcal{F}}(G) \subseteq \mathbf{Z}_{\mathcal{F}}(G)$ takes the form

$$0 \oplus \mathbb{R}^{\mathfrak{r}} \oplus \operatorname{Hom}\left(G^*, \pi^1(X)\right) \subseteq \mathbb{R}^{\mathfrak{r}} \oplus \mathbb{R}^{\mathfrak{r}} \oplus \operatorname{Hom}\left(G^*, \mathbb{Q} \otimes \pi^1(X)\right).$$

(As we mentioned above, if G is non-trivial and second countable then $\mathbb{R}^{\mathfrak{r}}$ is isomorphic to $\mathbb{R}$.) We also show that the group $\operatorname{Ext}(G^*, \pi^1(X))$ can be viewed as a direct summand in $\operatorname{Hom}(G^*, (\mathbb{Q}/\mathbb{Z}) \otimes \pi^1(X))$ and that there are isomorphisms of groups

$$\mathbf{H}_{\mathcal{F}}(G) \cong \mathbb{R}^{\mathfrak{r}} \oplus \frac{\operatorname{Hom}\left(G^*, \mathbb{Q} \otimes \pi^1(X)\right)}{\operatorname{Hom}\left(G^*, \pi^1(X)\right)} \cong \mathbb{R}^{\mathfrak{r}} \oplus \frac{\operatorname{Hom}\left(G^*, (\mathbb{Q}/\mathbb{Z}) \otimes \pi^1(X)\right)}{\operatorname{Ext}\left(G^*, \pi^1(X)\right)},$$

which simplify significantly under some additional assumptions on $\pi^1(X)$ or G (see Remark 6.24).

More on first cohomology groups. Let $\mathcal{F} \colon \Gamma \curvearrowright X$ be a minimal flow. In Theorem 4.39 we found splittings of the groups $\mathbf{Z}_{\mathcal{F}}(G)$ and $\mathbf{H}_{\mathcal{F}}(G)$ into subgroups formed by extensions with totally disconnected, respectively, connected sections, under the assumption of divisibility of $\mathbf{Z}_{\mathcal{F}}$ (which includes the case when Γ is simply connected). By using results from Theorems 6.14 and 6.19, we were able to obtain analogous splittings also in the case when the flow $\mathcal{F}$ is topologically free and has a free cycle. So let the assumptions of Theorems 6.14 and 6.19 be fulfilled, the notation introduced therein be fixed and let $G \in \mathsf{CAGp}$ be second countable and connected. In Theorem 6.25 we show that there are a direct sum and an isomorphism of groups

$$\mathbf{Z}_{\mathcal{F}}(G) = \mathbf{Z}_{\mathcal{F}}^{\mathrm{cn}}(G) \oplus \mathbf{Z}_{\mathcal{F}}^{\mathrm{td}}(G) \cong \mathbb{R}^{\mathfrak{r}} \oplus \left(\mathbb{R}^{\mathfrak{r}} \oplus \mathbb{Z}^{n\mathfrak{f}}\right),$$

where $\mathbf{Z}^{\mathrm{td}}_{\mathcal{F}}(G)$ is the subgroup of $\mathbf{Z}_{\mathcal{F}}(G)$ formed by the extensions $\mathcal{C}$ with $F(\mathcal{C})$ totally disconnected and $F(\mathcal{D})$ is connected for every $\mathcal{D} \in \mathbf{Z}^{\mathrm{cn}}_{\mathcal{F}}(G)$. Correspondingly, there are a direct sum and an isomorphism of groups

$$\mathbf{H}_{\mathcal{F}}(G) = \mathbf{H}^{\mathrm{cn}}_{\mathcal{F}}(G) \oplus \mathbf{H}^{\mathrm{td}}_{\mathcal{F}}(G) \cong \mathbb{R}^{\mathfrak{r}} \oplus \left(\mathbb{Q}^{m\mathfrak{t}} \oplus (\mathbb{Q}/\mathbb{Z})^{m\mathfrak{f}} \oplus \left(\bigoplus_{i=1}^{n} \mathbb{Z}_{d_i}\right)^{\mathfrak{f}}\right).$$

In Theorem 6.28 we study the topological morphism

$$\Psi_G \colon \mathbf{H}_{\mathcal{F}}(G) \to \mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$$

under the assumptions from the preceding paragraph. First we show that with the usual identification $\mathbf{H}_{\mathcal{F}}(G) \cong \mathrm{im}(\Psi_G)$, the former group is a direct summand in $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$ in the algebraic sense and there is a direct sum

$$\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}}) \cong \mathbf{H}_{\mathcal{F}}(G) \oplus \mathrm{Ext}(\mathfrak{t}(G^*), \mathbb{Z}^m) \oplus \left(\bigoplus_{i=1}^{n} \mathrm{tor}_{d_i}(\mathfrak{t}(G))\right).$$

The splitting is, in general, not topological. In fact, we show that the following conditions are equivalent:

- $\mathbf{H}_{\mathcal{F}}(G)$ is a topological direct summand in $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$,
- $\mathbf{H}_{\mathcal{F}}(G)$ is a closed subgroup of $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$,
- $\mathbf{H}_{\mathcal{F}}(G)$ coincides with $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$,
- G^* is a free abelian group, that is, $\mathfrak{t}(G^*) = 0$,
- G is a torus.

We conclude by proving that the closure $\overline{\mathbf{H}_{\mathcal{F}}(G)}$ of $\mathbf{H}_{\mathcal{F}}(G)$ in $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$ is a topological direct summand in $\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}})$ and the corresponding topological direct sum is

$$\mathrm{Hom}(G^*, \mathbf{H}_{\mathcal{F}}) \cong \overline{\mathbf{H}_{\mathcal{F}}(G)} \oplus \left(\bigoplus_{i=1}^{n} \mathrm{tor}_{d_i}(\mathfrak{t}(G))\right).$$

Dense groups of minimal extensions. In the last part of our work we are interested in the topological size of the sets of minimal group extensions. Similarly to the preceding sections, we consider minimal flows $\mathcal{F} \colon \Gamma \curvearrowright X$ of two types:

(1) Γ is a simply connected Lie group and X is a compact second countable space with $\pi^1(X) \neq 0$,
(2) Γ is a connected Lie group, X is a compact manifold, the flow $\mathcal{F}$ is topologically free and has a free cycle.

In Theorem 6.30 we show that in both situations, the minimal extensions from $\mathbf{Z}_{\mathcal{F}}(G)$ form a dense G_δ subset of $\mathbf{Z}_{\mathcal{F}}(G)$ for every connected second countable group $G \in \mathsf{CAGp}$. (As usual, $\mathbf{Z}_{\mathcal{F}}(G)$ is assumed to carry the topology of u.c.s. convergence.)

Since $\mathbf{Z}_{\mathcal{F}}(G)$ is an (abelian) Polish topological group, Theorem 6.30 shows that minimal extensions form a dense subset of $\mathbf{Z}_{\mathcal{F}}(G)$. Now is it possible that $\mathbf{Z}_{\mathcal{F}}(G)$ has a dense subgroup contained in $\mathbf{MinZ}_{\mathcal{F}}(G)$? To see what kind of abelian groups can be dense subgroups of $\mathbf{Z}_{\mathcal{F}}(G)$, we need to know the topological-algebraic structure of $\mathbf{Z}_{\mathcal{F}}(G)$. In the two cases listed above,

(1) $\mathbf{Z}_{\mathcal{F}}(G)$ is the additive topological group of a real separable Fréchet space (this is proved in Corollary 3.41),

(2) if $n = \text{rank}(H_1^w(\mathcal{F}))$ and $\mathfrak{f} = \text{rank}(\mathfrak{f}(G^*))$ then $\mathbf{Z}_{\mathcal{F}}(G)$ splits into a topological direct sum

$$\mathbf{Z}_{\mathcal{F}}(G) = \text{Div}(\mathbf{Z}_{\mathcal{F}}(G)) \oplus \mathbb{Z}^{n\mathfrak{f}},$$

where the divisible subgroup $\text{Div}(\mathbf{Z}_{\mathcal{F}}(G))$ of $\mathbf{Z}_{\mathcal{F}}(G)$ is the additive topological group of a real separable Fréchet space and $\mathbb{Z}^{n\mathfrak{f}}$ carries the product topology (this is proved in Proposition 6.21).

Having these information about $\mathbf{Z}_{\mathcal{F}}(G)$, we can prove in Theorem 6.32 that for every torsion-free abelian group R with $\aleph_0 \leq \text{rank}(R) \leq \mathfrak{c}$, the following statements hold in both situations (1) and (2):

- $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of the group R, which forms a dense subset of $\text{Div}(\mathbf{Z}_{\mathcal{F}}(G))$,
- $\mathbf{MinZ}_{\mathcal{F}}(G)$ contains an isomorphic copy of the group $R \oplus \mathbb{Z}^{(\aleph_0)}$, which forms a dense subset of $\mathbf{Z}_{\mathcal{F}}(G)$.