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Symmetries and conservation laws of differential equations

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Part I

INTRODUCTION

1 Summary of the thesis

The beginnings of the theory of Lie groups and Lie algebras were inseparably linked to the group analysis of differential equations and, in particular, to problems of group classification of differential equations. Inspired by the idea of creating a universal theory of integration of ordinary differential equations similar to the Galois theory of solving algebraic equations, S. Lie developed the theory of continuous transformation groups, classified locally non-singular transformation groups acting on the complex and real plane, described their differential invariants and then performed the group classification of second-order ordinary differential equations. S. Lie also solved the problems of group classification for two-dimensional linear partial differential equations and for nonlinear Klein–Gordon equations. Therefore, the main objects of study within the framework of group analysis of differential equations in Lie's time were continuous (both point and contact) symmetry or equivalence transformations of differential equations as well as algebraic and geometric structures related to such transformations that are called, in the modern terminology, local transformation groups and Lie algebras of (local) vector fields. In the seminal paper [41], Noether was the first to consider generalized symmetries of differential equations and to relate variational symmetries of a Lagrangian to local conservation laws of the associated system of Euler–Lagrange equations, thus extending the scope of group analysis of differential equations to these kinds of mathematical objects. Later, other kinds of symmetries of differential equations arose in the literature, including approximate, conditional, nonclassical and nonlocal symmetries, some of which are at most indirectly related to Lie groups and Lie algebras. For this reason this branch of mathematics is now often called symmetry analysis instead of group analysis. In addition to symmetries, many other objects encoding geometric properties of differential equations, like local and potential conservation laws, coverings, recursion operators, Lagrangian and Hamiltonian structures, are studied within the framework of symmetry analysis of differential equations.

The subject of this habilitation thesis fits into the scope of symmetry analysis of differential equations. The main part of the thesis consists of the following five papers:

[T1] Kunzinger M. and Popovych R.O., Singular reduction operators in two dimensions, *J. Phys. A: Math. Theor.* **41** (2008), 505201, 24 pp., arXiv:0808.3577.

[T2] Popovych R.O., Reduction operators of linear second-order parabolic equations, *J. Phys. A: Math. Theor.* **41** (2008), 185202, 31 pp., arXiv:0712.2764.

[T3] Kunzinger M. and Popovych R.O., Generalized conditional symmetries of evolution equations, *J. Math. Anal. Appl.* **379** (2011), 444–460, arXiv:1011.0277.

[T4] Kunzinger M. and Popovych R.O., Potential conservation laws, *J. Math. Phys.* **49** (2008), 103506, 34 pp., arXiv:0803.1156.

[T5] Popovych R.O. and Bihlo A., Symmetry preserving parameterization schemes, *J. Math. Phys.* **53** (2010), 073102, 36 pp., arXiv:1010.3010.

As one could infer from the titles of these papers, they are devoted to

- the study of nonclassical (or conditional) symmetries (including generalized ones) and nonclassical reductions of differential equations, especially, the analysis of no-go cases in finding such symmetries and reductions [T1, T2, T3];
- the development the general theory of potential conservation laws and finding criteria for determining whether a potential conservation law is nontrivial [T4];
- the application of methods of group classification of differential equation to construction of invariant parameterization schemes [T5].

The presentation in the selected papers is essentially self-contained. Each of them includes a comprehensive review of the related literature as well as necessary definitions and assertions. Nevertheless, below we review the results of these papers in the context of modern development of the respective subjects.

2 Reduction modules of differential equations

The "nonclassical" approach to finding solutions of differential equations in closed form was proposed in [16] using the particular example of the (1+1)-dimensional heat equation in order to extend the range of applicability of symmetry methods. Since the end of the 1980s this method has been applied to many particular differential equations modeling real-world phenomena, see, e.g., examples in [7, 24, 25, 58] and reviews in [29, 47]. Related objects, which are similar to subalgebras of Lie symmetry algebras, are named in the literature in different ways: nonclassical [36], Q-conditional [29], conditional [32], or partial [66] symmetries for short, or involutive families/modules of nonclassical/conditional symmetry operators [47, 71] in a more complete form. The main feature which is inherited by nonclassical symmetries from Lie symmetries is that they allow to construct ansatzes for the unknown function which reduce the differential equation under study to differential equations with a smaller number of independent variables [5, 44, 59, 66, 71]. This feature relates nonclassical symmetries to the direct method by Clarkson and Kruskal [23] and the general ansatz method [29]. In fact, however, the properties of nonclassical symmetries are more closely related to the theories of differential constraints and formal compatibility of systems of differential equations [34, 44, 59]. As there is plenty of arguments in favor of this point of view, we mostly use the term "reduction modules" (of vector fields) instead of "involutive families of conditional symmetry operators" and say that an involutive module of vector fields reduces a differential equation if the equation is reduced by an associated ansatz.

In this section, based on [T1, 31, 32, 34, 47, 58, 71] and, especially, [20], we present the revised and enhanced framework of nonclassical (conditional) symmetries of differential equations.

Given a foliated space of n independent variables $x = (x_1, \dots, x_n)$ and a single dependent variable u, consider a finite-dimensional involutive module Q of vector fields on this space, and suppose that the module dimension p of Q (over the ring of smooth functions of (x, u)) is not greater than n, $0 < p \leqslant n$. We additionally assume that the module Q satisfies the *rank condition*, i.e., for each fixed value of (x, u) the projection of Q to the space of x is p-dimensional. The attribute '*involutive*' means that the commutator of any two vector fields from Q belongs to Q. It is obvious that any one-dimensional module is involutive. Therefore, in the case $p = 1$ we can omit the attribute 'involutive' and talk only about modules.

In what follows the indices i and j run from 1 to n, the index s runs from 1 to p, the index σ runs from 1 to $n - p$, and we use the standard summation convention for repeated indices. Angular brackets $\langle \dots \rangle$ are used for denoting linear spans over the ring of smooth functions of (x, u). Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_i = \partial/\partial x_i$ and $\partial_u = \partial/\partial u$. Any function is considered as the zero-order derivative of itself. All considerations are local. The notion of functional independence will be understood below in the sense of total functional independence, which means that the functions are in fact functionally independent on each open subset of their common domain.

Suppose that the vector fields $Q_s = \xi^{si}(x, u)\partial_i + \eta^s(x, u)\partial_u$ form a basis of Q, i.e., $Q = \langle Q_1, \dots, Q_p \rangle$. Then the rank condition is equivalent to the equality $\operatorname{rank}(\xi^{si}) = p$ for each (x, u). The condition that the commutator of any pair of basis elements belongs to Q, $[Q_s, Q_{s'}] \in Q$, suffices for the module Q to be involutive. If the vector fields $\tilde{Q}_1, \dots, \tilde{Q}_p$ form another basis of Q, then there exists a nondegenerate $p \times p$ matrix-function $(\lambda^{ss'}(x, u))$ such that $\tilde{Q}_s = \lambda^{ss'} Q_{s'}$.

The first-order differential function $Q_s[u] := \eta^s(x, u) - \xi^{si}(x, u)u_i$ is called the *characteristic* of the vector field Q_s. In view of the Frobenius theorem, involutivity of Q is equivalent to

the fact that the characteristic system $\mathcal{Q}$ of PDEs $Q_s[u] = 0$, also called the *invariant surface condition*, has $n + 1 - p$ functionally independent integrals $I^0(x, u)$, ..., $I^{n-p}(x, u)$. Therefore, the general solution of this system can implicitly be represented in the form $F(I^0, \dots, I^{n-p}) = 0$, where F is an arbitrary smooth function of its arguments.

A differential function $G = G[v]$ of the dependent variables $v = (v^1, \dots, v^m)$ which in turn are functions of a tuple of independent variables $y = (y_1, \dots, y_l)$ will be viewed as a smooth function of y and a finite number of derivatives of v with respect to y. More rigorously, a differential function G is defined as a smooth function on a domain of the jet space $\mathrm{J}^r = \mathrm{J}^r(y|v)$ of some order r with independent variables y and dependent variables v [43]. The order $r = \operatorname{ord} G$ of the differential function G is defined to be equal to the maximal order of derivatives (resp. jet variables) involved in G, and $\operatorname{ord} G = -\infty$ if G depends only on y. Each set of differential functions of a fixed positive order as well as the set of differential functions of nonpositive order are invariant with respect to point transformations of (y, v).

Using another basis of Q gives just another representation of the characteristic system $\mathcal{Q}$ with the same set of solutions. This is why the characteristic system $\mathcal{Q}$ is associated with the module Q rather than with a fixed basis of Q. And vice versa, any family of $n+1-p$ functionally independent functions of x and u is a complete set of integrals of the characteristic system of an involutive p-dimensional module. Therefore, there exists a one-to-one correspondence between the set of involutive p-dimensional modules and the set of families of $n+1-p$ functionally independent functions of x and u, which is factorized with respect to the corresponding equivalence. (We consider two families of the same number of functionally independent functions of the same arguments as equivalent if any function from one of the families is functionally dependent on functions from the other family.)

A function $u = f(x)$ is called *invariant with respect to the involutive module* Q (or, briefly, Q-*invariant*) if it is a solution of the characteristic system $\mathcal{Q}$. This notion is justified by the following facts. In view of the rank condition, we can choose a basis of Q that spans, over the ground field, a p-dimensional (Abelian) Lie algebra $\mathfrak{g}$ of vector fields in the space (x, u).[1] The graph of each solution of the characteristic system $\mathcal{Q}$ is obviously invariant with respect to the p-parameter local transformation group generated by the algebra $\mathfrak{g}$.

We choose a basis of Q that consists of commuting vector fields Q_1, ..., Q_p and, for each fixed s, consider a solution $J^s = J^s(x, u)$ of the system $Q_{s'} J^s = \delta_{ss'}$, where $\delta_{ss'}$ is the Kronecker delta. Since the functions I^0, ..., I^{n-p}, J^1, ..., J^p of (x, u) are functionally independent, one can make the change of variables

$$\varphi = I^0(x, u), \quad \omega_\sigma = I^\sigma(x, u), \quad \omega'_s = J^s(x, u),$$

where $\omega = (\omega_1, \dots, \omega_{n-p})$ and $\omega' = (\omega'_1, \dots, \omega'_p)$ are considered as the new independent variables and φ is the new dependent variable. The variables ω and φ are called Q-invariant, and the variables ω' are called parametric for the module Q. In the new variables, the basis elements Q_s take the form $Q_s = \partial_{\omega'_s}$.

Next, consider an rth order differential equation $\mathcal{L}$ of the form $L(x, u_{(r)}) = 0$ for a single unknown function u of the independent variables $x = (x_1, \dots, x_n)$. Here, $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to x of order not greater than r, including u as the derivative of order zero. We assume that the order r of the equation $\mathcal{L}$ is essential, i.e., it is minimal among the orders of equations equivalent to $\mathcal{L}$ up to nonvanishing multipliers that are differential functions of u. In the local approach the equation $\mathcal{L}$ can be viewed as an

[1] Such a basis is constructed in the following way: We take an arbitrary basis of Q consisting of vector fields $Q_s = \xi^{si}(x, u)\partial_i + \eta^s(x, u)\partial_u$. Up to permutation of the independent variables and basis elements of Q, we can suppose in view of the rank condition that $\operatorname{rank}(\xi^{ss'}) = p$ and change the basis to $(\hat{Q}_s = \partial_s + \hat{\xi}^{s\iota}\partial_\iota + \hat{\eta}^s\partial_u)$, where the index ι runs from $p+1$ to n and the matrices $(\hat{\xi}^{s\iota})$ and $(\hat{\eta}^s)$ are the products of the matrix $(\xi^{ss'})^{-1}$ by the matrices $(\xi^{s\iota})$ and (η^s), respectively. Since the module Q is involutive, the vector fields $\hat{Q}_1$, ..., $\hat{Q}_p$ commute.

algebraic equation on the rth order jet space $\mathrm{J}^r = \mathrm{J}^r(x|u)$ and is identified with the manifold of its solutions in J^r,

$$\mathcal{L} = \{(x, u_{(r)}) \in \mathrm{J}^r \mid L(x, u_{(r)}) = 0\}.$$

We use the same symbol $\mathcal{L}$ for this manifold and also write $\mathcal{Q}_{(r)}$ both for the system consisting of the independent differential consequences of the characteristic system $\mathcal{Q}$ up to equation order r and for the manifold defined by the system $\mathcal{Q}_{(r)}$ in J^r, i.e.,

$$\mathcal{Q}_{(r)} = \{(x, u_{(r)}) \in \mathrm{J}^r \mid \mathrm{D}^\alpha Q_s[u] = 0,\ |\alpha| < r\},$$

where $\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \cdots \mathrm{D}_n^{\alpha_n}$, $\mathrm{D}_i = \partial_{x_i} + u_{\alpha+\delta_i}\partial_{u_\alpha}$ is the operator of total differentiation with respect to the variable x_i, $\alpha = (\alpha_1, \dots, \alpha_n)$ is an arbitrary multi-index, $\alpha_i \in \mathbb{N}\cup\{0\}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and δ_i is the multi-index whose ith entry equals 1 and whose other entries are zero. The variable u_α of the jet space J^r corresponds to the derivative $\partial^{|\alpha|}u/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, and $u_i \equiv u_{\delta_i}$, $u_{ij} \equiv u_{\delta_i+\delta_j}$, etc.

Without loss of generality, we can assume $F_{I^0} \neq 0$ in the representation $F(I^0, \dots, I^{n-p}) = 0$ of the general solution of the characteristic system $\mathcal{Q}$ and, considering this representation as an algebraic equation for I^0, …, I^{n-p}, resolve it with respect to I^0: $I^0 = \varphi(I^1, \dots, I^{n-p})$. In view of the rank condition, this gives the representation (in general, also implicit)

$$\mathcal{A}\colon \quad I^0(x, u) = \varphi(\omega), \quad \omega_\sigma = I^\sigma(x, u), \tag{1}$$

for solutions of the characteristic system $\mathcal{Q}$, where $\varphi = \varphi(\omega)$ is an arbitrary smooth function of $\omega = (\omega_1, \dots, \omega_{n-p})$. The latter representation is called an *ansatz* for u constructed with the module Q.

Making the ansatz $\mathcal{A}$ for u we can express all derivatives of u in terms of ω, ω' and derivatives of φ, then substitute these expressions to the differential function $L[u]$ and replace the remaining x's by their expressions in the new variables. Alternatively, we can change the variables to $(\omega, \omega', \varphi)$ from the outset and then take into account the constraints $\varphi_{\omega'_s} = 0$. The function obtained by the above procedure is denoted by $L|_{\mathcal{A}}$. It depends at most on ω, ω' and $\varphi_{(r)}$, where $\varphi_{(r)}$ denotes the tuple of derivatives of φ with respect to ω up to order r.

Definition 1. The ansatz $\mathcal{A}$ constructed using the module Q *reduces* the equation $\mathcal{L}$ if there exist smooth functions $\check\lambda = \check\lambda(\omega, \omega', \varphi_{(r)})$ and $\check L = \check L(\omega, \varphi_{(r)})$ such that the function $\check\lambda$ does not vanish and

$$L|_{\mathcal{A}} = \check\lambda(\omega, \omega', \varphi_{(r)})\check L(\omega, \varphi_{(r)}).$$

Then the module Q is called a *reduction module* of $\mathcal{L}$, and the equation $\check L(\omega, \varphi_{(r)}) = 0$ is a *reduced equation* associated with the ansatz $\mathcal{A}$.

The reduction procedure should additionally be specified in the case of reduction to algebraic equations when $p < n$, see the proof of Theorem 34 in [20].

The set of p-dimensional reduction modules of the equation $\mathcal{L}$ will be denoted by $\mathcal{R}^p(\mathcal{L})$.

Basis elements of one-dimensional reduction modules are called *reduction operators* [64].

Consider the following conditions on the (rth order) differential equation $\mathcal{L}$ and the involutive module Q satisfying the rank condition:

(C1) Q is a reduction module of the equation $\mathcal{L}$;

(C2) $V_{(r)}L[u] \in \langle L[u], \mathrm{D}^\alpha Q_s[u], |\alpha| < r\rangle$ for any $V \in Q$;

(C3) $V_{(r)}L[u]\big|_{\mathcal{L}\cap\mathcal{Q}_{(r)}} = 0$ for any $V \in Q$.

Here $V_{(r)}$ denotes the standard rth prolongation of a vector field $V = \xi^i(x,u)\partial_i + \eta(x,u)\partial_u$ [43, 49]: $V_{(r)} = V + \sum_{0<|\alpha|\leqslant r}\eta^\alpha\partial_{u_\alpha}$, where $\eta^\alpha = \mathrm{D}^\alpha V[u] + \xi^i u_{\alpha+\delta_i}$ and $V[u] = \eta - \xi^i u_i$.

In the conditions (C2) and (C3) it suffices to require that V runs through a basis $(Q_1,\dots,Q_p)$ of Q. Which basis is chosen for representing the characteristic system $\mathcal{Q}$ and checking the conditions (C2) and (C3) is not essential; cf. [32, 71].

All these conditions are preserved by point transformations of the variables (x,u).

Theorem 2. *The conditions* (C1) *and* (C2) *are equivalent and imply* (C3). *If the tuple of differential functions* $(L[u],\mathrm{D}^\alpha Q_s[u],|\alpha|<r)$ *is of maximal rank on* $\mathcal{L}\cap\mathcal{Q}_{(r)}$, *then the condition* (C3) *implies* (C2) *(and thereby also* (C1)*).*

Besides the case of maximal rank, there are other, more specific, cases when the conditions (C1)–(C3) are simultaneously satisfied, e.g., if $\mathcal{L}\cap\mathcal{Q}_{(r)}=\mathcal{Q}_{(r)}$.

A proof of Theorem 2 relies on the following assertion.

Lemma 3. *Let smooth functions* f, Λ^1, …, Λ^p, *and an involutive module* $Q=\langle Q_1,\dots,Q_p\rangle$ *of vector fields that are defined on a neighborhood* O_{z^0} *of a point* $z^0\in\mathbb{R}^l$ *for some* $l\in\mathbb{N}$ *satisfy the conditions* $Q_sf(z)=\Lambda^s(z)f(z)$ *for any* $z\in O_{z^0}$, $s=1,\dots,p$, *and* $\dim Q|_{z^0}=p$. *Then there exist a neighborhood* $\breve{O}_{z^0}\subset O_{z^0}$ *of* z^0 *and smooth functions* $\breve{f}$ *and* λ *defined on* $\breve{O}_{z^0}$ *such that* $\lambda(z)\neq0$, $Q^s\breve{f}(z)=0$ *and* $f(z)=\lambda(z)\breve{f}(z)$ *for any* $z\in\breve{O}_{z^0}$.

In earlier papers on reduction modules, a different terminology was used (see, e.g., [16, 29, 31, 45, 71]). Usually the condition (C3) was considered as the main one and was called the *conditional invariance criterion*. Then the differential equation $\mathcal{L}$ is called *conditionally invariant* with respect to the involutive module Q, whereas the module Q is called an *involutive module of conditional symmetry* (or Q-conditional symmetry, or nonclassical symmetry, etc.) operators of the equation $\mathcal{L}$. A version of the condition (C2) for systems of differential equations for several unknown functions appeared in [33]. In contrast to the case of a single differential equation for a single unknown function, the version of the condition (C2) for systems is sufficient but not necessary for an ansatz constructed with the module Q to reduce the system under consideration, cf. [34, Section 5]. An alternative approach to conditional invariance is to require that the joint system of $\mathcal{L}$ and $\mathcal{Q}_{(r)}$ is formally compatible in the sense of the absence of nontrivial differential consequences [44, 47]. But a subtle point here is which representation of the joint system should in fact be considered to decide formal compatibility, cf. [34, footnote 1] and [20, Section 6]. If the conditional invariance criterion is not satisfied but nevertheless the equation $\mathcal{L}$ has Q-invariant solutions, then one talks about weak invariance of the equation $\mathcal{L}$ with respect to the module Q [45, 47, 59].

There are reduction modules related to classical Lie symmetries. Let $\mathfrak{g}$ be a p-dimensional Lie invariance algebra of the equation $\mathcal{L}$, whose basis vector fields Q_1, …, Q_p satisfy the condition $\operatorname{rank}(\xi^{si})=\operatorname{rank}(\xi^{si},\eta^s)=p'$, where $p'\leqslant p$. Then the span of Q_1, …, Q_p over the ring of smooth functions of (x,u) is a p'-dimensional involutive module which belongs to $\mathcal{R}^{p'}(\mathcal{L})$. Modules of this kind are called *Lie reduction modules*. Other reduction modules are called *non-Lie*.

The following assertion is important for the study of reduction modules (cf. [71]).

Lemma 4. *Given an* r*th order differential equation* $\mathcal{L}$: $L[u]=0$, *a* p*-dimensional* $(0<p\leqslant n)$ *involutive module* Q *satisfying the rank condition and differential functions* $\tilde{L}[u]$ *and* $\lambda[u]\neq0$ *of an order not greater than* r *such that* $(L-\lambda\tilde{L})|_{\mathcal{Q}_{(r)}}=0$, *the module* Q *is a reduction module of* $\mathcal{L}$ *if and only if it is a reduction module of the equation* $\tilde{\mathcal{L}}$: $\tilde{L}[u]=0$. *An ansatz constructed using the module* Q *reduces* $\mathcal{L}$ *and* $\tilde{\mathcal{L}}$ *to equations that may differ at most by a nonvanishing multiplier.*

The classification of reduction modules can be notably enhanced and simplified by involving Lie symmetry and equivalence transformations of (classes of) differential equations. By $\mathfrak{M}^p$ we denote the set of p-dimensional modules of vector fields in the space of (x,u). Any point transformation of (x,u) induces a one-to-one mapping of $\mathfrak{M}^p$ into itself via push-forward of

vector fields. Namely, the transformation g: $\tilde{x}_i = X^i(x,u)$, $\tilde{u} = U(x,u)$ generates the mapping $g_*\colon \mathfrak{M}^p \to \mathfrak{M}^p$ such that for any $Q \in \mathfrak{M}^p$ and $V \in Q$ the vector field $V = \xi^i(x,u)\partial_i + \eta(x,u)\partial_u$ is mapped to the vector field $g_*V = \tilde{\xi}^i\partial_{\tilde{x}_i} + \tilde{\eta}\partial_{\tilde{u}}$, where $\tilde{\xi}^i(\tilde{x},\tilde{u}) = VX^i(x,u)$, $\tilde{\eta}(\tilde{x},\tilde{u}) = VU(x,u)$.

Given a group G of point transformations in the space of (x,u), the modules Q and $\tilde{Q}$ (of the same dimension) are called *equivalent* with respect to G if there exists some $g \in G$ such that $\tilde{Q} = g_*Q$.

Lemma 5. *Suppose that a point transformation g maps a differential equation $\mathcal{L}$ to a differential equation $\tilde{\mathcal{L}}$, $Q \in \mathcal{R}^p(\mathcal{L})$, and the image g_*Q satisfies the rank condition. Then $g_*Q \in \mathcal{R}^p(\tilde{\mathcal{L}})$.*

Corollary 6. *Let G be the point symmetry group of a differential equation $\mathcal{L}$. Then the equivalence of p-dimensional modules of vector fields with respect to the group G generates an equivalence relation in $\mathcal{R}^p(\mathcal{L})$.*

Next, we consider a class $\mathcal{L}|_{\mathcal{S}}$ of differential equations $\mathcal{L}_\theta$: $L(x,u_{(r)},\theta_{(q)}) = 0$. Here L is a fixed function of x, $u_{(r)}$ and $\theta_{(q)}$, where by θ we denote the tuple of arbitrary (parametric) differential functions $\theta(x,u_{(r)}) = (\theta^1(x,u_{(r)}),\dots,\theta^k(x,u_{(r)}))$, traversing the set $\mathcal{S}$ of solutions of an auxiliary system, and $\theta_{(q)}$ stands for the set of all the derivatives of θ of order not greater than q with respect to x and $u_{(r)}$. The auxiliary system consists of differential equations $S(x,u_{(r)},\theta_{(q')}(x,u_{(r)})) = 0$ and differential inequalities $\Sigma(x,u_{(r)},\theta_{(q')}(x,u_{(r)})) \neq 0$ (> 0, < 0, $\dots$) on θ, where both x and $u_{(r)}$ play the role of independent variables. Henceforth we call the functions θ *arbitrary elements*. We write $G^\sim$ and $\mathcal{G}^\sim$ for the equivalence group and the equivalence groupoid of the class $\mathcal{L}|_{\mathcal{S}}$, respectively. Roughly speaking, the group $G^\sim$ consists of the transformations of $(x,u_{(r)},\theta)$ that preserve the form of the equations from $\mathcal{L}|_{\mathcal{S}}$ and are point transformations with respect to (x,u) when θ is fixed. In fact, there are various kinds of equivalence groups [56, Section 2.3]. The groupoid $\mathcal{G}^\sim$ is the set $\{(\theta,\tilde{\theta},g) \mid \theta,\tilde{\theta} \in \mathcal{S},\ g \in \mathrm{T}(\theta,\tilde{\theta})\}$ naturally equipped with the groupoid structure via the composition of transformations. Here $\mathrm{T}(\theta,\tilde{\theta})$ denotes the set of point transformations of (x,u) that map the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\tilde{\theta}}$. See [10, 56] for rigorous definitions of notions related to classes of differential equations.

By P we denote the set of all pairs of the form $(\mathcal{L}_\theta, Q)$, where $\mathcal{L}_\theta$ is an equation from $\mathcal{L}|_{\mathcal{S}}$ and a module Q from $\mathcal{R}^p(\mathcal{L}_\theta)$. It follows from Lemma 5 that the action of transformations from the equivalence group $G^\sim$ or from the equivalence groupoid $\mathcal{G}^\sim$ on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathcal{R}^p(\mathcal{L}_\theta) \mid \theta \in \mathcal{S}\}$ induces an equivalence relation on P [58].

Definition 7. Let $\theta,\theta' \in \mathcal{S}$, $Q \in \mathcal{R}^p(\mathcal{L}_\theta)$, $Q' \in \mathcal{R}^p(\mathcal{L}_{\theta'})$. The pairs $(\mathcal{L}_\theta, Q)$ and $(\mathcal{L}_{\theta'}, Q')$ are called *$G^\sim$-equivalent* if there exists a transformation $\mathcal{T} \in G^\sim$ mapping the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\theta'}$, and $Q' = (\mathcal{T}^\theta)_*Q$. Here $\mathcal{T}^\theta$ is the point transformation of (x,u) obtained from $\mathcal{T}$ by fixing θ. The pairs $(\mathcal{L}_\theta, Q)$ and $(\mathcal{L}_{\theta'}, Q')$ are called *$\mathcal{G}^\sim$-equivalent* (or, simply, *pointwise equivalent*) if there exists a transformation $g \in \mathrm{T}(\theta,\tilde{\theta})$ such that $Q' = g_*Q$.

We will interpret the classification of p-dimensional reduction modules of equations in the class $\mathcal{L}|_{\mathcal{S}}$ with respect to $G^\sim$ (or $\mathcal{G}^\sim$) as the classification in P up to the corresponding equivalence relation. This problem can be investigated similarly to the usual group classification in classes of differential equations. Namely, at first we construct the modules that belong to $\mathcal{R}^p(\mathcal{L}_\theta)$ for any $\theta \in \mathcal{S}$. Then we classify, with respect to $G^\sim$ (or $\mathcal{G}^\sim$), the values of θ for which the equation $\mathcal{L}_\theta$ admits additional reduction modules.

3 No-go results on nonclassical reductions of differential equations

Involving the associated invariant surface condition in the conditional invariance criterion gives rise to a few significant complications of nonclassical symmetries in comparison with Lie symmetries. Given a differential equation $\mathcal{L}$, elements of its different reduction modules do not form

objects of a nice algebraic or geometrical structure. Hence it is not possible to compose single reduction operators in reduction modules as this is done for the maximal Lie invariance algebra of $\mathcal{L}$ and its subalgebras, which consist of vector fields generating one-parameter Lie symmetry (pseudo)groups of $\mathcal{L}$. Whereas the system of determining equations for Lie symmetries is linear, similar systems for reduction modules are nonlinear and should additionally be supplemented, in the course of considering modules of dimension greater than one, by the condition of involutivity, i.e., the closure of modules with respect to commutation of vector fields. Moreover, there is no single system of determining equations even for reduction modules of a fixed dimension. Instead, the entire set of such modules is partitioned into subsets associated with systems of determining equations which are rather different from each other. Solving some of these systems may be equivalent to solving the initial equation, which gives no-go cases of looking for reduction modules. Such no-go cases were known for a number of particular (1+1)-dimensional evolution equations including the linear heat equation [28, 29, 30, 39, 68], the Burgers equation [6, 39], linear second-order evolution equations [53, T2] as well as for the entire classes of (1+1)-dimensional evolution equations [70], multi-dimensional evolution equations [52] and even systems of such equations [64]. Note that in the course of the study of Lie symmetries a similar no-go situation arises for first-order ordinary differential equations [35, Theorem 10, p. 130], see also [43, Section 2.5]. In fact, all the above no-go cases of reduction operators are occurrences of a no-go case common to evolution equations and one more no-go case specific to linear second-order evolution equations. The causes giving rise to the partition of the module set and to no-go cases for nonclassical symmetries have not been investigated in the literature until recently. It was not understood in which way the results on no-go cases can be extended to reduction modules of other, non-evolution, equations.

In [T1] the partition of the set of reduction modules of a differential equation was related to lowering the order of this equation on the manifolds determined by the associated invariant surface conditions in the appropriate jet space. As a result, studying singular modules of vector fields which lower the order of the equation was included as the initial step in the procedure of finding nonclassical symmetries. In order to illustrate the main ideas of the proposed framework, we considered only the case of single partial differential equations in one dependent and two independent variables and single reduction operators. The notion of singular reduction operators was introduced. The weak singularity co-order of a reduction operator Q was shown to be equal to the essential order of the corresponding reduced equation and to the number of essential parameters in the family of Q-invariant solutions. No-go assertions on singular reduction operators of (1+1)-dimensional evolution and wave equations were derived and then generalized to parameterized families of vector fields which reduce partial differential equations in two independent variables to first-order ordinary differential equations.

In [20] we extended results of [T1] to the case of a greater number of independent variables. After revising and enhancing the framework of nonclassical symmetries, we introduced the concepts of singular and meta-singular modules of vector fields for differential functions. Any meta-singular module of dimension greater than two turns out to be necessarily involutive, in contrast to two-dimensional meta-singular modules. We described, up to point transformations, differential functions possessing meta-singular modules. The analogous notions of weakly singular and meta-singular modules for differential equations were introduced. The characterization of differential equations admitting weakly meta-singular modules that we obtained implies that instead of such modules it suffices to study meta-singular modules of the corresponding differential functions. A connection between the weak singularity co-order of reduction modules, the essential order of the corresponding reduced equations and, in the case of reduction to ordinary differential equations, the number of parameters in the corresponding families of invariant solutions was established. It was shown that the relation between the reducibility of a differential equation $\mathcal{L}$ by an involutive module Q and the formal compatibility of the joint system of $\mathcal{L}$ and the characteristic system associated to Q essentially involves the weak singularity co-order

of Q for $\mathcal{L}$. Revisiting results of [33] within the framework of singular reduction modules, we considered the specific case of reduction modules of dimension equal to the number of independent variables, which results in the reduction to algebraic equations. We reformulated and extended no-go results from [52] on modules reducing evolution equations to ordinary differential equations with time as the single independent variable. This motivated further consideration of reduction modules of singularity co-order one. Supposing that a differential equation $\mathcal{L}$ admits an n-dimensional meta-singular module M of singularity co-order one, where n is the number of independent variables in $\mathcal{L}$, we proved no-go assertions establishing a connection between $(n-1)$-dimensional reduction modules of $\mathcal{L}$ contained in M and solutions of $\mathcal{L}$. In particular, it was shown that the system of determining equations for such modules is reduced to the initial equation $\mathcal{L}$ by a composition of a differential substitution and a hodograph transformation. We also studied singular modules for quasi-linear second-order PDEs, where the dimension of modules was assumed to be less than the number of independent variables. It turned out that elliptic equations possess no singular modules. Any second-order evolution equation whose matrix of coefficients of second-order derivatives is non-degenerate possesses only singular modules of the kind that is common for general evolution equations. Generalized wave equations are much more complicated from this point of view. In particular, they may admit families of singular modules which have no interpretation in terms of meta-singular modules, which makes a further development of the framework of singular modules desirable.

Reduction modules of codimension one and singularity co-order one do not exhaust possible no-go cases for finding reduction modules. Other kinds of no-go cases for finding reduction modules were obtained for linear second-order evolution equations [30, 53, T2] and the Burgers equation [6, 39]. In particular, the system of determining equations for regular reduction operators of any (1+1)-dimensional linear second-order evolution equation $\mathcal{L}$ is reduced by a nonlocal transformation to a system of three copies of $\mathcal{L}$ [T2]. Therefore, the regular reduction operators of $\mathcal{L}$ constitute a no-go case different from the no-go case of singular reduction operators, which is common for all (1+1)-dimensional evolution equations. A similar phenomenon occurs for the Burgers equation $u_t+uu_x-\mu u_{xx}=0$, where a no-go case arises for regular reduction operators of the form $\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u$ with $\xi_u=1/2$ [6, 39, 51]. We believe that these no-go cases arise through the coupling of several properties of related equations such as the evolutionary form, the second order, and the linearity or linearizability. For a class of differential equations, the study of reduction modules may lead to no-go cases due to the appearance of arbitrary elements parameterizing equations of the class in the corresponding determining equations. We refer to [50] for the calculation of one-dimensional regular reduction modules spanned by vector fields of the form $\partial_t+\xi(t,x)\partial_x+\eta(t,x,u)\partial_u$ with $\xi_{xx}\neq 0$ for the class of generalized Burgers equations $u_t+uu_x+f(t,x)u_{xx}=0$.

The chain of "no-go" assertions on reduction operators of a linear (1+1)-dimensional evolution equation $\mathcal{L}$, which was presented in [T2], is not exhausted by the theorems on reduction of both the systems of determining equations for regular and singular reduction operators to the initial equation $\mathcal{L}$. We have also shown that application of conventional methods to solving of the determining equations for coefficients of such operators cannot lead to reduction operators giving new exact solutions of $\mathcal{L}$. In both the regular and singular cases, the determining equations form well-determined systems solving which is in fact equivalent to solving the equation $\mathcal{L}$. All transformation and symmetry properties of the systems of determining equations are induced by the corresponding properties of the initial equation $\mathcal{L}$. Reduction operators constructed via Lie reductions of the systems of determining equations are also connected to Lie invariance properties of $\mathcal{L}$. Nevertheless, it was demonstrated that involvement of ingenious empiric approaches different from the Lie one can give reduction operators which are useful for the construction of non-Lie exact solutions of linear $(1+1)$-dimensional evolution equations.

In [T3], the above no-go results were extended to generalized conditional symmetries of (1+1)-dimensional evolution equations. We discussed prerequisites for introducing the notion of

generalized conditional symmetries in symmetry analysis of differential equations and presented different versions of the corresponding invariance criterion for scalar evolution equations. The relationship of generalized conditional symmetries of evolution equations to the formal compatibility and passivity of systems of differential equations as well as to systems of vector fields in involution was analyzed. For this purpose we employed a weight of derivatives instead of the usual order and a ranking of derivatives, which are associated with evolution equations of a fixed order. We studied reductions of evolution equations with special ansatzes, where new unknown functions depend only on the variable t. The Zhdanov theorem [72, 73] (see also [8]) on the connection of generalized conditional symmetries of an evolution equation to ansatzes of the above special form reducing this equation was also revisited. This led to new results on the correspondence between generalized conditional symmetries, ansatzes and parametric families of solutions of evolution equations. It was also shown that up to certain equivalences there exist one-to-one correspondences between objects of these kinds. We proved a no-go theorem on determining equations for generalized conditional symmetries of evolution equations. Roughly speaking, it was shown that solving the determining equation for generalized conditional symmetries of an evolution equation $\mathcal{L}$ is equivalent to solving the original equation $\mathcal{L}$. More precisely, there is a nonlocal transformation between systems respectively including the equation $\mathcal{L}$ and the corresponding determining equation. An interpretation of usual conditional symmetries of evolution equations as specific generalized conditional symmetries was given and was then illustrated by a new nontrivial example.

4 Potential conservation laws

Conservation laws play a distinguished role in mathematical physics. They have many applications in several areas related to differential equations, including integrable systems, asymptotic integrability, and the construction of geometric numerical integration schemes.

There is a vast body of literature devoted to the study of local conservation laws. Here one is given a system of differential equations and aims to find its space of conservation laws, or at least a subspace of this space singled out by additional constraints, such as a prescribed upper bound for the order of conservation laws to be considered. Standard tools for the solution of the direct problem on conservation laws include Noether's theorem, different variations of the direct method and techniques based on co-symmetries, see [2, 3, 15, 19, 43, 57, 65, 69] and references therein. For a class of (systems of) differential equations, one should tackle the direct problem on conservation laws as a classification problem since then the space of conservation laws in general depends on the arbitrary elements parameterizing systems of the class.

The notion of potential conservation laws arises as a natural generalization of the notion of local conservation laws of differential equations. Given a system $\mathcal{L}$ of differential equations, we call a *potential conservation law* of $\mathcal{L}$ any local conservation law of a potential system associated with $\mathcal{L}$; the latter is constructed from $\mathcal{L}$ via introducing potentials for local conservation laws of $\mathcal{L}$; cf. [54]. This term first appeared in [17]. Potential conservation laws can be trivial in the sense that they are induced by local conservation laws of the initial system [T4, 54]. The idea of iterative introduction of potentials by using local conservation laws of a potential system obtained on the previous step was first suggested in the famous paper [67] and later formalized in the form of the notion of *universal Abelian covering* of differential equations [19, 40, 61]. (Quasi)potentials can also be introduced using general coverings. Gauged potential systems and general foliated systems can be investigated in the same framework [T4]. Although potential conservation laws of differential equations are interesting and important objects for study within the framework of symmetry analysis, nontrivial and complete results on such conservation laws were obtained only for a few classes of differential equations. See related reviews and references in [T4, 15, 54].

In [T4] we proved criteria of nontriviality of potential conservation laws. In particular, usual potential conservation laws have characteristics depending only on local variables if and only if they are induced by local conservation laws. Therefore, characteristics of pure potential conservation laws have to essentially depend on potential variables. Moreover, we presented extensions of the above results to gauged potential systems, Abelian and general coverings and general foliated systems of differential equations. An example illustrating possible applications of these results was given. A special version of the Hadamard lemma for fiber bundles and the notions of weighted jet spaces were proposed as new tools for the investigation of potential conservation laws.

It is appropriate to collect here a few results related to local conservation laws of differential equations and to give a rough outline for the procedure of iterative construction of potential systems. A more extensive account of this material can be found e.g. in [T4, 15, 43, 57].

Here and below we denote by $\mathcal{L}$ a system of differential equations under study. The system $\mathcal{L}$ consists of l equations of the form $L^\mu(x,u_{(r)})=0$, $\mu=1,\dots,l$, where $x=(x_1,\dots,x_n)$ are the n independent variables, $u=(u^1,\dots,u^m)$ are the m unknown functions (the dependent variables) and the symbol $u_{(r)}$ denotes all derivatives of the functions u with respect to x of order not greater than r. By definition the components of u are included in $u_{(r)}$ as the derivatives of order zero. Within the local approach, which is employed here, differential equations can be interpreted as algebraic[2] equations in the jet space $\mathrm{J}^\infty(x|u)$, where both the independent variables x and the derivatives of u with respect to x are assumed as usual variables. Recall that a smooth function f depending on x and a finite number of derivatives of u (i.e., a smooth function on an open set of $\mathrm{J}^\infty(x|u)$ with finite number of arguments and with values in the ground field) is called a *differential function* of u, which is denoted by $f=f[u]$. The order $\operatorname{ord} f$ of the differential function f is the highest order of derivatives involved in f, and, if f does not depend on derivatives of u, we set $\operatorname{ord} f=-\infty$.

Definition 8. A *conserved current* for the system $\mathcal{L}$ is an n-tuple of differential functions $F=(F^1[u],\dots,F^n[u])$ the total divergence of which vanishes on the solutions of $\mathcal{L}$,

$$(\operatorname{Div} F)\big|_{\mathcal{L}}=0. \tag{2}$$

Notation. In Definition 8 and in what follows, the total divergence operator is defined by $\operatorname{Div} F=\mathrm{D}_iF^i$. Similarly to Section 2, $\mathrm{D}_i=\mathrm{D}_{x_i}$ denotes the operator of total derivative with respect to the variable x_i. In other words, $\mathrm{D}_i=\partial_i+u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$, where $\alpha=(\alpha_1,\dots,\alpha_n)$ is an arbitrary multi-index, $\alpha_i\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the index i runs from 1 to n, the index a runs from 1 to m, the variable u^a_α of the jet space $\mathrm{J}^\infty(x|u)$ is identified with the derivative $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, $|\alpha|=\alpha_1+\dots+\alpha_n$, and δ_i is the multi-index with zeros everywhere except the ith entry, which equals 1. $\partial_i:=\partial/\partial x_i$ and $\partial_{u^a_\alpha}:=\partial/\partial u^a_\alpha$. The standard summation convention for sums over repeated indices is used. By $(\dots)\big|_{\mathcal{L}}=0$ we mean that the corresponding expression only vanishes for solutions of the system $\mathcal{L}$.

The validity of (2) on the solution set of $\mathcal{L}$ is significant for relating the conserved current F to $\mathcal{L}$. A conserved current F is *trivial* if it can be represented as $F=\hat F+\check F$, where $\hat F$ and $\check F$ are n-tuples of differential functions such that the components of $\hat F$ vanish on the solutions of $\mathcal{L}$ and $\check F$ is a null divergence. By null divergence it is meant that $\operatorname{Div}\check F=0$ holds unrestricted of the system $\mathcal{L}$.

Two conserved currents F and F' are called *equivalent* if their difference $F-F'$ is a trivial conserved current. It is obvious that for any system $\mathcal{L}$ its set of conserved currents, denoted by $\mathrm{CC}(\mathcal{L})$, is a linear space. Likewise, the subset of trivial conserved currents, denoted by $\mathrm{CC}_0(\mathcal{L})$, is a linear subspace of $\mathrm{CC}(\mathcal{L})$. The set of equivalence classes of $\mathrm{CC}(\mathcal{L})$ with respect to the

[2]Here the adjective "algebraic" is only used in the sense "non-differential"; it does not mean that equations are polynomial; cf. [43, Section 2.1].

above equivalence relation on conserved currents is the quotient space $\mathrm{CC}(\mathcal{L})/\mathrm{CC}_0(\mathcal{L})$, which is denoted by $\mathrm{CL}(\mathcal{L})$.

Definition 9. The linear space $\mathrm{CL}(\mathcal{L})$ is called the *space of (local) conservation laws* of the system $\mathcal{L}$. Its elements are called *(local) conservation laws* of the system $\mathcal{L}$.

In other words, equivalent conserved currents correspond to the same conservation law.

If the system $\mathcal{L}$ is totally nondegenerate [43] or weakly totally nondegenerate [T4], then it is possible to use the Hadamard lemma and 'integration by parts' to represent the definition of conserved current (2) in the form

$$\mathrm{Div}\, F = \lambda^1 L^1 + \cdots + \lambda^l L^l, \tag{3}$$

where in general the initial conserved current F should be replaced by one differing from F in a trivial conserved current, $F + \hat{F} \to F$, where the components of $\hat{F}$ vanish on the solutions of $\mathcal{L}$.

Definition 10. The l-tuple of differential functions $\lambda = (\lambda^1, \dots, \lambda^l)$ is called the *characteristic* and Eq. (3) is the *characteristic form* of the conservation law corresponding to the conserved current F.

The *Euler operator* $\mathsf{E} = (\mathsf{E}^1, \dots, \mathsf{E}^m)$ is the m-tuple of differential operators defined by

$$\mathsf{E}^a = (-\mathrm{D})^\alpha \partial_{u^a_\alpha}, \quad a = 1, \dots, m, \quad \text{where} \quad (-\mathrm{D})^\alpha = (-\mathrm{D}_1)^{\alpha_1} \cdots (-\mathrm{D}_n)^{\alpha_n}.$$

It is well known [43, Theorem 4.7] that a differential function f is (locally) a total divergence, meaning that $f = \mathrm{Div}\, F$ for some n-tuple of differential functions F, if and only if it is annihilated by the Euler operator, $\mathsf{E}^a f = 0$. In other words, $\mathrm{im}\,\mathrm{Div} = \ker \mathsf{E}$ (locally). Using this property of the Euler operator and applying it to the characteristic form of conservation laws (3), one obtains

$$\mathsf{E}^a(\lambda^1 L^1 + \cdots + \lambda^l L^l) = 0, \tag{4}$$

which is both a necessary and sufficient condition for the tuple λ to be a conservation-law characteristic of the system $\mathcal{L}$. The characteristic approach to conservation laws is particularly suitable for the automatic computation of low-order conservation laws for systems of differential equations in the extended Kovalevskaya form using computer algebra systems, see e.g. [22, 69].

The notion of triviality extends to conservation-law characteristics as well. A characteristic λ is called trivial if it vanishes for all solutions of $\mathcal{L}$. The existence of trivial characteristics makes it necessary to introduce the notion of equivalent characteristics. If the difference between the two characteristics λ and $\tilde{\lambda}$ is trivial, then λ and $\tilde{\lambda}$ are called *equivalent*. Just as for conserved currents, the set of characteristics, denoted by $\mathrm{Ch}(\mathcal{L})$, is a linear space with the subset $\mathrm{Ch}_0(\mathcal{L})$ of trivial characteristics being a linear subspace thereof. For a normal totally nondegenerate system $\mathcal{L}$ the characteristic form of conservation laws (3) then induces a one-to-one correspondence between the factor spaces $\mathrm{CC}(\mathcal{L})/\mathrm{CC}_0(\mathcal{L})$ and $\mathrm{Ch}(\mathcal{L})/\mathrm{Ch}_0(\mathcal{L})$. This correspondence forms the basis of both Noether's theorem and the direct method for construction of conservation laws as found in [2, 3, 15, 19, 43, 65].

If local conservation laws of a system $\mathcal{L}$ of differential equations are known, we can apply the lemma on null divergences to these conservation laws considered on the set of solutions of $\mathcal{L} = \mathcal{L}^0$. In this way we introduce potentials as additional dependent variables. Then we attach the equations connecting the potentials with the components of the corresponding conserved vectors to $\mathcal{L}^0$. If $n > 2$, then the attached equations of this kind form an underdetermined system with respect to the potentials. Therefore, we can also add gauge conditions on the potentials to $\mathcal{L}^0$. In fact, such additional conditions are absolutely necessary in the case $n > 2$. It was proved in Theorem 2.7 of [1] for a quite general situation that every local symmetry of a potential system

with potentials that are not additionally constrained is projectable to a local symmetry of the initial system, i.e., such a potential system gives no nontrivial potential symmetries. Moreover, each conservation law of such a system is invariant with respect to gauge transformations of the potentials [4].

We have to use linearly independent conservation laws since otherwise the introduced potentials will be *dependent* in the following sense: there exists a linear combination of the potential tuples that is a tuple of differential functions of u only.

Then we exclude the superfluous equations (i.e., the equations that are dependent on equations from $\mathcal{L}^0$ and the attached equations simultaneously) from the extended (potential) system $\mathcal{L}^1$, which will be called a *potential system of the first level*. Any conservation law of $\mathcal{L}^0$ is one of $\mathcal{L}^1$. We iterate the above procedure for $\mathcal{L}^1$ to find its conservation laws which are linearly independent from those from the previous iteration and will be called *potential conservation laws of the first level*.

We continue this process as long as possible (i.e., the iteration procedure has to be stopped if all the conservation laws of a *potential system* $\mathcal{L}^{k+1}$ *of the* $(k+1)$*-st level* are linearly dependent from the ones of $\mathcal{L}^k$) or inductively construct infinite chains of conservation laws. This procedure may yield *purely potential* conservation laws of the initial system $\mathcal{L}$, which are linearly independent from local conservation laws and depend explicitly on potential variables. As mentioned above, the idea of this iteration procedure can be traced back to the well-known paper by Wahlquist and Estabrook [67].

Any conservation law from the previous step of the iteration procedure will be a conservation law for the next step. Conservation laws which are obtained at the next step and depend only on variables of the previous step are linearly dependent from conservation laws from the previous step. It is also obvious that the conservation laws used for the construction of a potential system of the next level are trivial on the manifold of this system.

Since gauge conditions on potentials can be chosen in many different ways, an exhaustive implementation of the above iteration procedure is improbable if $n > 2$.

5 The method of invariant parameterization

Averaging a nonlinear model of fluid motion leads to the *closure problem*. There are terms arising in the averaged model that cannot be specified using just the information contained in the mean prognostic variables. In order to close the differential equations for these dependent variables, a subgrid-scale closure model has to be employed, i.e. it is necessary to specify a relation between the unresolved subgrid-scale values and the resolved mean fields. Such a subgrid-scale closure model parameterizes the effects of the unresolved subgrid-scale values, e.g., the unresolved geostrophic ocean eddies, in the averaged model.

As reconstructing these effects only from the information contained in the mean fields themselves can never be perfectly accurate, there is no single canonical closure model. Nevertheless, there are a few general accepted parameterization strategies, which are formulated, e.g., in [63]. Such strategies include maintaining of the correct dimensionality of the parameterized terms, the preservation of tensorial properties or invariance of the closure model under the change of the coordinate systems, just to name a few.

Unfortunately, these parameterization strategies are seldom sufficient to obtain a unique closure model for the subgrid-scale processes. This is why in [T5] a new strategy was proposed to restrict the multitude of parameterization models using geometric properties of differential equations, which is the framework of *invariant parameterization*.

The motivation to tackle the invariant-parameterization problem emerged from the work by Speziale, who emphasized the importance of Galilean invariance in subgrid-scale closure models for the filtered Navier–Stokes equations [62]. Later, Oberlack extended the work of Speziale

by establishing conditions that enable the construction of general subgrid-scale closure models for the filtered Navier–Stokes equations that respect all the Lie symmetries admitted by the unfiltered Navier–Stokes equations [42]. In practice, this leads to conditions both on the filter kernel to be used for the filtering and on the form of the subgrid-scale closure that can be taken.

The systematic extension of this idea to the construction of general *invariant parameterization schemes* for subgrid-scale terms of *any* averaged model was given in [T5], see also [11]. The main contribution here was to identify the problem of invariant parameterization as a group classification problem. Group classification aims to describe symmetries for system of differential equations from certain classes, i.e., for systems that include constant or functional parameters. By assuming a functional relation between the unknown subgrid-scale terms to be parameterized and the known grid-scale quantities to be used for the parameterization scheme and plugging this ansatz into the averaged or filtered governing equations, these equations then readily constitute a class of differential equations. In other words, the invariant-parameterization problem is brought into the form of a group classification problem, that can subsequently be tackled using standard methods in the field of group analysis of differential equations, such as the algebraic method of group classification [10, 11, 56].

A similar strategy was pursued in [9, 11] to find general parameterization schemes that preserve conservation laws of systems of differential equations in averaged models of these systems.

From the practical point of view, the only models so far investigated using the methods of invariant parameterization employ first-order closure schemes. That is, for the parameterization of the unknown subgrid-scale quantities, only the grid-scale variables and their derivatives have been used. In view of the state-of-the-art applications, first-order (local) closure models are certainly inadequate, as there is no way to include information on, e.g., variances of turbulent perturbation quantities in such models. The reason for this is that these variances are at once replaced by the grid-scale quantities, which is more than often insufficient from a physical point of view.

Suppose we are given a system of differential equations of the form

$$\Delta^l(x, u_{(n)}) = 0, \quad l = 1, \dots, m. \tag{5}$$

Here and in the following we denote by $x = (x^1, \dots, x^p)$ the independent variables and by $u_{(n)}$ the dependent variables $u = (u^1, \dots, u^q)$ as well as all derivatives of u with respect to x up to order n. We use the subscript notation to denote differentiation with respect to the associated variables.

When preparing system (5) for a numerical implementation, one first chooses a suitably fine grid on which the numerical solution will live. Effectively, this means that the unknown functions u are decomposed into $u = \bar{u} + u'$, where $\bar{u}$ is the resolved part of u (the grid-scale part) living on the discretization mesh, and u' is the unresolved part of u (the subgrid-scale part). Thus, practically speaking, $\bar{u}$ can be computed using a numerical model for system (5), whereas u' cannot be resolved by the model.

An alternative interpretation is that we regard $\bar{u}$ as the function obtained by averaging (or filtering) u over each grid cell. Various averaging or filtering operations can be used, but in the following we restrict ourselves to classical *Reynolds averaging*. The Reynolds averaging satisfies the Reynolds property $\overline{\bar{u}^i u^j} = \bar{u}^i \bar{u}^j$, which implies the *Reynolds averaging rule* $\overline{u^i u^j} = \bar{u}^i \bar{u}^j + \overline{u^{i'} u^{j'}}$. Here the overbar is to be interpreted as averaging in both space and time. We should like to stress though that the precise form of averaging (filtering) or discretization methodology chosen to obtain a system of equations for $\bar{u}$ is not essential from the theoretical point of what will follow.

To implement system (5) numerically we thus have to convert it into a system for $\bar{u}$. Assuming system (5) to be nonlinear, Reynolds-averaging it yields a system of the form

$$\bar{\Delta}^{1l}\big(x, \bar{u}_{(n)}, w^1_{(n')}\big) = 0, \quad l = 1, \dots, m, \tag{6}$$

where $\bar{\Delta}$'s are smooth functions of their arguments. The precise form of these functions and the maximal order n' of involved derivatives of w^1 depends on the equations in the original system (5) as well as on the averaging rule invoked. The tuple $w^1 = (w^{11}, \dots, w^{1k})$ collects all subgrid-scale terms originating from applying the averaging rule to nonlinear combinations of terms in the original system (5). These terms are those that need to be expressed in terms of the resolved quantities in order to close the system (6).

The most straightforward way of closing (6) is to set $w^{1i} = f^i(x, \bar{u}_{(r)})$, $i = 1, \dots, k$, where $f = (f^1, \dots, f^k)$ are parameterization functions, which should be specified, $r \in \mathbb{N} \cup \{0\}$. The associated parameterization scheme is then *local* and of *first order* [63]. System (6) is then closed to

$$\bar{\Delta}^{1l}\big(x, \bar{u}_{(n)}, f_{[n']}(x, \bar{u}_{(r)})\big) = 0, \quad l = 1, \dots, m, \tag{7}$$

which is then a class of differential equations, where the parameterization functions f play the role of arbitrary elements. Here and in what follows we denote by $f_{[n']}$ the collection of f and all total derivatives of f with respect to x up to order n'. Symmetry-preserving parameterization schemes are found upon solving a *group classification problem* for the class of equations (7). There are two main paradigms for solving a group classification problem: using *inverse group classification* or *direct group classification*.

Direct group classification proceeds by investigating for which specific values of the parameterization functions f, systems from the class (7) admit more symmetries than in the case of generic f. To facilitate an efficient classification, it is necessary to compute the so-called *equivalence group* of the given class first. Equivalence transformations in the class (7) are point transformations in the extended space with coordinates (x, u, f) that map systems from the this class to (other) systems of the same class. The classification of values of the parameter functions f such that the associated systems admit more symmetries than the generic case is then done up to equivalence. Several methods have been proposed to efficiently solve the group classification problem, among which the *algebraic method* is the most powerful for the classes of differential equations that typically arise within the problem of invariant parameterization. See [T5, 12, 48, 56] for the use of the algebraic method to solving group classification problems.

Inverse group classification relies on fixing a specific local Lie group acting on some manifold $M = X \times U$, where $x \in X$ are the independent variables and $u \in U$ are the dependent variables. The crucial step is to compute *differential invariants*, invariant functions on jet spaces $X \times U_{(n)}$, where $u_{(n)} \in U_{(n)}$, which when combined together yield systems of at most n-th order differential equations that are invariant under the selected Lie group. In the context of the invariant-parameterization problem, inverse group classification is the approach originally used in [42], where it was determined that any subgrid-scale closure scheme for the Navier–Stokes equations should admit the same symmetries as the original Navier–Stokes equations. In practice, within the framework of inverse group classification invariant parameterization schemes are most conveniently computed using the method of moving frames [21, 26, 27, 46], which allows one to readily obtain differential invariants using an *invariantization map*. In [11], it was proposed that invariant parameterization schemes can be constructed by applying a suitable invariantization map directly to a given closed, non-invariant system of the form (7).

It can also make sense to relax the assumption that the closed model (7) should preserve all the symmetries of the original model (5). In particular, the closed model (7) only captures the resolved part of the dynamics of the original model (5) and thus imposing invariance of this model under exactly the same symmetries as admitted by the parent model may be overly restrictive or even unphysical. The presence of particular boundary conditions imposed on the closed model (7) may also yield natural restrictions on which symmetries should be preserved by this model, see e.g. [13, 14, 18]. Lastly, not all symmetries of the original model (5) may be physically relevant to begin with. An example for this is provided by the system of

(1+1)-dimensional shallow-water equations which admits an infinite-dimensional maximal Lie invariance pseudogroup. In turn, the system of (1+2)-dimensional shallow-water equations admits only a finite-dimensional maximal Lie invariance group. The presence of the infinite-dimensional maximal Lie invariance pseudogroup of the (1+1)-dimensional shallow-water equations is an artefact since as a system of two first-order quasilinear equations in two variables this system can be linearized using a hodograph transformation. Thus, in constructing invariant parameterization schemes for the (1+1)-dimensional shallow-water equations one should restrict oneself to preserving only those symmetries that are counterparts of symmetries of the (1+2)-dimensional shallow-water equations, see [13] for further discussions.

In general, higher-order parameterization schemes aim to improve upon the overly simplistic assumption that the unresolved subgrid-scale fields w^1 can be directly expressed in terms of the resolved grid-scale quantities $\bar{u}_{(r)}$. Instead, one proceeds by deriving equations for the subgrid-scale fields w^1 themselves. The resulting equations will again be unclosed, depending on higher-order fields w^2 (triple correlation quantities, such as $\overline{u'u'_xu'_{xx}}$). One can repeat this construction and thus derive a system of the form

$$\bar{\Delta}^{jl_j}\big(x,\bar{u}_{(n)},w^1_{(n'_j)},\dots,w^j_{(n'_j)}\big)=0,\quad l_j=1,\dots,m_j,\quad j=1,\dots,s, \tag{8}$$

where $m_j\in\mathbb{N}$, $m_1:=m$. Note that this system is still not closed, as there are no prognostic equations for $w^s=(w^{s1},\dots,w^{sk_s})$. A *local s-th order parameterization scheme* is obtained by setting $w^s=f\big(x,\bar{u}_{(r)},w^1_{(r)},\dots,w^{s-1}_{(r)}\big)$ for some $r\in\mathbb{N}\cup\{0\}$ and some smooth function f of its arguments which will close system (8), yielding

$$\begin{aligned}&\bar{\Delta}^{jl_j}\big(x,\bar{u}_{(n)},w^1_{(n'_j)},\dots,w^j_{(n'_j)}\big)=0,\quad l_j=1,\dots,m_j,\quad j=1,\dots,s-1,\\&\bar{\Delta}^{sl_s}\big(x,\bar{u}_{(n)},w^1_{(n'_s)},\dots,w^{s-1}_{(n'_s)},f_{[n'_s]}\big(x,\bar{u}_{(r)},w^1_{(r)},\dots,w^{s-1}_{(r)}\big)\big)=0.\end{aligned} \tag{9}$$

We call a local s-th order parameterization scheme $w^s=f\big(x,\bar{u}_{(r)},w^1_{(r)},\dots,w^{s-1}_{(r)}\big)$ with a specific f *invariant* if system (9) is invariant under a prolongation of the maximal Lie invariance group (or a suitable subgroup) of system (5) to the subgrid-scale fields w^1, …, w^s or, more generally, under a reasonable point transformation group acting in the space with the coordinates $(x,\bar{u},w^1,\dots,w^s)$.

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Part II

PAPERS

Paper 1

Singular reduction operators in two dimensions

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Singular reduction operators in two dimensions

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Abstract
The notion of singular reduction operators, i.e., of singular operators of nonclassical (conditional) symmetry, of partial differential equations in two independent variables is introduced. All possible reductions of these equations to first-order ODEs are exhaustively described. As examples, properties of singular reduction operators of (1 + 1)-dimensional evolution and wave equations are studied. It is shown how to favourably enhance the derivation of nonclassical symmetries for this class by an in-depth prior study of the corresponding singular vector fields.

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1. Introduction

Distinctions in kind between Lie symmetries and nonclassical symmetries became apparent in the first presentation of nonclassical symmetries in [3] by the example of the (1 + 1)-dimensional linear heat equation and a particular class of operators. In contrast to classical Lie symmetries (see, e.g., [17]), the system of determining equations on the coefficients of nonclassical symmetry operators of the heat equation was found to be nonlinear and less overdetermined, and the set of such operators does not possess the structure of an algebra or even a vector space.

Another difference appears in the procedure of deriving the determining equations. Namely, *deriving systems of determining equations for nonclassical symmetries crucially depends on the interplay between the operators and the equations under consideration.* Thus, for the linear heat equation $u_t = u_{xx}$ the general form of nonclassical symmetry operators is $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$, where $(\tau, \xi) \neq (0, 0)$, and there are two essentially different cases of nonclassical symmetries: the *regular* case $\tau \neq 0$ and the *singular* case $\tau = 0$. The factorization up to nonvanishing functional multipliers gives the two respective cases for the further investigation: (1) $\tau = 1$ and (2) $\tau = 0, \xi = 1$.

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The problem of determining the nonclassical symmetries of the linear heat equation was completely solved in [10]. In the regular case $\tau = 1$, after partial integration of the corresponding determining equations, we obtain $\xi = g^1(t, x)$ and $\eta = g^2(t, x)u + g^3(t, x)$. The functions g^1, g^2 and g^3 satisfy a coupled nonlinear system of partial differential equations [3], which is linearized by a nonlocal transformation to a system of three uncoupled copies of the initial equation [9, 10, 24]. The underlying reason for this phenomenon lies in the interaction between the linearity and the evolution structure in the linear heat equation. Hence similar results can be obtained only for linear evolution equations [7, 18, 21] or related linearizable equations [13].

The singular case $(\tau, \xi) = (0, 1)$ was not considered in [3]. In this case the system of determining equations for nonclassical symmetries consists of a single $(1 + 2)$-dimensional nonlinear evolution equation for the unknown function η and, therefore, is not overdetermined. The determining equation is reduced by a nonlocal transformation to the initial equation with an additional implicit independent variable which can be assumed as a parameter [10]. The linearity of the heat equation is inessential here. Hence after the case of linear evolution equations [7, 18] this result was extended to general $(1 + 1)$-dimensional evolution equations [25], multi-dimensional evolution equations [19] and even systems of such equations [23]. Moreover, it was proved [19], that, e.g., in the $(1 + 1)$-dimensional case there exists a one-to-one correspondence between one-parametric families of solutions of an evolution equation and its reduction operators with $(\tau, \xi) = (0, 1)$.

The above results raise a number of interesting questions, to wit: what are possible causes for the existence of singular cases for reduction operators? Is the conventional partition of sets of reduction operators with the conditions of vanishing and nonvanishing coefficients of operators universal or is it appropriate only for certain classes of differential equations, e.g., evolution equations? Can partitions of sets of reduction operators, different from the conventional one, be useful? Does there exist an algorithmic way of singling out singular cases for reduction operators before deriving determining equations? What properties of a partial differential equation and a subset of its reduction operators lead to a 'no-go' situation (i.e., a single determining equation equivalent, in a certain sense, to the initial equation)? What is the optimal way of obtaining the determining equation for nonclassical symmetries? The purpose of the present paper is to answer these and other related questions.

Algorithms for deriving the determining equations for nonclassical symmetries were discussed, e.g., in [2, 6] but the focus of these works was quite different.

The conditional invariance of a differential equation with respect to an operator is equivalent to any ansatz associated with this operator reducing the equation to a differential equation with one less independent variables [26]. That is why we use the shorter and more natural term 'reduction operators' instead of 'operators of conditional symmetry' or 'operators of nonclassical symmetry' and say that an operator reduces a differential equation if the equation is reduced by the corresponding ansatz. The direct method of reduction with ansatzes of a special form was first explicitly applied in [4] to the Boussinesq equation although reductions by non-Lie ansatzes were already discussed, e.g., in [8]. A connection between the reduction by generalized ansatzes and compatibility with respect to higher-order constraints was found in [15].

To clarify the main ideas of the proposed framework of singular reduction operators, in this first presentation of the subject we consider only the case of a single partial differential equation in two independent and one dependent variables and a single reduction operator. We note, however, that more general cases can be included and will be the subject of forthcoming papers.

Some of the main conclusions of the present paper are:

- Singular cases of reduction operators of a partial differential equation are connected with the possibility of lowering the order of this equation on the manifolds determined by the corresponding invariant surface conditions in the appropriate jet space. Hence the first step of the procedure of finding nonclassical symmetries has to consist in studying singular modules of vector fields which lower the order of the equation. This step is entirely algorithmic, hence is especially suited to a direct implementation in symmetry-finding computer algebra programs. The structure of singular modules of vector fields has to be taken into account under splitting the set of reduction operators for factorization.
- The weak singularity co-order of a reduction operator Q coincides with the essential order of the corresponding reduced equation and the number of essential parameters in the family of Q-invariant solutions.
- If a single partial differential equation $\mathcal{L}$ in two independent variables admits a first co-order singular module S of vector fields then it necessarily possesses first co-order singular reduction operators belonging to S. The system of determining equations for such operators consists of a single partial differential equation DE in three independent variables of the same order as $\mathcal{L}$. The equation DE is reduced to $\mathcal{L}$ by a nonlocal transformation.

The paper is organized as follows: the main notions and statements on nonclassical symmetries are presented in section 2. Singular vector fields of differential functions and differential equations are defined and studied in sections 3 and 4, respectively. Singular reduction operators of $(1 + 1)$-dimensional evolution and nonlinear wave equations are exhaustively investigated in sections 5 and 6. It is shown that the conventional partition of sets of reduction operators is natural for evolution equations, in contrast to the case of nonlinear wave equations. A connection between the singularity co-order of reduction operators and the number of parameters in the corresponding families of invariant solutions is established in section 7. Section 8 is devoted to first co-order singular reduction operators of general partial differential equations in two independent and one dependent variables.

2. Reduction operators of differential equations

Following [11, 12, 22, 26], in this section we briefly collect the required notions and results on nonclassical (conditional) symmetries of differential equations. Also, we argue for the use of the name 'reduction operators' instead of 'nonclassical (conditional) symmetry operators'. In accordance with the aims of this paper we restrict our considerations to the case of two independent variables and a single reduction operator.

The set of (first-order) differential operators (or vector fields) of the general form

$$Q = \xi^i(x, u)\partial_i + \eta(x, u)\partial_u, \qquad (\xi^1, \xi^2) \neq (0, 0)$$

will be denoted by $\mathfrak{Q}$. In what follows, x denotes the pair of independent variables (x_1, x_2) and u is treated as the unknown function. The index i runs from 1 to 2, and we use the summation convention for repeated indices. Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_i = \partial/\partial x_i$ and $\partial_u = \partial/\partial u$. Any function is considered as its zero-order derivative. All our considerations are carried out in the local setting.

Two differential operators $\widetilde{Q}$ and Q are called *equivalent* if they differ by a multiplier which is a non-vanishing function of x and u: $\widetilde{Q} = \lambda Q$, where $\lambda = \lambda(x, u)$, $\lambda \neq 0$. The equivalence of operators will be denoted by $\widetilde{Q} \sim Q$. Factoring $\mathfrak{Q}$ with respect to this equivalence relation we arrive at $\mathfrak{Q}_{\mathrm{f}}$. Elements of $\mathfrak{Q}_{\mathrm{f}}$ will be identified with their representatives in $\mathfrak{Q}$.

The first-order differential function

$$Q[u] := \eta(x,u) - \xi^i(x,u)u_i$$

is called the *characteristic* of the operator Q. The characteristic PDE $Q[u] = 0$ (also known as the *invariant surface condition*) has two functionally independent solutions $\zeta(x,u)$ and $\omega(x,u)$. Therefore, the general solution of this equation can be implicitly represented in the form $F(\zeta,\omega) = 0$, where F is an arbitrary function.

A differential function $\Theta = \Theta[z]$ of the dependent variables $z = (z^1,\dots,z^m)$ which in turn are functions of a tuple of independent variables $y = (y_1,\dots,y_n)$ will be considered as a smooth function of y and derivatives of z with respect to y. The order $r = \operatorname{ord}\Theta$ of the differential function Θ equals the maximal order of derivatives involved in Θ. More precisely, the differential function Θ is defined as a function on a subset of the jet space $J^r(y|z)$ [14].

The characteristic equations of equivalent operators have the same set of solutions. Conversely, any family of two functionally independent functions of x and u is a complete set of integrals of the characteristic equation of a differential operator. Therefore, there exists a one-to-one correspondence between $\mathfrak{Q}_{\mathrm{f}}$ and the set of families of two functionally independent functions of x and u, which is factorized with respect to the corresponding equivalence relation. (Two families of the same number of functionally independent functions of the same arguments are considered equivalent if any function from one of the families is functionally dependent on functions from the other family.)

Since $(\xi^1,\xi^2) \neq (0,0)$ we can assume without loss of generality that $\zeta_u \neq 0$ and $F_\zeta \neq 0$ and resolve the equation $F = 0$ with respect to ζ: $\zeta = \varphi(\omega)$. This implicit representation of the function u is called an *ansatz* corresponding to the operator Q.

Consider an rth-order differential equation $\mathcal{L}$ of the form $L(x,u_{(r)}) = 0$ for the unknown function u of two independent variables $x = (x_1,x_2)$. Here $L = L[u] = L(x,u_{(r)})$ is a fixed differential function of order r and $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to x of order not greater than r, including u as the derivative of order zero. Within the local approach the equation $\mathcal{L}$ is treated as an algebraic equation in the jet space $J^r = J^r(x|u)$ of order r and is identified with the manifold of its solutions in J^r:

$$\mathcal{L} = \{(x,u_{(r)}) \in J^r \mid L(x,u_{(r)}) = 0\}.$$

Denote the manifold defined by the set of all the differential consequences of the characteristic equation $Q[u] = 0$ in J^r by $\mathcal{Q}_{(r)}$, i.e.,

$$\mathcal{Q}_{(r)} = \big\{(x,u_{(r)}) \in J^r \mid D_1^\alpha D_2^\beta Q[u] = 0, \alpha,\beta \in \mathbb{N}\cup\{0\}, \alpha+\beta < r\big\},$$

where $D_1 = \partial_1 + u_{\alpha+1,\beta}\partial_{u_{\alpha\beta}}$ and $D_2 = \partial_2 + u_{\alpha,\beta+1}\partial_{u_{\alpha\beta}}$ are the operators of total differentiation with respect to the variables x_1 and x_2, and the variable $u_{\alpha\beta}$ of the jet space J^r corresponds to the derivative $\partial^{\alpha+\beta}u/\partial x_1^\alpha \partial x_2^\beta$.

A precise and rigorous definition of nonclassical (or conditional) symmetry was first suggested in [11] (see also [12, 26]).

Definition 1. *The differential equation $\mathcal{L}$ is called conditionally invariant with respect to the operator Q if the relation $Q_{(r)}L(x,u_{(r)})\big|_{\mathcal{L}\cap\mathcal{Q}^{(r)}} = 0$ holds, which is called the conditional invariance criterion. Then Q is called an operator of conditional symmetry (or Q-conditional symmetry, nonclassical symmetry, etc) of the equation $\mathcal{L}$.*

In definition 1 the symbol $Q_{(r)}$ stands for the standard rth prolongation of the operator Q [14, 17]:

$$Q_{(r)} = Q + \sum_{0<\alpha+\beta\leqslant r} \eta^{\alpha\beta}\partial_{u_{\alpha\beta}}, \qquad \eta^{\alpha\beta} := D_1^\alpha D_2^\beta Q[u] + \xi^1 u_{\alpha+1,\beta} + \xi^2 u_{\alpha,\beta+1}.$$

The equation $\mathcal{L}$ is conditionally invariant with respect to Q if and only if the ansatz $\zeta = \varphi(\omega)$ constructed with Q reduces $\mathcal{L}$ to an ordinary differential equation $\check{\mathcal{L}}$: $\check{L}[\varphi] = 0$ [26]. Namely, there exist differential functions $\check{\lambda} = \check{\lambda}[\varphi]$ and $\check{L} = \check{L}[\varphi]$ of an order not greater than r (i.e., functions of ω and derivatives of φ with respect to ω up to order r) such that $L|_{u=\varphi(\omega)} = \check{\lambda}\check{L}$. The function $\check{\lambda}$ does not vanish and may depend on θ as a parameter, where the value $\theta = \theta(x, u)$ is functionally independent of ζ and ω. The differential function $\check{L}$ is assumed to be of minimal order $\check{r}$ which is possibly reached up to the equivalence generated by nonvanishing multipliers. Then the reduced equation $\check{\mathcal{L}}$ is of essential order $\check{r}$.

This is why we will also call operators of conditional symmetry *reduction operators* of $\mathcal{L}$.

Another treatment of conditional invariance is that the system $\mathcal{L} \cap \mathcal{Q}_{(r)}$ is compatible in the sense of not involving any nontrivial differential consequences [15, 16].

The property of conditional invariance is compatible with the equivalence relation on $\mathfrak{Q}$ [12, 26]:

Lemma 1. *If the equation $\mathcal{L}$ is conditionally invariant with respect to the operator Q then it is conditionally invariant with respect to any operator which is equivalent to Q.*

The set of reduction operators of the equation $\mathcal{L}$ is a subset of $\mathfrak{Q}$ and so will be denoted by $\mathfrak{Q}(\mathcal{L})$. In view of lemma 1, $Q \in \mathfrak{Q}(\mathcal{L})$ and $\widetilde{Q} \sim Q$ imply $\widetilde{Q} \in \mathfrak{Q}(\mathcal{L})$, i.e., $\mathfrak{Q}(\mathcal{L})$ is closed under the equivalence relation on $\mathfrak{Q}$. Therefore, the factorization of $\mathfrak{Q}$ with respect to this equivalence relation can be naturally restricted to $\mathfrak{Q}(\mathcal{L})$, resulting in the subset $\mathfrak{Q}_{\mathrm{f}}(\mathcal{L})$ of $\mathfrak{Q}_{\mathrm{f}}$. As in the whole set $\mathfrak{Q}_{\mathrm{f}}$, we identify elements of $\mathfrak{Q}_{\mathrm{f}}(\mathcal{L})$ with their representatives in $\mathfrak{Q}(\mathcal{L})$. In this approach the problem of completely describing all reduction operators for $\mathcal{L}$ is equivalent to finding $\mathfrak{Q}_{\mathrm{f}}(\mathcal{L})$.

The conditional invariance criterion admits the following useful reformulation [26].

Lemma 2. *Given a differential equation $\mathcal{L}$: $L[u] = 0$ of order r and differential functions $\tilde{L}[u]$ and $\lambda[u] \neq 0$ of an order not greater than r such that $L|_{\mathcal{Q}_{(r)}} = \lambda\tilde{L}|_{\mathcal{Q}_{(r)}}$, an operator Q is a reduction operator of $\mathcal{L}$ if and only if the relation $Q_{(\tilde{r})}\tilde{L}\big|_{\tilde{\mathcal{L}}\cap\mathcal{Q}_{(\tilde{r})}} = 0$ holds, where $\tilde{r} = \operatorname{ord}\tilde{L} \leqslant r$ and the manifold $\tilde{\mathcal{L}}$ is defined in $J^{\tilde{r}}$ by the equation $L[u] = 0$.*

The classification of reduction operators can be considerably enhanced and simplified by considering Lie symmetry and equivalence transformations of (classes of) equations.

Lemma 3. *Any point transformation of x and u induces a one-to-one mapping of $\mathfrak{Q}$ into itself. Namely, the transformation g: $\tilde{x}_i = X^i(x, u), \tilde{u} = U(x, u)$ generates the mapping $g_*: \mathfrak{Q} \to \mathfrak{Q}$ such that the operator Q is mapped to the operator $g_*Q = \tilde{\xi}^i\partial_{\tilde{x}_i} + \tilde{\eta}\partial_{\tilde{u}}$, where $\tilde{\xi}^i(\tilde{x}, \tilde{u}) = QX^i(x, u), \tilde{\eta}(\tilde{x}, \tilde{u}) = QU(x, u)$. If $Q' \sim Q$ then $g_*Q' \sim g_*Q$. Therefore, the corresponding factorized mapping $g_{\mathrm{f}}: \mathfrak{Q}_{\mathrm{f}} \to \mathfrak{Q}_{\mathrm{f}}$ also is well defined and bijective.*

Definition 2 ([20])**.** *Differential operators Q and $\widetilde{Q}$ are called equivalent with respect to a group G of point transformations if there exists $g \in G$ for which the operators Q and $g_*\widetilde{Q}$ are equivalent. We denote this equivalence by $Q \sim \widetilde{Q} \bmod G$.*

Lemma 4. *Given any point transformation g of an equation $\mathcal{L}$ to an equation $\tilde{\mathcal{L}}$, g_* maps $\mathfrak{Q}(\mathcal{L})$ to $\mathfrak{Q}(\tilde{\mathcal{L}})$ bijectively. The same is true for the factorized mapping g_{f} from $\mathfrak{Q}_{\mathrm{f}}(\mathcal{L})$ to $\mathfrak{Q}_{\mathrm{f}}(\tilde{\mathcal{L}})$.*

Corollary 1. *Let G be the point symmetry group of an equation $\mathcal{L}$. Then the equivalence of operators with respect to the group G generates equivalence relations in $\mathfrak{Q}(\mathcal{L})$ and in $\mathfrak{Q}_{\mathrm{f}}(\mathcal{L})$.*

Consider the class $\mathcal{L}|_{\mathcal{S}}$ of equations $\mathcal{L}_\theta$: $L(x, u_{(r)}, \theta) = 0$ parameterized with the parameter-functions $\theta = \theta(x, u_{(r)})$. Here L is a fixed function of $x, u_{(r)}$ and θ. The symbol θ denotes the tuple of arbitrary (parametric) differential functions $\theta(x, u_{(r)}) = (\theta^1(x, u_{(r)}), \dots, \theta^k(x, u_{(r)}))$ running through the set $\mathcal{S}$ of solutions of the system $S(x, u_{(r)}, \theta_{(q)}(x, u_{(r)})) = 0$. This system consists of differential equations on θ, where x and $u_{(r)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the derivatives of θ of order not greater than q. In what follows we call the functions θ arbitrary elements. Denote the point transformation group preserving the form of the equations from $\mathcal{L}|_{\mathcal{S}}$ by $G^{\sim}$.

Let P denote the set of the pairs consisting of an equation $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ and an operator Q from $\mathfrak{Q}(\mathcal{L}_\theta)$. In view of lemma 4, the action of transformations from the equivalence group $G^{\sim}$ on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathfrak{Q}(\mathcal{L}_\theta) \mid \theta \in \mathcal{S}\}$ together with the pure equivalence relation of differential operators naturally generates an equivalence relation on P.

Definition 3. *Let $\theta, \theta' \in \mathcal{S}$, $Q \in \mathfrak{Q}(\mathcal{L}_\theta)$, $Q' \in \mathfrak{Q}(\mathcal{L}_{\theta'})$. The pairs $(\mathcal{L}_\theta, Q)$ and $(\mathcal{L}_{\theta'}, Q')$ are called $G^{\sim}$-equivalent if there exists $g \in G^{\sim}$ such that g transforms the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\theta'}$, and $Q' \sim g_* Q$.*

The classification of reduction operators with respect to $G^{\sim}$ will be understood as the classification in P with respect to this equivalence relation, a problem which can be investigated similar to the usual group classification in classes of differential equations. Namely, we construct firstly the reduction operators that are defined for all values of θ. Then we classify, with respect to $G^{\sim}$, the values of θ for which the equation $\mathcal{L}_\theta$ admits additional reduction operators.

3. Singular vector fields of differential functions

Consider a vector field $Q = \xi^i(x, u)\partial_i + \eta(x, u)\partial_u$ with $(\xi^1, \xi^2) \neq (0, 0)$, defined in the space (x, u), and a differential function $L = L[u]$ of order $\operatorname{ord} L = r$ (i.e., a smooth function of $x = (x_1, x_2)$ and derivatives of u of orders up to r).

Definition 4. *The vector field Q is called singular for the differential function L if there exists a differential function $\tilde{L} = \tilde{L}[u]$ of an order less than r such that $L|_{\mathcal{Q}_{(r)}} = \tilde{L}|_{\mathcal{Q}_{(r)}}$. Otherwise Q is called a regular vector field for the differential function L. If the minimal order of differential functions whose restrictions on $\mathcal{Q}_{(r)}$ coincide with $L|_{\mathcal{Q}_{(r)}}$ equals k $(k < r)$ then the vector field Q is said to be of singularity co-order k for the differential function L. The vector field Q is called ultra-singular for the differential function L if $L|_{\mathcal{Q}_{(r)}} \equiv 0$.*

For convenience, the singularity co-order of ultra-singular vector fields and the order of identically vanishing differential functions are defined to equal -1. Regular vector fields for the differential function L are defined to have singularity co-order $r = \operatorname{ord} L$. The singularity co-order of a vector field Q for a differential function L will be denoted by $\operatorname{sco}_L Q$.

If Q is a singular vector field for L then any vector field equivalent to Q is singular for L with the same co-order of singularity.

A function $\tilde{L}$ satisfying the conditions of definition 4 can be constructively found. Namely, without loss of generality we can suppose that the coefficient ξ^2 of ∂_2 in Q is nonzero. Then any derivative of u of order not greater than r can be expressed, on the manifold $\mathcal{Q}_{(r)}$, via

derivatives of u with respect to x_1 only. For example, for the first- and second-order derivatives we have

$$\begin{aligned} u_2 &= \hat{\eta} - \hat{\xi} u_1, \\ u_{12} &= \hat{\eta}_1 - \hat{\xi}_1 u_1 + \hat{\eta}_u u_1 - \hat{\xi}_u u_1^2 - \hat{\xi} u_{11}, \\ u_{22} &= \hat{\eta}_2 - \hat{\xi}_2 u_1 + (\hat{\eta}_u - \hat{\xi}_u u_1)(\hat{\eta} - \hat{\xi} u_1) - \hat{\xi}\big(\hat{\eta}_1 - \hat{\xi}_1 u_1 + \hat{\eta}_u u_1 - \hat{\xi}_u u_1^2 - \hat{\xi} u_{11}\big), \end{aligned} \tag{1}$$

where $\hat{\xi} = \xi^1/\xi^2$ and $\hat{\eta} = \eta/\xi^2$. After substituting the expressions for the derivatives into L, we obtain a differential function $\hat{L}$ depending only on x, u and derivatives of u with respect to x_1. We will call $\hat{L}$ a *differential function associated with L on the manifold* $\mathcal{Q}_{(r)}$. The vector field Q is singular for the differential function L if and only if the order of $\hat{L}$ is less than r. The co-order of singularity of Q equals the order of $\hat{L}$. The vector field Q is ultra-singular if and only if $\hat{L} \equiv 0$. Therefore, testing that a vector field is singular for a differential function with two independent variables is realized in an entirely algorithmic procedure and can be easily included in existing programs for symbolic calculations of symmetries.

Consider the two-dimensional module $\{Q^\theta = \theta^i Q^i\}$ of vector fields over the ring of smooth functions of (x, u) generated by the vector fields $Q^i = \xi^{ij}(x, u)\partial_j + \eta^i(x, u)\partial_u$, where $\operatorname{rank}(\xi^{i1}, \xi^{i2}, \eta^i) = 2$. In the remainder of this section the parameter tuple $\theta = (\theta^1, \theta^2)$ runs through the set of pairs of smooth functions depending on (x, u), and i and j run from 1 to 2.

Definition 5. *The module $\{Q^\theta\}$ is called singular for the differential function L if for any θ with $(\theta^i\xi^{i1}, \theta^i\xi^{i2}) \neq (0, 0)$ the vector field Q^θ is singular for L. The singularity co-order of the module $\{Q^\theta\}$ coincides with the maximum of the singularity co-orders of its elements.*

By a point transformation, one of the basis vector fields, e.g. Q^2, can be reduced to ∂_u (transforming L simultaneously with Q^1 and Q^2.) Then $(\xi^{11}, \xi^{12}) \neq (0, 0)$, and up to permutation of independent variables we can assume $\xi^{12} \neq 0$ and, therefore, set $\eta^1 = 0$ and $\xi^{12} = 1$ by a change of basis. Any vector field from the module $\{Q^\theta\}$ with a nonzero value of θ^1 is equivalent to the vector field $Q^1 + \zeta Q^2$, where $\zeta = \theta^2/\theta^1$. All the other vector fields from $\{Q^\theta\}$ (which have $\theta^1 = 0$ and, therefore, are equivalent to ∂_u) can be neglected since each of them leads to the equation $\theta^2(x, u) = 0$ which completely determines u and therefore, does not give an ansatz for u.

This justifies why, up to point transformations, it suffices to study only singular sets of vector fields of the form $\{Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u\}$, with ξ a fixed smooth function of (x, u) and ζ running through all such functions. The latter form of singular sets of vector fields will be called *reduced*.

Further simplification depends on whether the module is closed under the Lie bracket. In case it is, it can be assumed to be generated by two commuting vector fields which can be simultaneously reduced by a point transformation to shift operators, e.g., $Q^1 = \partial_2$ and $Q^2 = \partial_u$. Thus in the reduced form ξ can be put to 0. If the module is not closed under the Lie bracket, we have $\xi_u \neq 0$ in the reduced form. After the point transformation $\tilde{x}_i = x_i$ and $\tilde{u} = \xi$ and a change of basis, we obtain the bases $\tilde{Q}^1 = \tilde{u}\partial_{\tilde{1}} + \partial_{\tilde{2}}$ and $\tilde{Q}^2 = \partial_{\tilde{u}}$. Hence:

Proposition 1. *In any two-dimensional module of vector fields in the space of three variables (x_1, x_2, u), any basis vector fields Q^1 and Q^2 can be locally reduced, by point transformations, to the form $Q^1 = \partial_2$ (resp. $Q^1 = u\partial_1 + \partial_2$) and $Q^2 = \partial_u$ if the module is closed (resp. not closed) with respect to the Lie bracket of vector fields.*

Theorem 1. *A differential function L with one dependent and two independent variables possesses a k th co-order singular two-dimensional module of vector fields if and only if it can be represented, up to point transformations, in the form*

$$L = \check{L}(x, \Omega_{r,k}), \tag{2}$$

where $\Omega_{r,k} = \big(\omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2} u, \alpha_1 \leqslant k, \alpha_1 + \alpha_2 \leqslant r\big), \xi \in \{0, u\}$, *and* $\check{L}_{\omega_\alpha} \neq 0$ *for some* ω_α *with* $\alpha_1 = k$.

Proof. Suppose that a differential function L possesses a k th co-order singular two-dimensional module of vector fields $\{Q^\theta = \theta^i Q^i\}$. By a point transformation and a change of basis, we represent the basis elements in the reduced form $Q^1 = \xi\partial_1 + \partial_2$ and $Q^2 = \partial_u$, where $\xi \in \{0, u\}$, and choose the subset $\{Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u\}$ in $\{Q^\theta\}$, where ζ runs through the set of smooth functions of (x, u). The initial differential function also will be changed by these transformations but throughout we will use the old notations for all new values.

We fix an arbitrary point $z^0 = \big(x^0, u^0_{(r)}\big) \in J^r$ and consider the vector fields from $\{Q^\zeta\}$ for which $z^0 \in \mathcal{Q}^\zeta_{(r)}$. This condition implies that the values of the derivatives of ζ with respect to only x_1 and x_2 in the point (x^0, u^0) are expressed via $u^0_{(r)}$ and values of derivatives of ζ in (x^0, u^0), containing differentiation with respect to u. The latter values are not constrained.

We introduce the new coordinates $\big\{x_i, \omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2} u, |\alpha| \leqslant r\big\}$ in J^r instead of the standard ones $\{x_i, u_\alpha, |\alpha| \leqslant r\}$. This is a valid change of coordinates since the Jacobian matrix $(\partial\omega_\alpha/\partial u_{\alpha'})$ is nondegenerate. Indeed, it is a triangular matrix with all diagonal entries equal to 1 if the following order of multi-indices is implemented: $\alpha < \beta :\Leftrightarrow |\alpha| < |\beta| \vee (|\alpha| = |\beta| \wedge \alpha_2 < \beta_2)$. Note that $\omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2} u = D_1^{\alpha_1}(Q^\zeta)^{\alpha_2} u$ on $\mathcal{Q}^\zeta_{(r)}$.

Consider the differential function $\hat{L}$ obtained from L by the above procedure of excluding, on the manifold $\mathcal{Q}^\zeta_{(r)}$, the derivatives of u involving differentiations with respect to x_2 (see (1)). Since Q^ζ is a k th co-order singular vector field for L, the function $\hat{L}$ does not depend on the derivatives $u_{(\kappa,0)}, \kappa = k + 1, \ldots, r$. We use this condition step-by-step, starting from the greatest value of κ and re-writing the derivatives in the new coordinates of J^r and in terms of L.

Thus, in the new coordinates the equation $\hat{L}_{u_{(r,0)}}(z^0) = 0$ has the form $L_{\omega_{(r,0)}}(z^0) = 0$. This completes the first step. Then in the second step the equation $\hat{L}_{u_{(r-1,0)}}(z^0) = 0$ implies that

$$L_{\omega_{(r-1,0)}}(z^0) + L_{\omega_{(r-1,1)}}(z^0)\zeta_u(x^0, u^0) = 0.$$

We split with respect to the value $\zeta_u(x^0, u^0)$ since it is unconstrained. As a result, we obtain the equations $L_{\omega_{(r-1,0)}}(z^0) = 0$ and $L_{\omega_{(r-1,1)}}(z^0) = 0$.

Iterating this procedure, before the μth step, $\mu \in \{1, \ldots, r - k\}$, we derive the equations $L_{\omega_{(r-\mu',\nu)}}(z^0) = 0, \mu' = 0, \ldots, \mu - 2, \nu = 0, \ldots, \mu'$. Then the equation $\hat{L}_{u_{(r-\mu+1,0)}}(z^0) = 0$ implies that

$$\sum_{\nu=0}^{\mu-1} L_{\omega_{(r-\mu+1,\nu)}}(z^0)(\partial_u(Q^\zeta)^\nu u)|_{(x,u)=(x^0,u^0)} = 0.$$

The values $\partial_u^{\nu+1}\zeta(x^0, u^0), \nu = 0, \ldots, \mu - 1$, are unconstrained. Splitting with respect to them, which is equivalent to splitting with respect to $(\partial_u(Q^\zeta)^\nu u)|_{(x,u)=(x^0,u^0)}, \nu = 0, \ldots, \mu - 1$, gives the equations $L_{\omega_{(r-\mu+1,\nu)}}(z^0) = 0, \nu = 0, \ldots, \mu - 1$.

Finally, after the $(r - k)$th step we derive the system $L_{\omega_{(r-\mu',\nu)}}(z^0) = 0, \mu' = 0, \ldots, r - k + 1, \nu = 0, \ldots, \mu'$, which implies condition (2).

Conversely, let an r th-order differential function L be of the form (2) (after a point transformation). For an arbitrary smooth function $\zeta = \zeta(x, u)$ we consider the vector field $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$ and the differential function $\tilde{L} = \check{L}(x, \tilde{\Omega}_{r,k})$ where

$$\tilde{\Omega}_{r,k} = \big(\omega_\alpha = D_1^{\alpha_1}(Q^\zeta)^{\alpha_2}u, \alpha_1 \leqslant k, \alpha_1 + \alpha_2 \leqslant r\big).$$

Then $\operatorname{ord}\tilde{L} = k$ and

$$L|_{\mathcal{Q}^\zeta_{(r)}} = \tilde{L}|_{\mathcal{Q}^\zeta_{(r)}},$$

i.e., $\{Q^\zeta = Q^1 + \zeta Q^2\}$, where $Q^1 = \xi\partial_1 + \partial_2$, $Q^2 = \partial_u$ and ζ runs through the set of smooth functions of (x, u), is a k th co-order singular set for the differential function L in the new variables. We complete the set by the vector fields equivalent to its elements or ∂_u and return to the old variables. As a result, for the differential function L we construct a k th co-order singular two-dimensional module of vector fields $\{Q^\theta = \theta^i Q^i\}$. □

Corollary 2. *A differential function with one dependent and two independent variables admits a k th co-order singular two-dimensional module generated by commuting vector fields if and only if it can be reduced by a point transformation of the variables to a differential function in which all differentiations with respect to one of the independent variables are of order* $\leqslant k$.

Corollary 3. *Any differential function with one dependent and two independent variables (not identically vanishing) admits no ultra-singular two-dimensional module of singular vector fields.*

Note 1. It is obvious that a singular module may contain vector fields whose singularity co-orders are less than the singularity co-order of the whole module. Suppose that $\{Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u\}$ is a singular set of vector fields for a differential function L, and its singularity co-order equals k. Then the values of ζ for which $\operatorname{sco}_L Q^\zeta < k$ are solutions of the equation

$$\sum_{\nu=0}^{r-k} \check{L}_{\omega_{(k,\nu)}}(x, \tilde{\Omega}_{r,k})(\partial_u(Q^\zeta)^\nu u) = 0,$$

where $\tilde{\Omega}_{r,k} = \big(D_1^{\alpha_1}(Q^\zeta)^{\alpha_2}u, \alpha_1 \leqslant k, \alpha_1 + \alpha_2 \leqslant r\big)$ and $\check{L}$ is defined in theorem 1. In other words, the regular values of ζ associated with the vector fields of the maximal singularity co-order k in $\{Q^\zeta\}$ satisfy the inequality

$$\sum_{\nu=0}^{r-k} \check{L}_{\omega_{(k,\nu)}}(x, \tilde{\Omega}_{r,k})(\partial_u(Q^\zeta)^\nu u) \neq 0.$$

4. Singular vector fields of differential equations

We will say that a vector field Q is *(strongly) singular for a differential equation* $\mathcal{L}$ if it is singular for the differential function $L[u]$ which is the left-hand side of the canonical representation $L[u] = 0$ of the equation $\mathcal{L}$. Usually we will omit the attribute 'strongly'.

Since left-hand sides of differential equations are defined up to multipliers which are nonvanishing differential functions, the conditions from definition 4 can be weakened when considering differential equations.

Definition 6. *A vector field Q is called weakly singular for the differential equation $\mathcal{L}$: $L[u] = 0$ if there exists a differential function $\tilde{L} = \tilde{L}[u]$ of an order less than r and*

a nonvanishing differential function $\lambda = \lambda[u]$ of an order not greater than r such that $L|_{\mathcal{Q}_{(r)}} = \lambda\tilde{L}|_{\mathcal{Q}_{(r)}}$. Otherwise Q is called a weakly regular vector field for the differential equation $\mathcal{L}$. If the minimal order of differential functions whose restrictions on $\mathcal{Q}_{(r)}$ coincide, up to nonvanishing functional multipliers, with $L|_{\mathcal{Q}_{(r)}}$ is equal to k $(k < r)$ then the vector field Q is said to be weakly singular of co-order k for the differential equation $\mathcal{L}$.

The notions of ultra-singularity in the weak and the strong sense coincide. Analogous to the case of strong regularity, weakly regular vector fields for the differential equation $\mathcal{L}$ are defined to have weak singularity co-order $r = \operatorname{ord} L$. The weak singularity co-order of a vector field Q for an equation $\mathcal{L}$ will be denoted by $\operatorname{wsco}_{\mathcal{L}} Q$.

Note that strong singularity implies weak singularity and hence weak regularity implies strong regularity. It is obvious that the weak singularity co-order is never greater and may be less than the strong singularity co-order. In particular, strongly regular vector fields may be singular in the weak sense. For example, the equation $u_{ttt} = \mathrm{e}^{u_{xx}}(u_x+u)$ possesses the singular vector field ∂_t whose strong and weak singularity co-order equal 2 and 1, respectively. The same vector field ∂_t is strongly regular and is of weak singularity co-order 1 for the equation $u_t = \mathrm{e}^{u_{xx}}(u_x + u)$.

If Q is a weakly singular vector field for $\mathcal{L}$ then any vector field equivalent to Q is weakly singular for $\mathcal{L}$ with the same co-order of weak singularity.

Weakly singular vector fields are related to characteristic directions (cf [14] concerning characteristic directions and characteristic hypersurfaces): Given a vector field $Q = \xi^i(x,u)\partial_i + \eta(x,u)\partial_u$ weakly singular for a differential equation $\mathcal{L}$, in each point of the manifold $\mathcal{L}$ the vector (ξ^1, ξ^2) is orthogonal to a characteristic direction of the equation $\mathcal{L}$ in this point.

Let $\hat{L}$ be a differential function associated with L on the manifold $\mathcal{Q}_{(r)}$, namely, obtained from L via excluding those derivatives of u which contain differentiations with respect to x_2 in view of equations defining $\mathcal{Q}_{(r)}$. Suppose additionally that $\hat{L}$ is of maximal rank in the derivative u of the highest order k appearing in this differential function, i.e., $\hat{L}_{u_{(k,0)}} \neq 0$ on the solution manifold of the equation $\hat{L} = 0$. Then the weak singularity co-order of Q for the equation $\mathcal{L}$: $L = 0$ equals the order k of $\hat{L}$ and, therefore, the strong singularity co-order of Q. Hence in this case testing that a vector field is weakly singular for a partial differential equation with two independent variables can be implemented via an entirely algorithmic procedure.

Theorem 2. *An r th-order differential equation $\mathcal{L}$: $L[u] = 0$ of maximal rank with one dependent and two independent variables possesses a k th co-order weakly singular two-dimensional module of vector fields if and only if L can be represented, up to point transformations, in the form*

$$L = \Lambda[u]\check{L}(x, \Omega_{r,k}), \tag{3}$$

where Λ is a nonvanishing differential function of order not greater than r, $\Omega_{r,k} = \big(\omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2}u, \alpha_1 \leqslant k, \alpha_1+\alpha_2 \leqslant r\big)$, $\xi \in \{0, u\}$ and $\check{L}_{\omega_\alpha} \neq 0$ for some ω_α with $\alpha_1 = k$.

Proof. We will freely use the notations and definitions from the proof of theorem 1.

Suppose first that a differential equation $\mathcal{L}$: $L[u] = 0$ is of maximal rank and admits a kth co-order weakly singular two-dimensional module of vector fields. Up to point transformations and changes of module basis, we may consider only a set $\{Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u\}$ of singular vector fields in the reduced form.

We fix an arbitrary point $z^0 = \big(x^0, u^0_{(r)}\big) \in \mathcal{L} \subset J^r$ and choose the vector fields from $\{Q^\zeta\}$ for which $z^0 \in \mathcal{Q}^\zeta_{(r)}$. This condition implies that the values of derivatives of ζ with respect

to only x_1 and x_2 in the point (x^0, u^0) are expressed via $u^0_{(r)}$ and values of derivatives of ζ in (x^0, u^0), containing differentiation with respect to u. The latter values are not constrained.

The differential function $\hat{L}$ is obtained from L by excluding, on the manifold $\mathcal{Q}^\zeta_{(r)}$, derivatives of u involving differentiations with respect to x_2 (see (1)). k th co-order weak singularity of Q^ζ for L leads to $\hat{L}_{u_{(\kappa,0)}}(z_0) = 0, \kappa = k+1, \dots, r$. We use this condition step-by-step as in the proof of theorem 1, starting from the greatest value of κ and re-writing the derivatives in the new coordinates $\{x_i, \omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2} u, |\alpha| \leqslant r\}$ of J^r and in terms of L. Therefore,

$$L_{\omega_{(r-\mu',\nu)}}(z^0) = 0, \qquad \mu' = 0, \dots, r-k+1, \quad \nu = 0, \dots, \mu',$$

which is satisfied for any $z^0 \in \mathcal{L}$. Applying the Hadamard lemma to each of these equations and then simultaneously integrating them, we obtain (3) (cf the proof of theorem 1 in [26]).

Conversely, let an r th-order differential function L be of form (3) (after a point transformation). For an arbitrary smooth function $\zeta = \zeta(x, u)$ we consider the vector field $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$ and the differential function $\tilde{L} = \check{L}(x, \tilde{\Omega}_{r,k})$, where

$$\tilde{\Omega}_{r,k} = \big(\omega_\alpha = D_1^{\alpha_1}(Q^\zeta)^{\alpha_2} u, \alpha_1 \leqslant k, \alpha_1 + \alpha_2 \leqslant r\ .$$

Then $\operatorname{ord} \tilde{L} = k$ and $L|_{\mathcal{Q}^\zeta_{(r)}} = \Lambda\tilde{L}|_{\mathcal{Q}^\zeta_{(r)}}$, i.e., $\{Q^\zeta = Q^1 + \zeta Q^2\}$, where $Q^1 = \xi\partial_1 + \partial_2$, $Q^2 = \partial_u$ and ζ runs through the set of smooth functions of (x, u), is a k th co-order weakly singular set for the differential equation $\mathcal{L}$ in the new variables. We complete the set by the vector fields equivalent to its elements or ∂_u and return to the old variables, thereby constructing a kth co-order weakly singular two-dimensional module of vector fields $\{Q^\theta = \theta^i Q^i\}$ for the differential equation $\mathcal{L}$. □

Corollary 4. *A differential equation $\mathcal{L}$: $L[u] = 0$ of maximal rank with one dependent and two independent variables possesses a k th co-order weakly singular two-dimensional module of vector fields if and only if this module is k th co-order strongly singular for $\mathcal{L}$ (possibly in a representation differing from $L[u] = 0$ in multiplication by a nonvanishing differential function of u).*

Definition 7. *A vector field Q is called a singular reduction operator of a differential equation $\mathcal{L}$ if Q is both a reduction operator of $\mathcal{L}$ and a weakly singular vector field of $\mathcal{L}$.*

5. Example: evolution equations

In this section we investigate singular reduction operators of (1 + 1)-dimensional evolution equations of the form

$$u_t = H(t, x, u_{(r,x)}), \tag{4}$$

where $r > 1, u_0 := u, u_k = \partial^k u/\partial x^k, u_{(r,x)} = (u_0, u_1, \dots, u_r)$ and $H_{u_r} \neq 0$. (We revert to the notation t and x for x_1 and x_2, respectively, and change the notations of the corresponding derivatives.) Evolution equations are quite specific from the point of view of singular vector fields and singular reduction operators.

Proposition 2. *A vector field $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$ is singular for the differential function $L = u_t - H(t, x, u_{(r,x)})$ of order $r > 1$ if and only if $\tau = 0$. The co-order of singularity of any singular vector field for any such differential function equals 1.*

Proof. Suppose that $\tau \neq 0$. Excluding the derivative u_t from L according to the equation $u_t = \eta/\tau - \xi u_x/\tau$ results in a differential function $\tilde{L} = \eta/\tau - \xi u_x/\tau - H(t, x, u_{(r,x)})$. Since

ord $\tilde{L} = r = \operatorname{ord} L$, the vector field Q is not singular in this case. Therefore, for the vector field Q to be singular, the coefficient τ has to vanish.

If $\tau = 0$ and therefore $\xi \neq 0$, all the derivatives u_k, $k = 1, \dots, r$, can be expressed, on the manifold $\mathcal{Q}_{(r)}$ via t, x and u: $u_k = (\partial_x + \zeta\partial_u)^{k-1}\zeta$, $k = 1, \dots, r$, where $\zeta = \eta/\xi$. Using these expressions for excluding the derivatives u_k, $k = 1, \dots, r$ from L, we obtain the differential function

$$\tilde{L} = u_t - \tilde{H}(t, x, u), \qquad \tilde{H} := H(t, x, u, \zeta, \zeta_x + \zeta\zeta_u, \dots, (\partial_x + \zeta\partial_u)^{r-1}\zeta),$$

whose order equals 1. Hence the vector field Q is singular for the differential function L, and its singularity co-order equals 1. □

Corollary 5. *For any* $(1 + 1)$*-dimensional evolution equation, the corresponding differential function possesses exactly one set of singular vector fields in the reduced form, namely,* $S = \{\partial_x + \zeta(x, u)\partial_u\}$. *The singularity co-order of S equals 1.*

It is obvious that under the condition $H_{u_r} \neq 0$ a vector field is singular for the differential function $u_t - H(t, x, u_{(r,x)})$ if and only if it is weakly singular for the differential equation $u_t = H(t, x, u_{(r,x)})$. Hence we do not distinguish between strong and weak singularity (cf corollary 4).

The vector fields ∂_2 and ∂_u generating the singular module associated with S commute and the differential function L contains only first-order differentiation with respect to t (namely, in the form of the derivative u_t). This perfectly agrees with corollary 2.

We fix an arbitrary equation $\mathcal{L}$ of form (4) and denote by $\mathcal{Q}_0(\mathcal{L})$ the set of reduction operators of $\mathcal{L}$, belonging to S. For the equation $\mathcal{L}$ and $Q \in \mathcal{Q}_0(\mathcal{L})$, the conditional invariance criterion implies only the single rth-order equation

$$\zeta_t + \zeta_u\tilde{H} = \tilde{H}_x + \zeta\tilde{H}_u, \qquad \tilde{H} := H(t, x, u, \zeta, \zeta_x + \zeta\zeta_u, \dots, (\partial_x + \zeta\partial_u)^{r-1}\zeta),$$

with respect to the single unknown function ζ with three independent variables t, x and u, which we will denote by $\mathrm{DE}_0(\mathcal{L})$. In other words, the system of determining equations in this case consists of the single equation $\mathrm{DE}_0(\mathcal{L})$ and, therefore, is not overdetermined. $\mathrm{DE}_0(\mathcal{L})$ is the compatibility condition of the equations $u_x = \zeta$ and $\mathcal{L}$.

Theorem 3. *Up to the equivalences of operators and solution families, for any equation of form (4) there exists a one-to-one correspondence between one-parametric families of its solutions and reduction operators with zero coefficients of* ∂_t. *Namely, each operator of this kind corresponds to the family of solutions which are invariant with respect to this operator. The problems of the construction of all one-parametric solution families of equation (4) and the exhaustive description of its reduction operators with zero coefficients of* ∂_t *are completely equivalent.*

Proof. Let $\mathcal{L}$ be an equation from class (4) and $Q = \partial_x + \zeta\partial_u \in \mathcal{Q}_0(\mathcal{L})$, i.e., the coefficient $\zeta = \zeta(t, x, u)$ satisfies the equation $\mathrm{DE}_0(\mathcal{L})$. An ansatz constructed with Q has the form $u = f(t, x, \varphi(\omega))$, where $f = f(t, x, \varphi)$ is a given function, $f_\varphi \neq 0, \varphi = \varphi(\omega)$ is the new unknown function and $\omega = t$ is the invariant independent variable. This ansatz reduces $\mathcal{L}$ to a first-order ordinary differential equation $\mathcal{L}'$ in φ, solvable with respect to φ'. The general solution of the reduced equation $\mathcal{L}'$ can be represented in the form $\varphi = \varphi(\omega, \varkappa)$, where $\varphi_\varkappa \neq 0$ and $\varkappa$ is an arbitrary constant. Substituting this solution into the ansatz results in the one-parametric family $\mathcal{F}$ of solutions $u = \tilde{f}(t, x, \varkappa)$ of $\mathcal{L}$ with $\tilde{f} = f(t, x, \varphi(t, \varkappa))$. Expressing the parameter $\varkappa$ from the equality $u = \tilde{f}(t, x, \varkappa)$, we obtain that $\varkappa = \Phi(t, x, u)$, where $\Phi_u \neq 0$. Then $\zeta = u_x = -\Phi_x/\Phi_u$ for any $u \in \mathcal{F}$, i.e., for any admissible value of $(t, x, \varkappa)$. This implies that $\zeta = -\Phi_x/\Phi_u$ for any admissible value of (t, x, u).

Conversely, suppose that $\mathcal{F} = \{u = f(t, x, \varkappa)\}$ is a one-parametric family of solutions of $\mathcal{L}$. The derivative $f_\varkappa$ is nonzero since the parameter $\varkappa$ is essential. We express $\varkappa$ from the equality $u = f(t, x, \varkappa)$: $\varkappa = \Phi(t, x, u)$ for some function $\Phi = \Phi(t, x, u)$ with $\Phi_u \neq 0$. Consider the operator $Q = \partial_x + \zeta\partial_u$, where the coefficient $\zeta = \zeta(t, x, u)$ is defined by $\zeta = -\Phi_x/\Phi_u$. $Q[u] = 0$ for any $u \in \mathcal{F}$. The ansatz $u = f(t, x, \varphi(\omega))$, where $\omega = t$, associated with Q, reduces $\mathcal{L}$ to the equation $\varphi_\omega = 0$. Therefore [26], $Q \in \mathcal{Q}_0(\mathcal{L})$ and hence the function ζ satisfies $\mathrm{DE}_0(\mathcal{L})$. □

Corollary 6. *The nonlinear* $(1+2)$*-dimensional evolution equation* $\mathrm{DE}_0(\mathcal{L})$ *is reduced by the composition of the nonlocal substitution* $\zeta = -\Phi_x/\Phi_u$*, where* Φ *is a function of* (t, x, u)*, and the hodograph transformation*

$$\begin{aligned} &\textit{the new independent variables:} && \tilde{t} = t, && \tilde{x} = x, && \varkappa = \Phi,\\ &\textit{the new dependent variable:} && \tilde{u} = u \end{aligned}$$

to the initial equation $\mathcal{L}$ *in the function* $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}, \varkappa)$ *with* $\varkappa$ *playing the role of a parameter.*

Note 2. One-parametric families $u = f(t, x, \varkappa)$ and $u = \tilde{f}(t, x, \tilde{\varkappa})$ are defined to be equivalent if they consist of the same functions and differ only by parameterizations, i.e., if there exists a function $\zeta = \zeta(\varkappa)$ such that $\zeta_\varkappa \neq 0$ and $\tilde{f}(t, x, \zeta(\varkappa)) = f(t, x, \varkappa)$. Equivalent one-parametric families of solutions are associated with the same operator from $\mathcal{Q}_0(\mathcal{L})$ and have to be identified.

Note 3. The triviality of the above ansatz and the reduced equation results from the above special representation for the solutions of the determining equation. Under this approach difficulties in the construction of ansatzes and the integration of the reduced equations are replaced by difficulties in obtaining the representation for the coefficients of the reduction operators.

The above consideration shows that for any evolution equation $\mathcal{L}$ the conventional partition of the set $\mathfrak{Q}(\mathcal{L})$ of its reduction operators with the conditions $\tau \neq 0$ and $\tau = 0$ is natural since it coincides with the partition of $\mathfrak{Q}(\mathcal{L})$ into the singular and regular subsets. *This is a specific property of evolution equations which does not hold for general partial differential equations in two independent variables.* After factorizing the subsets of $\mathfrak{Q}(\mathcal{L})$ with respect to the usual equivalence relation of reduction operators, we obtain two different cases of inequivalent reduction operators (the regular case $\tau = 1$ and the singular case $\tau = 0$ and $\xi = 1$), which have to be studied separately.

Singular reduction operators of $\mathcal{L}$ are described in a unified 'no-go' way. All singular reduction operators of $\mathcal{L}$ have the same singularity co-order equal to 1 and hence reduce $\mathcal{L}$ to first-order ordinary differential equations. The coincidence of the singularity co-orders guarantees the existence of a bijection between the set of singular reduction operators of $\mathcal{L}$ and the set of one-parametric families of its solutions (up to the natural equivalence relations in these sets). As a result, in the case $\tau = 0$ and $\xi = 1$ the determining equation for a single coefficient of ∂_u is reduced, with no additional assumptions and conditions, to the initial equation $\mathcal{L}$ by a nonlocal transformation (cf corollary 6).

The regular case $\tau = 1$ is more complicated than the singular one. It essentially depends on the structure of the equation including the order, the kind of nonlinearities, etc. Up to now there are no exhaustive results on regular reduction operators even for second-order evolution equations. Only certain subclasses of such equations were investigated. See, e.g., [1, 5, 21, 22] for the complete classifications of regular reduction operators for some subclasses of second-order evolution equations parameterized by functions of single arguments. For example, even

for the class of nonlinear diffusion equations of the general form $u_t = (f(u)u_x)_x$ (a classical example of solving a group classification problem for partial differential equations [17]), the set of values of the parameter-function f which correspond to equations possessing non-Lie regular reduction operators has not yet been found. Most evolution equations have no regular reduction operators. A simple example is

$$u_t = u_{xx} + u\,\mathrm{e}^{u_x} + x\,\mathrm{e}^{2u_x} + t\,\mathrm{e}^{3u_x} + \mathrm{e}^{4u_x} + \mathrm{e}^{5u_x}.$$

Some evolution equations (the linear ones [10, 21], Burgers' equation [13], etc) possess so many regular reduction operators that 'no-go' statements like those for singular reduction operators are true for them, but the nature of this 'no-go' differs from the 'no-go' of the singular case and is related to the property of linearity or linearizability of the corresponding evolution equations.

6. Example: nonlinear wave equations

The next example which we study in detail within the framework of singular reduction operators is given by the class of nonlinear wave equations (in the characteristic, or light-cone, variables) of the general form

$$u_{12} = F(u). \tag{5}$$

Here F is an arbitrary smooth function of u. This class essentially differs from the class of evolution equations within the framework of singular vector fields. The main differences are that each differential function corresponding to an equation from class (5) has two singular sets of vector fields and these sets contain vector fields of lower singularity co-orders than the singularity co-orders of the whole sets. Thus, for any F the vector field $Q = \xi^i(x,u)\partial_i + \eta(x,u)\partial_u$ is singular for the corresponding differential function $L = u_{12} - F(u)$ if and only if $\xi^1\xi^2 = 0$. Moreover, it is obvious that there are no differences between strong and weak singularity of vector fields for equations from class (5). Indeed, suppose that $\xi^2 \neq 0$. Excluding the derivatives u_2 and u_{12} from L according to (1), we obtain a differential function $\tilde{L}$ with the coefficient $-\xi^1/\xi^2$ of u_{11}. We have $\operatorname{ord}\tilde{L} < 2$ if and only if $\xi^1 = 0$.

Therefore, *for any F the differential function $L = u_{12} - F(u)$ possesses exactly two sets of singular vector fields in the reduced form*, $S = \{\partial_2 + \zeta(x,u)\partial_u\}$ and $S^* = \{\partial_1 + \zeta^*(x,u)\partial_u\}$. The vector fields equivalent to ∂_u are not suitable as reduction operators. Any singular vector field of L is equivalent to one of the above fields. Moreover, each equation of the form (5) admits the discrete symmetry transformation permuting the variables x_1 and x_2. This transformation generates a one-to-one mapping between S and S^* (cf corollary 1). Hence it suffices, up to equivalence of vector fields (and permutation of x_1 and x_2), to investigate only singular reduction operators from the set S.

For an equation $\mathcal{L}$ from class (5) and an operator $Q = \partial_2 + \zeta\partial_u$ the conditional invariance criterion takes the form

$$(\zeta_{12} + \zeta_{1u}u_2 + \zeta_{2u}u_1 + \zeta_{uu}u_1u_2 + \zeta_u u_{12})|_{\mathcal{L}\cap\mathcal{Q}_{(2)}} = \zeta F_u.$$

The intersection $\mathcal{L}\cap\mathcal{Q}_{(2)}$ is singled out from J^2 by the equations $u_2 = \zeta$, $\zeta_1 + \zeta_u u_1 = F$ and $u_{12} = F$. Our further considerations therefore depend on the values of ζ_u and F_u. We analyse all the possible cases.

Let $\zeta_u = 0$ and $F_u = 0$. Then Q is an ultra-singular vector field for the differential function L. The third equation defining $\mathcal{L}\cap\mathcal{Q}_{(2)}$ takes the form $\zeta_1 = F$ and contains no derivatives of u. It should be assumed as a condition with respect to ζ and hence the conditional invariance

criterion is identically satisfied in this case. An ansatz constructed with the operator Q is $u = \varphi(\omega) + \int \zeta \mathrm{d}x_2$, where $\omega = x_1$. It reduces equation (5) to an identity. This is explained by the ultra-singularity of the reduction operator Q.

If $\zeta_u = 0$ and $F_u \neq 0$, the singularity co-order of Q for the differential function L equals 0. The third equation defining $\mathcal{L} \cap \mathcal{Q}_{(2)}$ again takes the form $\zeta_1 = F$ but now can be solved with respect to u: $u = \breve{F}(\zeta_1)$, where $\breve{F}$ is the inverse to F. Then the conditional invariance criterion is equivalent to the equation $\zeta_{12} = \zeta F_u(\breve{F}(\zeta_1))$ with respect to ζ. The ansatz constructed with the operator Q reduces equation (5) to the algebraic equation $F(\varphi + \int \zeta \, \mathrm{d}x_2) = \zeta_1$ for the function φ in agreement with the singularity co-order 0 of Q. Indeed, inverting F, we obtain the equality $\varphi = \breve{F}(\zeta_1) - \int \zeta \mathrm{d}x_2$ whose right-hand side does not depend on x_2 in view of the equation on ζ. Conversely, let us fix a solution $u = f(x)$ of equation (5) and set $\zeta = f_2$. Then $\zeta_{12} = \zeta F_u(\breve{F}(\zeta_1))$, i.e., in view of the conditional invariance criterion $Q = \partial_2 + \zeta \partial_u$ is a reduction operator of equation (5), and $\zeta_u = 0$. The solution $u = f(x)$ is invariant with respect to Q. The above results can be summed up as follows:

Theorem 4. *For any equation from class (5) with $F_u \neq 0$ there exists a one-to-one correspondence between its solutions and reduction operators of the form $Q = \partial_2 + \zeta(x)\partial_u$ (resp. $Q^* = \partial_1 + \zeta^*(x)\partial_u$). Namely, each operator of this kind is of singularity co-order 0 and corresponds to the solution which is invariant with respect to this operator. The problems of solving an equation from class (5) with $F_u \neq 0$ and the exhaustive description of its reduction operators of the above form are completely equivalent.*

Corollary 7. *Any solution $u = f(x)$ of equation (5) with $F_u \neq 0$ is invariant with respect to two reduction operators $Q = \partial_2 + \zeta(x)\partial_u$ and $Q^* = \partial_1 + \zeta^*(x)\partial_u$ of equation (5), having singularity co-order 0. Here $\zeta = f_2$ and $\zeta^* = f_1$. The property of possessing the same invariant solution of equation (5) establishes a canonical bijection $Q \leftrightarrow Q^*$ between the sets of reduction operators of singularity co-order 0. The adjoint values of ζ and ζ^* are connected by the formulae*

$$\zeta^* = \frac{\zeta_{11}}{F_u(\breve{F}(\zeta_1))}, \qquad \zeta = \frac{\zeta^*_{22}}{F_u(\breve{F}(\zeta^*_2))}.$$

The regular values of ζ for which the singularity co-order of Q coincides with the singularity co-order of the whole family S (and equals 1) satisfy the condition $\zeta_u \neq 0$. The third equation defining $\mathcal{L} \cap \mathcal{Q}_{(2)}$ then provides the following expression for u_1:

$$u_1 = \frac{F - \zeta_1}{\zeta_u} =: \zeta^*.$$

The conditional invariance criterion implies only the single equation

$$\zeta_{12} + \zeta\zeta_{1u} + (\zeta_{2u} + \zeta\zeta_{uu})\frac{F - \zeta_1}{\zeta_u} + \zeta_u F = \zeta F_u \tag{6}$$

with respect to the single function ζ, i.e., in this case the system of determining equations consists of the single equation (6) and, therefore, is not overdetermined.

Equation (6) can be rewritten in the form of the compatibility condition

$$\zeta_1 + \zeta^*\zeta_u = \zeta^*_2 + \zeta\zeta^*_u = F$$

of the equations $u_1 = \zeta^*$, $u_2 = \zeta$ and $u_{12} = F$. It is obvious that $\zeta^*_u \neq 0$. Due to symmetry with respect to the permutation of x_1 and x_2, we obtain the following statement.

Proposition 3. *For any equation from class (5), there exists a canonical bijection $Q \leftrightarrow Q^*$ between sets of its singular reduction operators of the forms $Q = \partial_2 + \zeta(x,u)\partial_u$ and $Q^* = \partial_1 + \zeta^*(x,u)\partial_u$, where $\zeta_u \neq 0$ and $\zeta^*_u \neq 0$. This bijection is given by the formulae*

$$Q \to Q^*: \quad \zeta^* = \frac{F - \zeta_1}{\zeta_u}, \qquad Q^* \to Q: \quad \zeta = \frac{F - \zeta^*_2}{\zeta^*_u}.$$

A solution of equation (5) is invariant with respect to the operator Q if and only if it is invariant with respect to the operator Q^*.

Theorem 5. *Up to the equivalence of solution families, for any equation from class (5) with $F_u \neq 0$ there exists a one-to-one correspondence between one-parametric families of its solutions and reduction operators of the form $Q = \partial_2 + \zeta(x,u)\partial_u$, where $\zeta_u \neq 0$ (resp. $Q^* = \partial_1 + \zeta^*(x,u)\partial_u$, where $\zeta^*_u \neq 0$). Namely, any such operator corresponds to the family of solutions which are invariant with respect to this operator. The problems of the construction of all one-parametric solution families of an equation from class (5) with $F_u \neq 0$ and the exhaustive description of its reduction operators of the above form are completely equivalent.*

Proof. In view of proposition 3, it is sufficient to consider only operators with zero coefficient of ∂_1. Although the proof is similar to the proof of the analogous statement for evolution equations it differs from it in essential details and will therefore be presented completely.

An ansatz constructed with the operator $Q = \partial_2 + \zeta(x,u)\partial_u$ has the form $u = f(x, \varphi(\omega))$, where $f = f(x,\varphi)$ is a given function, $f_\varphi \neq 0$, $\varphi = \varphi(\omega)$ is the new unknown function and $\omega = x_1$ is the invariant independent variable. Here $\zeta_u \neq 0$ implies $f_{2\varphi} \neq 0$. Hence this ansatz reduces equation (5) to a first-order ordinary differential equation $\mathcal{L}'$ in φ, which is solvable with respect to φ'. The general solution of the reduced equation $\mathcal{L}'$ essentially depends on an arbitrary constant $\varkappa$: $\varphi = \varphi(\omega, \varkappa)$, where $\varphi_\varkappa \neq 0$. Substituting the general solution into the ansatz gives the one-parametric family $\mathcal{F}$ of solutions $u = \tilde{f}(x,\varkappa)$ of (5) with $\tilde{f} = f(x, \varphi(x_1, \varkappa))$.

Conversely, suppose that $F_u \neq 0$ and $\mathcal{F} = \{u = f(x,\varkappa)\}$ is a one-parametric family of solutions of (5). The derivative $f_\varkappa$ is nonzero since the parameter $\varkappa$ is essential. Therefore, $f_{12\varkappa} = f_\varkappa F_u(f) \neq 0$. We express $\varkappa$ from the equality $u = f(x,\varkappa)$: $\varkappa = \Phi(x,u)$ for some function $\Phi = \Phi(x,u)$ with $\Phi_u \neq 0$. Consider the operator $Q = \partial_2 + \zeta\partial_u$, where the coefficient $\zeta = \zeta(x,u)$ is defined by the formula $\zeta = -\Phi_2/\Phi_u$. $Q[u] = 0$ for any $u \in \mathcal{F}$. The ansatz $u = f(x, \varphi(\omega))$, where $\omega = x_1$, associated with Q, reduces (5) to the equation $\varphi_\omega = 0$ since $f_{2\varkappa} \neq 0$. Therefore [26], Q is a reduction operator of equation (5) and hence the function ζ satisfies equation (6). Moreover, we have $\zeta_u \neq 0$ since otherwise the operator Q would reduce (5) to an algebraic equation with respect to φ. □

Corollary 8. *Any adjoint singular reduction operators $Q = \partial_2 + \zeta(x,u)\partial_u$ and $Q^* = \partial_1 + \zeta^*(x,u)\partial_u$ of equation (5) (where necessarily $\zeta_u \neq 0$ and $\zeta^*_u \neq 0$) are associated with the same one-parametric family of solutions of this equation.*

Let ζ be an arbitrary solution of equation (6). Then $\zeta_u \neq 0$ and $Q = \partial_2 + \zeta(x,u)\partial_u$ is a reduction operator of equation (5). Consider a one-parametric family $\mathcal{F} = \{u = f(x,\varkappa)\}$ of solutions of (5), which are invariant with respect to Q. (Such a family exists in view of theorem 5.) Expressing the parameter $\varkappa$ from the equality $u = \tilde{f}(x,\varkappa)$, we obtain that $\varkappa = \Phi(x,u)$, where $\Phi_u \neq 0$. $\zeta = u_2 = -\Phi_2/\Phi_u$ for any $u \in \mathcal{F}$, i.e., for any admissible values of $(x,\varkappa)$. This implies that the representation $\zeta = -\Phi_2/\Phi_u$ is true for any admissible value of (x,u). This provides the background for the following statement.

Corollary 9. *The nonlinear three-dimensional equation (6) is reduced by composition of the Bäcklund transformation* $\zeta = -\Phi_2/\Phi_u$, $\zeta^* = -\Phi_1/\Phi_u$, *where* Φ *is a function of* (x, u), *and the hodograph transformation*

$$\begin{aligned} &\textit{the new independent variables:} \quad \tilde{x}_1 = x_1, \qquad \tilde{x}_2 = x_2, \qquad \varkappa = \Phi, \\ &\textit{the new dependent variable:} \quad \tilde{u} = u \end{aligned}$$

to equation (5) for the function $\tilde{u} = \tilde{u}(\tilde{x}, \varkappa)$ *with* $\varkappa$ *playing the role of a parameter.*

Proof. We take an arbitrary solution ζ of equation (6) (the condition $\zeta_u \neq 0$ is implicitly assumed to be satisfied) and set $\zeta^* = (F - \zeta_1)/\zeta_u$. In view of the Frobenius theorem, the system $\Phi_2 + \zeta\Phi_u = 0$, $\Phi_1 + \zeta^*\Phi_u = 0$ with respect to the function $\Phi = \Phi(x, u)$ is compatible since its compatibility condition $\zeta_1 + \zeta^*\zeta_u = \zeta_2^* + \zeta\zeta_u^*$ coincides with (6) and hence is identically satisfied. We choose a nonconstant solution Φ of this system. Then $\Phi_u \neq 0$, $\zeta = -\Phi_2/\Phi_u$ and $\zeta^* = -\Phi_1/\Phi_u$. After the hodograph transformation, the latter equations take the form $\tilde{u}_{\tilde{x}_2} = \zeta(\tilde{x}, \tilde{u})$ and $\tilde{u}_{\tilde{x}_1} = \zeta^*(\tilde{x}, \tilde{u})$. This directly implies that for any value of $\varkappa$ the function $\tilde{u} = \tilde{u}(\tilde{x}, \varkappa)$ satisfies equation (5). The parameter $\varkappa$ is essential in $\tilde{u}$ since $\tilde{u}_\varkappa = 1/\Phi_u \neq 0$.

It follows from the proof of theorem 5 that the application of the inverse transformations to a one-parametric family of solutions of equation (5) results in a solution of equation (6). □

Note 4. For any equation from class (5) with $F_u = 0$, reduction operators of the form $Q = \partial_2 + \zeta(x, u)\partial_u$, where $\zeta_u \neq 0$ (resp. $Q^* = \partial_1 + \zeta^*(x, u)\partial_u$, where $\zeta_u^* \neq 0$) also are bijectively associated with one-parametric families of its solutions, having the form $\{u = f(x, \varkappa)\}$ where $f_{1\varkappa} \neq 0$ (resp. $f_{2\varkappa} \neq 0$). The one-parametric families with $f_{1\varkappa} = 0$ (resp. $f_{2\varkappa} = 0$) necessarily existing in this case correspond to ultra-singular reduction operators with $\zeta_u = 0$ (resp. $\zeta_u^* = 0$), and the correspondence is not one-to-one.

The above investigation of singular reduction operators of nonlinear wave equations of the form (5) shows that for these equations the natural partition of the corresponding sets of reduction operators is into triples of subsets singled out by the conditions

$$(1)\ \xi^1 = 0; \qquad (2)\ \xi^2 = 0; \qquad (3)\ \xi^1\xi^2 \neq 0.$$

After the factorization with respect to the equivalence relation of vector fields, we obtain three subsets of reduction operators, which have to be investigated separately. The defining conditions for these subsets are, respectively,

$$(1)\ \xi^1 = 0, \quad \xi^2 = 1; \qquad (2)\ \xi^2 = 0, \quad \xi^1 = 1; \qquad (3)\ \xi^1 \neq 0, \quad \xi^2 = 1.$$

Since any equation from class (5) admits the point symmetry permuting x_1 and x_2, the second case is reduced to the first one and can be omitted. Finally we have two essentially different cases after factorization: the singular case $\xi^1 = 0$, $\xi^2 = 1$ and the regular case $\xi^1 \neq 0$, $\xi^2 = 1$. The gauge $\xi^2 = 1$ is not uniquely possible in the regular case and may be varied for optimizing the further consideration of this case.

Consider the other standard form

$$u_{11} - u_{22} = F(u) \tag{7}$$

of nonlinear wave equations, obtained from (5) via the point transformation $\tilde{x}_1 = x_1 - x_2$, $\tilde{x}_2 = x_1 + x_2$, $\tilde{u} = u$. Using this transformation, all the results derived for class (5) can easily be extended to class (7). Thus, any equation of form (7) possesses two singular sets of reduction operators, singled out by the conditions $\xi^1 = -\xi^2$ and $\xi^1 = \xi^2$, and one regular set of reduction operators, associated with the condition $\xi^1 \neq \pm\xi^2$. The singular sets are mapped to each other by alternating the sign of x_2 and hence one of them can be excluded from the

consideration. After factorization with respect to the equivalence relation of vector fields, we have two cases for our further study: the singular case $\xi^1 = \xi^2 = 1$ and the regular case $\xi^1 \neq \pm 1, \xi^2 = 1$.

For nonlinear wave equations of the general form $u_{11} - (G(u)u_2)_2 = F(u)$, where $G(u) > 0$, the natural partitions of the sets of reduction operators are determined by more complicated conditions depending on the parameter-function G. We will not discuss these equation here. We only remark that the singular sets of the corresponding reduction operators are associated with the conditions $\xi^2 = \sqrt{G}\xi^1$ and $\xi^2 = -\sqrt{G}\xi^1$, respectively.

The above examples underline that the application of the conventional partition for factorization of sets of reduction operators often leads to the splitting of uniform cases and to combining essentially different ones. As a result, the derived systems of determining equations for the coefficients of reduction operators is far from optimal and difficult to investigate. Therefore, natural partitions based on taking into account the structure of singular families of reduction operators offers a decisive advantage.

7. Reduction operators and parametric families of solutions

Proposition 4. *Let Q be a reduction operator of an equation $\mathcal{L}$. Then the weak singularity co-order of Q for $\mathcal{L}$ equals the essential order of the corresponding reduced ordinary differential equation.*

Proof. We carry out a point transformation in such a way that in the new variables the operator Q has the form $Q = \partial_{x_2}$. (For convenience, for the new variables we use the same notations as for the old ones.) Then an ansatz constructed with Q is $u = \varphi(\omega)$, where $\varphi = \varphi(\omega)$ is the new unknown function and $\omega = x_1$ is the invariant independent variable. The manifold $\mathcal{Q}_{(r)}$ is defined by the system $u_\alpha = 0$, where $\alpha = (\alpha_1, \alpha_2), \alpha_2 > 0, \alpha_1 + \alpha_2 \leqslant r = \operatorname{ord} L$.

Since $Q \in \mathcal{Q}(\mathcal{L})$, there exist differential functions $\check{\lambda} = \check{\lambda}[\varphi]$ and $\check{L} = \check{L}[\varphi]$ of an order not greater than r such that $L|_{u=\varphi(\omega)} = \check{\lambda}\check{L}$ (cf [26]). The function $\check{\lambda}$ does not vanish and may depend on x_2 as a parameter. The function $\check{L}$ is assumed to be of minimal order $\check{r}$ which may be attained up to the equivalence generated by nonvanishing multipliers. Then the reduced equation $\check{\mathcal{L}}$: $\check{L} = 0$ has essential order $\check{r}$.

The condition $\operatorname{wsco}_{\mathcal{L}} Q = k$ means that there exists a strictly k th-order differential function $\tilde{L} = \tilde{L}[u]$ and a nonvanishing differential function $\tilde{\lambda} = \tilde{\lambda}[u]$ of an order not greater than r, which depend at most on x and derivatives of u with respect to x_1, such that $L|_{\mathcal{Q}_{(r)}} = \tilde{\lambda}\tilde{L}|_{\mathcal{Q}_{(r)}}$.

If $\check{r}$ would be less than k, we could use $\tilde{\lambda}_{\text{new}} = \check{\lambda}|_{u\rightsquigarrow\varphi}$ and $\tilde{L}_{\text{new}} = \check{L}|_{u\rightsquigarrow\varphi}$ in the definition of weak singularity and would arrive at the contradiction $\operatorname{wsco}_{\mathcal{L}} Q \leqslant \operatorname{ord} \tilde{L}_{\text{new}} = \check{r} < k$. Therefore, $\check{r} \geqslant k$. (Here, '$y \rightsquigarrow z$' means that the value y should be substituted instead of the value z.)

Suppose that $\check{r} > k$. We have the equality $\check{\lambda}\check{L} = (\tilde{\lambda}\tilde{L})|_{u=\varphi(\omega)}$ in which the variable x_2 plays the role of a parameter. Fixing a value x_2^0 of x_2, we obtain the representation

$$\check{L} = \Lambda[\varphi]\tilde{L}\Big|_{u=\varphi(\omega),x_2=x_2^0}, \qquad \Lambda := \frac{\tilde{\lambda}|_{u=\varphi(\omega)}}{\check{\lambda}}\Big|_{x_2=x_2^0} \neq 0.$$

Since $\operatorname{ord} \tilde{L}|_{u=\varphi(\omega),x_2=x_2^0} \leqslant k < \check{r}$, this representation contradicts the condition that $\check{r}$ is the essential order of the reduced equation $\check{\mathcal{L}}$. Therefore, $\check{r} = k$. The inverse change of variables preserves the claimed property. □

Corollary 10. *Let Q be a reduction operator of an equation $\mathcal{L}$. Then the weak singularity co-order of Q for $\mathcal{L}$ equals the maximal number of essential parameters in families of Q-invariant solutions of $\mathcal{L}$.*

Proof. The essential order $\check{r}$ of the reduced ordinary differential equation $\check{\mathcal{L}}$ associated with Q coincides with the weak singularity co-order of Q for $\mathcal{L}$. The maximal number of essential parameters in solutions of $\check{\mathcal{L}}$ equals the order of $\check{\mathcal{L}}$. The substitution of these solutions into the corresponding ansatz leads to parametric families of Q-invariant solutions of $\mathcal{L}$, and all Q-invariant solutions of $\mathcal{L}$ are obtained in this way. Therefore, the maximal number of essential parameters in families of Q-invariant solutions of $\mathcal{L}$ equals $\check{r}$. □

Corollary 11. *Let Q be a k th co-order weakly singular reduction operator of an equation $\mathcal{L}$. Suppose additionally that a differential function of minimal order, associated with L on the manifold $\mathcal{Q}_{(r)}$ up to a nonvanishing multiplier, is of maximal rank in the derivative of u of the highest order k appearing in this differential function. Then $\mathcal{L}$ possesses a k-parametric family of Q-invariant solutions, and any Q-invariant solution of $\mathcal{L}$ belongs to this family.*

Proof. Under this assumption, the reduced ordinary differential equation $\check{\mathcal{L}}$ associated with Q can be written in normal form and hence has a k-parametric general solution which contains all solutions of $\check{\mathcal{L}}$. Substituting it into the corresponding ansatz, this solution gives a k-parametric family of Q-invariant solutions of $\mathcal{L}$. There are no other Q-invariant solutions of $\mathcal{L}$. □

Corollary 12. *Suppose that a differential function of minimal order, associated with L on the manifold $\mathcal{Q}_{(r)}$ up to a nonvanishing multiplier, is of maximal rank in the highest order derivative of u appearing in this differential function. If the maximal number of essential parameters in families of Q-invariant solutions of $\mathcal{L}$ is not less than the weak singularity co-order of Q for $\mathcal{L}$ then Q is a reduction operator of $\mathcal{L}$.*

Proof. Point transformations of the variables do not change the claimed property. We use the variables and notations from the proof of proposition 4. Consider the differential function $\hat{L}[\varphi] = \tilde{L}|_{u=\varphi(\omega)}$. It depends on x_2 as a parameter and $\operatorname{ord}\hat{L} = k$. Due to the condition of maximal rank, we can resolve the equation $\hat{L} = 0$ with respect to the highest order derivative $\varphi^{(k)}$: $\varphi^{(k)} = R[\varphi]$, where $\operatorname{ord} R < k$.

If $R_{x_2} \neq 0$, splitting with respect to x_2 in the equation $\hat{L} = 0$ results in an ordinary differential equation $\tilde{R}[\varphi] = 0$ of an order lower than k. Any Q-invariant solution of $\mathcal{L}$ has the form $u = \varphi(\omega)$, where the function φ satisfies, in particular, the equation $\tilde{R}[\varphi] = 0$. This contradicts the condition that the maximal number of essential parameters in families of Q-invariant solutions of $\mathcal{L}$ is not less than k.

Therefore, $R_{x_2} = 0$, i.e., the equation $\varphi^{(k)} = R[\varphi]$ is a reduced equation which is obtained from $\mathcal{L}$ by the substitution of the ansatz $u = \varphi(\omega)$ constructed with the operator $Q = \partial_2$. □

Note 5. For any operator Q, the maximal number of essential parameters in families of Q-invariant solutions of $\mathcal{L}$ cannot be greater than $\operatorname{wsco}_L Q$.

Summing up the above consideration, we can formulate the following statement.

Proposition 5. *Suppose that a differential function of minimal order, associated with the differential function $L[u]$ on the manifold $\mathcal{Q}_{(r)}$ ($r = \operatorname{ord} L$) up to a nonvanishing multiplier, is of maximal rank in the highest order derivative of u appearing in this differential function. Then any two of the following properties imply the third one:*

(1) Q is a reduction operator of the equation $\mathcal{L}$: $L = 0$.

(2) The weak singularity co-order of Q for $\mathcal{L}$ equals k $(0 \leqslant k \leqslant r)$.
(3) The equation $\mathcal{L}$ possesses a k-parametric family of Q-invariant solutions, and any Q-invariant solution of $\mathcal{L}$ belongs to this family.

The properties of ultra-singular vector fields as reduction operators are obvious.

Proposition 6.

(1) Any ultra-singular vector field Q of a differential equation $\mathcal{L}$ is a reduction operator of this equation. An ansatz constructed with Q reduces $\mathcal{L}$ to the identity. Therefore, the family of Q-invariant solutions of $\mathcal{L}$ is parameterized by an arbitrary function of a single Q-invariant variable.
(2) If the family of Q-invariant solutions of $\mathcal{L}$ is parameterized by an arbitrary function of a single Q-invariant variable then Q is an ultra-singular vector field for $\mathcal{L}$.

8. Reduction operators of singularity co-order 1

Encouraged by the above investigation of evolution and, especially, wave equations, we study co-order one singular reduction operators of general partial differential equations in two independent and one dependent variables.

Consider an equation $\mathcal{L}$: $L = 0$, where $L = L[u]$ is a differential function of order $r > 1$. Suppose that the function L admits a first co-order singular module of vector fields. (In view of corollary 4, we can restrict ourselves to considering only strong singularity of vector fields for differential equations.) Without loss of generality, up to changing variables we can assume that the module contains a first co-order singular set $S = \{Q^\zeta\}$ of vector fields in the reduced form, i.e., $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$ for any smooth function ζ of (x, u) and a fixed smooth function ξ. Additionally, we can assume $\xi \in \{0, u\}$.

By theorem 1, the differential function L can be written in the form $L = \check{L}(x, \Omega_{r,1})$, where

$$\Omega_{r,1} = \big(\omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2}u, \alpha_1 \leqslant 1, \alpha_1 + \alpha_2 \leqslant r\big),$$

and $\check{L}_{\omega_\alpha} \neq 0$ for some ω_α with $\alpha_1 = 1$. Then the restriction of L to $\mathcal{Q}^\zeta_{(r)}$ coincides with the restriction, to the same manifold $\mathcal{Q}^\zeta_{(r)}$, of the function $\tilde{L}^\zeta = \check{L}(x, \tilde{\Omega}_{r,1})$, where

$$\tilde{\Omega}_{r,1} = \big(D_1^{\alpha_1}(Q^\zeta)^{\alpha_2}u, \alpha_1 \leqslant 1, \alpha_1 + \alpha_2 \leqslant r\big).$$

Thus, the form of $\tilde{L}^\zeta$ is determined by the forms of L and ξ and a chosen value of the parameter-function ζ. Depending on the value of ζ, the differential function $\tilde{L}^\zeta$ may either identically vanish or be of order 0 or 1. This means that either the vector field Q^ζ is ultra-singular or $\mathrm{sco}_L\, Q^\zeta = 0$ or $\mathrm{sco}_L\, Q^\zeta = 1$, respectively. We investigate each of the above cases separately. Below we additionally suppose that the function $\tilde{L}^\zeta$ is of maximal rank with respect to u (resp. u_1) if $\mathrm{sco}_L\, Q^\zeta = 0$ (resp. $\mathrm{sco}_L\, Q^\zeta = 1$).

The values of ζ for which Q^ζ for $\mathcal{L}$ is ultra-singular are singled out by the condition $\tilde{L}^\zeta = 0$, where u and u_1 are considered as independent variables. Splitting this condition with respect to u_1 gives a system $\mathcal{S}_{-1}$ of partial differential equations in ζ of orders less than r, which may be incompatible in the general case. The incompatibility of this system means that the set S contains no ultra-singular vector fields. For example, evolution equations of orders greater than 1 and nonlinear wave equations of the form (5) with $F_u \neq 0$, in contrast to equations of the form (5) with $F_u = 0$, have no ultra-singular vector fields, see sections 5 and 6. ζ satisfying the ultra-singularity condition guarantees that $Q^\zeta \in \mathfrak{Q}(\mathcal{L})$ and the family of Q^ζ-invariant solutions of $\mathcal{L}$ is parameterized by an arbitrary function of a single Q^ζ-invariant variable.

If $\operatorname{sco}_L Q^\zeta = 0$, the parameter-function ζ satisfies the condition $\tilde{L}^\zeta_{u_1} = 0$ with u and u_1 viewed as independent variables, which is weaker than the ultra-singularity condition. Therefore, the corresponding system $\mathcal{S}_0$ of partial differential equations in ζ of orders less than r, obtained by splitting the zero co-order singularity condition with respect to u_1, has more chances of being compatible than $\mathcal{S}_{-1}$. Thus, any nonlinear wave equation of the form (5) with $F_u \neq 0$ admits zeroth co-order singular vector fields although this is not the case for ultra-singular vector fields. At the same time, evolution equations do not possess zeroth co-order singular vector fields.

Certain conditions which are sufficient for the compatibility of $\mathcal{S}_0$ can be formulated. Thus, if $\check{L}_{\omega_{(1,0)}} = 0$ and $\xi_u = 0$ then the system $\mathcal{S}_0$ is compatible since it is satisfied by any ζ with $\zeta_u = 0$. In other words, $\operatorname{sco}_L Q^\zeta \leqslant 0$ for any $\zeta = \zeta(x)$. Let us consider this particular case in more detail. (Recall that under the condition $\xi_u = 0$ the coefficient ξ can be assumed, up to point transformations, to equal 0 but we will not use this possibility.)

If additionally $\check{L}_{\omega_{(0,0)}} = 0$, the condition $\tilde{L}^\zeta = 0$ under the assumption $\zeta = \zeta(x)$ implies only a single partial differential equation with respect to ζ. Any of its solutions is a solution of $\mathcal{S}_{-1}$ and hence the corresponding vector field Q^ζ is ultra-singular for L.

Otherwise $\operatorname{sco}_L Q^\zeta = 0$ and we can resolve the equation $\tilde{L}^\zeta = 0$ with respect to u: $u = G^\zeta(x)$, where the expression for the function G^ζ depends on the parameter-function $\zeta = \zeta(x)$ and its derivatives up to order $r-1$. Then the conditional invariance criterion is equivalent to the r th-order partial differential equation $\zeta = \xi G^\zeta_1 + G^\zeta_2$ with respect to ζ. If ζ is a solution of this equation then Q^ζ is a reduction operator of $\mathcal{L}$. The ansatz constructed with the operator Q^ζ can be taken in the form $u = \varphi(\omega) + G^\zeta(x)$, where $\varphi = \varphi(\omega)$ is the new unknown function and $\omega = \omega(x)$ is the invariant independent variable satisfying the equation $\xi\omega_1 + \omega_2 = 0$. It reduces the initial equation $\mathcal{L}$ to a trivial algebraic equation $\varphi = 0$, i.e., the function $u = G^\zeta(x)$ is a unique Q^ζ-invariant solution of $\mathcal{L}$. Conversely, let us fix a solution $u = f(x)$ of the equation $\mathcal{L}$ and set $\zeta = \xi f_1 + f_2$. Then $f = G^\zeta(x)$ and hence $\zeta = \xi G^\zeta_1 + G^\zeta_2$, i.e., in view of the conditional invariance criterion $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$ is a reduction operator of $\mathcal{L}$, and $\zeta_u = 0$. The solution $u = f(x)$ is invariant with respect to Q^ζ by construction. Thus we obtain:

Theorem 6. *Suppose that an equation $\mathcal{L}$: $L = 0$ possesses a first co-order singular set $S = \{Q^\zeta\}$ of vector fields in the reduced form $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$ with $\xi_u = 0$, i.e., its right-hand side L is represented in the form $L = \check{L}(x, \Omega_{r,1})$, where*

$$\Omega_{r,1} = \big(\omega_\alpha = D_1^{\alpha_1}(\xi D_1 + D_2)^{\alpha_2}u, \alpha_1 \leqslant 1, \alpha_1 + \alpha_2 \leqslant r\big),$$

$\check{L}_{\omega_\alpha} \neq 0$ for some α with $\alpha_1 = 1$, and additionally $\check{L}_{\omega_{(1,0)}} = 0$ and $\check{L}_{\omega_{(0,0)}} \neq 0$. Then there exists a one-to-one correspondence between solutions of $\mathcal{L}$ and reduction operators from S with $\zeta_u = 0$. Namely, any such operator is of singularity co-order 0 and corresponds to the unique solution which is invariant with respect to this operator. The problems of solving the equation $\mathcal{L}$ and the exhaustive description of its reduction operators of the above form are completely equivalent.

Now we consider the regular values of ζ for which the singularity co-order of Q^ζ coincides with the singularity co-order of the whole family S (and equals 1). If $\operatorname{sco}_L Q^\zeta = 1$, the parameter-function ζ satisfies the regularity condition $\tilde{L}^\zeta_{u_1} \neq 0$. Therefore, the equation $\tilde{L}^\zeta = 0$ which is equivalent to $\mathcal{L}$ on the manifold $\mathcal{Q}^\zeta_{(r)}$ can be solved with respect to u_1: $u_1 = G^\zeta(x, u)$, where the expression for the function G^ζ depends on the parameter-function ζ and its derivatives up to order $r-1$. Applied to the equation $\mathcal{L}$ and the operator Q^ζ, the conditional invariance criterion implies only the equation

$$\zeta_1 + \zeta_u G^\zeta - (\xi_1 + \xi_u G^\zeta)G^\zeta = \xi G^\zeta_1 + G^\zeta_2 + \zeta G^\zeta_u \tag{8}$$

with respect to the function ζ. Therefore, in this case the system of determining equations consists of the single equation (8) and, therefore, is not overdetermined. This equation can be rewritten as the compatibility condition

$$\zeta_1 + \zeta_u G^\zeta - (\xi_1 + \xi_u G^\zeta) G^\zeta - \xi\left(G^\zeta_1 + G^\zeta_u G^\zeta\right) = G^\zeta_2 + (\zeta - \xi G^\zeta) G^\zeta_u$$

of the equations $u_1 = G^\zeta$ and $\xi u_1 + u_2 = \zeta$ with respect to u. The order of (8) equals r and hence is greater than the order of the system $\mathcal{S}_0$. This guarantees (under certain conditions of smoothness, e.g., in the analytical case) that equation (8) has solutions which are not solutions of $\mathcal{S}_0$. In other words, *the equation $\mathcal{L}$ necessarily possesses first co-order singular reduction operators which belong to S.*

The results of section 7 imply that for each first co-order singular reduction operator Q of the equation $\mathcal{L}$ there exists a one-parametric family of Q-invariant solutions of $\mathcal{L}$. If the equation $\mathcal{L}$ admits a co-order one singular module of vector fields, the converse statement is true as well.

Theorem 7. *Suppose that an equation $\mathcal{L}$: $L = 0$ possesses a co-order one singular set $S = \{Q^\zeta\}$ of vector fields in the reduced form $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$. Then for any one-parametric family $\mathcal{F}$ of solutions of $\mathcal{L}$ there exists a value of the parameter-function $\zeta = \zeta(x, u)$ such that Q^ζ is a reduction operator of $\mathcal{L}$ and each solution from $\mathcal{F}$ is invariant with respect to Q^ζ.*

Proof. Consider a one-parametric family $\mathcal{F} = \{u = f(x, \varkappa)\}$ of solutions of $\mathcal{L}$. The derivative $f_\varkappa$ is nonzero since the parameter $\varkappa$ is essential. From $u = f(x, \varkappa)$ we derive $\varkappa = \Phi(x, u)$ with some function $\Phi = \Phi(x, u)$, where $\Phi_u \neq 0$, and then define $\zeta = \zeta(x, u)$ by the formula

$$\zeta = -\frac{\xi\Phi_1 + \Phi_2}{\Phi_u}.$$

Since $f_i = -\Phi_i/\Phi_u|_{u=f}$, $i = 1, 2$, then $\xi f_1 + f_2 = \zeta|_{u=f}$, i.e., any solution from $\mathcal{F}$ is Q^ζ-invariant. Then either Q^ζ is an ultra-singular vector field for L or $\mathrm{sco}_L\, Q^\zeta = 1$. (The case $\mathrm{sco}_L\, Q^\zeta = 0$ is impossible since otherwise the equation $\mathcal{L}$ could not have a one-parametric family of Q^ζ-invariant solutions.) Any ultra-singular vector field for L is a reduction operator of $\mathcal{L}$. If $\mathrm{sco}_L\, Q^\zeta = 1$ then Q is a reduction operator of $\mathcal{L}$ in view of corollary 12. □

Corollary 13. *Suppose that an equation $\mathcal{L}$ possesses a first co-order singular set $S = \{Q^\zeta\}$ of vector fields in the reduced form $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$, and that no element of S is ultra-singular for $\mathcal{L}$. Then up to the equivalence of solution families there exists a bijection between one-parametric families of solutions of $\mathcal{L}$ and its first co-order singular reduction operators belonging to S. Namely, each operator of this kind corresponds to the family of solutions which are invariant under it. The problems of the construction of all one-parametric solution families of the equation $\mathcal{L}$ and the exhaustive description of its reduction operators of the above form are completely equivalent.*

This bijection is broken in the presence of ultra-singular vector fields.

The above relation between one-parametric families of solutions and first co-order singular reduction operators can be stated as a connection between the initial equation $\mathcal{L}$ and the determining equation (8).

Corollary 14. *Suppose that an equation $\mathcal{L}$: $L = 0$ possesses a first co-order singular set $S = \{Q^\zeta\}$ of vector fields in the reduced form $Q^\zeta = \xi\partial_1 + \partial_2 + \zeta\partial_u$. Then the determining equation for values of ζ corresponding to first co-order singular reduction operators of $\mathcal{L}$ is*

reduced by composition of the Bäcklund transformation $\xi\Phi_1+\Phi_2+\zeta\Phi_u=0$, $\Phi_1+G^\zeta\Phi_u=0$ *where* Φ *is a function of* (x,u)*, and the hodograph transformation*

$$\begin{aligned} &\textit{the new independent variables:} && \tilde{x}_1=x_1, && \tilde{x}_2=x_2, && \varkappa=\Phi,\\ &\textit{the new dependent variable:} && \tilde{u}=u \end{aligned}$$

to the initial equation $\mathcal{L}$ *for the function* $\tilde{u}=\tilde{u}(\tilde{x},\varkappa)$ *with* $\varkappa$ *playing the role of a parameter.*

Proof. We fix an arbitrary solution ζ of equation (8), which additionally satisfies the condition $\tilde{L}^\zeta_{u_1}\neq 0$. In view of the Frobenius theorem, the equations $\xi\Phi_1+\Phi_2+\zeta\Phi_u=0$ and $\Phi_1+G^\zeta\Phi_u=0$ are compatible with respect to the function $\Phi=\Phi(x,u)$ since their compatibility condition coincides with (8) and hence is identically satisfied. We choose a nonconstant solution Φ of both these equations. Then $\Phi_u\neq 0$ and

$$\zeta=-\xi\frac{\Phi_1}{\Phi_u}+\frac{\Phi_2}{\Phi_u},\qquad G^\zeta=-\frac{\Phi_1}{\Phi_u}.$$

After the hodograph transformation, the latter equations take the form $\xi\tilde{u}_{\tilde{x}_1}+\tilde{u}_{\tilde{x}_2}=\zeta(\tilde{x},\tilde{u})$ and $\tilde{u}_{\tilde{x}_1}=G^\zeta(\tilde{x},\tilde{u})$. This directly implies that for any value of $\varkappa$ the function $\tilde{u}=\tilde{u}(\tilde{x},\varkappa)$ satisfies the equation $\mathcal{L}$. The parameter $\varkappa$ is essential in $\tilde{u}$ since $\tilde{u}_\varkappa=1/\Phi_u\neq 0$.

It follows from the proof of theorem 7 that the application of the inverse transformations to a one-parametric family of solutions of the initial equation $\mathcal{L}$ results in a solution of equation (8) if the defined value of ζ satisfies the regularity condition $\tilde{L}^\zeta_{u_1}\neq 0$. □

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Paper 2

Reduction operators of linear second-order parabolic equations

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Reduction operators of linear second-order parabolic equations

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Abstract

The reduction operators, i.e. the operators of nonclassical (conditional) symmetry, of (1 + 1)-dimensional second-order linear parabolic partial differential equations and all the possible reductions of these equations to ordinary differential ones are exhaustively described. This problem proves to be equivalent, in some sense, to solving initial equations. The 'no-go' result is extended to the investigation of point transformations (admissible transformations, equivalence transformations, Lie symmetries) and Lie reductions of the determining equations for the nonclassical symmetries. Transformations linearizing the determining equations are obtained in the general case and under different additional constraints. A nontrivial example illustrating applications of reduction operators to finding exact solutions of equations from the class under consideration is presented. An observed connection between reduction operators and Darboux transformations is discussed.

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1. Introduction

The notion of nonclassical symmetry (also called Q-conditional or, simply, conditional symmetry) was introduced in [2] by the example of the (1 + 1)-dimensional linear heat equation and a particular class of operators. A precise and rigorous definition was suggested later (see, e.g., [8, 9, 44]). In contrast to classical Lie symmetry, the system of determining equations on the coefficients of conditional symmetry operators of the heat equation was found to be nonlinear and less overdetermined [2]. First, this system was investigated in [42] in detail, where it was partially linearized and its Lie symmetries were found. The problem on

conditional symmetries of the heat equation was completely solved in [7], see also [6]. Namely, the determining equations were obtained in both the cases arising under consideration and then studied from the Lie symmetry point of view and reduced to the initial equation with nonlocal transformations. The maximal Lie invariance algebras of both the sets of the determining equations appeared isomorphic to the maximal Lie invariance algebra of the initial equation. (Later few of these results were re-obtained in [15].) The results of [7] were extended in [5, 22, 23] to a class of linear transfer equations which generalize the heat equation. Thus, for these equations the 'no-go' theorems on linearization of determining equations for coefficients of conditional symmetry operators to the initial equations were proved in detail and wide multi-parametric families of exact solutions were constructed with non-Lie reductions. It was observed in [43] that the proof of the theorem from [7] on reducibility of determining equations to initial ones in the case of conditional symmetry operators with vanishing coefficients of ∂_t is extended to the class of $(1 + 1)$-dimensional evolution equations. This theorem was also generalized to multi-dimensional evolution equations [24] and even systems of such equations [40].

The conditional invariance of a differential equation with respect to an involutive family of l vector fields is equivalent to that any Ansatz associated with this family reduces the equation to a differential equation with the lesser by l number of independent variables [44]. That is why, we use the shorter and more natural term 'reduction operators' instead of 'operators of conditional symmetry' or 'operators of nonclassical symmetry' and say that a family of operators reduces a differential equation in case the equation is reduced by the associated Ansatz.

In this paper, we investigate the reduction operators of the second-order linear parabolic partial differential equations in two independent variables, which have the general form

$$Lu = u_t - A(t, x)u_{xx} - B(t, x)u_x - C(t, x)u = 0, \tag{1}$$

where the coefficients A, B and C are (real) analytic functions of t and x, $A \neq 0$. These coefficients form the entire tuple of arbitrary elements of class (1). We justify the partition of the sets of reduction operators into two subsets depending on vanishing or nonvanishing of the coefficients of ∂_t. Usually this point is missed in the literature on conditional symmetries. After factorization by the equivalence relation between reduction operators, we find the determining equations for the coefficients of operators from both the subsets. All the possible reductions of equations from class (1) to ordinary differential equations are described. Different kinds of 'no-go' statements on the reduction of study (including solution) of the determining equations to the corresponding initial ones are obtained for equations from class (1). In particular, the point transformations of all kinds in both the classes of determining equations (admissible transformations, transformations from the associated equivalence groups, Lie symmetry transformations) are induced by the corresponding point transformations in class (1). Lie solutions of the determining equations first prove to admit nontrivial interpretations in terms of Lie invariance properties of the initial equations. An example on the application of reduction operators is presented. It shows that in spite of the 'no-go' statements nonclassical symmetry is an effective tool for finding exact solutions of partial differential equations.

There are a number of motivations inducing us to carry out the above investigations. Class (1) contains important subclasses that are widely applied in different science (probability theory, physics, financial mathematics, biology, etc). The most famous examples are the Kolmogorov equations ($C = 0$) and adjoint to them the Fokker–Planck equations ($A_{xx} - B_x + C = 0$) which form a basis for analytical methods in the investigation of continuous-time continuous-state Markov processes. (The other names are Kolmogorov

backward and Kolmogorov forward equations, respectively.) The first use of the Fokker–Planck equation was the statistical description of Brownian motion of a particle in a fluid. Fokker–Planck equations with different coefficients also describe the evolution of one-particle distribution functions of a dilute gas with long-range collisions, problems of diffusion in colloids, population genetics, stock markets, quantum chaos, etc. Due to their importance and relative simplicity, equations from class (1) are conventional objects for studies in the framework of group analysis of differential equations. Lie symmetries of these equations were classified by Lie [14]. The $(1 + 1)$-dimensional linear heat equation is often used as an illustrative example in textbooks on the subject [17] and a benchmark example for computer programs calculating symmetries of differential equations [11]. It is the equation that is connected with the invention of nonclassical symmetries [2]. First, discussions on weak symmetries also involved the linear heat equation and a Fokker–Planck equation [19, 37]. At the same time, all previous studies of nonclassical symmetries of equations (1) were not systematic. Only a few equations and single properties were considered.

The results of [5, 7, 22, 27] are extended in the present paper mainly in two directions. First, the entire class (1) is regularly investigated with the nonclassical symmetry point of view and, second, non-evident properties of point transformations and Lie reductions of the determining equations are found via involving admissible transformations in the framework of nonclassical symmetries.

Our paper is organized as follows. Necessary notions and statements on nonclassical symmetries are presented in section 2. The notion of equivalence of nonclassical symmetries with respect to a transformation group or a set of admissible transformations plays a crucial role in our consideration and therefore is separately given in section 3. Section 4 is devoted to reviewing the known results on admissible transformations, point symmetries and equivalences in class (1), including discrete ones. The presentation of these results is important since they form a basis for the application of our technique involving transformations between equations and are extended in the paper to both the classes of determining equations. Moreover, Lie symmetry operators are special cases of reduction operators. The determining equations are derived in section 5 for both the cases of nonvanishing and vanishing coefficients of ∂_t. It is proved in section 6 via description of all possible reductions that solving the determining equations is equivalent to the construction of parametric families of solution of the corresponding initial equations. As a result, nonlocal transformations reducing the determining equations to the initial ones are found. Point transformations and Lie reductions of the determining equations are studied in sections 7 and 8, respectively. The results on Lie reductions of the determining equations corresponding to reduction operators with zero coefficients of ∂_t are presented in such a form that they are directly extended to the general class of $(1 + 1)$-dimensional evolution equations. In section 9 we investigate the determining equations along with some non-Lie additional constraints. A nontrivial application of reduction operators to finding exact solutions of equations from class (1), arising under Lie reductions of the Navier–Stokes equations, is presented in section 10. In the last section we discuss possible extensions of obtained results, in particular, via study of the observed connection between reduction operators and the Darboux transformations of equations from class (1).

To check the results on Lie invariance of differential equations appearing in the paper, we used the unique program LIE by Head [11].

2. Reduction operators of differential equations

Following [8, 9, 35, 44], in this section we shortly adduced necessary notions and results on nonclassical (conditional) symmetries of differential equations. After substantiating with

different arguments, we use the name 'families of reduction operators' instead of 'involutive families of nonclassical (conditional) symmetry operators'.

Consider an involutive family $Q = \{Q^1, \dots, Q^l\}$ of l ($l \leqslant n$) first-order differential operators

$$Q^s = \xi^{si}(x,u)\partial_i + \eta^s(x,u)\partial_u, \qquad s = 1, \dots, l$$

in the space of the variables x and u, satisfying the condition $\operatorname{rank} \|\xi^{si}(x,u)\| = l$.

Hereafter, x denote the n-tuple of independent variables $(x_1, \dots, x_n)$ and u is treated as the unknown function. The index i runs from 1 to n, the indices s and σ run from 1 to l, and we use the summation convention for repeated indices; $\partial_i = \partial/\partial x_i$, $\partial_u = \partial/\partial u$. Subscripts of functions denote differentiation with respect to the corresponding variables. The local consideration is assumed.

The requirement of involution for the family Q means that the commutator of any pair of operators from Q belongs to the span of Q over the ring of smooth functions of the variables x and u, i.e.,

$$\forall\, s, s' \qquad \exists\, \zeta^{ss'\sigma} = \zeta^{ss'\sigma}(x,u): \qquad [Q^s, Q^{s'}] = \zeta^{ss'\sigma} Q^\sigma .$$

The set of such families will be denoted by $\mathcal{Q}^l$.

If the operators $Q^1, \dots, Q^l$ form an involutive family Q, then the family $\widetilde{Q}$ of differential operators

$$\widetilde{Q}^s = \lambda^{s\sigma} Q^\sigma, \qquad \text{where} \quad \lambda^{s\sigma} = \lambda^{s\sigma}(x,u), \qquad \det\|\lambda^{s\sigma}\| \neq 0$$

is also involutive and is called *equivalent* to the family Q. This will be denoted by $\widetilde{Q} = \{\widetilde{Q}^s\} \sim Q = \{Q^s\}$. (In the case $l = 1$ the functional matrix $(\lambda^{s\sigma})$ becomes a single nonvanishing multiplier $\lambda = \lambda(x,u)$.) Denote also the result of factorization of $\mathcal{Q}^l$ with respect to this equivalence relation by $\mathcal{Q}^l_{\rm f}$. Elements of $\mathcal{Q}^l_{\rm f}$ will be identified with their representatives in $\mathcal{Q}^l$.

If a family consists of a single operator ($l = 1$), the involution condition degenerates to an identity. Therefore, in this case we can omit the words 'involutive family' and talk only about operators. Thus, two differential operators are equivalent if they differ on a multiplier being a non-vanishing function of x and u.

The first-order differential function $Q^s[u] := \eta^s(x,u) - \xi^{si}(x,u)u_i$ is called the *characteristic* of the operator Q^s. In view of the Frobenius theorem, the above involution condition is equivalent to that the characteristic system $Q[u] = 0$ of PDEs $Q^s[u] = 0$ (also called the *invariant surface condition*) has $n + 1 - l$ functionally independent integrals $\omega^0(x,u), \dots, \omega^{n-l}(x,u)$. Therefore, the general solution of this system can be implicitly presented in the form $F(\omega^0, \dots, \omega^{n-l}) = 0$, where F is an arbitrary function of its arguments.

The characteristic systems of equivalent families of operators have the same set of solutions. And vice versa, any family of $n + 1 - l$ functionally independent functions of x and u is a complete set of integrals of the characteristic system of an involutive family of l differential operators. Therefore, there exists the one-to-one correspondence between $\mathcal{Q}^l_{\rm f}$ and the set of families of $n + 1 - l$ functionally independent functions of x and u, which is factorized with respect to the corresponding equivalence. (Two families of the same number of functionally independent functions of the same arguments are considered equivalent if any function from one of the families is functionally dependent on functions from the other family.)

A function $u = f(x)$ is called *invariant with respect to the involutive operator family* Q (or, briefly, Q-*invariant*) if it is a solution of the characteristic system $Q[u] = 0$. This notion is justified by the following facts. Any involutive family of l operators is equivalent to a basis $\widetilde{Q} = \{\widetilde{Q}^s\}$ of an l-dimensional (Abelian) Lie algebra $\mathfrak{g}$ of vector fields in the space (x,u). Each solution $u = f(x)$ of the associated characteristic system satisfies the characteristic

system $\widetilde{Q}[u] = 0$. Therefore, the graph of the function $u = f(x)$ is invariant with respect to the l-parametric local transformation group generated by the algebra $\mathfrak{g}$.

Since rank $\|\xi^{si}(x,u)\| = l$, we can assume without loss of generality that $\omega^0_u \neq 0$ and $F_{\omega^0} \neq 0$ and resolve the equation $F = 0$ with respect to ω^0: $\omega^0 = \varphi(\omega^1, \dots, \omega^{n-l})$. This representation of the function u is called an *Ansatz* corresponding to the family Q.

Consider an rth-order differential equation $\mathcal{L}$ of the form $L(x, u_{(r)}) = 0$ for the unknown function u of n independent variables $x = (x_1, \dots, x_n)$. Here, $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to x of order not greater than r, including u as the derivative of the zero order. Within the local approach the equation $\mathcal{L}$ is treated as an algebraic equation in the jet space $J^{(r)}$ of the order r and is identified with the manifold of its solutions in $J^{(r)}$. Denote this manifold by the same symbol $\mathcal{L}$ and the manifold defined by the set of all the differential consequences of the characteristic system $Q[u] = 0$ in $J^{(r)}$ by $\mathcal{Q}^{(r)}$, i.e.,

$$\mathcal{Q}^{(r)} = \left\{(x, u_{(r)}) \in J^{(r)} \middle| D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u] = 0,\ \alpha_i \in \mathbb{N} \cup \{0\},\ |\alpha| := \alpha_1 + \dots + \alpha_n < r\right\},$$

where $D_i = \partial_{x_i} + u_{\alpha+\delta_i}\partial_{u_\alpha}$ is the operator of total differentiation with respect to the variable x_i, $\alpha = (\alpha_1, \dots, \alpha_n)$ is an arbitrary multi-index, δ_i is the multi-index whose ith entry equals 1 and whose other entries are zero. The variable u_α of the jet space $J^{(r)}$ corresponds to the derivative $\partial^{|\alpha|}u \big/ \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$.

Definition 1. *The differential equation $\mathcal{L}$ is called conditionally invariant with respect to the involutive family Q if the relation $Q^s_{(r)} L(x, u_{(r)})\big|_{\mathcal{L}\cap\mathcal{Q}^{(r)}} = 0$ holds, which is called the conditional invariance criterion. Then Q is called an involutive family of conditional symmetry (or Q-conditional symmetry, nonclassical symmetry, etc) operators of the equation $\mathcal{L}$. Here the symbol $Q^s_{(r)}$ stands for the standard rth prolongation of the operator Q^s [17, 21]: $Q^s_{(r)} = Q^s + \sum_{|\alpha|\leqslant r} \eta^{s\alpha}\partial_{u_\alpha}$, where $\eta^{s\alpha} = D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u] + \xi^{si} u_{\alpha+\delta_i}$.*

The equation $\mathcal{L}$ is conditionally invariant with respect to the family Q if and only if the Ansatz constructed with this family reduces $\mathcal{L}$ to a differential equation with $n-l$ independent variables [44]. So, we will also call involutive families of conditional symmetry operators the *families of reduction operators* of $\mathcal{L}$. Another treatment of conditional invariance is that the system $\mathcal{L} \cap \mathcal{Q}^{(r)}$ is compatible in the sense of absence of nontrivial differential consequences [18, 20]. If the infinitesimal invariance condition is not satisfied but nevertheless the equation $\mathcal{L}$ has $\mathcal{Q}$-invariant solutions then $\mathcal{Q}$ is called a family of weak symmetry operators of the equation $\mathcal{L}$ [19, 20]. Nonclassical symmetries are often defined as generators of parametric groups of transformations preserving the solutions of $\mathcal{L}$ which additionally satisfy the corresponding invariant surface condition [12]. It is necessary to precisely interpret all the terms involved in this definition since otherwise it leads to the conclusion that, roughly speaking, any operator is a nonclassical symmetry of any partial differential equation. See also [1, 4, 20] for the discussion of connections between different kinds of symmetries.

Lemma 1 ([9, 44])**.** *If a differential equation is conditionally invariant with respect to an operator family Q, then it is conditionally invariant with respect to any family of operators, which is equivalent to Q.*

The set of involutive families of l reduction operators of the equation $\mathcal{L}$ is a subset of $\mathcal{Q}^l$ and so will be denoted by $\mathcal{Q}^l(\mathcal{L})$. In view of lemma 1, $Q \in \mathcal{Q}^l(\mathcal{L})$ and $\widetilde{Q} \sim Q$ imply $\widetilde{Q} \in \mathcal{Q}^l(\mathcal{L})$, i.e., $\mathcal{Q}^l(\mathcal{L})$ is closed under the equivalence relation on $\mathcal{Q}^l$. Therefore, the factorization of $\mathcal{Q}^l$ with respect to this equivalence relation can be naturally restricted on $\mathcal{Q}^l(\mathcal{L})$ that results in the subset $\mathcal{Q}^l_{\rm f}(\mathcal{L})$ of $\mathcal{Q}^l_{\rm f}$. As in the whole set $\mathcal{Q}^l_{\rm f}$, we identify elements of $\mathcal{Q}^l_{\rm f}(\mathcal{L})$ with their representatives in $\mathcal{Q}^l(\mathcal{L})$. In this approach, the problem of complete description of families of l reduction operators for the equation $\mathcal{L}$ is nothing but the problem of finding $\mathcal{Q}^l_{\rm f}(\mathcal{L})$.

A different terminology can be used to call elements of $\mathcal{Q}^l_{\rm f}$. Namely, it is possible to consider each element of $\mathcal{Q}^l_{\rm f}$ as a C^∞-module of the module dimension l, closed with respect to commutation [20, 41].

There are families of reduction operators related to classical Lie symmetries. Let $\mathfrak{g}$ be an l-dimensional Lie invariance algebra of the equation $\mathcal{L}$, whose basis operators satisfy the condition rank $\|\xi^{si}\| = $ rank $\|\xi^{si}, \eta^s\| (= l' \leqslant l)$. The subsets consisting of l' elements of $\mathfrak{g}$, which are linearly independent over the ring of smooth functions of x and u, belong to $\mathcal{Q}^{l'}(\mathcal{L})$ and are equivalent to each other. The families of similar kind and ones equivalent to them will be called *Lie families of reduction operators*. The other families of reduction operators will be called *non-Lie*.

3. Equivalence of families of reduction operators with respect to transformation groups

We can essentially simplify and order the investigation of reduction operators, additionally taking into account Lie symmetry transformations in the case of a single equation [25] and transformations from the equivalence group or the whole set of admissible transformations in the case of a class of equations [35]. Then the problem becomes similar to the group classification of differential equations.

Lemma 2. *Any point transformation of x and u induces a one-to-one mapping of $\mathcal{Q}^l$ into itself. Namely, the transformation g: $\tilde{x} = X(x,u)$, $\tilde{u} = U(x,u)$ generates the mapping g^l_*: $\mathcal{Q}^l \to \mathcal{Q}^l$ such that the involutive family Q is mapped to the involutive family $g^l_* Q$ consisting from the operators $g_* Q^s = \tilde{\xi}^{si}\partial_{\tilde{x}_i} + \tilde{\eta}^s \partial_{\tilde{u}}$, where $\tilde{\xi}^{si}(\tilde{x},\tilde{u}) = Q^s X^i(x,u)$, $\tilde{\eta}^s(\tilde{x},\tilde{u}) = Q^s U(x,u)$. If $Q' \sim Q$ then $g^l_* Q' \sim g^l_* Q$. Therefore, the corresponding factorized mapping $g^l_{\rm f}$: $\mathcal{Q}^l_{\rm f} \to \mathcal{Q}^l_{\rm f}$ also is well defined and one-to-one.*

Definition 2 ([25, 33])**.** *Involutive families Q and $\widetilde{Q}$ of the same number l of differential operators are called equivalent with respect to a group G of point transformations ($Q \sim \widetilde{Q}$ mod G) if there exists a transformation g from G for which the families Q and $g^l_* \widetilde{Q}$ are equivalent.*

Lemma 3. *Given any point transformation g of the equation $\mathcal{L}$ to an equation $\tilde{\mathcal{L}}$, g^l_* maps $\mathcal{Q}^l(\mathcal{L})$ to $\mathcal{Q}^l(\tilde{\mathcal{L}})$ in a one-to-one manner. The same statement is true for the factorized mapping $g^l_{\rm f}$ from $\mathcal{Q}^l_{\rm f}(\mathcal{L})$ to $\mathcal{Q}^l_{\rm f}(\tilde{\mathcal{L}})$.*

Corollary 1. *Let G be a Lie symmetry group of the equation $\mathcal{L}$. Then the equivalence of involutive families of l differential operators with respect to the group G generates equivalence relations in $\mathcal{Q}^l(\mathcal{L})$ and in $\mathcal{Q}^l_{\rm f}(\mathcal{L})$.*

Consider a class $\mathcal{L}|_{\mathcal{S}}$ of equations $\mathcal{L}_\theta$: $L(x, u_{(r)}, \theta(x, u_{(r)})) = 0$ parameterized by θ. Here, L is a fixed function of $x, u_{(r)}$ and θ. The symbol θ denotes the tuple of arbitrary (parametric) functions $\theta(x, u_{(r)}) = (\theta^1(x, u_{(r)}), \dots, \theta^k(x, u_{(r)}))$ running through the solution set $\mathcal{S}$ of the system $S(x, u_{(r)}, \theta_{(q)}(x, u_{(r)})) = 0$. This system consists of differential equations on θ, where x and $u_{(r)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order not greater than q. In what follows we call the functions θ *arbitrary elements*. By $G^\sim$ we denote the point transformations group preserving the form of the equations from $\mathcal{L}|_{\mathcal{S}}$.

For a fixed value $l \leqslant n$, consider the set $P = P(L, S)$ of all pairs each of which consists of an equation $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ and a family Q from $\mathcal{Q}^l(\mathcal{L}_\theta)$. In view of lemma 3, the action of transformations from $G^\sim$ on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathcal{Q}^l(\mathcal{L}_\theta) | \theta \in \mathcal{S}\}$ together with the pure equivalence

relation of involutive families of l differential operators naturally generates an equivalence relation on P.

Definition 3. *Let $\theta, \theta' \in \mathcal{S}$, $Q \in \mathcal{Q}^l(\mathcal{L}_\theta)$, $Q' \in \mathcal{Q}^l(\mathcal{L}_{\theta'})$. The pairs $(\mathcal{L}_\theta, Q)$ and $(\mathcal{L}_{\theta'}, Q')$ are called $G^{\sim}$-equivalent if there exists a transformation $g \in G^{\sim}$ which maps the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\theta'}$, and $Q' \sim g^l_* Q$.*

The classification of families of reduction operators with respect to $G^{\sim}$ will be understood as classification in P with respect to the above equivalence relation. This problem can be investigated in a way similar to the usual group classification in classes of differential equations. Namely, we construct first the reduction operators which are defined for all values of the arbitrary elements. Then we classify, with respect to the equivalence group, the values of arbitrary elements for which the corresponding equations admit additional families of reduction operators.

In an analogous way, we can also introduce equivalence relations on P, which are generated by either generalizations of usual equivalence groups or all admissible point transformations [30] (also called form-preserving ones [13]) in pairs of equations from $\mathcal{L}|_{\mathcal{S}}$.

Note 1. The consideration of the previous and this sections and known examples of studying reduction operators lead to the empiric conclusion that possessing a wide Lie symmetry group by a differential equation $\mathcal{L}$ complicates, in some way, finding nonclassical symmetries of $\mathcal{L}$. Indeed, any subalgebra of the corresponding maximal Lie invariance algebra, satisfying the transversality condition, generates a class of equivalent Lie families of reduction operators. A non-Lie families of reduction operators existing, the action of symmetry transformations on it results in a series of non-Lie families of reduction operators, which are inequivalent in the usual sense. Therefore, for any fixed value of l the system of determining equations on coefficients of operators from $\mathcal{Q}^l(\mathcal{L})$ is not sufficiently overdetermined to be completely integrated in an easy way, even after factorized with respect to the equivalence relation in $\mathcal{Q}^l(\mathcal{L})$. To produce essentially different non-Lie reductions, one have to exclude the solutions of determining equations, which give Lie families of reduction operators and non-Lie families being equivalent to others with respect to the Lie symmetry group of $\mathcal{L}$. As a result, the ratio of efficiency of such reductions to expended efforts can be vanishingly small.

4. Lie group analysis of linear second-order parabolic equations

Group classification in class (1) was first performed by Lie [14] as a part of his classification of general linear second-order PDEs in two independent variables. (See also a modern treatment of this subject in [21].) We shortly adduce these classical results, extending them for our purposes with using the notions of admissible transformations and normalized classes of differential equations. First, normalization properties of different classes of linear second-order parabolic equations were simultaneously analyzed in [34] in detail.

Roughly speaking, an admissible transformation in a class of systems of differential equations is a point transformation connecting at least two systems from this class (in the sense that one system is transformed into the other by the transformation). The equivalence group of the class is the set of admissible transformations which can be applied to every system from the class. The class is called *normalized* if any admissible transformation in this class belongs to its equivalence group and is called *strongly normalized* if additionally the equivalence group is generated by transformations from the point symmetry groups of systems from the class. The set of admissible transformations of a *semi-normalized class* is generated by the transformations from the equivalence group of the whole class and the

transformations from the point symmetry groups of initial or transformed systems. Strong semi-normalization is defined in the same way as strong normalization. Any normalized class is semi-normalized. Two systems from a semi-normalized class are transformed into one another by a point transformation iff they are equivalent with respect to the equivalence group of this class. See [26, 28, 30, 32] for precise definitions and statements.

Any point transformation $\mathcal{T}$ in the space of variables (t, x, u) has the form $\tilde{t} = \mathcal{T}^t(t, x, u), \tilde{x} = \mathcal{T}^x(t, x, u), \tilde{u} = \mathcal{T}^u(t, x, u)$, where the Jacobian $|\partial(\mathcal{T}^t, \mathcal{T}^x, \mathcal{T}^u)/\partial(t, x, u)|$ does not vanish.

Lemma 4. *A point transformation $\mathcal{T}$ connects two equations from class (1) if and only if $\mathcal{T}^t_x = \mathcal{T}^t_u = 0, \mathcal{T}^x_u = 0, \mathcal{T}^u_{uu} = 0$, i.e.,*

$$\tilde{t} = T(t), \qquad \tilde{x} = X(t, x), \qquad \tilde{u} = U^1(t, x)u + U^0(t, x), \tag{2}$$

where T, X, U^1 and U^0 are arbitrary smooth functions of their arguments such that $T_t X_x U^1 \neq 0$ and additionally U^0/U^1 is a solution of the initial equation. The arbitrary elements are transformed by the formulae

$$\tilde{A} = \frac{X_x^2}{T_t} A, \qquad \tilde{B} = \frac{X_x}{T_t}\left(B - 2\frac{U^1_x}{U^1}A\right) - \frac{X_t - AX_{xx}}{T_t}, \qquad \tilde{C} = -\frac{U^1}{T_t} L \frac{1}{U^1}. \tag{3}$$

Here, $L = \partial_t - A\partial_{xx} - B\partial_x - C$ is the second-order linear differential operator associated with the initial (non-tilde) equation.

Corollary 2. *Class (1) is strongly semi-normalized. The equivalence group $G^{\sim}$ of class (1) is formed by the transformations determined in the space of variables and arbitrary elements by formulae (2), (3), where T, X and U^1 are arbitrary smooth functions of their arguments such that $T_t X_x U^1 \neq 0$ and $U^0 = 0$ additionally.*

Note 2. Due to the presence of the linear superposition principle, class (1) is not normalized because it is formed by linear homogeneous equations. The minimal normalized superclass of class (1) is the associated class of inhomogeneous equations of the general form

$$u_t = A(t, x)u_{xx} + B(t, x)u_x + C(t, x)u + D(t, x).$$

Using transformations from $G^{\sim}$, the arbitrary elements A and B can be simultaneously gauged to 1 and 0, respectively. Hence, any equation from class (1) can be reduced by a transformation from $G^{\sim}$ to an equation of the general form

$$u_t - u_{xx} + V(t, x)u = 0. \tag{4}$$

The admissible transformations in subclass (4) are those admissible transformations in class (1) which preserve the gauges $A = 1$ and $B = 0$, i.e., which additionally satisfy the conditions $\mathcal{T}^t_t = \left(\mathcal{T}^x_x\right)^2$ and $2\mathcal{T}^x_x\mathcal{T}^u_{xu} = -\mathcal{T}^x_t\mathcal{T}^u_u$.

Corollary 3. *A point transformation $\mathcal{T}$ connects two equations from class (4) if and only if it has the form*

$$\tilde{t} = \int \sigma^2 \,\mathrm{d}t, \qquad \tilde{x} = \sigma x + \zeta, \qquad \tilde{u} = U^1 u + U^0, \qquad U^1 := \theta \exp\left(-\frac{\sigma_t}{4\sigma}x^2 - \frac{\zeta_t}{2\sigma}x\right),$$
$$\tilde{V} = \frac{1}{\sigma^2}\left(V + \frac{\sigma\sigma_{tt} - 2{\sigma_t}^2}{4\sigma^2}x^2 + \frac{\sigma\zeta_{tt} - 2\sigma_t\zeta_t}{2\sigma^2}x - \frac{\theta_t}{\theta} - \frac{\sigma_t}{2\sigma} - \frac{{\zeta_t}^2}{4\sigma^2}\right), \tag{5}$$

where $\sigma = \sigma(t), \zeta = \zeta(t), \theta = \theta(t)$ and $U^0 = U^0(t, x)$ are arbitrary smooth functions of their arguments such that $\sigma\theta \neq 0$ and U^0/U^1 is a solution of the initial equation. Class (4)

is strongly semi-normalized. Any transformation from the equivalence group $G^{\sim}_{\rm r}$ of class (4) has form (5), where $U^0 = 0$ additionally.

The narrower equivalence group under preserving certain normalization properties suggests class (4) as the most convenient one for group classification. Moreover, solving the group classification problem for class (1) is reduced to solving the group classification problem for class (4). The results on the group classification of class (1) (resp. (4)) can be formulated in the form of the following theorem [14, 21].

Theorem 1. *The kernel Lie algebra of class (1) (resp. (4)) is $\langle u\partial_u\rangle$. Any equation from class (1) (resp. (4)) is invariant with respect to the operators $f\partial_u$, where the parameter-function $f = f(t,x)$ runs through the solution set of this equation. All possible $G^{\sim}$-inequivalent (resp. $G^{\sim}_{\rm r}$-inequivalent) cases of extension of the maximal Lie invariance algebra are exhausted by the following ones (the values of V are given together with the corresponding maximal Lie invariance algebras):*

(1) $V = V(x)$: $\langle \partial_t, u\partial_u, f\partial_u\rangle$;
(2) $V = \mu x^{-2}$, $\mu \neq 0$: $\langle \partial_t, D, \Pi, u\partial_u, f\partial_u\rangle$;
(3) $V = 0$: $\langle \partial_t, \partial_x, G, D, \Pi, u\partial_u, f\partial_u\rangle$.

Here, $D = 2t\partial_t + x\partial_x$, $\Pi = 4t^2\partial_t + 4tx\partial_x - (x^2+2t)u\partial_u$, $G = 2t\partial_x - xu\partial_u$.

Let $\mathcal{L}$ be an equation from class (1), $\mathfrak{g}(\mathcal{L})$ denote its maximal Lie invariance algebra and $\mathfrak{g}^{\infty}(\mathcal{L})$ be the infinite-dimensional ideal of this algebra, consisting of the operators of the form $f\partial_u$, where the parameter-function $f = f(t,x)$ runs through the solution set of $\mathcal{L}$. The quotient algebra $\mathfrak{g}(\mathcal{L})/\mathfrak{g}^{\infty}(\mathcal{L})$ is identified with the finite-dimensional subalgebra $\mathfrak{g}^{\rm ess}(\mathcal{L})$ of $\mathfrak{g}(\mathcal{L})$, spanned by the 'essential' Lie invariance operators of $\mathcal{L}$, which do not contain summands of the form $f(t,x)\partial_u$. Each operator from $\mathfrak{g}(\mathcal{L})$ is similar to an operator from $\mathfrak{g}^{\rm ess}(\mathcal{L})$ under a trivial linear-superposition transformation $\tilde t = t$, $\tilde x = x$, $\tilde u = u + f(t,x)$.

Corollary 4. *For every equation $\mathcal{L}$ from class (1)* $\dim \mathfrak{g}^{\rm ess}(\mathcal{L}) \in \{1, 2, 4, 6\}$.

It will be shown below that for every equation $\mathcal{L}$ from class (1) the number of reduction operators being inequivalent with respect to the group of linear-superposition transformations, roughly speaking, is significantly greater that the number of 'essential' Lie invariance operators.

5. Determining equations for reduction operators of linear second-order parabolic equations

In the case of two independent variables t and x and one dependent variable u, each reduction operator is written as $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$, where $(\tau,\xi) \neq (0,0)$. The conditional invariance criterion for an equation $\mathcal{L}$ from class (1) and the operator Q has the form [8]

$$Q_{(2)}Lu\,\big|_{Lu=0,\,Q[u]=0,\,D_tQ[u]=0,\,D_xQ[u]=0} = 0,$$

where $Q_{(2)}$ is the standard second prolongation of Q, $Q[u] = \eta - \tau u_t - \xi u_x$ is the characteristic of Q and D_t and D_x denote the total differentiation operators with respect to t and x, respectively:

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots,$$
$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots.$$

All equalities hold true as algebraic relations in the second-order jet space $J^{(2)}$ over the space of the independent variables (t, x) and the dependent variable u.

Since $\mathcal{L}$ is an evolution equation, there are two principally different cases of finding its reduction operators: $\tau \neq 0$ and $\tau = 0$. The investigation of these cases results in the preliminary description of the reduction operators.

Lemma 5. *Every reduction operator of an equation $\mathcal{L}$ from class (1) is equivalent to either an operator*

$$\partial_t + g^1(t,x)\partial_x + (g^2(t,x)u + g^3(t,x))\partial_u,$$

where the functions $g^1 = g^1(t,x)$, $g^2 = g^2(t,x)$ and $g^3 = g^3(t,x)$ satisfy the system

$$\begin{aligned}
&g^1_t - Ag^1_{xx} - Bg^1_x + \left(2g^1_x - \frac{A_x}{A}g^1 - \frac{A_t}{A}\right)(g^1 + B) + B_x g^1 + 2Ag^2_x + B_t = 0,\\
&g^2_t - Ag^2_{xx} - Bg^2_x + \left(2g^1_x - \frac{A_x}{A}g^1 - \frac{A_t}{A}\right)(g^2 - C) - C_x g^1 - C_t = 0,\\
&g^3_t - Ag^3_{xx} - Bg^3_x + \left(2g^1_x - \frac{A_x}{A}g^1 - \frac{A_t}{A}\right)g^3 - Cg^3 = 0,
\end{aligned} \tag{6}$$

or an operator $\partial_x + \eta(t,x,u)\partial_u$, where the function $\eta = \eta(t,x,u)$ is a solution of the equation

$$\eta_t = A(\eta_{xx} + 2\eta\eta_{xu} + \eta^2\eta_{uu}) + A_x(\eta_x + \eta\eta_u) + (B\eta)_x + C(\eta - u\eta_u) + C_x u. \tag{7}$$

Example 1. Each equation from class (1) with $C = 0$ possesses the reduction operator ∂_x.

We denote the set of reduction operators of the equation $\mathcal{L}$ from class (1) by $\mathcal{Q}(\mathcal{L})$, omitting the superscript 1. The corresponding set factorized with respect to the equivalence of reduction operators is denoted by $\mathcal{Q}_{\rm f}(\mathcal{L})$. Consider the subsets $\mathcal{Q}_1(\mathcal{L})$ and $\mathcal{Q}_0(\mathcal{L})$ of $\mathcal{Q}(\mathcal{L})$, which consist of the operators constrained by the conditions $\tau = 1$ and $(\tau, \xi) = (0, 1)$, respectively. The factor-set $\mathcal{Q}_{\rm f}(\mathcal{L})$ can be identified with $\mathcal{Q}_1(\mathcal{L}) \cup \mathcal{Q}_0(\mathcal{L})$. This union represents the canonical partition of $\mathcal{Q}_{\rm f}(\mathcal{L})$. The systems of form (6) and equations of form (7) associated with the equation $\mathcal{L}$ (and being the determining equations for the operators from $\mathcal{Q}_1(\mathcal{L})$ and $\mathcal{Q}_0(\mathcal{L})$) are denoted by $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_0(\mathcal{L})$, respectively. It is obvious that the rules $\mathcal{L} \to \mathrm{DE}_1(\mathcal{L})$ and $\mathcal{L} \to \mathrm{DE}_0(\mathcal{L})$ define one-to-one mappings of class (1) onto classes (6) and (7).

Note 3. The partition of sets of reduction operators according to the condition of (non-) vanishing of the coefficient τ is natural for equations from class (1) (as well as the whole class of evolution equations) and agrees with their transformational properties. See section 7 for details.

Note 4. For certain reasons, here reduction operators are studied for equations of the non-reduced form (1). At the same time, it is enough, up to the equivalence relation generated by the equivalence group of class (1) on the set of pairs '(an equation of form (1), its reduction operator)', to investigate only subclass (4) of equations with $A = 1$ and $B = 0$. The determining equations (6) and (7) for equations from class (4) have the simpler general form

$$\begin{aligned}
&g^1_t - g^1_{xx} + 2g^1_x g^1 + 2g^2_x = 0,\\
&g^2_t - g^2_{xx} + 2g^1_x(g^2 + V) + V_x g^1 + V_t = 0,\\
&g^3_t - g^3_{xx} + 2g^1_x g^3 + Vg^3 = 0
\end{aligned} \tag{8}$$

and

$$\eta_t = \eta_{xx} + 2\eta\eta_{xu} + \eta^2\eta_{uu} - V(\eta - u\eta_u) - V_x u. \tag{9}$$

6. Linearization of determining equations to initial ones

There are connections between solution families of an equation $\mathcal{L}$ from class (1) and its reduction operators. This generates connections of the system $\mathrm{DE}_1(\mathcal{L})$ and the equation $\mathrm{DE}_0(\mathcal{L})$ with the initial equation $\mathcal{L}$ via nonlocal transformations.

Consider at first reduction operators from $\mathcal{Q}_1(\mathcal{L})$. Below the indices i and j run from 1 to 3. The indices p and q run from 1 to 2. The summation convention over repeated indices is used.

Theorem 2. *Up to the equivalences of operators and solution families, for any equation from class (1) there exists a one-to-one correspondence between its reduction operators with nonzero coefficients of ∂_t and two-parametric families of its solutions of the form*

$$u = c_1 v^1(t,x) + c_2 v^2(t,x) + v^3(t,x), \tag{10}$$

where c_1 and c_2 are constant parameters. Namely, each operator of such kind corresponds to the family of solutions which are invariant with respect to this operator. The problem of the construction of all two-parametric solution families of equation (1), which are linear in parameters, is completely equivalent to the problem of the exhaustive description of its reduction operators with nonzero coefficients of ∂_t.

Corollary 5. *Nonlinear coupled system (6) is reduced by the transformation*

$$\begin{aligned} &g^1 = -A\frac{v^1 v^2_{xx} - v^1_{xx} v^2}{v^1 v^2_x - v^1_x v^2} - B, \qquad g^2 = -A\frac{v^1_x v^2_{xx} - v^1_{xx} v^2_x}{v^1 v^2_x - v^1_x v^2} + C,\\ &g^3 = \frac{A}{v^1 v^2_x - v^1_x v^2}\begin{vmatrix} v^1 & v^1_x & v^1_{xx}\\ v^2 & v^2_x & v^2_{xx}\\ v^3 & v^3_x & v^3_{xx}\end{vmatrix} \end{aligned} \tag{11}$$

to the uncoupled system of three copies of equation (1) for the functions $v^i = v^i(t,x)$:

$$Lv^i = v^i_t - Av^i_{xx} - Bv^i_x - Cv^i = 0, \tag{12}$$

and the functions v^1 and v^2 being linearly independent.

Note 5. Let $W(\varphi^1,\dots,\varphi^n)$ denote the Wronskian of the functions $\varphi^k = \varphi^k(t,x), k = 1,\dots,n$, with respect to the variable x, i.e. $W(\varphi^1,\dots,\varphi^n) = \det(\partial^l\varphi^k/\partial x^l)^n_{k,l=1}$. Then transformation (11) can be rewritten as

$$g^1 = -A\frac{(W(v^1,v^2))_x}{W(v^1,v^2)} - B, \qquad g^2 = -A\frac{W(v^1_x,v^2_x)}{W(v^1,v^2)} + C, \qquad g^3 = A\frac{W(v^1,v^2,v^3)}{W(v^1,v^2)}.$$

The solutions $\varphi^k = \varphi^k(t,x), k = 1,\dots,n$, of an equation from class (1) are linearly independent if and only if $W(\varphi^1,\dots,\varphi^n) \neq 0$. See, e.g., lemma 6 in [31]. Therefore, formulae (11) are well defined.

Proof. Let $\mathcal{L}$ be an equation from class (1) and $Q = \partial_t + g^1\partial_x + (g^2u + g^3)\partial_u \in \mathcal{Q}_1(\mathcal{L})$, i.e., the coefficients $g^i = g^i(t,x)$ satisfy the system $\mathrm{DE}_1(\mathcal{L})$. An Ansatz associated with Q has the form $u = f^1(t,x)\varphi(\omega) + f^0(t,x)$, where $f^1 = f^1(t,x)$ and $f^0 = f^0(t,x)$ are given coefficients, $f^1 \neq 0, \varphi = \varphi(\omega)$ is the new unknown function, $\omega = \omega(t,x)$ is the invariant-independent variable and $\omega_x \neq 0$. This Ansatz reduces $\mathcal{L}$ to an (in general, inhomogeneous) linear second-order ordinary differential equation in φ, which we denote by $\mathcal{L}'$. The general solution of $\mathcal{L}'$ is represented in the form $\varphi = c_p\varphi^p(\omega) + \varphi^3(\omega)$, where φ^3 is a particular solution of $\mathcal{L}'$, φ^1 and φ^2 are linearly independent solutions of the corresponding

homogeneous equation and c_1 and c_2 are arbitrary constants. Substituting the general solution of $\mathcal{L}'$ into the Ansatz, we obtain the two-parametric family of solutions of $\mathcal{L}$, having form (10) with $v^p = f\varphi^p$ and $v^3 = f\varphi^3 + g$. The split in the equations $Lu = 0$ and $Q[u] = 0$ with respect to the constants c_1 and c_2 implies that each of the functions v^i is a solution of $\mathcal{L}$ and

$$(g_1 + B)v^p_x - (g^2 - C)v^p = -Av^p_{xx}, \qquad (g_1 + B)v^3_x - (g^2 - C)v^3 - g^3 = -Av^3_{xx}.$$

Since $v^1v^2_x - v^1_x v^2 \neq 0$, the last system is a well-defined linear system of algebraic equations with respect to (g^1, g^2, g^3), whose solution is represented by (11).

Conversely, suppose that $\mathcal{F}$ is a two-parametric family of solutions of $\mathcal{L}$, having form (10). This means that each of the functions v^i is a solution of $\mathcal{L}$. The functions v^1 and v^2 are linearly independent since both the parameters c_1 and c_2 are essential. Consider the operator $Q = \partial_t + g^1\partial_x + (g^2u + g^3)\partial_u$, where the coefficients g^i are defined by (11). $Q[u] = 0$ for any $u \in \mathcal{F}$. The Ansatz $u = v^1\varphi(\omega) + v^3$, where $\omega = v^2/v^1$, constructed with Q, reduces $\mathcal{L}$ to the equation $\varphi_{\omega\omega} = 0$ since $(v^2/v^1)_x = W(v^1, v^2)/(v^1)^2 \neq 0$. Therefore [44], $Q \in \mathcal{Q}_1(\mathcal{L})$ and the functions g^i have to satisfy the system $\mathrm{DE}_1(\mathcal{L})$. □

Corollary 6. *Let $\mathcal{L}$ be an equation from class (1) and $G^\infty(\mathcal{L})$ denote the trivial Lie invariance group of $\mathcal{L}$, consisting of the linear superposition transformations of the form $\tilde t = t, \tilde x = x$ and $\tilde u = u + f(t, x)$, where the parameter-function $f = f(t, x)$ runs through the solution set of $\mathcal{L}$. Every reduction operator of the equation $\mathcal{L}$ with a nonvanishing coefficient of ∂_t is $G^\infty(\mathcal{L})$-equivalent to an operator $\partial_t + g^1\partial_x + g^2u\partial_u$, where the functions $g^1 = g^1(t, x)$ and $g^2 = g^2(t, x)$ satisfy the first two equations of* $\mathrm{DE}_1(\mathcal{L})$.

Proof. Suppose that a reduction operator Q of the equation $\mathcal{L}$ has a nonvanishing coefficient of ∂_t. In view of lemma 5, the operator Q is equivalent to an operator $\hat Q$ of the form $\partial_t + g^1\partial_x + (g^2u + g^3)\partial_u$, where the functions $g^1 = g^1(t, x)$, $g^2 = g^2(t, x)$ and $g^3 = g^3(t, x)$ satisfy the system $\mathrm{DE}_1(\mathcal{L})$. It follows from the proof of theorem 2 that the coefficient g^3 possesses the representation $g^3 = v^3_t + g^1v^3_x - g^2v^3$, where $v^3 = v^3(t, x)$ is a solution of $\mathcal{L}$. Then the transformation from $G^\infty(\mathcal{L})$ with $f = -v^3$ maps the operator $\hat Q$ to the operator $\tilde Q = \partial_t + g^1\partial_x + (g^2\tilde u + \tilde g^3)\partial_{\tilde u}$, where $\tilde g^3 = g^3 - v^3_t - g^1v^3_x + g^2v^3 = 0$. □

Note 6. The functions v^i satisfying the system (12) and the additional conditions (11) with fixed values of the coefficients g^j are defined up to the transformation

$$\tilde v^p = \mu_{pq}v^q, \qquad \tilde v^3 = v^3 + \mu_{3q}v^q, \tag{13}$$

where $\mu_{iq} = \mathrm{const}$, and $\det(\mu_{pq}) \neq 0$. Transformation (13) induces the transformation of the constants c_1 and c_2: $\tilde c_p = \tilde\mu_{pq}(c_q - \nu_q)$, where $(\tilde\mu_{pq}) = (\mu_{p'q'})^{-1}$. It is obvious that the families of solutions (10) and $u = \tilde c_1\tilde v^1 + \tilde c_2\tilde v^2 + \tilde v^3$ coincides up to re-parameterization and can be identified.

Consider reduction operators from $\mathcal{Q}_0(\mathcal{L})$.

Theorem 3. *Up to the equivalences of operators and solution families, for any equation of form (1) there exists a one-to-one correspondence between one-parametric families of its solutions and reduction operators with zero coefficients of ∂_t. Namely, each operator of such kind corresponds to the family of solutions which are invariant with respect to this operator. The problems of the construction of all one-parametric solution families of equation (1) and the exhaustive description of its reduction operators with zero coefficients of ∂_t are completely equivalent.*

Corollary 7. *The nonlinear* $(1+2)$*-dimensional equation (7) is reduced by composition of the nonlocal substitution* $\eta = -\Phi_x/\Phi_u$*, where* Φ *is a function of* (t, x, u)*, and the hodograph transformation*

$$\begin{aligned} &\text{the new independent variables:} && \tilde{t} = t, && \tilde{x} = x, && \varkappa = \Phi, \\ &\text{the new dependent variable:} && \tilde{u} = u \end{aligned} \tag{14}$$

to the initial equation $L\tilde{u} = 0$ *in the function* $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}, \varkappa)$ *with* $\varkappa$ *playing the role of a parameter.*

Proof. Let $\mathcal{L}$ be an equation from class (1) and $Q = \partial_x + \eta\partial_u \in \mathcal{Q}_0(\mathcal{L})$, i.e., the coefficient $\eta = \eta(t, x, u)$ satisfies the equation $\mathrm{DE}_0(\mathcal{L})$. An Ansatz associated with Q has the form $u = f(t, x, \varphi(\omega))$, where $f = f^1(t, x, \varphi)$ is a given function, $f_\varphi \neq 0, \varphi = \varphi(\omega)$ is the new unknown function and $\omega = t$ is the invariant-independent variable. This Ansatz reduces $\mathcal{L}$ to a first-order ordinary differential equation in φ, which we denote by $\mathcal{L}'$. The general solution of $\mathcal{L}'$ is represented in the form $\varphi = \varphi(\omega, \varkappa)$, where $\varphi_\varkappa \neq 0$ and $\varkappa$ is an arbitrary constant. The substitution of the general solution of $\mathcal{L}'$ into the Ansatz results in the one-parametric family $\mathcal{F}$ of solutions $u = \tilde{f}(t, x, \varkappa)$ of $\mathcal{L}$ with $\tilde{f} = f(t, x, \varphi(t, \varkappa))$. Expressing the parameter $\varkappa$ from the equality $u = \tilde{f}(t, x, \varkappa)$, we obtain that $\varkappa = \Phi(t, x, u)$, where $\Phi_u \neq 0$. Then $\eta = u_x = -\Phi_x/\Phi_u$ for any $u \in \mathcal{F}$, i.e., for any admissible values of $(t, x, \varkappa)$. This implies that $\eta = -\Phi_x/\Phi_u$ for any admissible values of (t, x, u).

Conversely, suppose that $\mathcal{F} = \{u = f(t, x, \varkappa)\}$ is a one-parametric family of solutions of $\mathcal{L}$. The derivative $f_\varkappa$ is nonzero since the parameter $\varkappa$ is essential. We express $\varkappa$ from the equality $u = f(t, x, \varkappa)$: $\varkappa = \Phi(t, x, u)$ for some function $\Phi = \Phi(t, x, u)$ with $\Phi_u \neq 0$. Consider the operator $Q = \partial_x + \eta\partial_u$, where the coefficient $\eta = \eta(t, x, u)$ is defined by the formula $\eta = -\Phi_x/\Phi_u$. $Q[u] = 0$ for any $u \in \mathcal{F}$. The Ansatz $u = f(t, x, \varphi(\omega))$, where $\omega = t$, associated with Q, reduces $\mathcal{L}$ to the equation $\varphi_\omega = 0$. Therefore [44], $Q \in \mathcal{Q}_0(\mathcal{L})$ and hence the function η satisfies the equation $\mathrm{DE}_0(\mathcal{L})$. □

Note 7. One-parametric families of solutions $u = f(t, x, \varkappa)$ and $u = \tilde{f}(t, x, \tilde{\varkappa})$ of $\mathcal{L}$ are assumed equivalent if they consist of the same solutions and differ only by parameterizations, i.e., if there exists a function $\zeta = \zeta(\varkappa)$ such that $\zeta_\varkappa \neq 0$ and $\tilde{f}(t, x, \zeta(\varkappa)) = f(t, x, \varkappa)$. Equivalent one-parametric families of solutions are associated with the same operator from $\mathcal{Q}_0(\mathcal{L})$ and have to be identified.

Note 8. The supposed triviality of the above Ansätze and reduced equations is connected with the usage of the special representations for the solutions of the determining equations. Under this approach, difficulties in the construction of Ansätze and the integration of reduced equations are replaced by difficulties in obtaining the representations for coefficients of reduction operators.

7. Admissible transformations, the equivalence groups and Lie symmetries of determining equations

The 'no-go' results of the previous section can be extended with the investigation of point transformations, Lie symmetries and Lie reductions of determining equations (6) and (7). Thus, the maximal Lie invariance algebras of (6) and (7) are isomorphic to the maximal Lie invariance algebras of equation (1) in a canonical way. (Before this result was known only for the linear heat equation [7].) Moreover, the similar statements are true for the complete point symmetry groups including discrete symmetry transformations as well as the equivalence groups and sets of admissible transformations of classes of the above equations.

All these statements are justified by lemmas 3 and 4. Indeed, each point transformation $\mathcal{T}$ between equations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ from class (1) has form (2) and induces the one-to-one mappings $\mathcal{T}_*\colon \mathcal{Q}(\mathcal{L}) \to \mathcal{Q}(\tilde{\mathcal{L}})$ and $\mathcal{T}_{\rm f}\colon \mathcal{Q}_{\rm f}(\mathcal{L}) \to \mathcal{Q}_{\rm f}(\tilde{\mathcal{L}})$. Due to the conditions $\mathcal{T}^t_x = 0$ and $\mathcal{T}^t_u = 0$, the transformation $\mathcal{T}_*$ preserves the constraint $\tau = 0$ (resp. $\tau \neq 0$) for coefficients of reduction operators. Therefore, the transformation $\mathcal{T}_{\rm f}$ is split into the one-to-one mappings $\mathcal{T}_{{\rm f},1}\colon \mathcal{Q}_1(\mathcal{L}) \to \mathcal{Q}_1(\tilde{\mathcal{L}})$ and $\mathcal{T}_{{\rm f},0}\colon \mathcal{Q}_0(\mathcal{L}) \to \mathcal{Q}_0(\tilde{\mathcal{L}})$ according to the canonical partitions of $\mathcal{Q}_{\rm f}(\mathcal{L})$ and $\mathcal{Q}_{\rm f}(\tilde{\mathcal{L}})$. This implies that there exist the transformations $\mathcal{T}_1$ and $\mathcal{T}_0$ in the spaces of the variables (t, x, g^1, g^2, g^3) and (t, x, u, η), which are induced by the transformation $\mathcal{T}$ in a canonical way. It is evident that

$$\mathcal{T}_1({\rm DE}_1(\mathcal{L})) = {\rm DE}_1(\tilde{\mathcal{L}}), \qquad \mathcal{T}_0({\rm DE}_0(\mathcal{L})) = {\rm DE}_0(\tilde{\mathcal{L}}).$$

The procedure of deriving the explicit formulae for $\mathcal{T}_1$ is the following: acting on the operator $\partial_t + g^1\partial_x + (g^2u + g^3)\partial_u$ by $\mathcal{T}_*$ and then normalizing the coefficient of $\partial_{\tilde t}$ to 1, we obtain the operator $\partial_{\tilde t} + \tilde g^1\partial_{\tilde x} + (\tilde g^2\tilde u + \tilde g^3)\partial_{\tilde u}$, where the new coefficients $\tilde g^i = \tilde g^i(\tilde t, \tilde x)$, $i = 1, 2, 3$, are calculated by the formulae

$$\begin{aligned}
\tilde g^1 &= \frac{X_x}{T_t}g^1 + \frac{X_t}{T_t},\\
\tilde g^2 &= \frac{1}{T_t}g^2 + \frac{U^1_x}{T_tU^1}g^1 + \frac{U^1_t}{T_tU^1},\\
\tilde g^3 &= \frac{U^1}{T_t}g^3 - \frac{U^0}{T_t}g^2 + \frac{U^0_xU^1 - U^0U^1_x}{T_tU^1}g^1 + \frac{U^0_tU^1 - U^0U^1_t}{T_tU^1}.
\end{aligned} \tag{15}$$

Formulae (15) describe the action of $\mathcal{T}_1$ on the dependent variables (g^1, g^2, g^3). The independent variables t and x and the arbitrary elements A, B and C are transformed by the same formulae (2) and (3) as ones of the transformation $\mathcal{T}$. The transformation of u is neglected.

If the transformation $\mathcal{T}$ belongs to the equivalence group $G^{\sim}$ of class (1) then it is defined for all values of arbitrary elements. Therefore, the same statement is true for $\mathcal{T}_1$, i.e., $\mathcal{T}_1$ belongs to the equivalence group $G^{\sim}_1$ of class (6). In other words, the equivalence group of the initial class induces a subgroup of the equivalence group of the class of determining equations for the case $\tau = 1$.

Suppose that the transformation $\mathcal{T}$ is parameterized by the parameter ε and this family of transformations form a one-parametric Lie symmetry group of the equation $\mathcal{L}$, generated by an operator $Q = \tau\partial_t + \xi\partial_x + (\zeta^1u + \zeta^0)\partial_u$. We differentiate formulae (15) with respect to ε and then put $\varepsilon = 0$, taking into account the conditions

$$\begin{aligned}
&\tau = \tau(t) = T_\varepsilon|_{\varepsilon=0}, && T|_{\varepsilon=0} = t, && \xi = \xi(t,x) = X_\varepsilon|_{\varepsilon=0}, && X|_{\varepsilon=0} = x,\\
&\zeta^1 = \zeta^1(t,x) = U^1_\varepsilon\big|_{\varepsilon=0}, && U^1|_{\varepsilon=0} = 1, && \zeta^0 = \zeta^0(t,x) = U^0_\varepsilon\big|_{\varepsilon=0}, && U^0|_{\varepsilon=0} = 0.
\end{aligned}$$

As a result, we obtain the expressions for the coefficients θ^i of the Lie symmetry operator $Q_1 = \tau\partial_t + \xi\partial_x + \theta^i\partial_{g^i}$ of the system ${\rm DE}_1(\mathcal{L})$, associated with the operator Q:

$$\begin{aligned}
\theta^1 &= (\xi_x - \tau_t)g^1 + \xi_t,\\
\theta^2 &= -\tau_tg^2 + \eta^1_xg^1 + \eta^1_t,\\
\theta^3 &= (\eta^1 - \tau_t)g^3 - \eta^0g^2 + \eta^0_xg^1 + \eta^0_t.
\end{aligned} \tag{16}$$

The explicit formulae for $\mathcal{T}_0$ are derived in the analogous way. The action of $\mathcal{T}_*$ on the operator $\partial_x + \eta\partial_u$ and the normalization of the coefficient of $\partial_{\tilde x}$ to 1, result in the operator $\partial_{\tilde x} + \tilde\eta\partial_{\tilde u}$, where

$$\tilde\eta = \frac{U^1}{X_x}\eta + \frac{U^1_x}{X_x}u + \frac{U^0_x}{X_x}. \tag{17}$$

Formula (17) represents the expression for the dependent variable η transformed by $\mathcal{T}_0$. The transformations of independent variables t, x and u and the arbitrary elements A, B and C are given by formulae (2) and (3). The unique difference from the transformation $\mathcal{T}$ is that the variable u is assumed independent. This implies that each transformation from the equivalence group $G^{\sim}$ of class (1) induces a transformation from the equivalence group $G_0^{\sim}$ of class (7).

Under the infinitesimal approach, each Lie invariance operator $Q = \tau\partial_t + \xi\partial_x + (\zeta^1 u + \zeta^0)\partial_u$ of $\mathcal{L}$ is prolonged to the Lie invariance operator $Q_0 = Q + \theta\partial_\eta$ of $\mathrm{DE}_0(\mathcal{L})$, where the coefficient θ is determined by the formula

$$\theta = (\zeta^1 - \xi_x)\eta + \zeta^1_x u + \zeta^0_x. \tag{18}$$

The problem is to prove that the induced objects (resp. admissible transformations, point equivalences, point symmetries and Lie invariance operators) exhaust all possible objects of the corresponding kinds for determining equations.

Lemma 6. *If a point transformation connects two systems* $\mathrm{DE}_1(\mathcal{L})$ *and* $\mathrm{DE}_1(\tilde{\mathcal{L}})$ *from class (6) then it has the form*

$$\tilde{t} = T(t), \qquad \tilde{x} = X(t,x), \qquad \tilde{g}^i = G^{ii'}(t,x)g^{i'} + G^{i0}(t,x), \tag{19}$$

where T, X, G^{33} *and* G^{32} *are smooth functions of their arguments such that* $T_t X_x G^{33} \neq 0$ *and additionally* G^{32}/G^{33} *is a solution of the associated equation* $\mathcal{L}$*;* $i, i' = 1, 2, 3$*. The other parameter-functions in (19) are explicitly defined:*

$$\begin{aligned}
&G^{10} = \frac{X_t}{T_t}, && G^{11} = \frac{X_x}{T_t}, && G^{12} = 0, && G^{13} = 0,\\
&G^{20} = \frac{(T_t G^{33})_t}{{T_t}^2 G^{33}}, && G^{21} = \frac{G^{33}_x}{T_t G^{33}}, && G^{22} = \frac{1}{T_t}, && G^{23} = 0,\\
&G^{30} = \frac{(T_t G^{33})_t}{{T_t}^2 G^{33}}, && G^{31} = \frac{G^{33}_x}{G^{33}} G^{32} - G^{32}_x.
\end{aligned} \tag{20}$$

The arbitrary elements are transformed by the formulae

$$\tilde{A} = \frac{X_x^2}{T_t} A, \qquad \tilde{B} = \frac{X_x}{T_t}\left(B - 2\frac{G^{33}_x}{G^{33}}A\right) - \frac{X_t - AX_{xx}}{T_t}, \qquad \tilde{C} = -G^{33} L \frac{1}{T_t G^{33}}. \tag{21}$$

Here, $L = \partial_t - A\partial_{xx} - B\partial_x - C$ *is the second-order linear differential operator associated with the equation* $\mathcal{L}$*.*

Proof. The systems $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_1(\tilde{\mathcal{L}})$ consist of second-order evolution equations which are linear in the derivatives, and coefficients of second derivatives form the nonsingular matrices $\mathrm{diag}(A, A, A)$ and $\mathrm{diag}(\tilde{A}, \tilde{A}, \tilde{A})$, respectively. In view of corollary 13 of [34] each transformation between such systems necessarily has form (19). We apply the direct method with taking into account conditions (19) and find more conditions which can be split by g^i and g^i_x. The system of determining equations on parameters of the transformation, obtained after the split, implies equations (20) and expressions (3) for transformations of the arbitrary elements. □

Theorem 4. *There exists a canonical one-to-one correspondence between the sets of admissible transformations of classes (1) and (6). Namely, each point transformation between equations* $\mathcal{L}$ *and* $\tilde{\mathcal{L}}$ *from class (1) induces a point transformation between the associated systems* $\mathrm{DE}_1(\mathcal{L})$ *and* $\mathrm{DE}_1(\tilde{\mathcal{L}})$ *according to formulae (15). In both the transformations the independent variables are transformed in the same way. The induced transformations exhaust the sets of admissible transformation in class (6).*

Proof. It only remains to prove that every admissible transformation in class (6) is induced by an admissible transformation in class (1) in the above way. We fix two point-equivalent systems from class (6). They necessarily are systems of determining equations for reduction operators with the unit coefficients of ∂_t for some equations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ from class (1). Therefore, these systems can be denoted by $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_1(\tilde{\mathcal{L}})$, respectively. Consider a point transformation $\breve{\mathcal{T}}$ mapping the system $\mathrm{DE}_1(\mathcal{L})$ to the system $\mathrm{DE}_1(\tilde{\mathcal{L}})$. In view of lemma 6, the transformation $\breve{\mathcal{T}}$ has form (19), where G^{32}/G^{33} is a solution of $\mathcal{L}$ and the other parameter-functions $G^{ii'}$ and G^{i0} are explicitly expressed by (20). Formulae (21) describe connections between the arbitrary elements of $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_1(\tilde{\mathcal{L}})$. We associate the transformation $\breve{\mathcal{T}}$ with the transformation $\mathcal{T}$ in the space of the variables (t, x, u), having form (2), where $U^1 = T_t G^{33}$ and $U^0 = T_t G^{32}$. By the construction, U^1/U^0 is a solution of $\mathcal{L}$. Since the pairs $(\mathrm{DE}_1(\mathcal{L}), \mathrm{DE}_1(\tilde{\mathcal{L}}))$ and $(\mathcal{L}, \tilde{\mathcal{L}})$ have the same tuples of arbitrary elements, lemma 4 and formulae (21) imply that $\mathcal{T}$ is a point transformation from $\mathcal{L}$ to $\tilde{\mathcal{L}}$. The comparison of (20) with (15) allows us to conclude that $\breve{\mathcal{T}}$ is induced by $\mathcal{T}$, i.e., $\breve{\mathcal{T}} = \mathcal{T}_1$. □

Note 9. It follows from the proof of theorem 4 that 'if . . . then . . .' in lemma 6 can be replaced by '. . . if and only if . . .', i.e., the presented conditions are necessary and sufficient.

Corollary 8. *The equivalence group $G^{\sim}_1$ of class (6) is isomorphic to the equivalence group $G^{\sim}$ of class (1). The canonical isomorphism is established by formulae (15), where $U^0 = 0$.*

Corollary 9. *For each equation $\mathcal{L}$ from class (1), the maximal point symmetry groups (resp. the maximal Lie invariance algebras) of the equation $\mathcal{L}$ and the system $\mathrm{DE}_1(\mathcal{L})$ are isomorphic. A Lie symmetry operator $Q = \tau\partial_t + \xi\partial_x + (\zeta^1 u + \zeta^0)\partial_u$ of $\mathcal{L}$ induces the Lie symmetry operator $Q_1 = \tau\partial_t + \xi\partial_x + \theta^i\partial_{g^i}$ of the system $\mathrm{DE}_1(\mathcal{L})$, where the coefficients θ^i, $i = 1, 2, 3$, are defined by formulae (16).*

Corollaries 8 and 9 along with theorem 1 give the group classification of class (6).

Corollary 10. *The kernel Lie algebra of class (6) is $\langle I_1\rangle$, where $I_1 = g^3\partial_{g^3}$. Any system from class (6) is invariant with respect to the operators of the form $Z_1(f) = (f_t + f_x g^1 - f g^2)\partial_{g^3}$, where the parameter-function $f = f(t, x)$ runs through the solution set of the associated equation $f_t = Af_{xx} + Bf_x + Cf$. All possible $G^{\sim}_1$-inequivalent cases of extension of the maximal Lie invariance algebra are exhausted by the following systems of the reduced form (8) (the values of V are given together with the corresponding maximal Lie invariance algebras):*

(1) $V = V(x)$: $\langle\partial_t, I_1, Z_1(f)\rangle$;
(2) $V = \mu x^{-2}$, $\mu \neq 0$: $\langle\partial_t, D_1, \Pi_1, I_1, Z_1(f)\rangle$;
(3) $V = 0$: $\langle\partial_t, \partial_x, G_1, D_1, \Pi_1, I_1, Z_1(f)\rangle$.

Here,

$$D_1 = 2t\partial_t + x\partial_x - g^1\partial_{g^1} - 2g^2\partial_{g^2},$$
$$\Pi_1 = 4t^2\partial_t + 4tx\partial_x + 4(x - tg^1)\partial_{g^1} - (8tg^2 + 2xg^1 + 2)\partial_{g^2} - (x^2 + 10t)g^3\partial_{g^3},$$
$$G_1 = 2t\partial_x + 2\partial_{g^1} - g^1\partial_{g^2} - xg^3\partial_{g^3}.$$

Note 10. It is obvious that corollaries 8, 9 and 10 can be reformulated for subclass (4) of the initial equations in the reduced form and subclass (8) of the corresponding determining equations of the first kind (the case $\tau \neq 0$).

A specific question for class (6) is what transformations of the functions (v^1, v^2, v^3) defined in corollary 5 are induced by admissible transformations in class (6). It is clear that each induced transformation is admissible in class (12). Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be equations from

class (1). Denote the corresponding systems of form (12) by $3\mathcal{L}$ and $3\tilde{\mathcal{L}}$ and the corresponding sets of formulae (11) by $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively. It is proved analogously to lemma 6 that any point transformation connecting the systems $3\mathcal{L}$ and $3\tilde{\mathcal{L}}$ has the form

$$\tilde{t} = T(t), \qquad \tilde{x} = X(t,x), \qquad \tilde{v}^i = U^1(t,x)\mu_{ij}v^j + U^{i0}(t,x),$$

where $\mu_{ij} = \text{const}, \det(\mu_{ij}) \neq 0, i, j = 1,2,3$; T, X, U^1 and U^{i0} are arbitrary smooth functions of their arguments such that $T_t X_x U^1 \neq 0$ and additionally U^{i0}/U^1 are solutions of the equation $\mathcal{L}$. The arbitrary elements are transformed by formulae (3), where $L = \partial_t - A\partial_{xx} - B\partial_x - C$ is the second-order linear differential operator associated with the equation $\mathcal{L}$. The agreement of transformations between $3\mathcal{L}$ and $3\tilde{\mathcal{L}}$ with transformations between $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_1(\tilde{\mathcal{L}})$ via formulae (11) implies the additional conditions

$$\mu_{13} = \mu_{23} = 0, \qquad U^{10} = U^{20} = 0, \qquad \mu_{33} = 1, \qquad U^1 = T_t G^{33}, \qquad U^{30} = T_t G^{30}$$

for the admissible transformations between the systems $3\mathcal{L} \cap \mathcal{G} \cap \mathrm{DE}_1(\mathcal{L})$ and $3\tilde{\mathcal{L}} \cap \tilde{\mathcal{G}} \cap \mathrm{DE}_1(\tilde{\mathcal{L}})$. To derive these conditions, we express all the tilde variables in $\tilde{\mathcal{G}}$ via the non-tilde ones, then substitute the expressions for g^i given by $\mathcal{G}$ into $\tilde{\mathcal{G}}$ and split with respect to v^j and their derivatives. Combining the obtained result with theorem 4 and omitting the systems $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_1(\tilde{\mathcal{L}})$ as differential consequences of the systems $3\mathcal{L} \cap \mathcal{G}$ and $3\tilde{\mathcal{L}} \cap \tilde{\mathcal{G}}$, respectively, we get that the point transformation $\mathcal{T}$ of form (2) between the equations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces the point transformation

$$\tilde{t} = T(t), \qquad \tilde{x} = X(t,x), \qquad \tilde{v}^p = U^1(t,x)\mu_{pq}v^q, \qquad \tilde{v}^3 = U^1(t,x)\mu_{3q}v^q + U^0(t,x),$$

where $\det(\mu_{pq}) \neq 0$, $p, q = 1,2$, between the system $3\mathcal{L} \cap \mathcal{G}$ and $3\tilde{\mathcal{L}} \cap \tilde{\mathcal{G}}$. The appearance of the additional constants μ_{iq} in the induced transformation is explained by uncertainty (13) under determining the function v^i. The consideration of a one-parametric Lie symmetry group of the equation $\mathcal{L}$ instead of a single transformation between the (possibly different) equations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ results in a formula for the extension of Lie symmetry operators of $\mathcal{L}$ to Lie symmetry operators of $3\mathcal{L}$. Namely, the following statement is true.

Lemma 7. *Each Lie symmetry operator* $Q = \tau\partial_t + \xi\partial_x + (\zeta^1 u + \zeta^0)\partial_u$ *of the equation* $\mathcal{L}$ *generates the family*

$$\{\tau\partial_t + \xi\partial_x + \zeta^1 v^i \partial_{v^i} + \zeta^0 \partial_{v^3} + \lambda_{iq} v^q \partial_{v^i} \mid \lambda_{iq} = \text{const}\}$$

of Lie symmetry operators of the associated system $3\mathcal{L}$ *with the additional conditions* $\mathcal{G}$. *Here,* $i, j = 1,2,3, q = 1,2$. *The functions* g^j *satisfy the system* $\mathrm{DE}_1(\mathcal{L})$ *being the compatibility condition of* $3\mathcal{L} \cap \mathcal{G}$.

The chain of similar statements is also obtained for class (7).

Lemma 8. *If a point transformation in the space of the variables* (t, x, u, η) *connects two equations* $\mathrm{DE}_0(\mathcal{L})$ *and* $\mathrm{DE}_0(\tilde{\mathcal{L}})$ *from class (7) then it has the form given by formulae (2) and (17), where* T, X, U^1 *and* U^0 *are arbitrary smooth functions of their arguments such that* $T_t X_x U^1 \neq 0$ *and additionally* U^0/U^1 *is a solution of the equation* $\mathcal{L}$. *The arbitrary elements are transformed by formulae (3), where* $L = \partial_t - A\partial_{xx} - B\partial_x - C$ *is the second-order linear differential operator associated with the equation* $\mathcal{L}$.

Proof. The matrices formed by the coefficients of the second derivations in the equations $\mathrm{DE}_0(\mathcal{L})$ and $\mathrm{DE}_0(\tilde{\mathcal{L}})$ are singular. That is why we cannot use the results of [36] on admissible transformations in classes of parabolic equations having positively defined matrices of the coefficients of the second derivations. All determining equations have to be obtained independently.

We use the direct method. Consider a point transformation $\mathcal{T}$ from the equation $\mathrm{DE}_0(\mathcal{L})$ to the equation $\mathrm{DE}_0(\tilde{\mathcal{L}})$ of the general form $[\tilde t,\tilde x,\tilde u,\tilde\eta]=[T,X,U,H](t,x,u,\eta)$ with the nonvanishing Jacobian. Sometimes we will also assume that the old variables (t,x,u,η) are functions of the new variables $(\tilde t,\tilde x,\tilde u,\tilde\eta)$ and do a simultaneous split with respect to both the old and new variables. This trick is correct under certain conditions. We introduce the notations $Q:=D_x+\eta D_u$, $\tilde Q:=D_{\tilde x}+\tilde\eta D_{\tilde u}$ and $F:=\tilde Q\tilde\eta$. In the old variables, the function F is expressed via t,x,u,η,η_t,η_x and η_u, and moreover $(F_{\eta_t},F_{\eta_x},F_{\eta_u})\neq(0,0,0)$. (Indeed, the condition $F_{\eta_t}=F_{\eta_x}=F_{\eta_u}=0$ means that the function F depends only on (t,x,t,η) in the old variables and, therefore, is a function of only $(\tilde t,\tilde x,\tilde u,\tilde\eta)$ in the new variables. Then we could split the equation $F=\tilde\eta_{\tilde x}+\tilde\eta\tilde\eta_{\tilde u}$ defining F with respect to the derivatives of $\tilde\eta$ and obtain the contradiction $0=1$.)

The equation $\mathrm{DE}_0(\tilde{\mathcal{L}})$ can be written in the form $\tilde QF=\cdots$, where the right-hand side contains derivatives only up to order 1. We return to the old variables in $\mathrm{DE}_0(\tilde{\mathcal{L}})$ and confine it to the manifold of the equation $\mathrm{DE}_0(\mathcal{L})$, expressing the derivative η_{xx} from $\mathrm{DE}_0(\mathcal{L})$ and substituting the found expression into $\mathrm{DE}_0(\tilde{\mathcal{L}})$. Then we split the obtained equation DE_0' step-by-step with respect to different subsets of the other derivatives of η (or $\tilde\eta$ alternatively). To optimize this procedure, we start from the subsets of derivatives giving the simplest determining equations and take into account found equations for the further split. Note that the expression $\tilde QF$ has the representation $\tilde QF=(\tilde Qt)D_tF+(\tilde Qx)D_xF+(\tilde Qu)D_uF$.

After collecting the coefficients of η_{tt}, η_{tx} and η_{tu} in DE_0', we derive the system

$$(\tilde Qt)F_{\eta_t}=0,\qquad (\tilde Qt)F_{\eta_x}+(\tilde Qx)F_{\eta_t}=0,\qquad (\tilde Qt)F_{\eta_u}+(\tilde Qu)F_{\eta_t}=0$$

which implies the equation $\tilde Qt=0$ since $\big(F_{\eta_t},F_{\eta_x},F_{\eta_u}\big)\neq(0,0,0)$. We expand the expression $\tilde Qt$, assuming t a function of $(\tilde t,\tilde x,\tilde u,\tilde\eta)$: $\tilde Qt=t_{\tilde x}+t_{\tilde\eta}\tilde\eta_{\tilde x}+\tilde\eta(t_{\tilde u}+t_{\tilde\eta}\tilde\eta_{\tilde u})$. The split of the equation $\tilde Qt=0$ with respect to the new jet variables $\tilde\eta_{\tilde x}$ and $\tilde\eta_{\tilde u}$ results in the equations $t_{\tilde\eta}=0$ and $t_{\tilde x}+\tilde\eta t_{\tilde u}=0$. Then the subsequent split with respect to the new variable $\tilde\eta$ gives the equations $t_{\tilde x}=0$ and $t_{\tilde u}=0$. Therefore, t is a function of only $\tilde t$, i.e., $\tilde t$ depends only on t, $\tilde t=T(t)$. Under this condition, the function F expressed in the old variables does not depend on η_t, i.e., $F_{\eta_t}=0$ and hence $\big(F_{\eta_x},F_{\eta_u}\big)\neq(0,0)$.

Collecting the coefficients of η_{uu} and η_{xu} in DE_0' gives the system

$$(\tilde Qu)F_{\eta_u}-\eta^2(\tilde Qx)F_{\eta_x}=0,\qquad (\tilde Qx)F_{\eta_u}+(\tilde Qu)F_{\eta_x}-2\eta(\tilde Qx)F_{\eta_x}=0.$$

Since $\big(F_{\eta_x},F_{\eta_u}\big)\neq(0,0)$, the determinant of the matrix of this system considered as a system of linear algebraic equations with respect to $\big(F_{\eta_x},F_{\eta_u}\big)$ has to vanish, i.e., $(\tilde Qu-\eta\tilde Qx)^2=0$ that implies $\tilde Qu=\eta\tilde Qx$. Assuming x and u the functions of $(\tilde t,\tilde x,\tilde u,\tilde\eta)$, we expand the expressions $\tilde Qx$ and $\tilde Qu$ similarly to $\tilde Qt$ and split the equation $\tilde Qu=\eta\tilde Qx$ with respect to the new jet variables $\tilde\eta_{\tilde x}$ and $\tilde\eta_{\tilde u}$. This results to the equations $u_{\tilde\eta}=\eta x_{\tilde\eta}$ and $u_{\tilde x}+\tilde\eta u_{\tilde u}=\eta(x_{\tilde x}+\tilde\eta x_{\tilde u})$. Alternating the old and new variables in any derived equation gives a correct equation. Therefore, we also have the equations $U_\eta=HX_\eta$, $U_x+\eta U_u=H(X_x+\eta X_u)$.

The next term for collecting coefficients in DE_0' is η_t. The equation obtained by this split is presented as $AG=\tilde A(\tilde Qx)F_{\eta_x}$, where G denotes the coefficient of η_t in $\tilde\eta_{\tilde t}$. Under the above-stated conditions, the expressions appearing in this equation take the form

$$F=\frac{1}{\Delta}\left(\frac{D(H,U)}{D(x,u)}+H\frac{D(X,H)}{D(x,u)}\right),\qquad G=\frac{1}{T_t\Delta}\frac{\partial(H,X,U)}{\partial(\eta,x,u)},\qquad \tilde Qx=\frac{U_u-HX_u}{\Delta}.$$

Hereafter $\Delta=D(X,U)/D(x,u)(\neq0)$, and

$$\frac{\partial(Z^1,\dots,Z^k)}{\partial(z_1,\dots,z_k)}\quad\text{and}\quad\frac{D(Z^1,\dots,Z^k)}{D(z_1,\dots,z_k)}$$

denote the usual and total Jacobians of the functions $Z^1,\dots,Z^k$ with respect to the variables $z_1,\dots,z_k$, respectively. Note that in the case of a single dependent variable each total Jacobian

is, at most, a first-order polynomial in the derivatives of this dependent variable. Removing the denominators from the equations $AG = \tilde{A}(\tilde{Q}x)F_{\eta_x}$ results in the equation

$$A\Delta^2\frac{\partial(H,X,U)}{\partial(\eta,x,u)} = \tilde{A}(U_u - HX_u)\left[\Delta\left(\frac{\partial(H,U)}{\partial(\eta,u)} + H\frac{\partial(X,H)}{\partial(\eta,u)}\right)\right.$$
$$\left. - \frac{\partial(X,U)}{\partial(\eta,u)}\left(\frac{D(H,U)}{D(x,u)} + H\frac{D(X,H)}{D(x,u)}\right)\right]$$

the right-hand side of which is at most a first-order polynomial in η_x and η_u. In view of nonvanishing A and $\partial(H,X,U)/\partial(\eta,x,u)$, this implies that the coefficients of η_x and η_u in Δ equal zero, i.e., $\partial(X,U)/\partial(\eta,u) = 0$ and $\partial(X,U)/\partial(x,\eta) = 0$. Then $\partial(X,U)/\partial(x,u) \neq 0$ since otherwise the transformation $\mathcal{T}$ is singular. Hence $X_\eta = U_\eta = 0$.

Collecting the coefficients of η_x^2 in DE_0' leads to the equation $H_{\eta\eta}(U_u - HX_x)^2 = 0$. Note that $U_u - HX_x = (\tilde{Q}x)\Delta \neq 0$ since $\Delta \neq 0$ and $\tilde{Q}x \neq 0$. (Via the split with respect to unconstrained tilde variables, vanishing $\tilde{Q}x$ implies the condition $x_{\tilde{x}} = x_{\tilde{u}} = x_{\tilde{\eta}} = 0$ which contradict the nonsingularity of the inverse of $\mathcal{T}$.) Therefore, $H_{\eta\eta} = 0$, i.e., $H = H^1(t,x,u)\eta + H^0(t,x,u)$, where $H^1 = H_\eta \neq 0$. Knowing the explicit dependence of H on η allows us to additionally split all equations with respect to η. Thus, splitting the equation $U_x + \eta U_u = H(X_x + \eta X_u)$ gives the condition $X_u = 0$ (hence $X_x U_u \neq 0$) and, then, the conditions $H^1 = U_u/X_x$ and $H^0 = U_x/X_x$. The equation DE_0' contains only a single term including $\eta^2\eta_u$. Equating the corresponding coefficient to zero, we derive the condition $U_{uu} = 0$.

The whole set of the above found conditions on T, X, U and H implies that the form of the transformation $\mathcal{T}$ is described by formulae (2) and (17). Then the operator Q is transformed in a simple way: $\tilde{Q} = X_x^{-1}Q$. This gives us the idea to rewrite the equations $\mathrm{DE}_0(\mathcal{L})$ and $\mathrm{DE}_0(\tilde{\mathcal{L}})$ in terms of the operators Q and $\tilde{Q}$, respectively. Thus, the equation $\mathrm{DE}_0(\mathcal{L})$ has the form

$$\eta_t + \eta_u(AQ\eta + B\eta + Cu) = AQ^2\eta + (A_x + B)Q\eta + (B_x + C)\eta + C_x u.$$

All derivatives of η containing the differentiation with respect to x are excluded from DE_0' by the substitution $\eta_x = Qu - \eta\eta_u$, and hence DE_0' can be split with respect to $Q^2\eta$, η_u, $Q\eta$, η and u. Collecting the coefficients of the terms $\eta_u Q\eta$, $\eta_u\eta$, $\eta_u u$ and η_u, we obtain formulae (3) for transformations of the arbitrary elements A, B and C and the condition $L(U^1/U^0) = 0$. □

Note 11. We do not split under deriving determining equations in the proof of lemma 8 as much as possibly since the resulting system would be too cumbersome and, moreover, the proof of theorem 5 implies that in fact this complete system is reduced to the set of conditions presented in lemma 8.

Theorem 5. *There exists a canonical one-to-one correspondence between the sets of admissible transformations of classes (1) and (7). Namely, each point transformation between equations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ from class (1) is extended to a point transformation between the associated equations* $\mathrm{DE}_0(\mathcal{L})$ *and* $\mathrm{DE}_0(\tilde{\mathcal{L}})$ *according to formula (17). In both the transformations the variables (t,x,u) and the arbitrary elements are transformed in the same way. The extended transformations exhaust the sets of admissible transformation in class (7).*

Proof. The extension of each admissible transformation in class (1) by formula (17) gives an admissible transformation in class (7). Therefore, it is enough to check that every admissible transformation in class (7) coincides with the extension of an admissible transformation in class (1). We take two equations from class (6) which are connected via a point transformation.

They necessarily are determining equations for reduction operators with the zero coefficients of ∂_t and the unit coefficients of ∂_x for some equations $\mathcal{L}$ and $\tilde{\mathcal{L}}$ from class (1). Therefore, these equations can be denoted by $\mathrm{DE}_0(\mathcal{L})$ and $\mathrm{DE}_0(\tilde{\mathcal{L}})$, respectively. Consider a point transformation $\breve{\mathcal{T}}$ mapping $\mathrm{DE}_0(\mathcal{L})$ to $\mathrm{DE}_0(\tilde{\mathcal{L}})$. In view of lemma 8, the transformation $\breve{\mathcal{T}}$ has the form given by formulae (2) and (17) and, therefore, is projectable on the space of the variables (t, x, u). Denote its projection by $\mathcal{T}$. The pairs $(\mathrm{DE}_0(\mathcal{L}), \mathrm{DE}_0(\tilde{\mathcal{L}}))$ and $(\mathcal{L}, \tilde{\mathcal{L}})$ have the same tuples of arbitrary elements transformed by the same formulae (3). That is why lemmas 4 and 8 imply that $\mathcal{T}$ is a point transformation from $\mathcal{L}$ to $\tilde{\mathcal{L}}$. It is clear that the transformation $\breve{\mathcal{T}}$ is the extension of $\mathcal{T}$ by formula (17), i.e., $\breve{\mathcal{T}} = \mathcal{T}_0$. □

Corollary 11. *The equivalence group $G_0^{\sim}$ of class (7) is isomorphic to the equivalence group $G^{\sim}$ of class (1). The canonical isomorphism is established by the extension of transformations from $G_0^{\sim}$ to the variable η via formula (17), where $U^0 = 0$.*

Corollary 12. *For any equation $\mathcal{L}$ from class (1), the maximal point symmetry groups (resp. the maximal Lie invariance algebras) of the equations $\mathcal{L}$ and $\mathrm{DE}_0(\mathcal{L})$ are isomorphic. The canonical isomorphism between the algebras is realized via the extension of each Lie symmetry operator $Q = \tau\partial_t + \xi\partial_x + (\zeta^1 u + \zeta^0)\partial_u$ of $\mathcal{L}$ to the Lie symmetry operator $Q_1 = Q + \big((\zeta^1 - \xi_x)\eta + \zeta^1_x u + \zeta^0_x\big)\partial_\eta$ of $\mathrm{DE}_0(\mathcal{L})$.*

In view of corollaries 11 and 12, the results on the group classification of class (7) follow from theorem 1.

Corollary 13. *The kernel Lie algebra of class (7) is $\langle I_0\rangle$, where $I_0 = u\partial_u + \eta\partial_\eta$. Any equation from class (7) is invariant with respect to the operators of the form $Z_0(f) = f\partial_u + f_x\partial_\eta$, where the parameter-function $f = f(t, x)$ runs through the solution set of the associated equation $f_t = Af_{xx} + Bf_x + Cf$. All possible $G_0^{\sim}$-inequivalent cases of extension of the maximal Lie invariance algebra are exhausted by the following equations of the reduced form (9) (the values of V are given together with the corresponding maximal Lie invariance algebras):*

(1) $V = V(x)$: $\langle\partial_t, I_0, Z_0(f)\rangle$;
(2) $V = \mu x^{-2}$, $\mu \neq 0$: $\langle\partial_t, D_0, \Pi_0, I_0, Z_0(f)\rangle$;
(3) $V = 0$: $\langle\partial_t, \partial_x, G_0, D_0, \Pi_0, I_0, Z_0(f)\rangle$

Here,

$$D_0 = 2t\partial_t + x\partial_x - \eta\partial_\eta,$$
$$\Pi_0 = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)u\partial_u - (x\eta + 6t\eta + 2xu)\partial_\eta,$$
$$G_0 = 2t\partial_x - xu\partial_u - (x\eta + u)\partial_\eta.$$

8. Lie reductions of determining equations

Suppose that an equation $\mathcal{L}$ from class (1) admits a Lie symmetry operator $Q = \tau\partial_t + \xi\partial_x + \zeta\partial_u$. The coefficients of Q necessarily satisfy the conditions $\tau_x = \tau_u = 0$, $\xi_u = 0$ and $\zeta_{uu} = 0$, i.e., $\tau = \tau(t)$, $\xi = \xi(t, x)$ and $\zeta = \zeta^1(t, x)u + \zeta^0(t, x)$, and ζ^0 is a solution of $\mathcal{L}$.

In view of corollaries 9 and 12, the determining equations $\mathrm{DE}_1(\mathcal{L})$ and $\mathrm{DE}_0(\mathcal{L})$, respectively, possess the Lie symmetry operators Q_1 and Q_0 associated with Q, which can be applied to reduce the determining equations and construct their exact solutions. The found solutions of the determining equations give the reduction operators of a special kind for the initial equation $\mathcal{L}$, implicitly connected with Lie invariance properties of $\mathcal{L}$. The question is what properties the solutions of $\mathcal{L}$, invariant with respect to such reduction operators, possess, e.g., whether these solutions necessarily are Lie invariant or they are not.

An admissible transformation $\mathcal{T}$ of the equation $\mathcal{L}$ in class (1) has form (2) and maps the pair $(\mathcal{L}, Q)$ to a pair $(\mathcal{L}', Q')$, where the equation $\mathcal{L}'$ also belongs to class (1) and Q' is a nontrivial (resp. trivial) Lie symmetry operator of $\mathcal{L}'$ if Q is a nontrivial (resp. trivial) Lie symmetry operator of $\mathcal{L}$. Up to the equivalence generated by the set of all admissible transformations of class (1) (see lemma 4) in the set of pairs (equation of form (1), its Lie symmetry operator), we can assume that $Q \in \{\partial_t, \partial_x\}$ or $Q \in \{u\partial_u, \partial_u\}$ if Q is a nontrivial or trivial Lie symmetry operator of $\mathcal{L}$, respectively. $Q \sim \partial_t$ if $\tau \neq 0$ and $Q \sim \partial_x$ if $\tau = 0$ and $\xi \neq 0$.

If $Q \in \{\partial_t, \partial_x\}$, the Lie symmetry operator Q_1 of the system $\mathrm{DE}_1(\mathcal{L})$ and the Lie symmetry operator Q_0 of the equation $\mathrm{DE}_0(\mathcal{L})$, which are associated with the operator Q, formally have the same form as the operator Q but are defined in different spaces of variables.

Proposition 1. *Suppose that an equation $\mathcal{L}$ from class (1) possesses a Lie symmetry operator $Q = \tau\partial_t + \xi\partial_x + \zeta\partial_u$, where necessarily $\tau = \tau(t)$, $\xi = \xi(t, x)$ and $\zeta = \zeta^1(t, x)u + \zeta^0(t, x)$ and additionally $\tau \neq 0$. Let Q_1 be the associated Lie symmetry operator of the system $\mathrm{DE}_1(\mathcal{L})$, a solution (g^1, g^2, g^3) of $\mathrm{DE}_1(\mathcal{L})$ be Q_1-invariant and $R = \partial_t + g^1\partial_x + (g^2u + g^3)\partial_u \in \mathcal{Q}_1(\mathcal{L})$ be the corresponding reduction operator. Then the functions g^1, g^2 and g^3 are expressed, according to formulae (11), via a solution (v^1, v^2, v^3) of the uncoupled system $3\mathcal{L}$, which is invariant with respect to the Lie symmetry operator*

$$\tau\partial_t + \xi\partial_x + \zeta^1 v^1\partial_{v^1} + \zeta^1 v^2\partial_{v^2} + (\zeta^1 v^3 + \zeta^0)\partial_{v^3} + \lambda_{iq}v^q\partial_{v^i}$$

of this system for some constants λ_{iq}, $i = 1, 2, 3$, $q = 1, 2$. Here the functions v^1 and v^2 have to be linearly independent. Each R-invariant solution of $\mathcal{L}$ is a linear combination, with the unit coefficient of v^3, of the components of the Lie invariant solution (v^1, v^2, v^3) of the system $3\mathcal{L}$.

Proof. It is sufficient to consider only the reduced form of Lie symmetry operators, which is $Q = \partial_t$ in the case $\tau \neq 0$. Then $Q_1 = \partial_t$. The equation $\mathcal{L}$ is Lie invariant with respect to the operator ∂_t if and only if $A_t = B_t = C_t = 0$. Consider an operator $R = \partial_t + g^1\partial_x + (g^2u + g^3)\partial_u \in \mathcal{Q}_1(\mathcal{L})$, where the coefficient tuple (g^1, g^2, g^3) is a Q_1-invariant solution of $\mathrm{DE}_1(\mathcal{L})$, i.e., it additionally satisfies the condition $g^1_t = g^2_t = g^3_t = 0$. An Ansatz constructed with the operator R has the form $u = f^1(x)\varphi(\omega) + f^0(x)$, where $f^1 = f^1(x) \neq 0$ and $f^0 = f^0(x)$ are given coefficients, $\varphi = \varphi(\omega)$ is the new unknown function, $\omega = t + \varrho(x)$ is the invariant-independent variable and $\varrho_x \neq 0$. This Ansatz reduces $\mathcal{L}$ to a (in general, inhomogeneous) linear second-order constant-coefficient ordinary differential equation in φ, which we denote by $\mathcal{L}'$. The general solution of $\mathcal{L}'$ is represented in the form $\varphi = c_p\varphi^p(\omega) + \varphi^3(\omega)$, where φ^3 is a particular solution of $\mathcal{L}'$, φ^1 and φ^2 are linearly independent solutions of the corresponding homogeneous equation and c_1 and c_2 are arbitrary constants. Let us recall that $p, q = 1, 2$. Substituting the general solution of $\mathcal{L}'$ into the Ansatz, we obtain the two-parametric family of solutions of $\mathcal{L}$, having form (10) with $v^p = f\varphi^p$ and $v^3 = f\varphi^3 + g$. Due to $\mathcal{L}'$ is a constant-coefficient equation, the functions v^i admit the representation $v^p = \psi^{pq}(t)\theta^q(x)$ and $v^3 = \psi^{3q}(t)\theta^q(x) + \theta^3(x)$, where $\psi^{iq}_t = \lambda_{ip}\psi^{pq}$ for some constants λ_{ip} depending on the coefficients of $\mathcal{L}'$. Therefore, (v^1, v^2, v^3) is a solution of the system $3\mathcal{L}$, which is invariant with respect to the Lie symmetry operator $\partial_t + \lambda_{iq}v^q\partial_{v^i}$ of this system. □

Proposition 2. *Suppose that the system $\mathrm{DE}_1(\mathcal{L})$ associated with an equation $\mathcal{L}$ from class (1) possesses a Lie invariance operator Q_1 with the vanishing coefficient of ∂_t and a nonvanishing coefficient of ∂_x. Let a solution (g^1, g^2, g^3) of $\mathrm{DE}_1(\mathcal{L})$ be invariant with respect to Q_1. Then the associated reduction operator $\partial_t + g^1\partial_x + (g^2u + g^3)\partial_u$ of the equation $\mathcal{L}$ is necessarily equivalent to a Lie invariance operator of $\mathcal{L}$.*

Proof. Consider the case $Q = \partial_x$. The equation $\mathcal{L}$ possesses the Lie symmetry operator ∂_x if and only if $A_x = B_x = C_x = 0$. Then the equivalence transformation $\tilde{t} = T(t), \tilde{x} = x + \varphi(t)$ and $\tilde{u} = \psi(t)u$, where $T_t = A, \varphi_t = B, \psi_t = C\psi$ and $\psi \neq 0$, maps Q to $\partial_{\tilde{x}}$ and reduces $\mathcal{L}$ to the linear heat equation $\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}$ associated with the values $\tilde{A} = 1$ and $\tilde{B} = \tilde{C} = 0$. That is why without loss of generality we can assume that $A = 1$ and $B = C = 0$. An Ansatz constructed for the system $\mathrm{DE}_1(\mathcal{L})$ by the operator $Q_1 = \partial_x$ is $g^i = g^i(t)$ and the corresponding reduced system has the form $g^i_t = 0$, i.e., $g^i = \text{const}$. The operator $\partial_t + g^1\partial_x + (g^2u + g^3)\partial_u$ with constant coefficients belongs to the maximal Lie invariance algebra of the equation $\mathcal{L}$ which coincides under our suppositions with the linear heat equation. The obtained statement is reformulated for the general form of Q with the vanishing coefficient of ∂_t. □

Results on Lie solutions of the determining equation $\mathrm{DE}_0(\mathcal{L})$ can be presented as a single statement without split into different cases depending on the structure of the corresponding Lie symmetry operators. To formulate them in a compact form, we need to introduce at first the auxiliary notion of *one-parametric solution families* of the equation $\mathcal{L}$, *associated with the Lie symmetry operator* Q of $\mathcal{L}$. The set of such families is partitioned into two subsets which are, respectively, formed by the *singular associated families* consisting of Q-invariant solutions of $\mathcal{L}$ and the *regular associated families* obtained via acting on fixed non-Q-invariant solutions of $\mathcal{L}$ by the one-parametric transformation group generated by Q.

Let us recall that Q_0 denotes the Lie symmetry operator of $\mathrm{DE}_0(\mathcal{L})$, associated with Q. Equivalent families of solutions, which differ only by parameterization, are identified. In particular, regular one-parametric families associated with the same operator are equivalent if and only if they differ only by parameter shifts. Such families are obtained by the action of the same one-parametric transformation group on fixed solutions which are similar with respect to this group. A neighborhood of a nonsingular point of Q is considered. (Otherwise, the one-to-one correspondence in the following theorem may be broken. In some cases it can saved by taking into account discrete symmetry transformations, see note 14 of [29].)

Formulae (17) and (18) imply the following statement which will be used below.

Proposition 3. *Let an equation $\mathcal{L}$ from class (1) be invariant with respect to a point transformation $\mathcal{T}$ (resp. an operator Q) and the function $\eta = \eta(t, x, u)$ be a solution of the associated determining equation* $\mathrm{DE}_0(\mathcal{L})$. *Then the equations $u_x = \eta(t, x, u)$ admit the transformation $\mathcal{T}$ (resp. the operator Q) as a point symmetry transformation (resp. a Lie symmetry operator) if and only if the function η is an invariant of the associated point symmetry transformation $\mathcal{T}_0$ (resp. the associated Lie symmetry operator Q_0) of the equation* $\mathrm{DE}_0(\mathcal{L})$.

Theorem 6. *For each equation $\mathcal{L}$ from class (1) and each Lie symmetry operator Q of $\mathcal{L}$, there exists a one-to-one correspondence between Q_0-invariant solutions of the determining equation* $\mathrm{DE}_0(\mathcal{L})$ *and one-parametric families of solutions of $\mathcal{L}$, associated with Q. Namely, the reduction of the equation $\mathcal{L}$ by an operator $\partial_x + \eta\partial_u$, where the coefficient η is a Q_0-invariant solution of* $\mathrm{DE}_0(\mathcal{L})$, *gives a one-parametric solution family of $\mathcal{L}$, associated with Q. And vice versa, each family of the above kind consists of solutions invariant with respect to an operator $\partial_x + \eta\partial_u$, where the coefficient η is a Q_0-invariant solution of* $\mathrm{DE}_0(\mathcal{L})$.

Proof. Suppose that an equation $\mathcal{L}$ from class (1) admits a Lie symmetry operator Q. We denote the one-parametric transformation group with the infinitesimal operator Q by G. Let a solution η of the equation $\mathrm{DE}_0(\mathcal{L})$ be invariant with respect to the associated operator Q_0. Then the system $\mathcal{L}_\eta$ of the equation $\mathcal{L}$ with the additional constraint $u_x = \eta$ possesses Q as a Lie symmetry operator. The general solution $\mathcal{F}$ of $\mathcal{L}_\eta$ is a one-parametric solution family of

$\mathcal{L}$. There are two different cases of the structure of $\mathcal{F}$. In the first case the family $\mathcal{F}$ consists of Q-invariant solutions of $\mathcal{L}$ and, therefore, is a singular one-parametric solution family associated with the operator Q. In the second case the family $\mathcal{F}$ contains a solution $u = u^0(t, x)$ of $\mathcal{L}$, which is not Q-invariant. A one-parametric family of solutions of $\mathcal{L}_\eta$ obtained via acting on the solution u^0 by transformations from G is equivalent to $\mathcal{F}$. Therefore, $\mathcal{F}$ is a regular one-parametric solution family associated with the operator Q.

Vice versa, if a one-parametric solution family of the equation $\mathcal{L}$ is associated with the operator Q then the corresponding additional constraint $u_x = \eta$ with a solution η of $\mathrm{DE}_0(\mathcal{L})$ admits Q as a Lie symmetry operator. In view of proposition 3, this implies that the function η is Q-invariant. □

Since the determining equation $\mathrm{DE}_0(\mathcal{L})$ has three independent variables, it also admits Lie reductions with respect to two-dimensional subalgebras of its maximal Lie invariance algebras to ordinary differential equations and, therefore, possesses the corresponding invariant solutions. To formulate the statement on such solutions analogously to theorem 6, we need to define *one-parametric families of solutions* of the equation $\mathcal{L}$, *associated with the two-dimensional Lie invariance algebra* $\mathfrak{g}$ of $\mathcal{L}$. The whole set of associated families is also partitioned into the subsets of the *singular* and *regular* families. Each singular associated family consists of $\mathfrak{g}$-invariant solutions of $\mathcal{L}$. Each regular associated family is obtained via acting on fixed Q^1-invariant and non-Q^2-invariant solution of $\mathcal{L}$ by the one-parametric transformation group generated by Q^2. Here, Q^1 and Q^2 are arbitrary linearly independent elements of $\mathfrak{g}$.

Theorem 7. *Suppose that a two-dimensional Lie invariance algebra* $\mathfrak{g}$ *of an equation* $\mathcal{L}$ *from class (1) induces the Lie invariance algebra* $\mathfrak{g}_0$ *of the corresponding determining equation* $\mathrm{DE}_0(\mathcal{L})$*, which is appropriate for the Lie reduction of* $\mathrm{DE}_0(\mathcal{L})$*. Then there exists a one-to-one correspondence between* $\mathfrak{g}_0$*-invariant solutions of* $\mathrm{DE}_0(\mathcal{L})$ *and one-parametric families of solutions of* $\mathcal{L}$*, associated with* $\mathfrak{g}$*. Namely, the reduction of* $\mathcal{L}$ *by an operator* $\partial_x + \eta\partial_u$*, where the coefficient* η *is a* $\mathfrak{g}_0$*-invariant solution of* $\mathrm{DE}_0(\mathcal{L})$*, gives a one-parametric family of solutions of* $\mathcal{L}$*, associated with* $\mathfrak{g}$*. And vice versa, each family of this kind consists of solutions invariant with respect to an operator* $\partial_x + \eta\partial_u$*, where the coefficient* η *is a* $\mathfrak{g}_0$*-invariant solution of* $\mathrm{DE}_0(\mathcal{L})$*.*

Proof. We denote by G the two-parametric transformation group with the Lie algebra $\mathfrak{g}$ and locally parameterize elements of G in a neighborhood of the identical transformation by the pair $(\varepsilon_1, \varepsilon_2)$: $g(\varepsilon_1, \varepsilon_2) \in G$. In particular, $g(0, 0)$ is the identical transformation and the infinitesimal operators $Q^i = g_{\varepsilon_i}(0, 0)$, $i = 1, 2$, form a basis of the algebra $\mathfrak{g}$. Let a solution η of the equation $\mathrm{DE}_0(\mathcal{L})$ be invariant with respect to the associated algebra $\mathfrak{g}_0$. Then $\mathfrak{g}$ is a Lie invariance algebra of the system $\mathcal{L}_\eta$ formed by the equation $\mathcal{L}$ and the additional constraint $u_x = \eta$. The general solution $\mathcal{F}$ of $\mathcal{L}_\eta$ is a one-parametric solution family of $\mathcal{L}$. We explicitly represent this family by the formula $u = f(t, x, \varkappa)$. There are two different cases of its possible structure. The family $\mathcal{F}$ can consist of $\mathfrak{g}$-invariant solutions of $\mathcal{L}$ and, therefore, be a singular one-parametric solution family associated with the algebra $\mathfrak{g}$. The other possibility is that the family $\mathcal{F}$ contains a solution $u = f(t, x, \varkappa_0)$ of $\mathcal{L}$, which is not $\mathfrak{g}$-invariant. Then the solution $u = f(t, x, \varkappa_0)$ is invariant with respect to the operator $\varkappa_{0,1}Q^2 - \varkappa_{0,2}Q^1 \in \mathfrak{g}$, where $\varkappa_{0,i} = (g(\varepsilon_1, \varepsilon_2)\varkappa_0)_{\varepsilon_i}|_{(\varepsilon_1,\varepsilon_2)=(0,0)}$. The action of the one-parametric subgroup G' of G with the infinitesimal operator $\varkappa_{0,1}Q^1 + \varkappa_{0,2}Q^2 \in \mathfrak{g}$ is (locally) transitive on $\mathcal{F}$. It means that $\mathcal{F}$ is a regular one-parametric solution family associated with the algebra $\mathfrak{g}$, which is obtained via acting by G' on the fixed solution $u = f(t, x, \varkappa_0)$.

Conversely, if a one-parametric solution family of the equation $\mathcal{L}$ is associated with the algebra $\mathfrak{g}$ then the corresponding additional constraint $u_x = \eta$, where $\eta = \eta(t, x, u)$ is a solution of $\mathrm{DE}_0(\mathcal{L})$, admits $\mathfrak{g}$ as a Lie symmetry algebra. In view of proposition 3, this implies that the function η is $\mathfrak{g}$-invariant. □

9. Particular cases of reductions and linearization

In this section we consider a few examples of typical additional conditions to the determining equations, which are different from Lie ones. A special attention is paid to an interpretation of the confinement of the linearizing transformations given in corollaries 5 and 7 to the particular cases under consideration. Presented examples also show that nontrivial reduction operators associated with nontrivial additional conditions to determining equations can finally lead to trivial solutions of equations from class (1).

We fix an equation $\mathcal{L}$ from class (1). The extension of possibilities for constraints of the determining equations in comparison with the initial equation $\mathcal{L}$ is connected with a greater number of unknown functions in $\mathrm{DE}_1(\mathcal{L})$ and the additional independent variable u in $\mathrm{DE}_0(\mathcal{L})$.

Consider at first reduction operators of $\mathcal{L}$ with the vanishing coefficients of ∂_t.

Example 2. Suppose that $Q_0 = \partial_x$ is a reduction operator of $\mathrm{DE}_0(\mathcal{L})$. It means that the arbitrary elements satisfy the condition $A_x = B_{xx} = C_x = 0$. The problem is to investigate the solutions of $\mathrm{DE}_0(\mathcal{L})$, which are invariant with respect to Q_0. We do an equivalence transformation of the form $\tilde{t} = T(t), \tilde{x} = X^1(t)x + X^0(t), \tilde{u} = U^1(t)u$, where the arbitrary elements A, B and C and the function η are transformed according to formulae (3) and (17). The parameter-functions T, X^1, X^0 and U^1 can be chosen in such a way that $\tilde{A} = 1$, $\tilde{B} = 0$ and $\tilde{C} = 0$. In the new variables the operator Q_0 equals $X^1\partial_{\tilde{x}}$ and hence is equivalent to $\partial_{\tilde{x}}$. This is why we can assume without loss of generality that $A = 1$, $B = 0$ and $C = 0$, i.e., $\mathcal{L}$ coincides with the linear heat equation. Then $Q_0 = \partial_x$ is a Lie symmetry operator of $\mathrm{DE}_0(\mathcal{L})$. The corresponding reduced equation $\eta_t = \eta\eta_{uu}$ for the function $\eta = \eta(t, u)$ is equivalent, on the subset of nonvanishing solutions, to the remarkable nonlinear diffusion equation $\zeta_t = (\zeta^{-2}\zeta_u)_u$, where $\zeta = 1/\eta$. It is well known that this diffusion equation is linearized to the linear heat equation [3, 39]. We derive this transformation via confining the transformation of $\mathrm{DE}_0(\mathcal{L})$ to, formally, $\mathcal{L}$, presented in corollary 7. We put $\Phi = \Psi(t, u) - x$, where $\Psi_u \neq 0$. Then $\eta = -\Phi_x/\Phi_u = 1/\Psi_u$, i.e., $\zeta = \Psi_u$. After integrating, we obtain the equation $\Psi_t = \Psi_{uu}/{\Psi_u}^2 + \beta(t)$ in the function $\Psi = \Psi(t, u)$. The 'integration constant' $\beta = \beta(t)$ can be assumed to vanish due to the ambiguity in the connection between ζ and Ψ. The confinement of transformation (14) is the hodograph transformation

$$\begin{aligned} &\text{the new independent variables:} && \tilde{t} = t, \qquad \tilde{x} = \Psi, \\ &\text{the new dependent variable:} && \tilde{u} = u \end{aligned}$$

since here the variable x has to be replaced by $\Psi = x + \Phi$. The application of this transformation results in the linear heat equation $\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}$. Note that the above interpretation of the confinement of transformation (14) differs from the interpretation in the proof of theorem 9 of [29].

Example 3. Let the function η satisfies the additional condition $\eta_{uu} = 0$, i.e., $\eta = \eta^1(t, x)u + \eta^0(t, x)$. Then the equation $\mathrm{DE}_0(\mathcal{L})$ is reduced to the system

$$\begin{aligned} &\eta^1_t = \left(A\eta^1_x + A(\eta^1)^2 + B\eta^1 + C\right)_x, \\ &\eta^0_t = A\left(\eta^0_{xx} + 2\eta^0\eta^1_x\right) + A_x\left(\eta^0_x + \eta^0\eta^1\right) + (B\eta^0)_x + C\eta^0. \end{aligned} \tag{22}$$

Putting $\Phi = \Phi^1(t,x)u + \Phi^0(t,x)$, we rewrite the transformation described in corollary 7 in terms of η^1 and η^0. The condition $\eta = -\Phi_x/\Phi_u$ implies that $\eta^1 = -\Phi^1_x/\Phi^1$ and $\eta^0 = -\Phi^0_x/\Phi^1$. The hodograph transformation (14) is equivalent to expressing u from the formula for Φ:

$$u = \frac{\Phi - \Phi^0}{\Phi^1} = \Psi^1(t,x)\varkappa + \Psi^0(t,x),$$

where $\Psi^1 = 1/\Phi^1$ and $\Psi^0 = \Phi^0/\Phi^1$. Since the expression for u has to be the solution family of $\mathcal{L}$ with the parameter $\varkappa = \Phi$, Ψ^1 and Ψ^0 are solutions of $\mathcal{L}$, $\Psi^1 \neq 0$. Finally, we derive the representation

$$\eta^1 = \frac{\Psi^1_x}{\Psi^1}, \qquad \eta^0 = \Psi^0_x - \frac{\Psi^1_x}{\Psi^1}\Psi^0, \tag{23}$$

where Ψ^1 and Ψ^0 are solutions of the initial equation $\mathcal{L}$. In other words, transformation (23) reduces the nonlinear system (22) in η^1 and η^0 to the system of two uncoupled copies of $\mathcal{L}$. The expression for η^1 in (23) coincides, up to sign, with the well-known Cole–Hopf substitution linearizing the Burgers equation. (If $A = 1$ and $B = C = 0$, the first equation of (22) coincides, up to signs, with the Burgers equation.) The expression for η^1 in (23) is obtained as the Darboux transformation of the solution Ψ^0, associated with the solution Ψ^1. It follows from (23) that the reduction operator $R = \partial_x + (\eta^1 u + \eta^0)\partial_u$ is $G^\infty(\mathcal{L})$-equivalent to the operator $\partial_x + \eta^1 u\partial_u$. Indeed, the transformation $\tilde t = t, \tilde x = x, \tilde u = u - \Psi^0$ belongs to $G^\infty(\mathcal{L})$ and maps the operator R to the operator $\tilde R = \partial_{\tilde x} + \eta^1 \tilde u\partial_{\tilde u}$.

An Ansatz constructed with R has the form $u = \Psi^1(t,x)\varphi(\omega) + \Psi^0(t,x)$, where $\varphi = \varphi(\omega)$ is an invariant unknown function of the invariant-independent variable $\omega = t$. The associated reduced equation is $\varphi_\omega = 0$, i.e., $\varphi = \text{const}$. Therefore, $u = \Psi^1\varkappa + \Psi^0$ is the family of R-invariant solutions of $\mathcal{L}$.

Vice versa, the solution family $u = \Psi^1(t,x)\varkappa + \Psi^0(t,x)$ of the equation $\mathcal{L}$ is necessarily invariant with respect to the reduction operator $\partial_x + (\eta^1(t,x)u + \eta^0(t,x))\partial_u$, where the coefficients η^1 and η^0 are determined by formulae (23).

As a result, we obtain the following statement.

Proposition 4. *For any equation of form (1), there exists a one-to-one correspondence between one-parametric families of its solutions, linearly depending on parameters, and reduction operators of the form $\partial_x + \eta(t,x,u)\partial_u$, where $\eta_{uu} = 0$. Namely, each operator of such kind corresponds to the family of solutions which are invariant with respect to this operator.*

Example 4. At first sight, the additional condition $\eta_x + \eta\eta_u = 0$ seems much more complicated than the conditions studied in the previous examples. In fact, it leads only to solutions of the initial equation $\mathcal{L}$, which are first-order polynomials with respect to x. To see this, we carry out the transformation described in corollary 7 and, as a result, obtain the condition $\tilde u_{\tilde x\tilde x} = 0$. In contrast to the solutions of $\mathcal{L}$, the associated solutions of $\mathrm{DE}_0(\mathcal{L})$ have a complex structure and are difficult to construct.

The system S consisting of the equations $\mathrm{DE}_0(\mathcal{L})$ and $\eta_x + \eta\eta_u = 0$ has the compatibility condition $(B_{xx} + 2C_x)\eta + C_{xx}u = 0$. Before considering the possible cases, we note that the equation $\eta_x + \eta\eta_u = 0$ is invariant with respect to the transformations from the equivalence group $G_0^{\sim}$ of class (7), which additionally satisfy the conditions $\left(U^1_x/(U^1)^2\right)_x = 0$ and $(X_x/(U^1)^2)_x = 0$. Denote the subgroup of $G_0^{\sim}$, consisting of these transformations, by $\breve G_0^{\sim}$. The solutions of the system S are constructed up to $\breve G_0^{\sim}$-equivalence.

If $B_{xx} + 2C_x \neq 0$, the function η has the form $\eta = \eta^1(t,x)u$. Then $\eta^1 = 0$ and $C_x = 0$ up to $\breve G_0^{\sim}$-equivalence. The interpretation of this solution is obvious. An associated Ansatz

for $\mathcal{L}$ and the corresponding reduced equation are $u = \varphi(\omega)$, where $\omega = t$ and $\varphi_\omega = 0$. The family of the associated invariant solutions of $\mathcal{L}$ is formed by the constant functions.

The condition $B_{xx} + 2C_x = 0$ implies $C_{xx} = 0$. Up to $\check{G}_0^{\sim}$-equivalence we can assume that $B = C = 0$. Then the system S is reduced to the system $\eta_t = 0, \eta_x + \eta\eta_u = 0$. Its nonzero solutions are implicitly determined by the formula $u = x\eta + w(\eta)$, where $w = w(\eta)$ is an arbitrary smooth function of η. An associated Ansatz for the equation $\mathcal{L}$ is found from the condition $u = xu_x + w(u_x)$ which is the Clairaut's equation with the implicit parameter t. We choose the Ansatz $u = \varphi(\omega)x + w(\varphi(\omega))$, where $\omega = t$. The corresponding reduced equation is $\varphi_\omega = 0$, i.e., the associated invariant solutions of $\mathcal{L}$ has the form $u = cx + w(c)$, where c is an arbitrary constant.

Let us emphasize that the obtained results have a compact form only due to the consideration up to $\check{G}_0^{\sim}$-equivalence.

Now we present a single example concerning the system $\mathrm{DE}_1(\mathcal{L})$. In view of corollary 6 we can assume without loss of generality that $g^3 = 0$ and, therefore, consider only the first two equations of the system $\mathrm{DE}_1(\mathcal{L})$. The $G_1^{\sim}$-invariance of the equation $g^3 = 0$ additionally justifies this assumption.

Example 5. The constraint $g^2 = 0$ is invariant with respect to the transformations from the equivalence group $G_1^{\sim}$, in which $U^1 = 1$. These transformations are presented by formulae (2), (3) and (15), where $U^1 = 1$ and $U^0 = 0$, and form the subgroup of $G_1^{\sim}$, denoted by $\check{G}_1^{\sim}$. Up to the $\check{G}_1^{\sim}$-equivalence, the coefficient A can be assumed equal to 1. Imposing the conditions $g^2 = g^3 = 0$ and $A = 1$, we reduce $\mathrm{DE}_1(\mathcal{L})$ to the system

$$g^1_t - g^1_{xx} + 2g^1 g^1_x + (Bg^1)_x + B_t = 0, \tag{24}$$

$$C_t + g^1 C_x + 2g^1_x C = 0. \tag{25}$$

Equation (24) is linearized to the equation $w_t = w_{xx} + (Bw)_x$ by the generalization $g^1 = -w_x/w - B$ of the Cole–Hopf substitution and then to the equation $v_t = v_{xx} + Bv_x$ by the subsequent substitution $w = v_x$. In the case $C = 0$, the resulting substitution $g^1 = -v_{xx}/v_x - B$ is the confinement of transformation (11) under the assumptions $v^3 = 0$, $v^2 = 1$ and $v^1 = v$, where v is a nonconstant solution of $\mathcal{L}$.

Equation (25) admits a double interpretation depending on a reading of the phrase 'the equation $\mathcal{L}$ possesses the reduction operator $\partial_t + g^1\partial_x$'. It can be considered either as an additional constraint for the function g^1 or an equation in the coefficient C. Choosing the second alternative, we obtain $C = v_x^2\Phi(v)$ for some function $\Phi = \Phi(v)$.

If $C = 0$, equation (25) is an identity. Therefore, the equation $\mathcal{L}$ admits any reduction operators of the form $\partial_t - (v_{xx}/v_x + B)\partial_x$, where $v = v(t, x)$ runs through the set of nonconstant solutions of $\mathcal{L}$. The corresponding two-parametric solution family of $\mathcal{L}$ is $u = c_1 v(t, x) + c_2$.

Note 12. Since we do not initially specify values of the arbitrary elements and derive conditions on arbitrary elements depending on possessed reduction operators, the above examples have features of inverse problems of group analysis. Namely, we simultaneously describe both reduction operators with certain properties and values of arbitrary elements for which the corresponding equations admit such reduction operators. A similar inverse problem for generalized conditional symmetries of evolution equations is investigated in [38]. Due to possibilities on the variation of arbitrary elements and application of equivalence transformations, the problems of this kind essentially differ from the problem of finding reduction operators of a fixed equation.

10. Applications

In sections 6–8 'no-go' statements of different kinds have been proved for the reduction operators of the equations from class (1). The term 'no-go' has to be treated only as the impossibility of exhaustive solving of the problem or the inefficiency of finding Lie symmetries and Lie reductions of the determining equations. At the same time, imposing additional (non-Lie) constraints on coefficients of reduction operators, one can construct particular examples of reduction operators and then apply them to the construction of exact solutions of an initial equation. Since the determining equations have more dependent or independent variables and, therefore, more degrees of freedom than the initial ones, it is more convenient often to guess a simple solution or a simple Ansatz for the determining equations, which can give a parametric set of complicated solutions of the initial equations. (A similar situation is for Lie symmetries of first-order ordinary differential equations.) It is the approach that was used, e.g., in [10] to construct exact solutions of a (nonlinear) fast diffusion equation with reduction operators having the zero coefficients of ∂_t. Earlier this approach was applied to the interesting subclass of class (1), consisting of the linear transfer equations of the general form

$$u_t = u_{xx} + \frac{h(t)}{x}u_x. \tag{26}$$

These equations arise, in particular, under symmetry reduction of the Navier–Stokes equations [5, 22, 23]. The investigation of reduction operators allowed us to construct a series of multi-parametric solutions of equations (26) and, as a result, wide solution families of the Navier–Stokes equations, parameterized by constants and functions of t.

We consider class (26) as an example showing possible ways of imposing nontrivial additional constraints to determining equations. This subclass is singled out from the whole class (1) by the conditions on arbitrary elements $A = 1$, $(xB)_x = 0$ and $C = 0$.

We fix an equation $\mathcal{L}$ from class (26). The maximal Lie invariance algebra of $\mathcal{L}$ is the algebra

(1) $\langle u\partial_u,\ f\partial_u\rangle$ if $h \neq \text{const}$;
(2) $\langle \partial_t,\ D,\ \Pi_h,\ u\partial_u,\ f\partial_u\rangle$ if $h = \text{const},\ h \notin \{0, 2\}$;
(3) $\langle \partial_t,\ D,\ \Pi_h,\ 2\partial_x - hx^{-1}u\partial_u,\ G_h,\ u\partial_u,\ f\partial_u\rangle$ if $h \in \{0, 2\}$.

Here, $D = 2t\partial_t + x\partial_x$, $\Pi_h = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2(1+h)t)u\partial_u$, $G_h = 2t\partial_x - (x + htx^{-1})u\partial_u$. The function $f = f(t, x)$ runs through the set of solutions of $\mathcal{L}$. The case $h = 2$ is reduced to the linear heat equation ($h = 0$) by the transformation $\tilde{t} = t, \tilde{x} = x$ and $\tilde{u} = xu$, cf theorem 1. The intersection of the maximal Lie invariance algebras of equations from class (26) coincides with $\langle u\partial_u,\ \partial_u\rangle$, i.e., the kernel Lie symmetry group of class (26) consists of scalings and translations of u. It is easy to see that the equation $\mathcal{L}$ possesses no nontrivial Lie symmetries and, therefore, no Lie reductions if $h \neq \text{const}$. At the same time, non-Lie reduction operators can be found for an arbitrary value of h.

Any reduction operator of $\mathcal{L}$ with the nonzero coefficient of ∂_t is $G^\infty(\mathcal{L})$-equivalent to an operator $\partial_t + g^1\partial_x + g^2u\partial_u$, where the functions $g^1 = g^1(t, x)$ and $g^2 = g^2(t, x)$ satisfy the first two equations of the corresponding determining system $\mathrm{DE}_1(\mathcal{L})$. Following example 5, we impose the additional constraint $g^2 = 0$. Then the second equation of $\mathrm{DE}_1(\mathcal{L})$ is identically satisfied. The first equation of $\mathrm{DE}_1(\mathcal{L})$ is rewritten in the form

$$(g^1 + hx^{-1})_t = \left(g^1_x - g^1(g^1 + hx^{-1})\right)_x.$$

We put the left- and right-hand sides equal to 0. Then $g^1 = \chi(x) - hx^{-1}$ and $g^1_x - g^1(g^1 + hx^{-1}) = \psi(t)$. The compatibility of these equations implies that $\chi = -x^{-1}$ and $\psi = 0$, i.e., $g^1 = -(h(t) + 1)x^{-1}$, and the corresponding reduction operator is

$$Q = \partial_t - (h(t) + 1)x^{-1}\partial_x.$$

As a result, the equation $\mathcal{L}$ possesses the family of Q-invariant solutions

$$u = c_2\left(x^2 + 2\int (h(t)+1)\,\mathrm{d}t\right) + c_1. \tag{27}$$

Each reduction operator of $\mathcal{L}$ with the zero coefficient of ∂_t is equivalent to an operator $\partial_x + \eta\partial_u$, where the coefficient $\eta = \eta(t,x,u)$ satisfies the corresponding determining equation $\mathrm{DE}_0(\mathcal{L})$:

$$\eta_t = \eta_{xx} + 2\eta\eta_{xu} + \eta^2\eta_{uu} + h(x^{-1}\eta)_x. \tag{28}$$

Suppose that the same operator $\partial_x + \eta\partial_u$ is a reduction operator of all equations from class (26), i.e., the function η is a solution of (28) for any value of h. This demand leads to the additional constraint $(x^{-1}\eta)_x = 0$ implying that $\eta = x\zeta(t,u)$. We substitute the expression for η into (28) and split with respect to x. Integrating the obtained system $\zeta_{uu} = 0$, $\zeta_t = 2\zeta\zeta_u$, we construct all its solutions:

$$\zeta = -\frac{u+\mu}{2(t+\varkappa)} \quad \text{or} \quad \zeta = \nu,$$

where μ, $\varkappa$ and ν are arbitrary constants. In other words, the common reduction operators of equations from class (26) are exhausted, up to equivalence with respect to the kernel Lie symmetry group (more precisely, up to translations of u), by the operators of the form

$$G_\varkappa = (2t+\varkappa)\partial_x - xu\partial_u \qquad \text{and} \qquad \partial_x + \nu\partial_u.$$

(It is obvious that there are no common reduction operators with nonzero coefficients of ∂_t.) The constant $\varkappa$ cannot be put equal to 0 similarly to the constant μ since translations of t do not belong to the kernel Lie symmetry group of class (26) and the classification up to the equivalence group of class (26) is not convenient for the consideration. The operator $G_\varkappa$ is represented as the linear combination $G + \varkappa\partial_x$ of the Galilean operator $G = 2t\partial_x - xu\partial_u$ and translational operator ∂_x. The non-reduced form for the coefficient of ∂_x in $G_\varkappa$ is chosen to obtain this representation. For any equation $\mathcal{L}$ from class (26), the reduction operator $R = \partial_x + \nu\partial_u$ is $G^\infty(\mathcal{L})$-equivalent to the operator ∂_x which is trivial since the arbitrary element C equals 0 in class (26). Another formulation of the above result is the following: each equation from class (26) is conditionally invariant with respect to arbitrary linear combinations of the Galilean operator G and the translational operator ∂_x. The family of $G_\varkappa$-invariant solutions of an equation of form (26) consists of the functions

$$u = c_1 \exp\left\{-\frac{x^2}{2(2t+\varkappa)} - \int \frac{h(t)+1}{2t+\varkappa}\,\mathrm{d}t\right\}.$$

The corresponding family for the operator $\partial_x + \nu\partial_u$ has form (27) with $c_2 = \nu$.

The constructed exact solutions are generalized to a series of similar solutions:

$$u = \sum_{k=0}^{N} T^k(t)x^{2k}, \qquad u = \sum_{k=0}^{N} S^k(t)\left(\frac{x}{2t+\varkappa}\right)^{2k} \exp\left\{-\frac{x^2}{2(2t+\varkappa)} - \int \frac{h(t)+1}{2t+\varkappa}\,\mathrm{d}t\right\}.$$

The functions $T^k = T^k(t)$ and $S^k = S^k(t)$, respectively, satisfy the systems of ODEs

$$T^k_t = 2(k+1)(h(t)+2k+1)T^{k+1}, \qquad k = \overline{0,N-1}, \qquad T^N_t = 0,$$
$$S^k_t = 2(k+1)(h(t)+2k+1)(2t+\varkappa)^{-2}S^{k+1}, \qquad k = \overline{0,N-1}, \qquad S^N_t = 0,$$

which are easily integrated. These series of exact solutions can also be found using different techniques connected with reduction operators and their generalizations, in particular, via nonlocal transformations in class (26), associated with reduction operators [5, 22].

11. Discussion

The main result of the present paper is the chain of 'no-go' statements on reduction operators of linear (1 + 1)-dimensional parabolic equations. These statements show that the application of conventional methods to solving the determining equations for coefficients of such operators cannot lead to reduction operators giving new exact solutions of initial equations. In both cases naturally arising under the consideration, the determining equations form well-determined systems whose solving is in fact equivalent to solving of the corresponding equations from class (1). All transformational and symmetry properties of the determining equations are induced by the corresponding properties of the initial equations. Reduction operators constructed via Lie reductions of the determining equations are also connected with Lie invariance properties of the initial equations. Nevertheless, it is demonstrated in section 10 that the involvement of ingenious empiric approaches different from the Lie one can give reduction operators which are useful for the construction of non-Lie exact solutions of equations from class (1).

Techniques developed in this paper can be applied to the general class of (1 + 1)-dimensional evolution equations. We also plan to consider generalized reduction operators of linear (1 + 1)-dimensional parabolic equations, whose coefficients depend on the derivatives of u. An interesting subject related to this is the connection between (generalized) reduction operators and Darboux transformations. Here we give some hints on this connection.

Consider a fixed tuple of linearly independent functions $(\psi^1, \dots, \psi^p)$ of t and x, and the linear independence is assumed over the ring of smooth functions of t. The *Darboux transformation* constructed with the tuple $(\psi^1, \dots, \psi^p)$ is denoted by $\mathrm{DT}[\psi^1, \dots, \psi^p]$ and is defined by formula [16, 34]

$$\tilde{u} = \mathrm{DT}[\psi^1, \dots, \psi^p](u) = \frac{W(\psi^1, \dots, \psi^p, u)}{W(\psi^1, \dots, \psi^p)}.$$

Here, $W(\varphi^1, \dots, \varphi^s)$ denote the Wronskian of the functions $\varphi^1, \dots, \varphi^s$ with respect to the variable x, i.e., $W(\varphi^1, \dots, \varphi^s) = \det(\partial^{i-1}\varphi^j/\partial x^{i-1})_{i,j=1}^s$. The initial (u) and, therefore, obtained $(\tilde{u})$ functions also depend on t and x.

The transformation $\mathrm{DT}[\psi^1, \dots, \psi^p]$ is represented as the action of a linear p-order differential operator with differentiations with respect to only x, $\mathrm{DT}[\psi^1, \dots, \psi^p](u) = \mathrm{DT}[\psi^1, \dots, \psi^p]u$. The operator will be denoted by the same symbol as the transformation and called the *Darboux operator* associated with the tuple $(\psi^1, \dots, \psi^p)$. In the cases $p = 1$ and $p = 2$, the expressions of the Darboux operators, respectively, are

$$\mathrm{DT}[\psi^1] = \partial_x - \frac{\psi_x}{\psi}, \qquad \mathrm{DT}[\psi^1, \psi^2] = \partial_{xx} - \frac{(W(\psi^1, \psi^2))_x}{W(\psi^1, \psi^2)}\partial_x + \frac{W(\psi^1_x, \psi^2_x)}{W(\psi^1, \psi^2)}.$$

If the functions $\psi^1, \dots, \psi^p$ are linearly independent solutions of an equation $\mathcal{L}$ from class (1), then they are linearly independent over the ring of smooth functions of t [31, 34]. The Darboux transformation $\mathrm{DT}[\psi^1, \dots, \psi^p]$ maps the equation $\mathcal{L}$ to the equation $\tilde{\mathcal{L}}$ also belonging to class (1) and having the following values of arbitrary elements [16, 34]:

$$\tilde{A} = A, \qquad \tilde{B} = B + pA_x, \qquad \tilde{C} = C + pB_x + \frac{p(p+1)}{2}A_{xx} + \frac{W_x}{W}A_x + 2\left(\frac{W_x}{W}\right)_x A,$$

where the abbreviation $W = W(\psi^1, \dots, \psi^p)$ is used.

Suppose that a reduction operator Q of $\mathcal{L}$ has the canonical form and is associated with a first-order linear differential operator $\widetilde{Q}$ acting on the functions of t and x. It means that either $Q = \partial_t + g^1\partial_x + g^2 u\partial_u$ if $Q \in \mathcal{Q}_1(\mathcal{L})$ or $Q = \partial_x + \eta^1 u\partial_u$ if $Q \in \mathcal{Q}_0(\mathcal{L})$. (Here, g^1, g^2

and η^1 are the functions of t and x.) In the first case the operator $\widetilde{Q} = -\partial_t - g^1\partial_x + g^2$ equals the operator $-A\,\mathrm{DT}[v^1, v^2]$ on the solution set of the equation $\mathcal{L}$, where the solutions $v^i = v^i(t, x)$, $i = 1, 2$, of $\mathcal{L}$ are determined according to corollary 5. In the second case the coefficient η^1 admits the representation $\eta^1 = \Psi_x/\Psi$, where $\Psi = \Psi(t, x)$ is a solution of $\mathcal{L}$. Therefore, $\widetilde{Q} = -\mathrm{DT}[\Psi]$. Finally, we have the following statement.

Proposition 5. *Let a reduction operator Q of an equation $\mathcal{L}$ from class (1) be associated, up to the equivalence relations of operators, with a first-order linear differential operator acting on the functions of t and x. Then it is equivalent to a Darboux operator constructed with one (resp. two) linearly independent solution of this equation in the case of vanishing (resp. nonvanishing) coefficient of ∂_t.*

The properties of single reduction operators of multi-dimensional equations essentially differ from that in the $(1 + 1)$-dimensional case. For example, all single reduction operators of $(1 + n)$-dimensional linear heat equations are exhaustively classified in [33] for arbitrary n without addressing the general solution of this equation that annuls the possibility of 'no-go' statements. At the same time, it is not the case for involutive families of reduction operators [24, 40].

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Paper 3

Generalized conditional symmetries of evolution equations

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Generalized conditional symmetries of evolution equations

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ABSTRACT

We analyze the relationship of generalized conditional symmetries of evolution equations to the formal compatibility and passivity of systems of differential equations as well as to systems of vector fields in involution. Earlier results on the connection between generalized conditional invariance and generalized reduction of evolution equations are revisited. This leads to a no-go theorem on determining equations for operators of generalized conditional symmetry. It is also shown that up to certain equivalences there exists a one-to-one correspondence between generalized conditional symmetries of an evolution equation and parametric families of its solutions.

1. Introduction

Generalized conditional symmetries provide an effective method for finding exact solutions of evolution equations. Similarly to other such methods [44], it can be viewed as an instance of the general method of differential constraints [60,65] (or "side conditions" [44]). Within the framework of empiric compatibility theory, generalized conditional symmetries as differential constraints compatible with an initial equation were investigated by Olver [43] in order to justify the method on "nonlinear separation" of variables by Galaktionov [20]. Another interpretation of generalized conditional symmetries of an evolution equation is to consider them as invariant manifolds of this equation, i.e., manifolds in appropriate jet spaces that are invariant under the flow generated by the equation. This is the terminology in which generalized conditional symmetries of systems of evolution equations were first studied by Kaptsov [1,32] although the importance of invariant manifolds of evolution equations was understood much earlier [36].

From the symmetry point of view, the notion of generalized conditional symmetry arises by merging the notions of generalized and conditional symmetries, cf. Section 2. The idea of significantly extending Lie symmetries of differential equations by including derivatives of the relevant dependent variables in the coefficients of the associated infinitesimal generators first appeared in the fundamental paper of Noether [40] in connection with her study of conservation laws. Symmetries of this kind are called, e.g., generalized [42], Lie–Bäcklund [8,27] or higher-order [6] symmetries in the literature. See additionally the excellent sketch on the history of generalized symmetries and relevant terminology in [42, pp. 374–377]. The concept of conditional symmetries arose much later. Its origin can be traced back to the thesis of Bluman [5] and the paper by Bluman and Cole [7], where it was presented in terms of "nonclassical groups" or the "nonclassical" method of finding similarity solutions, respectively, cf. the detailed discussion in [6, Section 5.2.2]. A version of the corresponding invariance criterion explicitly taking into account the differential consequences involved in the process was first proposed by Fushchych and Tsyfra in [18]. Combining results of [14,18] and other previous papers, in [13] Fushchych introduced the general concept of conditional invariance. Around this time the terms "conditional invariance" and "Q-conditional invari-

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ance" began to be used regularly in connection with the method of Bluman and Cole and soon evolved into the terms Q-conditional [16] or, simply, conditional [19] and nonclassical [37] symmetry. The notions of generalized and conditional symmetries were merged, within the framework of symmetry analysis of differential equations, by Fokas and Liu [12] in the special case when evolution equations and symmetries do not explicitly involve the time variable and by Zhdanov [67] in the general case.

The variety of possible interpretations and related notions and a number of different names for the parent notions of conditional and generalized symmetries leads to the diversity of names used for generalized conditional symmetry in the literature. We have already mentioned the terms "invariant manifold" [1,3,32] (resp. "invariant set" [21,24]) and "compatible differential constraint" [43]. Additionally, combining names of the parent notions of symmetries leads, in particular, to the terms "conditional Lie–Bäcklund symmetry" [29,30,67] and "higher (or higher-order) conditional symmetry" [4,68]. Sometimes special names are used for particular cases of generalized conditional symmetries. For example, linear compatible differential constraints for diffusion–reaction equations were called "additional generating conditions" in [9]. For uniformity, we will use the term "generalized conditional symmetry" [12,53,54] throughout the paper. This will additionally emphasize the relation of this notion to symmetry analysis although the nature even of usual conditional symmetries is in fact closer to compatibility theory, cf. [35].

The main purpose of this paper is to investigate basic problems concerning generalized conditional symmetry of $(1+1)$-dimensional evolution equations of the general form

$$u_t = H(t, x, u_{(r,x)}), \tag{1}$$

where $r \geqslant 1$, $u_t = \partial u/\partial t$, $u_0 := u$, $u_k = \partial^k u/\partial x^k$, $u_{(r,x)} = (u_0, u_1, \dots, u_r)$ and $H_{u_r} \neq 0$. Among these problems are the comparative analysis of different versions of the conditional invariance criterion, the study of the possibility of solving the corresponding determining equations as well as relating generalized conditional symmetries to the concept of reduction, multiparametric families of solutions and different notions of compatibility for overdetermined systems of partial differential equations. Most results of the paper can be extended to systems of $(1+1)$-dimensional evolution equations if certain restrictions for generalized conditional symmetries are imposed, cf. [1]. We restrict our consideration to single evolution equations for the sake of clarity of presentation.

Throughout the paper we denote by $\mathcal{E}$ a fixed equation of the form (1). The indices a and b run from 1 to ρ, and we use the summation convention for repeated indices. Bar over a letter denotes a tuple of ρ consecutive values. Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_u = \partial/\partial u$ and $u_{tk} = \partial^{k+1}u/\partial t\partial x^k$. We also will use another notation for derivatives: $u_\alpha = u_{\alpha_0,\alpha_1} = \partial^{|\alpha|}u/\partial t^{\alpha_0}\partial x^{\alpha_1}$, where $\alpha = (\alpha_0, \alpha_1)$ is a multiindex, $\alpha_0, \alpha_1 \in \mathbb{N} \cup \{0\}$ and $|\alpha| = \alpha_0 + \alpha_1$, so that $u_k = u_{0,k}$ and $u_{tk} = u_{1,k}$. Any function is considered as its zero-order derivative. $D_t = \partial_t + u_{\alpha_0+1,\alpha_1}\partial_{u_\alpha}$ and $D_x = \partial_x + u_{\alpha_0,\alpha_1+1}\partial_{u_\alpha}$ are the operators of total differentiation with respect to the variables t and x, respectively. All our considerations are carried out in the local setting.

In the next section we discuss prerequisites for introducing the notion of generalized conditional symmetries in symmetry analysis and present different versions of the corresponding invariance criterion for single evolution equations. Relations of generalized conditional symmetries to formal compatibility and passivity of certain overdetermined systems of partial differential equations as well as to involutivity of certain systems of vector fields are established in Sections 3, 4 and 5, respectively. For this purpose we employ a weight of derivatives instead of the usual order (Section 3) and a ranking of derivatives (Section 4), which are associated with evolution equations of a fixed order. Reductions of evolution equations with special ansatzes are studied in Section 6. The Zhdanov theorem [67,68] (see also [4]) on the connection of generalized conditional symmetries of an evolution equation with ansatzes of a special form reducing this equation is also revisited. This leads to new results on the correspondence between generalized conditional symmetries, ansatzes and parametric families of solutions of evolution equations. In Section 7 we prove a no-go theorem on determining equations for generalized conditional symmetries of evolution equations. Roughly speaking, it is shown that solving the determining equations is equivalent to solving the original equations. An interpretation of usual conditional symmetries of evolution equations as special generalized conditional symmetries is given in Section 8 and is then illustrated by a new nontrivial example. Properties of generalized conditional symmetries of evolution equations are summed up in the conclusion.

2. Different forms of the criterion of conditional invariance

The criterion of generalized conditional invariance of evolution equations arises as a natural extension of both the criterion of generalized invariance and the criterion of conditional invariance. This is why we at first analyze the latter criteria in the case of evolution equations.

By the conventional definition, an equation $\mathcal{E}$ of the form (1) is *conditionally invariant* with respect to the vector field $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$, where the coefficients τ, ξ and η are functions of t, x and u, if the relation $Q_{(r)}E|_{\mathcal{E}_r\cap\mathcal{Q}_r} = 0$ holds. Here $E := u_t - H$ and the symbol $Q_{(r)}$ stands for the standard rth prolongation of the operator Q [42,46]:

$$Q_{(r)} = Q + \sum_{0<|\alpha|\leqslant r} \big(D_t^{\alpha_0}D_x^{\alpha_1}Q[u] + \tau u_{\alpha_0+1,\alpha_1} + \xi u_{\alpha_0,\alpha_1+1}\big)\partial_{u_\alpha}, \tag{2}$$

where $Q[u] = \eta - \tau u_t - \xi u_x$ is the characteristic of the vector field Q, and $\mathcal{Q}_r$ denotes the manifold defined by the set of all the differential consequences of the characteristic equation $\mathcal{Q}$: $Q[u] = 0$ in the rth-order jet space J^r, i.e.,

$$\mathcal{Q}_r = \big\{(t,x,u_{(r)}) \in J^r \;\big|\; D_t^{\alpha_0} D_x^{\alpha_1} Q[u] = 0,\ \alpha_0+\alpha_1 < r\big\}.$$

The manifold defined by the equation $\mathcal{E}$ in J^r is denoted by $\mathcal{E}_r$. In comparison with classical Lie symmetries, the weakening of the invariance condition consists in equating $Q_{(r)}E$ to zero on the submanifold $\mathcal{E}_r \cap \mathcal{Q}_r$ but not on the entire manifold $\mathcal{E}_r$. As $\mathcal{E}$ is an evolution equation, only differential consequences of $\mathcal{Q}$ with respect to x are in fact essential when substituting into the expression $Q_{(r)}E$ (cf. the proof of Proposition 4). Hence the conditional invariance criterion can be rewritten in the form $Q_{(r)}E|_{\mathcal{E}_r \cap \mathcal{Q}_{(r,x)}} = 0$, where

$$\mathcal{Q}_{(r,x)} = \big\{(t,x,u_{(r)}) \in J^r \;\big|\; D_x^k Q[u] = 0,\ k = 0,\dots,r-1\big\},$$

and the bound r for orders of the occurring differential consequences of the equations $\mathcal{E}$ and $\mathcal{Q}$ is not essential.

Two vector fields $\tilde{Q}$ and Q are called equivalent if they differ by a multiplier which is a nonvanishing function of x and u: $\tilde{Q} = \lambda Q$, where $\lambda = \lambda(x,u)$, $\lambda \neq 0$. The property of conditional invariance matches nicely with this equivalence relation. Namely, if the equation $\mathcal{E}$ is conditionally invariant with respect to the vector field Q then it is conditionally invariant with respect to any operator which is equivalent to Q. Therefore the equivalence relation of vector fields has a well-defined restriction to the set of conditional symmetries of the equation $\mathcal{E}$.

In the case of generalized symmetries, the extension of the notion of Lie symmetries is to permit the dependence of coefficients of vector fields on derivatives of u [42]. A generalized vector field Q is a symmetry of $\mathcal{E}$ if and only if the associated evolutionary vector field $Q[u]\partial_u$ is. Hence it is sufficient to consider only evolutionary vector fields as generalized infinitesimal symmetries. Additionally, if an evolutionary vector field $Q = \eta\partial_u$ is a symmetry of $\mathcal{E}$ and the difference $\tilde{\eta} - \eta$ vanishes on solutions of $\mathcal{E}$ then the vector field $\tilde{Q} = \tilde{\eta}\partial_u$ also is a symmetry of $\mathcal{E}$. Such generalized symmetries are called equivalent. In view of the evolution form of $\mathcal{E}$ this means that we need to consider only generalized symmetries whose characteristics do not depend on derivatives containing differentiation with respect to t.

Merging the above extensions of classical Lie symmetries leads to the notion of generalized conditional symmetries. Consider a generalized vector field $Q = \eta\partial_u$ with $\eta = Q[u]$ being a differential function, i.e., a smooth function of t, x and a finite number of derivatives of u.

Definition 1. An evolution equation $\mathcal{E}$ is called *conditionally invariant* with respect to the generalized vector field $Q = \eta\partial_u$ if the condition

$$Q_{(r)}E|_{\mathcal{M}} = 0$$

holds, where the rth prolongation $Q_{(r)}$ of Q is defined by (2) and $\mathcal{M}$ denotes the set of all differential consequences of the equation $\mathcal{E}$ and differential consequences of the equation $\eta = 0$ with respect to x. In this case, Q is called an operator of *generalized conditional symmetry* of the equation $\mathcal{E}$ and the above condition is the *criterion of conditional invariance*.

As $Q_{(r)}E = D_t\eta - \sum_{k=0}^{r} H_{u_k} D_x^k \eta$ and the last sum identically vanishes in view of differential consequences of the equation $\eta = 0$ with respect to x, we obtain at once another form of the criterion of conditional invariance [67]:

$$D_t\eta|_{\mathcal{M}} = 0.$$

After calculating the orders of the occurring differential consequences, Definition 1 can be equivalently reformulated with a precise determination of the underlying jet space. To this end, it suffices to consider the criterion $D_t\eta|_{\mathcal{M}} = 0$ within the jet space J^m of order $m = \max\{r(\alpha_0+1)+\alpha_1 \mid \eta_{u_\alpha} \neq 0\}$ which coincides with the weight of $D_t\tilde{\eta}$ (cf. Section 3). Then the criterion takes the form $D_t\eta|_{\mathcal{M}_m} = 0$, where $\mathcal{M}_m$ is the manifold determined by $\mathcal{M}$ in J^m. All other similar conditions can be formalized in the same way.

There are two well-defined equivalence relations on the set of generalized conditional symmetries of the equation $\mathcal{E}$, which extend the above equivalence relations of conditional and generalized symmetries, respectively.

Suppose that $\tilde{\eta} = \lambda\eta$, where λ is a nonvanishing differential function, i.e., $Q = \eta\partial_u$ and $\tilde{Q} = \tilde{\eta}\partial_u$ are equivalent generalized vector fields. Then the vector field $Q = \eta\partial_u$ is a generalized conditional symmetry of the equation $\mathcal{E}$ if and only if the vector field $\tilde{Q} = \tilde{\eta}\partial_u$ is. Indeed, $D_t\tilde{\eta} = \lambda D_t\eta + \eta D_t\lambda$ vanishes assuming $\mathcal{M}$ if and only if $D_t\eta$ does. Moreover, $D_t\tilde{\eta}$ vanishes assuming $\mathcal{M}$ if and only if it vanishes assuming $\tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ denotes the set of all differential consequences of the equation $\mathcal{E}$ and differential consequences of the equation $\tilde{\eta} = 0$ with respect to x. This allows one to restrict the equivalence relation of generalized vector fields to the set of generalized conditional symmetries of the equation $\mathcal{E}$ in a well-defined way, analogously to the case for usual conditional symmetries. Hence we will say that generalized conditional symmetries $Q = \eta\partial_u$ and $Q = \tilde{\eta}\partial_u$ of $\mathcal{E}$ are *equivalent as vector fields* if there exists a nonvanishing differential function λ such that $\tilde{\eta} = \lambda\eta$.

If differential functions η and $\tilde{\eta}$ coincide on the manifold defined by differential consequences of $\mathcal{E}$ in a jet space of suitable order, then in view of the Hadamard lemma we have a representation $\tilde{\eta} = \eta + \chi^\alpha D_t^{\alpha_0} D_x^{\alpha_1} E$, where the summation is over a finite set of α's and the χ^α are differential functions. Hence the condition $D_t\eta|_{\mathcal{M}} = 0$ is equivalent to the condition $D_t\tilde{\eta}|_{\mathcal{M}} = 0$ and, therefore, the condition $D_t\tilde{\eta}|_{\tilde{\mathcal{M}}} = 0$. In other words, the vector field $Q = \eta\partial_u$ is a generalized conditional symmetry of $\mathcal{E}$ if and only if the vector field $\tilde{Q} = \tilde{\eta}\partial_u$ is. For this reason we will call the generalized conditional symmetries $Q = \eta\partial_u$ and $\tilde{Q} = \tilde{\eta}\partial_u$ *equivalent on solutions of* $\mathcal{E}$.

In contrast to the equivalence of generalized conditional symmetries as vector fields, the equivalence on solutions does not agree with the reduction procedure. Some vector fields from a set of generalized conditional symmetries equivalent on solutions of $\mathcal{E}$ cannot be used for reduction of $\mathcal{E}$, while some of them are appropriate for reduction but the corresponding reduction procedures differ in the number of invariant independent and dependent variables in the associated ansatzes and, therefore, the structure of the reduced systems, cf. Section 8.

We can merge the above two equivalence relations of generalized conditional symmetries into a single notion. Namely, generalized conditional symmetries $Q=\eta\partial_u$ and $\tilde{Q}=\tilde{\eta}\partial_u$ of $\mathcal{E}$ are called *equivalent* if there exists a nonvanishing differential function λ such that $\tilde{\eta}-\lambda\eta$ is equal to zero on solutions of $\mathcal{E}$.

Taking into account the equivalence on solutions of the evolution equation $\mathcal{E}$, we can restrict our considerations to generalized conditional symmetries of the *reduced form* $\hat{Q}=\hat{\eta}\partial_u$, where the characteristic $\hat{\eta}$ is a *reduced differential function*, i.e., it depends on t, x and derivatives of u with respect to only x. Generalized conditional symmetries in reduced form are equivalent if and only if their characteristics differ in a nonvanishing multiplier being a reduced differential function. Up to this equivalence, we can replace $\hat{Q}$ by the corresponding *canonical form*

$$\check{Q}=\big(u_\rho-\check{\eta}(t,x,u_{(\rho-1,x)})\big)\partial_u, \tag{3}$$

where ρ is the order of $\hat{\eta}$ and the condition of maximal rank of $\hat{\eta}$ with respect to u_ρ is additionally assumed to be satisfied. The function $\check{\eta}=\check{\eta}(t,x,u_{(\rho-1,x)})$ is obtained by solving the equation $\hat{\eta}=0$ with respect to u_ρ.

An evolution equation $\mathcal{E}$ is conditionally invariant with respect to a generalized evolution vector field $Q=\eta(t,x,u_{(\rho,x)})\partial_u$ in reduced form if

$$Q_{(r)}(u_t-H)|_{\mathcal{E}_{r+\rho}\cap\mathcal{Q}_{(r+\rho,x)}}=0,\quad \text{or}\quad D_t\eta|_{\mathcal{E}_{r+\rho}\cap\mathcal{Q}_{(r+\rho,x)}}=0, \tag{4}$$

where $Q_{(r)}$ is the rth prolongation of Q defined by (2), $\mathcal{E}_{r+\rho}$ (resp. $\mathcal{Q}_{(r+\rho,x)}$) is the manifold determined in the $(r+\rho)$th-order jet space by differential consequences of the equation $\mathcal{E}$ (resp. the equation $\eta=0$ only with respect to x). If Q is in canonical form, i.e. $\eta=u_\rho-\check{\eta}(t,x,u_{(\rho-1,x)})$, the criterion of conditional invariance of $\mathcal{E}$ with respect to Q reads

$$D_x^\rho H=D_t\check{\eta}\quad \text{on } \big\{u_{\rho+k}=D_x^k\check{\eta},\ k=0,\dots,r,\ u_{tl}=D_x^l H,\ l=0,\dots,\rho-1\big\}. \tag{5}$$

After making all necessary substitutions in (5), we obtain the single determining equation

$$\hat{D}_t\check{\eta}=\hat{D}_x^\rho\hat{H} \tag{6}$$

in $\check{\eta}$, where $\hat{H}=H(t,x,u_0,\dots,u_r)$ if $\rho>r$, $\hat{H}=H(t,x,u_0,\dots,u_{\rho-1},\check{\eta},\hat{D}_x\check{\eta},\dots,\hat{D}_x^{r-\rho}\check{\eta})$ if $\rho\leqslant r$, and

$$\hat{D}_t=\partial_t+\big(\hat{D}_x^{b-1}\hat{H}\big)\partial_{u_{b-1}},\qquad \hat{D}_x=\partial_x+\sum_{b=1}^{\rho-1}u_b\partial_{u_{b-1}}+\check{\eta}\partial_{u_{\rho-1}}$$

are the operators of total differentiation restricted to the manifold $\mathcal{E}_{r+\rho}\cap\mathcal{Q}_{(r+\rho,x)}$. Eq. (6) is a $(1+\rho)$-dimensional evolution equation in an unknown function $\check{\eta}$ of the independent variables t, x, $u_0,\dots,u_{\rho-1}$, and we have no possibilities for splitting with respect to unconstrained variables.

There also exist other forms and interpretations of the criterion of generalized conditional invariance of evolution equations in the literature. Suppose that the generalized evolution vector field Q is in reduced form. On the manifold $\mathcal{E}_{r+\rho}$ we have $Q_{(r)}(u_t-H)=\eta_t+\eta_*H-H_*\eta$, where f_* denotes the Fréchet derivative of a differential function f depending solely on t, x and derivatives of u with respect to x,

$$f_*=\sum_{i=0}^{\infty}f_{u_i}D_x^i.$$

Since the differential function $\eta_t+\eta_*H-H_*\eta$ does not involve derivatives with respect to t and mixed derivatives, we can rewrite (4) in the form

$$(\eta_t+\eta_*H-H_*\eta)|_{\mathcal{Q}_{(r+\rho,x)}}=0,\quad \text{or}\quad (\eta_t+\eta_*H)|_{\mathcal{Q}_{(r+\rho,x)}}=0.$$

If $\eta_t=\eta_x=0$, $H_t=H_x=0$ and η is of maximal rank with respect to u_ρ, in view of the Hadamard lemma the last condition is equivalent to the condition $\eta_*H-H_*\eta=F[u,\eta]$ presented in Definition 1.1 of [12]. Here $F[u,\eta]$ is a smooth function of derivatives of u with respect to x and total derivatives of η with respect to x such that $F[u,0]=0$.

Introducing the notation $\tilde{D}_t$ for the reduced operator of total differentiation with respect to t on the solution set of the equation $\mathcal{E}$,

$$\tilde{D}_t=\partial_t+\sum_{k=0}^{\infty}\big(D_x^kH\big)\partial_{u_k},$$

we represent $\eta_t + \eta_* H$ as $\tilde{D}_t \eta$ and obtain as another form of the criterion of generalized conditional invariance of evolution equations

$$\tilde{D}_t \eta|_{\mathcal{Q}_{(r+\rho,x)}} = 0,$$

which can be interpreted as the condition of invariance of the equation $\eta = 0$ with respect to the formal transformation group [27] generated by the generalized vector field $\tilde{D}_t$. Since the vector field $\tilde{D}_t$ is associated with the equation $\mathcal{E}$, the solution set of the equation $\eta = 0$ is called an *invariant set*, or, interpreted as a manifold in an appropriate jet space, an *invariant manifold* of the equation $\mathcal{E}$ [1, Section 3.1]. This interpretation is especially clear in the case $\eta_t = 0$ and $H_t = 0$. Then we can rewrite the criterion in the form

$$(H\partial_u)_{(\rho)} \eta|_{\mathcal{Q}_{(r+\rho,x)}} = 0,$$

consider t as the group parameter of the formal transformation group corresponding to the generalized vector field $H\partial_u$ in evolution form and interpret the equation $\mathcal{E}$ as the equation for finding this group.

Remark. Both symmetries and cosymmetries of an evolution equation are generalized conditional symmetries thereof but they obviously do not exhaust the entire set of its generalized conditional symmetries. For example, countable sets of independent symmetries and conservation laws had been known for the Sawada–Kotera equation $u_t = u_5 - 30uu_3 - 30u_1u_2 + 180u^2u_1$. Recently a series of generalized conditional symmetries of this equation, which are neither symmetries nor cosymmetries, was explicitly constructed in [3].

3. Formal compatibility and conditional symmetry

The relations between usual conditional (nonclassical) symmetries, reduction and compatibility of the combined system consisting of the initial equation and the corresponding invariant surface equation were discovered in [50] and were also studied and extended to the generalized framework in [43]. In particular, it was shown that the conditional invariance criterion is the compatibility condition of the combined system. This also was remarked, e.g., in [12]. At the same time, the rigorous formalization of this relation is nontrivial and was not considered so far even for evolution equations.

In this section we use the definition of formal compatibility as presented, e.g., in [47,56,57]. We temporarily employ notations compatible with these references, hence slightly different from the rest of the paper.

Let $\mathcal{L}_k$ be a system of l differential equations $L^1[u] = 0, \dots, L^l[u] = 0$ in n independent variables $x = (x_1, \dots, x_n)$ and m dependent variables $u = (u^1, \dots, u^m)$, which involves derivatives of u up to order k. The system $\mathcal{L}_k$ is interpreted as a system of algebraic equations in the jet space J^k and defines a manifold in J^k, which is also denoted by $\mathcal{L}_k$. The sth-order prolongation $\mathcal{L}_{k+s}$ of the system $\mathcal{L}_k$, $s \in \mathbb{N}$, is the system in J^{k+s} consisting of the equations $D_1^{\alpha_1} \dots D_n^{\alpha_n} L^j[u] = 0$, $j = 1, \dots, l$, $|\alpha| \leqslant s$. Here D_i is the total derivative operator with respect to the variable x_i. The projection of the corresponding manifold on J^{k+s-q}, where $q \in \mathbb{N}$ and $q \leqslant s$, is denoted by $\mathcal{L}^{(q)}_{k+s-q}$. The system $\mathcal{L}_k$ is called *formally compatible* (or *formally integrable*) if $\mathcal{L}^{(1)}_{k+s} = \mathcal{L}_{k+s}$ for any $s \in \mathbb{N} \cup \{0\}$ [47,56,57].

The first obstacle in harmonizing the above definition of formal compatibility and the definition of generalized conditional symmetry of evolution equations is that the equations $\mathcal{E}$ and $\eta = 0$ have, as a rule, different orders. Therefore, trivial differential consequences of these equations should be attached to the joint system of $\mathcal{E}$ and $\eta = 0$ before testing its compatibility.

The other obstacle is that the order of each of these equation may be lowered on the manifold of the other equation. To avoid this, we take the following steps.

Firstly, we replace the equation $\eta = 0$ by the equation $\hat{\eta} = 0$ which is equivalent to the equation $\eta = 0$ under the condition that $\mathcal{E}$ is satisfied, does not contain derivatives involving differentiation with respect to t and is of minimal order among equations possessing these properties. In other words, we convert the generalized vector field $Q = \eta\partial_u$ into its reduced form $\hat{Q} = \hat{\eta}\partial_u$, where $\hat{\eta}$ is of minimal order.

Secondly, instead of the usual order of derivatives and differential functions with the independent variables t and x we use the weight w defined by the rule:

$$\mathrm{w}(t) = \mathrm{w}(x) = 0, \qquad \mathrm{w}(u_\alpha) = [\alpha] := r\alpha_0 + \alpha_1.$$

The technique of working with a weight does not differ essentially from the order technique and so a number of analogous notions can be introduced. Thus, in the *weighted jet space* $J^k_{\mathrm{w}}(t, x|u)$ we include the variables whose weight is not greater than k. The weight $\mathrm{w}(L)$ of any differential function $L = L[u]$ equals the maximal weight of variables explicitly appearing in L. The weight of the equation $L[u] = 0$ equals $\mathrm{w}(L)$. In particular, $\mathrm{w}(u_t) = \mathrm{w}(H) = r$. This implies that the weight of the equation $\mathcal{E}$ cannot be lowered by using differential consequences of the equation $\hat{\eta} = 0$. The introduction of the weight also justifies the exclusion of the derivative u_t and mixed derivatives from η since in contrast to the usual order the weight cannot be raised under this exclusion. Note that the weight is also preserved by admissible transformations of evolution equations. As for any point or contact transformation between two evolution equations the expression of the transformed t depends only on t [33,38], and the weight of every differential function $L[u]$ is invariant with respect to such transformations.

Given a system $\mathcal{L}_k$ of l differential equations $L^j[u]=0$, $j=1,\dots,l$, in the independent variables (t,x) and the dependent variable u, which involves derivatives of u up to weight k, the sth weight prolongation $\mathcal{L}_{k+s}$ of the system $\mathcal{L}_k$, $s\in\mathbb{N}$, is the system in $J^{k+s}_{\mathrm{w}}(t,x|u)$ consisting of the equations $D_t^{\alpha_0}D_x^{\alpha_1}L^j[u]=0$, $[\alpha]\leqslant s$. The system $\mathcal{L}_{k+s}$ is constructed from the system $\mathcal{L}_{k+s-1}$ by attaching to $\mathcal{L}_{k+s-1}$ the equations $D_t^{\alpha_0}D_x^{\alpha_1}L^j[u]=0$, $[\alpha]=s$. The set of these attached equations can be viewed to consist of the equations obtained via acting by D_x on $D_t^{\alpha_0}D_x^{\alpha_1}L^j[u]=0$, $[\alpha]=s-1$, and, if r divides s, the equation obtained from $D_t^{s/r-1}L^j[u]=0$ via acting by D_t.

Let $s=\max(r,\rho)$, i.e., s is the weight of the joint system $\mathcal{S}$ of the differential equations $\mathcal{E}$ and $\hat\eta=0$, where $\rho=\mathrm{w}(\hat\eta)=\operatorname{ord}\hat\eta$. Denote by P_q and $\mathcal{P}_q$, where $q\geqslant s$, the system

$$D_x^k\hat\eta=0,\quad k=0,\dots,q-\rho,\qquad D_t^{\alpha_0}D_x^{\alpha_1}(u_t-H)=0,\quad [\alpha]\leqslant q-r$$

of algebraic equations in the jet space $J^q_{\mathrm{w}}(t,x|u)$ and the corresponding manifold, respectively. In particular, the system P_s is obtained via completing the reduced systems of $\mathcal{E}$ and $\hat\eta=0$ by trivial differential consequences which have, as equations, weights not greater than s.

Proposition 1. *The system P_s is formally compatible if and only if the evolution equation $\mathcal{E}$ is conditionally invariant with respect to the operator $Q=\eta\partial_u$.*

Proof. By R_q, where $q\geqslant s$, we denote the $(q-s)$th weight prolongation of the system P_s. Thus, the system R_s coincides with P_s. Additionally to the equations of P_q, the system R_q includes the equations $D_t^{\alpha_0}D_x^{\alpha_1}\hat\eta=0$, where $[\alpha]\leqslant q-\rho$ and $\alpha_0\neq0$.

Suppose that the system R_s is formally compatible. Consider the differential function

$$F=D_t\hat\eta-H_{u_r}D_x^r\hat\eta-\hat\eta_{u_\rho}D_x^\rho(u_t-H).$$

The equation $F=0$ is a consequence of $R_{r+\rho}$, and $\mathrm{w}(F)\leqslant r+\rho-1$. As $R^{(1)}_{r+\rho-1}=R_{r+\rho-1}$ by assumption, the equation $F=0$ also is a consequence of the system $R_{r+\rho-1}$ which coincides with the system $P_{r+\rho-1}$. We conclude that $F|_{\mathcal{P}_{r+\rho-1}}=0$ and, therefore, $D_t\hat\eta|_{\mathcal{P}_{r+\rho}}=0$. The last equality is nothing but a form of the conditional invariance criterion.

Conversely, let the evolution equation $\mathcal{E}$ be conditionally invariant with respect to the operator $Q=\eta\partial_u$. Then we prove by induction with respect to the value q that $R_q=P_q$. The equality is obvious for $q=s$. Supposing that the equality is true for a fixed q, let us prove it for $q+1$. As $R_q=P_q$, the prolonged system R_{q+1} includes P_{q+1} as a subsystem and additionally contains the equations $D_tD_x^l\hat\eta=0$, $l=0,\dots,q+1-\rho-r$, which are identities on $\mathcal{P}_{q+1}$ since $D_tD_x^l\hat\eta|_{\mathcal{P}_{r+\rho+l}}=D_x^lD_t\hat\eta|_{\mathcal{P}_{r+\rho+l}}=0$. (To prove this last equality, use the fact that $D_t\hat\eta|_{\mathcal{P}_{r+\rho}}=0$, apply the Hadamard lemma, and act by D_x^l on the resulting representation.) Hence $R_{q+1}=P_{q+1}$, completing the induction. Among the left hand sides of equations from P_{q+1} only the differential functions $D_x^{q+1-\rho}\hat\eta$ and $D_t^{\alpha_0}D_x^{\alpha_1}(u_t-H)$, $[\alpha]=q+1-r$ depend on variables of weight $q+1$, and they are functionally independent with respect to these variables. Hence $R^{(1)}_q=P_q=R_q$. □

4. Passivity and conditional symmetry

For the convenience of the reader, at first we briefly present basic notions of Riquier's compatibility theory. See, e.g., [39] and references therein for a more extended presentation of these notions and related results. We again use the notation from the beginning of the previous section. In what follows the indices a and b run from 1 to m, the indices i and j run from 1 to n, α and β run through the multiindex set $\{(\alpha_1,\dots,\alpha_n)\mid\alpha_i\in\mathbb{N}\cup\{0\}\}$.

Usually the set of derivatives $\{u^a_\alpha\}$ is assumed partially ordered. A derivative u^a_α is said to be lower (resp. strictly lower) than a derivative u^b_β, and we write $u^a_\alpha\leqslant u^b_\beta$ (resp. $u^a_\alpha\lneq u^b_\beta$), if $a=b$ and $\alpha_i\leqslant\beta_i$ (resp. $a=b$, $\alpha_i\leqslant\beta_i$ and $\alpha\neq\beta$). In contrast to this, the initial point of Riquier's theory is a suitable total ordering of derivatives, which is compatible with differentiations. Namely, a *ranking* is a total (or linear) ordering $\preccurlyeq$ of derivatives such that $u^a_\alpha\prec D_iu^a_\alpha$ and if $u^a_\alpha\prec u^b_\beta$ then $D_iu^a_\alpha\prec D_iu^b_\beta$. (As usual, $u^a_\alpha\prec u^b_\beta$ means that $u^a_\alpha\preccurlyeq u^b_\beta$ and $u^a_\alpha\neq u^b_\beta$.) In view of these properties of a ranking, the condition $u^a_\alpha\leqslant u^b_\beta$ implies $u^a_\alpha\preccurlyeq u^b_\beta$.

Suppose that a ranking of derivatives is fixed. By the *leading derivative* of a differential function $F[u]$ we mean the maximal element in the finite set of derivatives $\{u^a_\alpha\mid F_{u^a_\alpha}\neq0\}$ if this set is not empty. Consider a system $\mathcal{L}$ of finitely many differential equations resolved with respect to their leading derivatives:

$$u^{a_s}_{\alpha_s}=F^s[u],\quad s=1,\dots,l.$$

The set of leading derivatives of $\mathcal{L}$ consists of the leading derivatives of the above equations, i.e., it equals $\{u^b_\beta\mid\exists u^{a_s}_{\alpha_s}\colon\ u^b_\beta=u^{a_s}_{\alpha_s+\beta}\}$. The infinite prolongation $\mathcal{L}_\infty$ of the system $\mathcal{L}$ is formed by all possible differential consequences

$$u^{a_s}_{\alpha_s+\beta}=D_1^{\beta_1}\dots D_n^{\beta_n}F^s[u].$$

Each of the differential consequences is automatically resolved with respect to its leading derivative, which is called a *principal derivative* of the initial system $\mathcal{L}$. In other words, the set of principal derivatives of $\mathcal{L}$ consists of the derivatives of the leading derivatives of $\mathcal{L}$. The other derivatives are called *parametric derivatives* of $\mathcal{L}$.

Differential consequences of $\mathcal{L}$ involving only parametric derivatives are said to be *integrability* (or *compatibility*) *conditions*. A system $\mathcal{L}$ is *active* if it has unsatisfied integrability conditions, otherwise it is called a *passive* system.

A system $\mathcal{L}$ of equations resolved with respect to its leading derivatives is called

- *triangular* if every leading derivative of $\mathcal{L}$ is the leading derivative of only one equation,
- *autoreduced* if no principal derivative occurs on the right hand side of any equation of $\mathcal{L}$,
- *orthonomic* if it is triangular and autoreduced.

It is obvious that all of the above properties depend on the choice of ranking.

Let us return to evolution equations of the form (1). The basic idea for introducing a ranking is to assume that $u_r \prec u_t \prec u_{r+1}$. The extension of the last condition to all derivatives of u leads to the following ranking:

$$u_\alpha \preccurlyeq u_\beta \quad \Longleftrightarrow \quad [\alpha] < [\beta] \vee \big([\alpha] = [\beta] \wedge \alpha_0 \leqslant \beta_0\big).$$

We recall that $[\alpha] = r\alpha_0 + \alpha_1$. This ranking agrees well with the derivative weight introduced in the previous section. We rank derivatives by their weight and then use the lexicographic order for derivatives with the same weight.

After this ranking is fixed, the exclusion of derivatives involving differentiation with respect to t from the equation $\eta = 0$ by means of differential consequences of $\mathcal{E}$ and the subsequent solving of the resulting equation $\hat{\eta} = 0$ with respect to its leading derivative u_ρ can be viewed as replacing the joint system of $\mathcal{E}$ and $\eta = 0$ by the equivalent orthonomic system $\mathcal{S}$

$$u_t = \hat{H}, \qquad u_\rho = \check{\eta}$$

without mixed derivatives on the left hand side. Here the function $\hat{H}$ coincides with that defined after Eq. (6). The leading derivatives of this system are u_t and u_ρ; the principal derivatives are u_α, where $\alpha_0 \geqslant 1$ or $\alpha_1 \geqslant \rho$; and the other derivatives $u_0, \dots, u_\rho$ are parametric.

Proposition 2. *The equation $\mathcal{E}$ is conditionally invariant with respect to the operator $Q = \eta\partial_u$ if and only if the system $\mathcal{S}$ is passive with respect to the above ranking.*

Proof. The infinite prolongation of $\mathcal{S}$ is the system $\mathcal{S}_\infty$

$$u_{\alpha_0+1,\alpha_1} = D_t^{\alpha_0} D_x^{\alpha_1} \hat{H}, \qquad u_{\alpha_0,\alpha_1+\rho} = D_t^{\alpha_0} D_x^{\alpha_1} \check{\eta}.$$

The simplest possibility for deriving integrability conditions of $\mathcal{S}$ is to equate the expressions for mixed derivatives obtained by differentiating the first and second equations, respectively: $D_t^{\alpha_0} D_x^{\alpha_1+\rho} \hat{H} = D_t^{\alpha_0+1} D_x^{\alpha_1} \check{\eta}$. Each of the derived equations is an identity on equations of $\mathcal{S}_\infty$ involving only derivatives lower than the associated mixed derivative (and, consequently, there are no other differential consequences) if and only if the conditional invariance criterion is satisfied by the equation $\mathcal{E}$ and the operator $Q = \eta\partial_u$, cf. Eq. (6). □

5. Relation to involutivity of vector fields

A connection between generalized conditional symmetries of systems of evolution equations (in terms of invariant manifolds) and involutivity of certain system of vector fields was first noted by Kaptsov [32] (see also [1, p. 131]). For simplicity and uniformity, we restrict our considerations to the class (1).

Let the function u be a solution of the joint system $\mathcal{S}$ of the equations $\mathcal{E}$ and $u_\rho = \check{\eta}$. We introduce the new dependent variables $v^{a-1} = u_{a-1}$ and two vector fields

$$\check{D}_x = \partial_x + \sum_{a=1}^{\rho-1} v^a \partial_{v^{a-1}} + \check{\eta}\partial_{v^{\rho-1}}, \qquad \check{D}_t = \partial_t + \big(\check{D}_x^{a-1}\check{H}\big)\partial_{v^{a-1}},$$

where $\check{H} = H(t, x, v^0, \dots, v^r)$ if $\rho > r$, $\check{H} = H(t, x, v^0, \dots, v^{\rho-1}, \check{\eta}, \check{D}_x\check{\eta}, \dots, \check{D}_x^{r-\rho}\check{\eta})$ if $\rho \leqslant r$, and u_{a-1} is replaced by v^{a-1} in $\check{\eta}$.

In view of the equations for u, the functions v^{a-1} satisfy the system of differential equations

$$v_x^{a-1} = v^a, \quad a = 1, \dots, \rho-1, \qquad v_x^{\rho-1} = \check{\eta}\big(t, x, v^0, \dots, v^{\rho-1}\big), \qquad v_t^{b-1} = \check{D}_x^{b-1}\check{H} \tag{7}$$

which is associated with the system of vector fields $\{\check{D}_t, \check{D}_x\}$.

Proposition 3. *The equation $\mathcal{E}$ is conditionally invariant with respect to the operator $Q = \eta\partial_u$ if and only if the system of vector fields $\{\check{D}_t, \check{D}_x\}$ is in involution.*

Proof. $[\check{D}_t, \check{D}_x] = (\check{D}_x^\rho \check{H} - \check{D}_t\check{\eta})\partial_{v^{\rho-1}}$. Therefore, the system of vector fields $\{\check{D}_t, \check{D}_x\}$ is in involution if and only if these vector fields commute, i.e., $\check{D}_x^\rho \check{H} - \check{D}_t\check{\eta} = 0$. This last equation, after the inverse substitution $u_{a-1} = v^{a-1}$, is equivalent to Eq. (6). □

If the system of vector fields $\{\check{D}_t, \check{D}_x\}$ is in involution, the associated system (7) is completely integrable in the old terminology (see, e.g., [11, p. 1]).

Corollary 1. *A $(1+1)$-dimensional evolution equation $\mathcal{E}$ is conditionally invariant with respect to a ρth-order operator Q in reduced form if and only if it possesses a ρ-parametric family of Q-invariant solutions.*

Proof. Suppose that the equation $\mathcal{E}$ is conditionally invariant with respect to the operator Q. Then the system of vector fields $\{\check{D}_t, \check{D}_x\}$ is in involution and, therefore, is integrable by the Frobenius theorem. The dimension of the span of $\{\check{D}_t, \check{D}_x\}$ equals two for any fixed point $(t, x, v^0, \dots, v^{\rho-1})$. Therefore, the general solution of the system (7) is parameterized by $2 + \rho - 2 = \rho$ arbitrary constants. Its projection to v^0 necessarily contains all the arbitrary constants and gives the general solution of the joint system of $\mathcal{E}$ and $\hat{\eta} = 0$.

If the equation $\mathcal{E}$ is not conditionally invariant with respect to the operator Q then the system of vector fields $\{\check{D}_t, \check{D}_x\}$ is not in involution and can be iteratively completed for integrability by $[\check{D}_t, \check{D}_x]$ and the other subsequent commutators which do not lie in the span (over the ring of smooth functions) of the system of vector fields from the previous steps. The dimension of the span of the completed system is greater than two. (We consider a neighborhood of a point in which $\check{D}_x^\rho \check{H} - \check{D}_t\check{\eta} \neq 0$.) Therefore, the general solution of system (7) is parameterized by less than ρ arbitrary constants. □

Corollary 2. *The set of joint solutions of an equation $u_\rho = \check{\eta}(t, x, u_{(\rho-1,x)})$ and an evolution equation $\mathcal{E}$ is parameterized by at most ρ constants.*

6. Reduction and conditional symmetry

In this section we discuss ansatzes for the unknown function u, i.e., specific forms for finding families of solutions. We shall focus on the following class of (generalized) ansatzes:

$$u = F\big(t, x, \bar{\varphi}(\omega)\big), \qquad \bar{\varphi} = \big(\varphi^1, \dots, \varphi^\rho\big), \tag{8}$$

where $\varphi^1, \dots, \varphi^\rho$ are new unknown functions of the single invariant variable $\omega = t$, $\det \Phi \neq 0$. By Φ and $\hat{\Phi}$ we denote the matrices

$$\Phi = \big(\Phi^{ab}\big) = \frac{\partial(F_0, \dots, F_{\rho-1})}{\partial(\varphi^1, \dots, \varphi^\rho)} = (F_{a-1,\varphi^b}), \qquad \hat{\Phi} = \big(\hat{\Phi}^{ab}\big) = \Phi^{-1}. \tag{9}$$

Here $F_{a-1} = \partial^{a-1}F/\partial x^{a-1}$ and $F_{a-1,\varphi^b} = \partial^a F/\partial x^{a-1}\partial\varphi^b$.

Ansatzes $u = F^1(t, x, \bar{\varphi}^1(\omega))$ and $u = F^2(t, x, \bar{\varphi}^2(\omega))$ with the same number of new unknown functions and the same $\omega = t$ are called *equivalent* if there exists a vector-function $\bar{\zeta} = \bar{\zeta}(t, \bar{\varphi}^1)$ invertible with respect to $\bar{\varphi}^1$ such that $F^2(t, x, \bar{\zeta}(t, \bar{\varphi}^1)) = F^1(t, x, \bar{\varphi}^1)$. This notion of equivalence can be extended, e.g., by permitting dependence of $\bar{\varphi}^1$ and $\bar{\varphi}^2$ on different arguments $\omega_1 = \omega_1(t)$ and $\omega_2 = \omega_2(t)$, respectively, but we do not consider this possibility in order to retain the distinguished role of the variable t for evolution equations which is fundamental for the general line of argument in this paper.

Lemma 1. *Up to the equivalence of ansatzes, for any fixed ρ there exists a bijection between operators of the form* (3) *and ansatzes of the form* (8)*.*

Proof. An ansatz constructed with an operator Q of the form (3) is a representation of the general solution of the ordinary differential equation $u_\rho = \check{\eta}(t, x, u_{(\rho-1,x)})$ (with t playing the role of a parameter) and, therefore, has the form (8). Equivalent ansatzes only amount to different representations of the general solution. (This in fact is the reason for our notion of equivalence of ansatzes.)

The function $\check{\eta}$ from the constraint corresponding to an ansatz of the form (8) can be calculated by the standard method of reconstructing the right hand side of an ordinary differential equation from its general solution. Namely, differentiating the ansatz with respect to x up to order $\rho - 1$ and solving the resulting system $u_{a-1} = F_{a-1}$ with respect to $\bar{\varphi}$, we obtain expressions for $\bar{\varphi}$ as a function of t, x and $u_{(\rho-1,x)}$: $\varphi^a = \mathcal{I}^a(t, x, u_{(\rho-1,x)})$. (This is possible since $\det \Phi \neq 0$.) Then the ansatz corresponds to the constraint $u_\rho = \check{\eta}$, where $\check{\eta} = F_\rho(t, x, \bar{\varphi})|_{\varphi^a = \mathcal{I}^a(t,x,u_{(\rho-1,x)})}$. □

The *reduction procedure* with ansatz (8) is implemented in the following way. The substitution of (8) into $\mathcal{E}$ gives the equation $F_t + F_{\varphi^a}\varphi^a_t = \tilde{H}$, where $\tilde{H} = \tilde{H}(t, x, \bar{\varphi}) = H(t, x, F_{(r,x)}(t, x, \bar{\varphi}))$. We differentiate this equation with respect to x up to order $\rho - 1$ and solve the system so obtained with respect to $\bar{\varphi}_t$ (which is possible since $\det \Phi \neq 0$). This procedure results in the system

$$\varphi^a_t = G^a := \hat{\Phi}^{ab}(\tilde{H} - F_t)_{b-1}. \tag{10}$$

In general, the right hand sides G^a of the equations of this system will be functions of t, x and $\bar{\varphi}$.

Definition 2. If all the functions G^a are independent of x, the system $\varphi^a_t = G^a(t, \bar{\varphi})$ is a well-determined system of ordinary differential equations in $\bar{\varphi}$, which is called the *reduced system* associated with the equation $\mathcal{E}$ and ansatz (8). In this case we say that the ansatz (8) reduces the equation $\mathcal{E}$.

Remark. There also exists another notion of reduction in which a split with respect to the independent variables complementary to the invariant ones is possible after substituting ansatzes into the initial equations [45]. This kind of reduction is connected with the notion of weak symmetry [45,52] and may be called *weak reduction*. In contrast to it, Definition 2 gives a special case of the general notion of reduction which does not involve a split [43,67]. It generalizes the classical Lie reduction based on Lie symmetries [42,46] and the reduction procedures related to nonclassical [7,70] and generalized [27, Section 18.2] symmetries.

To allow for a smooth presentation of the subsequent results we now introduce some notions related to parametric families of functions and prove some auxiliary statements.

Definition 3. The parameters $\varkappa_1, \ldots, \varkappa_\rho$ are *essential* in a parametric family $\{f(t, x, \bar{\varkappa})\}$ of functions of t and x if there do not exist a function $\tilde{f}$ of $\tilde{\rho} + 2$ arguments, where $\tilde{\rho} < \rho$, and functions $\zeta^1, \ldots, \zeta^{\tilde{\rho}}$ of $\bar{\varkappa}$ such that $f(t, x, \bar{\varkappa}) = \tilde{f}(t, x, \zeta^1(\bar{\varkappa}), \ldots, \zeta^{\tilde{\rho}}(\bar{\varkappa}))$.

Lemma 2. *Let $\mathcal{F} = \{u = f(t, x, \bar{\varkappa})\}$ be a parametric family of solutions of $\mathcal{E}$. All the parameters $\varkappa_1, \ldots, \varkappa_\rho$ are essential in $\mathcal{F}$ (i.e., $\mathcal{F}$ is indeed a ρ-parametric family) if and only if*

$$\det \frac{\partial(f_0, \ldots, f_{\rho-1})}{\partial(\varkappa_1, \ldots, \varkappa_\rho)} \neq 0. \tag{11}$$

Proof. Suppose to the contrary that all the parameters in $\mathcal{F}$ are essential but condition (11) in not satisfied. The latter implies that the values t, x, $f_0, \ldots, f_{\rho-1}$ are functionally dependent. Thus there exists ρ' and a function η' of $\rho' + 2$ variables such that $\rho' < \rho$ and $f_{\rho'} = \eta'(t, x, f_{(\rho'-1,x)})$. This means that any solution of $\mathcal{E}$ from the family $\mathcal{F}$ also is a solution of the equation $u_{\rho'} = \eta'(t, x, u_{(\rho'-1,x)})$. Therefore, in view of Corollary 2 the number of essential parameters of $\mathcal{F}$ is not greater than ρ', contradicting our assumption.

Conversely, if some of the parameters $\varkappa_1, \ldots, \varkappa_\rho$ are inessential in $\mathcal{F}$ then the determinant from (11) must obviously vanish. □

Roughly speaking, the parameters in families of solutions of evolution equations are essential if and only if they are essential with respect to x. This provides further evidence that $(1 + 1)$-dimensional evolution equations are closely related to ordinary differential equations and in various aspects the variable t plays the role of a parameter.

Definition 4. Families $\{u = f(t, x, \bar{\varkappa})\}$ and $\{u = \tilde{f}(t, x, \bar{\varkappa}')\}$ of functions with the same number of parameters are defined to be *equivalent* if they consist of the same functions and differ only by parameterizations, i.e., if there exists an invertible vector-function $\bar{\zeta} = \bar{\zeta}(\bar{\varkappa})$ such that $\tilde{f}(t, x, \bar{\zeta}(\bar{\varkappa})) = f(t, x, \bar{\varkappa})$.

Now we present the main statements of this section.

Theorem 1. *Up to the re-parametrization equivalence of solution families and the equivalence of ansatzes, for any equation of the form* (1) *there exists a one-to-one correspondence between ρ-parametric families of its solutions and ansatzes reducing this equation.*

Proof. Suppose that an ansatz of the form (8) reduces $\mathcal{E}$. Since the reduced system is a normal system of ρ first-order ordinary differential equations in φ, its general solution can be represented in the form $\bar{\varphi} = \bar{\psi}(\omega, \bar{\varkappa})$, where $\bar{\varkappa} = (\varkappa_1, \ldots, \varkappa_\rho)$ are arbitrary constants and $\det(\psi^a_{\varkappa_b}) \neq 0$. This representation is unique up to re-parametrization. Substituting this solution into the ansatz results in the ρ-parametric family $\mathcal{F}$ of solutions $u = f(t, x, \bar{\varkappa})$ of $\mathcal{E}$ with $f = F(t, x, \bar{\psi}(t, \bar{\varkappa}))$. All the parameters $\varkappa_1, \ldots, \varkappa_\rho$ are essential in $\mathcal{F}$ by the chain rule since

$$\det(f_{a-1,x_b}) = \det\big(F_{a-1,\varphi^{b'}}\psi^{b'}_{x_b}\big)\big|_{\bar\varphi=\bar\psi(t,\bar x)} = \det\Phi|_{\bar\varphi=\bar\psi(t,\bar x)}\det\big(\psi^a_{x_b}\big) \neq 0. \tag{12}$$

Conversely, let $\mathcal{F} = \{u = f(t,x,\bar x)\}$ be a ρ-parametric family of solutions of $\mathcal{E}$. In view of Lemma 2 the expression $u = f(t,x,\bar\varphi(\omega))$, where $\omega = t$, defines an ansatz for u. This ansatz reduces $\mathcal{E}$ to the system $\varphi^a_\omega = 0$. Indeed, after substituting the ansatz into $\mathcal{E}$ we obtain

$$\big(f_t + f_{x_a}\varphi^a_t - H(t,x,f_{(r,x)})\big)\big|_{\bar x=\bar\varphi(t)} = f_{x_a}|_{\bar x=\bar\varphi(t)}\varphi^a_t = 0 \tag{13}$$

since $f_t = H(t,x,f_{(r,x)})$. We differentiate the last equality in (13) with respect to x up to order $\rho-1$ and solve the resulting system with respect to $\bar\varphi_t$. This system has only the zero solution since $\det(f_{a-1,x_b}) \neq 0$. □

Theorem 2. *A $(1+1)$-dimensional evolution equation $\mathcal{E}$ is conditionally invariant with respect to a ρth-order evolution vector field Q in reduced form if and only if an ansatz constructed with Q reduces the equation $\mathcal{E}$ to a normal system of ρ first-order ordinary differential equations in the ρ new unknown functions $\varphi^1,\dots,\varphi^\rho$.*

Proof. Suppose that the equation $\mathcal{E}$ is conditionally invariant with respect to the vector field Q. In view of Corollary 1, the equation $\mathcal{E}$ possesses a ρ-parametric family $\{u = f(t,x,\bar x)\}$ of Q-invariant solutions. Then the expression $u = f(t,x,\bar\varphi(\omega))$, where $\omega = t$, defines an ansatz for u associated with Q and reducing the equation $\mathcal{E}$, cf. the proof of Theorem 1.

Conversely, suppose that an ansatz of the form (8) reduces the equation $\mathcal{E}$. Let Q be the operator of the form (3) associated with this ansatz. Such an operator always exists (cf. Lemma 1). In view of Theorem 1 the ansatz gives a ρ-parametric family $\mathcal{F}$ of joint solutions of the equations $\mathcal{E}$ and $\mathcal{Q}$. Then Corollary 1 implies that the equation $\mathcal{E}$ is conditionally invariant with respect to the operator Q. □

Corollary 3. *Up to the re-parametrization equivalence of solution families, for any equation of the form* (1) *there exists a one-to-one correspondence between ρ-parametric families of its solutions and canonical ρth-order conditional symmetry operators. Namely, each operator of this kind corresponds to the family of solutions which are invariant with respect to this operator. The problems of the construction of all ρ-parametric solution families of Eq.* (1) *and the exhaustive description of its canonical ρth-order conditional symmetry operators are completely equivalent.*

Proof. It is enough to combine Theorems 1 and 2. Each solution constructed with an ansatz of the form (8) is invariant with respect to the canonical ρth-order reduction operator of $\mathcal{E}$ associated with the ansatz. □

Example 1. Analyzing the results from [28] on the group classification of $(1+1)$-dimensional variable-coefficient nonlinear diffusion–convection equations of the general form

$$f(x)u_t = \big(g(x)A(u)u_x\big)_x + h(x)B(u)u_x,$$

where $f(x)g(x)A(u) \neq 0$, we obtain only one essentially variable-coefficient equation

$$x^2u_t = \big(u^{-6/5}u_x\big)_x + x^2u_x \tag{14}$$

which is invariant with respect to a realization of the algebra $\mathrm{sl}(2,\mathbb{R})$. All the other $\mathrm{sl}(2,\mathbb{R})$-invariant equations from the class under consideration are similar (i.e., mapped by point transformations) to the well-known ("constant-coefficient") Burgers and $u^{-4/3}$-diffusion equations. Instead of Eq. (14) it is more convenient to study the equation

$$x^2v_t = vv_{xx} - \frac{5}{6}(v_x)^2 + x^2v_x \tag{15}$$

for the function $v = u^{-6/5}$, i.e., $u = v^{-5/6}$. The maximal Lie invariance algebra of Eq. (15) is $\mathfrak{g} = \langle \partial_t, t\partial_t + x\partial_x + 3v\partial_v, t^2\partial_t + (2tx+x^2)\partial_x + 6(t+x)v\partial_v\rangle$. Extending Lie ansatzes constructed by one-dimensional subalgebras of $\mathfrak{g}$, we derive the generalized ansatz

$$v = 2x^3 + \varphi^4(t)x^4 + \varphi^5(t)x^5 + \varphi^6(t)x^6, \tag{16}$$

which reduces Eq. (15) to the system of ordinary differential equations

$$\varphi^4_t = 7\varphi^5 - \frac{4}{3}\big(\varphi^4\big)^2, \qquad \varphi^5_t = 18\varphi^6 - \frac{4}{3}\varphi^4\varphi^5, \qquad \varphi^6_t = -\frac{5}{6}\big(\varphi^5\big)^2 + 2\varphi^4\varphi^6.$$

This ansatz represents the general solutions of the equation

$$x^3v_{xxx} - 12x^2v_{xx} + 60xv_x - 120v + 12x^3 = 0.$$

In view of Theorem 2, the reduction of Eq. (15) with the ansatz (16) is equivalent to the fact that (15) is conditionally invariant with respect to the third-order evolution vector field

$$\big(x^3v_{xxx} - 12x^2v_{xx} + 60xv_x - 120v + 12x^3\big)\partial_v.$$

7. No-go theorem on determining equation

In terms of local solutions, Corollary 3 means that there exists a (local) one-to-one correspondence between solutions of the determining equation (6) and ρ-parametric families of solutions of the initial equation (1). We show that this correspondence is realized by transformations between systems associated with these equations.

Theorem 3. *The system in the functions $\theta^a = \theta^a(t, x, u_0, \dots, u_{\rho-1})$, which consists of the partial differential equation* (6)*, where $\check\eta$ is identified with θ^ρ, and the algebraic equations $\theta^1 = u_1, \dots, \theta^{\rho-1} = u_{\rho-1}$, is reduced by the composition of the differential substitution*

$$\bar\theta = -\Psi^{-1}\bar{\mathcal{I}}_x, \tag{17}$$

where $\bar{\mathcal{I}} = \bar{\mathcal{I}}(t, x, u_0, \dots, u_{\rho-1})$ are the new unknown functions, $\Psi := (\mathcal{I}^a_{u_{b-1}})$ and $\det\Psi \neq 0$, and the hodograph transformation

$$\begin{aligned} &\textit{the new independent variables}: \quad \tilde t = t, \quad \tilde x = x, \quad \varkappa_a = \mathcal{I}^a,\\ &\textit{the new dependent variable}: \qquad v^{b-1} = u_{b-1} \end{aligned} \tag{18}$$

to the system formed by the initial equation $\mathcal{E}$ in the function $\tilde u = \tilde u(\tilde t, \tilde x, \bar x)$ and the equations $v^{b-1} = \partial^{b-1}\tilde u/\partial\tilde x^{b-1}$, $b = 2, \dots, \rho$, where $\bar x$ plays the role of a parameter tuple and $\tilde u$ is identified with v^0.

Proof. At first we construct a direct transformation. Extending Eq. (6), we introduce the notation $\theta^1 = u_1, \dots, \theta^{\rho-1} = u_{\rho-1}$, $\theta^\rho = \check\eta$. This notation is natural since in view of the definition of the operators $\hat D_t$ and $\hat D_x$ and Eq. (6), the functions θ^a satisfy the conditions $\hat D_t\theta^a = \hat D^a_x\hat H$. We consider the system consisting of the partial differential equation (6) and the algebraic equations $\theta^1 = u_1, \dots, \theta^{\rho-1} = u_{\rho-1}$ and carry out the differential substitution (17). In other words, $\bar{\mathcal{I}}$ is a tuple of solutions of the equation $\hat D_x\mathcal{I} = 0$ with $\det(\mathcal{I}^a_{u_{b-1}}) \neq 0$. It is determined by $\bar\theta$ up to the transformation $\bar{\mathcal{I}} \to \bar G(t, \bar{\mathcal{I}})$, where $G^a_{\mathcal{I}^b} \neq 0$. Then we carry out the hodograph transformation (18). In what follows, for convenience we denote the function v^0 by $\tilde u$ and the derivatives $\partial^k\tilde u/\partial\tilde x^k$ by $\tilde u_k$, $k = 1, 2, \dots$. Differentiating the equality $\bar x = \bar{\mathcal{I}}$ with respect to $\tilde x$, we obtain $\bar{\mathcal{I}}_x + v^{b-1}_{\tilde x}\bar{\mathcal{I}}_{u_{b-1}} = 0$. As $\hat D_x\bar{\mathcal{I}} = 0$ and $\det\Psi \neq 0$, this means that $v^{b-1}_{\tilde x} = v^b$, $b < \rho$, $v^{\rho-1}_{\tilde x} = \check\eta(\tilde t, \tilde x, v^0, \dots, v^{\rho-1})$, and therefore $v^{b-1} = \tilde u_{b-1}$ and $\tilde u_\rho = \tilde\eta = \check\eta(\tilde t, \tilde x, \tilde u_{(\rho-1,\tilde x)})$, i.e., $\theta^a = \tilde u_a$. In the new variables we also have that $\hat D_x = \partial_{\tilde x} + (\hat D_x\mathcal{I}^a)\partial_{\varkappa_a} = \partial_{\tilde x}$. This operator acts on the functions of $\tilde t$, $\tilde x$ and derivatives of $\tilde u$ as the operator $D_{\tilde x}$ of total derivation with respect to the variable $\tilde x$. Hence $\hat D^k_x\check\eta = \tilde u_{\rho+k}$ and $\hat H = \tilde H := H(\tilde t, \tilde x, \tilde u_{(r,\tilde x)})$. Analogously

$$\hat D_t = \partial_{\tilde t} + \big(\hat D_t\mathcal{I}^a\big)\partial_{\varkappa_a} = \partial_{\tilde t} - \big(\tilde u_{b-1,\tilde t} - D^{b-1}_{\tilde x}\tilde H\big)\mathcal{I}^a_{u_{b-1}}\partial_{\varkappa_a}$$

since $\mathcal{I}^a_t + \tilde u_{b-1,\tilde t}\mathcal{I}^a_{u_{b-1}} = 0$. Moreover, as $\tilde u_\rho = \tilde\eta$, we also have $\tilde u_{\rho\varkappa_a} = D_{\varkappa_a}\tilde\eta = \tilde\eta_{\tilde u_{b-1}}\tilde u_{b-1,\varkappa_a}$, and the matrix $(\tilde u_{b-1,\varkappa_a})$ is the inverse of the matrix $(\mathcal{I}^a_{u_{b-1}})$. This is why in the new variables Eq. (6) takes the form

$$D^\rho_{\tilde x}(\tilde u_{\tilde t} - \tilde H) = \tilde\eta_{\tilde u_{b-1}}D^{b-1}_{\tilde x}(\tilde u_{\tilde t} - \tilde H).$$

For a fixed function $\tilde u$, the equation $w_\rho = \tilde\eta_{\tilde u_{b-1}}w_{b-1}$ with respect to the function $w = w(\tilde t, \tilde x, \bar x)$ is a ρth-order ordinary differential equation, with $\tilde x$ as the independent variable and $\tilde t$ and $\bar x$ playing the role of parameters. The functions $\tilde u_{\varkappa_a}$ are linearly independent solutions of this equation since $\det(\tilde u_{\varkappa_a,b-1}) \neq 0$. Therefore, there exist functions $\zeta^a = \zeta^a(t, \bar x)$ such that $\tilde u_{\tilde t} - \tilde H = \zeta^a\tilde u_{\varkappa_a}$. In view of the indeterminacy of $\bar{\mathcal{I}}$, we can make the transformation $\bar x \to \bar G(t, \bar x)$ to transform the last equation to the equation of the same form with $\zeta^a = 0$.

Conversely, let $\tilde u = \tilde u(\tilde t, \tilde x, \bar x)$ be a ρ-parametric solution of Eq. (1). (We use the notation with tildes to be consistent with the first part of the proof.) Assuming $v^{b-1} = \tilde u_{b-1}$ as the unknown functions, we obtain the system $v^0_{\tilde t} = H(\tilde t, \tilde x, v^0_{(r,\tilde x)})$ and $v^{b-1}_{\tilde x} = v^b$, $b < \rho$. We successively carry out the inverse of the hodograph transformation (18) and the inverse of the differential substitution (17) and denote the function $\theta^\rho = \theta^\rho(t, x, u_0, \dots, u_{\rho-1})$ by $\check\eta$. By construction we have that $\theta^1 = u_1, \dots, \theta^{\rho-1} = u_{\rho-1}$ and for each $\bar x$ the solution $\tilde u = \tilde u(\tilde t, \tilde x, \bar x)$ of (1) is invariant with respect to the operator $Q = (u_\rho - \check\eta)\partial_u$. This means that Q is an operator of generalized conditional symmetry of (1) and, therefore, the function $\check\eta$ satisfies Eq. (6). □

We call Theorem 3 "a no-go theorem" since it basically states that solving the determining equation for generalized conditional symmetry operators is as difficult as solving the original equation. It generalizes the analogous no-go theorem on the determining equations for usual conditional symmetry operators of evolution equations, whose coefficient of ∂_t is equal to zero [17,34,48,49,62,69]. The main problem in generalizing that result was that the corresponding hodograph transformation should involve ρ independent variables. At the same time, both the initial and determining equations involve only a single dependent variable.

Note that the attribute "no-go" should be treated as impossibility of the exhaustive solution of the problem. At the same time, imposing additional constraints on the differential function $\check\eta = \check\eta(t, x, u_{(\rho-1,x)})$ or choosing a specific form for this

function, one can construct a number of particular examples of generalized conditional symmetries and then apply them to finding exact solutions of the original equation $\mathcal{E}$. Since the determining equation (6) has more independent variables and, therefore, more degrees of freedom, often it is more convenient to guess a simple solution or a simple ansatz for the determining equation, which may then provide a parametric set of more complicated solutions of the original equation $\mathcal{E}$.

This situation is similar to that of Lie symmetries of first-order ordinary differential equations. Indeed, the solution of the determining equation for Lie symmetries of a first-order ordinary differential equation $\mathcal{L}$ is a much more complicated problem than the solution of the original equation $\mathcal{L}$. Even if a Lie symmetry generator of $\mathcal{L}$ is known, it may be just as difficult to find an invariant of the associated one-parameter group (which is a necessary step of solving by the Lie method) as it was to integrate the original differential equation $\mathcal{L}$ [42, pp. 131–133]. At the same time, certain first-order ordinary differential equations (e.g., homogeneous ones) possess simple Lie symmetries which can easily be found by an educated guess and then effectively used for the integration of these equations.

The above approach to the construction of exact solutions using generalized conditional symmetries of special kinds was applied in the literature to a number of different classes of evolution equations, in particular to quasilinear second-order evolution equations. We recall only some of these results.

Generalized conditional symmetries of many particular cases of equations of the general form

$$u_t = g^3(t,x,u)u_{xx} + g^2(t,x,u)u_x^2 + g^1(t,x,u)u_x + g^0(t,x,u)$$

were looked for by a number of authors in a form similar to the right hand sides of the corresponding equations,

$$\eta = u_{xx} + \tilde{g}^2(t,x,u)u_x^2 + \tilde{g}^1(t,x,u)u_x + \tilde{g}^0(t,x,u),$$

or in the equivalent form $\eta = u_t + \hat{g}^2(t,x,u)u_x^2 + \hat{g}^1(t,x,u)u_x + \hat{g}^0(t,x,u)$, see, e.g., [22,30,29,53,54,67] and references therein. Another intensively investigated class of generalized conditional symmetries and generalized ansatzes is related to differential constraints which are equivalent to linear differential constraints with respect to point transformations, see, e.g., [9,20,25,67] and references therein and cf. also Example 1.

As shown in the next section, a generalized first-order conditional symmetry in canonical form $(u_x - \eta(t,x,u))\partial_u$ of an evolution equation is, up to sign, the evolution form of the singular nonclassical symmetry operator $\partial_x + \eta(t,x,u)\partial_u$ of the same equation. In [51] such symmetries of different classes of quasilinear second-order evolution equations were studied under the assumption of separation of variables in the coefficient η, $\eta = \zeta^0(t)\zeta^1(x)\zeta^2(u)$. Earlier the partial case $\eta = \zeta^1(x)\zeta^2(u)$ was investigated in [23] for equations of the form $u_t = u_x^\sigma u_{xx} + \mu u_x^{\sigma+1} + f(u)$. The important special subcases $\zeta^1(x) = x$ and $\zeta^1(x) = x^{-1}$ were separated therein. The latter subcase, which generalizes scale-invariant solutions, was considered within a more general framework in [21]. An extension of results obtained in [23] was presented in [55]. The ansatz $\check{\eta} = \eta^1(t,x)u^{\alpha+1} + \eta^0(t,x)u^\alpha$ was used in [26] for the fast diffusion equations of the form $u_t = (u^{-\alpha}u_x)_x$.

8. Usual and generalized reduction operators

It seems natural that usual conditional symmetry is a particular case of generalized conditional symmetry. On the other hand, the criterion of usual conditional symmetry restricted to the case of evolution equations is essentially different from (4). This is why we formulate the precise relation as a proposition.

Proposition 4. *The vector field $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$, where the coefficients τ, ξ and η are functions of t, x and u, is a usual conditional symmetry operator of an equation $\mathcal{E}$ of the form (1) if and only if the operator $\hat{Q} = \hat{\eta}\partial_u$, where $\hat{\eta} = \eta - \tau H - \xi u_x$, is a generalized conditional symmetry operator of the same equation.*

Proof. The first way of proving this is simpler but essentially involves statements on properties of the corresponding families of invariant solutions. A solution of $\mathcal{E}$ is Q-invariant if and only if it is $\hat{Q}$-invariant. Moreover, $\operatorname{ord}\hat{\eta} = \rho$, where $\rho = r$ if $\tau \neq 0$ and $\rho = 1$ if $\tau = 0$. Suppose that Q is a usual reduction operator of $\mathcal{E}$. Propositions 2 and 5 from [34] imply that the equation $\mathcal{E}$ possesses an r-parametric (resp. one-parametric) family of Q-invariant solutions if $\tau \neq 0$ (resp. $\tau = 0$). Then Corollary 1 implies that $\hat{Q}$ is a generalized conditional symmetry operator of $\mathcal{E}$. The proof of the converse is similar.

The second way is more direct and technical. We have to show that the corresponding invariance criteria are equivalent. In what follows $E = u_t - H$, $\tilde{\eta} = Q[u] = \eta - \tau u_t - \xi u_x$, $k = 0,\dots,r$, $\hat{k} = 0,\dots,\rho$, $j = 0,\dots,r-1$ and $\hat{j} = 0,\dots,\rho-1$. We have $(\hat{Q}_{(r)} - Q_{(r)})E = ED_t\tau - \xi D_x E - H_{u_k}D_x^k E$. The expression $Q_{(r)}E$ involves at most the derivatives u_k and $u_{t,\hat{j}}$. Hence the differential consequences which should be taken into account in the usual conditional invariance criterion are exhausted by $\mathcal{E}$ itself and $D_x^j\tilde{\eta} = 0$. Analogously, the expression $\hat{Q}_{(r)}(u_t - H)$ involves at most the derivatives u_m, $m = 0,\dots,r+\rho$, and $u_{t,\hat{k}}$. Therefore, the differential consequences which should be taken into account in the usual conditional invariance criterion are exhausted by $D_x^{\hat{k}}E = 0$ and $D_x^k\hat{\eta} = 0$. Finally, we have the chain of equivalences

$$Q_{(r)}E|_{\mathcal{E}\cap\mathcal{Q}_r} = 0 \iff Q_{(r)}E = 0 \text{ when } E = 0 \text{ and } D_x^j\tilde{\eta} = 0 \iff$$
$$Q_{(r)}E = 0 \text{ when } D_x^{\hat{k}}E = 0 \text{ and } D_x^j\tilde{\eta} = 0 \iff$$

$$\hat{Q}_{(r)}E=0 \text{ when } D_x^{\hat{k}}E=0 \text{ and } D_x^k\hat{\eta}=0 \iff$$
$$\hat{Q}_{(r)}E|_{\mathcal{E}_{r+\rho}\cap\hat{\mathcal{Q}}_{(r+\rho,x)}}=0,$$

and the result follows. □

Despite the fact that the sets of Q- and $\hat{Q}$-invariant solutions of $\mathcal{E}$ coincide, in the case $\tau\neq 0$ the procedures of the reduction of $\mathcal{E}$ with respect to the operators Q and $\hat{Q}$ are quite different. Thus, the operator Q reduces $\mathcal{E}$ to a single rth-order ODE in a single unknown function, where the invariant independent variable necessarily depends on x or u. The operator $\hat{Q}$ reduces $\mathcal{E}$ to a system of r first-order ODEs in r unknown functions, where t can be taken as the invariant independent variable. We illustrate this situation by the following example.

Example 2. Usual reduction operators of variable coefficient semilinear diffusion equations with power sources were investigated in [63,64]. Namely, the equations studied have the general form

$$f(x)u_t=\big(g(x)u_x\big)_x+h(x)u^m, \tag{19}$$

where f, g and h are arbitrary smooth functions of the variable x, $f(x)g(x)h(x)\neq 0$, and m is an arbitrary constant, $m\neq 0,1$. The most convenient approach to this problem, as it turns out, is to map the class (19) to the class

$$v_t=v_{xx}+H(x)v^m+F(x)v \tag{20}$$

by a family of point transformations parameterized by arbitrary elements f, g and h and then to investigate usual reduction operators of equations from the latter class. Under both the group classification and the classification of reduction operators the following equation is singled out from the class (20):

$$v_t=v_{xx}-\frac{v^3}{x^3}+\frac{9}{4}\frac{v}{x^2}. \tag{21}$$

Note that by the point transformation $\tilde{t}=t$, $\tilde{x}=\ln|x|$, $\tilde{v}=v/\sqrt{2|x|}$ Eq. (21) is reduced to the equation $e^{2\tilde{x}}\tilde{v}_{\tilde{t}}=\tilde{v}_{\tilde{x}\tilde{x}}-2\tilde{v}^3+2\tilde{v}$.

The maximal Lie invariance algebra of (21) is generated by the operators $D=4t\partial_t+2x\partial_x+v\partial_v$ and ∂_t. Inequivalent non-Lie usual reduction operators of (21), having nonzero coefficients of ∂_t are exhausted, up to the discrete symmetry transformation of alternating the sign of v, by the operator

$$Q=\partial_t+\left(\frac{3\sqrt{2}}{2}\frac{v}{x^{3/2}}-\frac{3}{x}\right)\partial_x-\frac{3}{2}\left(\frac{v^3}{x^3}-\frac{3\sqrt{2}}{2}\frac{v^2}{x^{5/2}}-\frac{v}{x^2}+\frac{2\sqrt{2}}{x^{3/2}}\right)\partial_v.$$

For all expressions to be well-defined, we have to restrict ourselves to values $x>0$. (Another way is to replace x by $|x|$.)

We discuss two ways of using the operator Q for finding exact solutions of Eq. (21).

First way. To construct an ansatz with the operator Q, we have to solve the quasilinear first-order partial differential equation $Q[v]=0$. The corresponding invariant independent variable necessarily involves the dependent variable v. For simplifying calculations, we suppose at first that $v_t\neq 0$ and carry out the hodograph transformation $\tilde{t}=v$, $\tilde{x}=x$, $\tilde{v}=t$ which maps Eq. (21) and the reduction operator Q to the equation

$$\tilde{v}_{\tilde{t}}{}^2\tilde{v}_{\tilde{x}\tilde{x}}+\tilde{v}_{\tilde{x}}{}^2\tilde{v}_{\tilde{t}\tilde{t}}-2\tilde{v}_{\tilde{t}}\tilde{v}_{\tilde{x}}\tilde{v}_{\tilde{t}\tilde{x}}+\tilde{v}_{\tilde{t}}{}^2+\frac{\tilde{t}^3}{\tilde{x}^3}\tilde{v}_{\tilde{t}}{}^3-\frac{9}{4}\frac{\tilde{t}}{\tilde{x}^2}\tilde{v}_{\tilde{t}}{}^3=0 \tag{22}$$

and its reduction operator

$$\tilde{Q}=-\frac{3}{2}\left(\frac{\tilde{t}^3}{\tilde{x}^3}-\frac{3\sqrt{2}}{2}\frac{\tilde{t}^2}{\tilde{x}^{5/2}}-\frac{\tilde{t}}{\tilde{x}^2}+\frac{2\sqrt{2}}{\tilde{x}^{3/2}}\right)\partial_{\tilde{t}}+\left(\frac{3\sqrt{2}}{2}\frac{\tilde{t}}{\tilde{x}^{3/2}}-\frac{3}{\tilde{x}}\right)\partial_{\tilde{x}}+\partial_{\tilde{v}},$$

respectively. An ansatz constructed with the operator $\tilde{Q}$ has the form

$$\tilde{v}=z(\omega)+\frac{1}{24}\tilde{x}^2\frac{\tilde{t}+\sqrt{2\tilde{x}}}{\tilde{t}-\sqrt{2\tilde{x}}}-\frac{1}{12}\tilde{x}^2,\quad \text{where } \omega=\tilde{x}^2\frac{\tilde{t}-\sqrt{2\tilde{x}}}{\tilde{t}+\sqrt{2\tilde{x}}},$$

and reduces (22) to the single second-order linear ordinary differential equation $\omega z_{\omega\omega}+2z_\omega=0$ in the function $z=z(\omega)$. After substituting to the ansatz, the general solution $z=\tilde{c}_1+\tilde{c}_2\omega^{-1}$ of the reduced equation gives the exact solution

$$\tilde{v}=\frac{\tilde{x}^4+24\tilde{c}_2}{24\tilde{x}^2}\frac{\tilde{t}+\sqrt{2\tilde{x}}}{\tilde{t}-\sqrt{2\tilde{x}}}-\frac{1}{12}\tilde{x}^2+\tilde{c}_1$$

of (22). Applying the inverse hodograph transformation, we construct the non-Lie solution

$$v=\sqrt{2x}\frac{3x^4+(24t+c_1)x^2-c_2}{x^4+(24t+c_1)x^2+c_2} \tag{23}$$

of Eq. (21), where $c_1=-24\tilde{c}_1$ and $c_2=-24\tilde{c}_2$. The constant $\tilde{c}_1$ can be canceled by a translation with respect to t. If $c_2\neq 0$, this constant can be set to 1 by a scale transformation generated by D. (Recall that the above transformations are Lie symmetries of Eq. (21).) The solution (23) with $c_2=0$ is a Lie solution invariant with respect to the scale symmetry operator D. However, it is much harder to find this solution by the reduction with respect to the operator D. The corresponding ansatz $v=\sqrt{2x}z(\omega)$, where $\omega=x/\sqrt{|t|}$, has a simple form but the reduced ODE $\omega^2 z_{\omega\omega}+\omega(1-\omega)z_\omega+2z-2z^3=0$ is nonlinear and complicated.

Under the condition $v_t=0$ the equation $Q[u]=0$ implies Eq. (21) and is reduced by the transformation $v=\sqrt{2x}z(x)$ to the equation $(z-1)(xz_x+z^2-1)=0$ which is equivalent to the Riccati equation $xz_x=1-z^2$. Therefore, the corresponding invariant solutions of (21) have the form

$$v=\sqrt{2x}\frac{c_1x^2-c_2}{c_1x^2+c_2}, \tag{24}$$

where only the ratio of the constants c_1 and c_2 is essential. Note that a function v is a stationary solution of (21) if and only if $v=\sqrt{2x}z(x)$, where $z=z(x)$ is a solution of the equation $z_{xx}=2(z^3-z)$ which is integrable in terms of elliptic functions.

Second way. Another way to use the operator Q for finding exact solutions of Eq. (21) is to consider the second-order reduction operator $\eta(t,x,v,v_x,v_{xx})\partial_v$, where the differential function coincides with the characteristic $Q[v]$ on the manifold determined by Eq. (21) in the corresponding second-order jet space. Here

$$\eta=-v_{xx}-\frac{3\sqrt{2}}{2}\frac{vv_x}{x^{3/2}}+\frac{3}{x}v_x-\frac{1}{2}\frac{v^3}{x^3}+\frac{9\sqrt{2}}{4}\frac{v^2}{x^{5/2}}-\frac{3}{4}\frac{v}{x^2}-\frac{3\sqrt{2}}{x^{3/2}}.$$

The associated invariant surface condition $\eta=0$ is a second-order ordinary differential equation, where x and v are independent and dependent variables, respectively, and t plays the role of a parameter. It is reduced by the differential substitution

$$v=\sqrt{2x^3}\frac{w_x}{w},$$

to the linear equation

$$x^3w_{xxx}-3xw_x+3w=0 \tag{25}$$

in the new unknown function $w=w(t,x)$, whose general solution is $w=\psi^0(t)x^3+\psi^1(t)x+\psi^2(t)x^{-1}$. Therefore, we have the following ansatz for the function v:

$$v=\sqrt{2x}\frac{3\psi^0(t)x^4+\psi^1(t)x^2-\psi^2(t)}{\psi^0(t)x^4+\psi^1(t)x^2+\psi^2(t)}, \tag{26}$$

where only two ratios of the functions ψ^μ, $\mu=1,2,3$, are essential.

To make a conventional reduction of Eq. (21) with ansatz (26), we would suppose that one of the functions ψ^μ, e.g. ψ^0, is nonvanishing. After substituting ansatz (26) into (21), we would obtain the reduced system of two first-order ODEs in the functions $\varphi^i=\psi^i/\psi^0$, $i=1,2$. Then it would be necessary to consider the case $\psi^0=0$ and $\psi^1\neq 0$ and to derive the reduced first-order ODE in $\varphi=\psi^2/\psi^1$. The condition $(\psi^0,\psi^1)=(0,0)$ leads to the single solution $v=-\sqrt{2x}$. This partition into different cases corresponds to the partition made in the first way.

We use a more advanced technique allowing us to avoid the consideration of different cases. The entire systems of the equation $\eta=0$ and Eq. (21) is equivalent to the system of Eqs. (25) and

$$w_t=3w_{xx}+\frac{3}{x}w_x-\frac{3}{x^2}w. \tag{27}$$

Moreover, $(x^3w_{xxx}-3xw_x+3w)\partial_w$ is an operator of generalized conditional symmetry of (27). Therefore, the associated ansatz $w=\psi^0(t)x^3+\psi^1(t)x+\psi^2(t)x^{-1}$ reduces Eq. (27), and the corresponding reduced system is $\psi^0_t=0$, $\psi^1_t=24\psi^0$, $\psi^2_t=0$ with the general solution $\psi^0=c_0$, $\psi^1=24c_0t+c_1$, $\psi^2=c_2$. As a result, we have the solution

$$v=\sqrt{2x}\frac{3c_0x^4+(24c_0t+c_1)x^2-c_2}{c_0x^4+(24c_0t+c_1)x^2+c_2}$$

of Eq. (27). The conditions $c_0\neq 0$ and $c_0=0$ correspond to the solutions (23) and (24), respectively.

In a similar way, the conversion of usual nonclassical symmetries into generalized ones was implicitly used, e.g., in [2,10,41] in the reduction of the nonlinear (constant coefficient) heat equations with source terms in the form of cubic polynomials, including the Fitzhugh–Nagumo equation.

9. Conclusion

In the study of generalized conditional symmetry of evolution equations we adhere to the following principles:

1. The property of an operator Q to be a conditional symmetry of a differential equation $\mathcal{L}$ is equivalent to the fact that the corresponding invariant surface equation $Q[u] = 0$ is formally compatible (in a certain sense) with $\mathcal{L}$, i.e., the joint system of the above two equations has no nontrivial differential consequences. This determines what differential consequences of these equations should be involved in the criterion of the conditional invariance of the equation $\mathcal{L}$ with respect to the operator Q. In the property of formal compatibility conditional symmetries differ from purely weak symmetries [45,52] for which the calculation of integrability conditions (resp. "the reduction to passive form") of the corresponding joint systems has to be carried out in each case. In fact, weak symmetries of $\mathcal{L}$ are associated with differential constraints whose solution sets at least intersect the solution set of $\mathcal{L}$.
2. Therefore, the criterion of conditional invariance in fact is nothing but the criterion of formal compatibility for a system associated with the pair of equations $\mathcal{L}$ and $Q[u] = 0$. This has two consequences: There does not exist a universal explicit criterion of conditional invariance similar to the criterion of Lie invariance, which would contain a priori the complete information which differential consequences to take into account and would be appropriate for any system of differential equations and any set of generalized vector fields. At the same time, for any fixed pair of a system of differential equations and a set of generalized vector fields the criterion can be formulated in different forms.
3. Single generalized conditional symmetries are assumed equivalent if they differ by multipliers being nonvanishing differential functions. Therefore it suffices to consider only symmetries with characteristic containing some isolated (e.g., highest-order) derivative of the unknown function.
4. In order to be usable, a conditional symmetry should correspond to an integrable differential constraint which admits a simple representation of its general solution. Such a representation is considered as an ansatz for the solution of the initial equation $\mathcal{L}$. The formal compatibility of the differential constraint with $\mathcal{L}$ should imply a (strong) reduction of $\mathcal{L}$ by the ansatz. In other words, after the substitution of the ansatz into $\mathcal{L}$ we should obtain a system of differential equations of a simpler structure, e.g., with a smaller number of independent variables. Symmetries equivalent as vector fields induce the same set of ansatzes and equivalent reductions. In fact there does not exist a universal precise definition of reduction which does not involve splitting with respect to parametric variables and covers all possible representations of solutions. In view of the above problems of integrability and defining reduction, it is still unclear in the general case what differential constraints formally compatible with the initial equation should be considered associated with reduction operators. This question becomes trivial and has a positive answer in the situation considered in the paper (single evolution equations and differential constraints depending only on derivatives with respect to x). Probably, in the general case it would be more natural to assume that the notion of reduction operator is narrower than the notion of conditional symmetry, cf. [43].
5. If the characteristics of operators coincide on the manifold determined by the initial equation $\mathcal{L}$ and one of the operators corresponds to a differential constraint formally compatible with $\mathcal{L}$ then the other operators have the same property. Such conditional symmetry operators can be considered equivalent in a weak sense since they are associated with the same set of invariant solutions of $\mathcal{L}$. At the same time, they are inequivalent, in general, from the point of view of their usefulness for finding solutions. In particular, they may give inequivalent ansatzes and reduced systems.

These general principles can be applied in other situations as well. We plan to complete soon our study on basic properties of usual (i.e., first-order quasilinear) conditional symmetries of systems of differential equations.

In spite of the no-go results presented in the paper, generalized conditional symmetries can be effectively applied to the construction of exact solutions of evolution equations. As it is impossible to exhaustively describe generalized conditional symmetries of a fixed evolution equation, they should be looked for under additional constraints or in special classes of differential functions, e.g., with separated variables. In this way, usual and generalized conditional symmetries were studied for a number of particular subclasses of evolution equations, cf. the discussion in the end of Section 7. Note that only in [43] generalized conditional symmetries which are not in reduced form were considered. Generalized conditional symmetries were also used for the exact solution of initial-value problems for evolution equations [4,68]. Another relevant direction of research is the related inverse problem, namely, the description of evolution equations possessing certain generalized conditional symmetries, see [31,58,59,61] and references therein.

A systematic investigation of generalized conditional symmetries of non-evolution equations in fact is not available in the literature at the moment. An exception is the paper [43] of Olver mentioned in the introduction, where the connection between the reduction of a partial differential equation by a generalized ansatz within the higher-order direct method of Galaktionov [20] and the compatibility of the associated differential constraint with this equation was discovered. At the same time, there exist a number of examples on the application of generalized ansatzes to finding exact solutions of non-evolution equations, which are collected, e.g., in [15,25]. It is obvious that all such examples can be interpreted within the framework proposed in [43]. Ansatzes of another kind with new unknown functions depending on different arguments arise under generalized separation of variables [1,66]. Theoretical aspects of this subject should certainly be further investigated.

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Paper 4

Potential conservation laws

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Potential conservation laws

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We prove that potential conservation laws have characteristics depending only on local variables if and only if they are induced by local conservation laws. Therefore, characteristics of pure potential conservation laws have to essentially depend on potential variables. This statement provides a significant generalization of results of the recent paper by Bluman *et al.* [J. Math. Phys. **47**, 113505 (2006)]. Moreover, we present extensions to gauged potential systems, Abelian and general coverings, and general foliated systems of differential equations. An example illustrating possible applications of these results is given. A special version of the Hadamard lemma for fiber bundles and the notions of weighted jet spaces are proposed as new tools for the investigation of potential conservation laws. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2993117]

I. INTRODUCTION

In a recent paper by Bluman *et al.*[6] a remarkable result on potential conservation laws was obtained. Namely, it was shown that for an arbitrary system of differential equations a conservation law of a potential system with a characteristic which depends only on the independent variables is induced by a local conservation law of the initial system. It appears that this statement was deduced after an in-depth investigation of important examples on potential symmetries which were considered in Ref. 6. This approach seems natural since, according to the famous Russian mathematician Vladimir Arnold, mathematics is an inductive and experimental science. In the present paper we show that this theorem admits a significant generalization and that, moreover, a converse statement is true as well. The possibility of deriving this result is suggested by recalling the rule of transforming conservation laws under point transformations between systems of differential equations.[22,24] The application of a hodograph-type transformation to a characteristic which exclusively depends on the independent variables may result in a characteristic including dependent variables. Generally, characteristics of induced conservation laws of potential systems can depend on derivatives of unknown functions of the initial system, and systems of other kinds related to standard potential systems (systems determining Abelian or general coverings, gauged potential systems, general foliated systems) can be investigated in the same framework.

More precisely, we rigorously prove a number of statements on this subject (Proposition 3 and Theorems 6–9), which can be summed up as follows.

Theorem 1: *The following statements on a conservation law of a two-dimensional potential system (a system determining an Abelian covering, a multidimensional standard potential system without gauges) are equivalent if the corresponding initial system is totally nondegenerate.*

(1) The conservation law is induced by a conservation law of the initial system.
(2) It contains a conserved vector which does not depend on potentials.
(3) Some of its extended characteristics are induced by characteristics of the initial system.
(4) It possesses a characteristic not depending on potentials.

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The equivalence of the first three statements is also true for conservation laws of general foliated systems, including multidimensional gauged potential systems and covering systems.

Further results on conservation laws of weakly gauged potential systems (Theorem 10) and general covering systems (Theorem 11) are established as well.

Theorem 1 allows us to formulate a criterion (Proposition 8) on purely potential conservation laws in terms of characteristics. Namely, *a conservation law of a system determining an Abelian covering (a potential system in the two-dimensional case) is not induced by a conservation law of the corresponding initial system if and only if it is associated with a completely reduced characteristic depending on potentials*. Here, a characteristic of a system of differential equations is called completely reduced if it does not depend on the derivatives of the unknown functions, which are assumed to be constrained to the solution set of the system. In particular, any completely reduced characteristic of a system determining an Abelian covering does not depend on the derivatives of potentials of orders greater than 0 since they are constrained due to differential consequences of the potential part of the system. Any conservation law possesses a completely reduced characteristic since expressing the constrained variables via the unconstrained ones in a characteristic results in an equivalent characteristic.

Our paper is organized as follows. Some basic notions and results on conservation laws are collected in Sec. II for the reader's convenience. Results on characteristics of conservation laws are singled out in Sec. III due to their particular importance for the paper. The exposition in these two sections follows, in general, the well-known textbook by Olver[21] while at the same time taking into account Refs. 22, 24, and 33. Two versions of the Hadamard lemma for fiber bundles, which play a crucial role for our further considerations, are formulated and proven in Sec. IV. Then we successively study conservation laws of general foliated systems (Sec. V), potential systems with two independent variables (Sec. VI), systems determining Abelian coverings (Sec. VII), standard and gauged potential systems in the multidimensional case (Sec. VIII), and general covering systems (Sec. IX). The criterion for purely potential conservation laws is formulated in Sec. X. Possible applications of the obtained results are illustrated by an example in the final section.

II. BASIC PROPERTIES OF CONSERVATION LAWS

Let $\mathcal{L}$ be a system $L(x,u_{(\rho)})=0$ of l differential equations $L^1=0,\dots,L^l=0$ for m unknown functions $u=(u^1,\dots,u^m)$ of n independent variables $x=(x_1,\dots,x_n)$. Here $u_{(\rho)}$ denotes the set of all the derivatives of the functions u with respect to x of order no greater than ρ, including u as the derivative of order zero. It is always assumed that the set of differential equations forming the system under consideration canonically represents this system and is minimal. The minimality of a set of equations means that no equation from this set is a differential consequence of the other equations. By $L_{(k)}$ we will always denote a maximal set of algebraically independent differential consequences of $\mathcal{L}$ that have, as differential equations, orders not greater than k. We identify $\mathcal{L}_{(k)}$ with the corresponding system of algebraic equations in $J^k(x|u)$ and associate it with the manifold $\mathcal{L}_{(k)}$ determined by this system.

Here $J^k(x|u)$ is the kth order jet space with the independent variables x and the dependent variables u. A smooth function defined on a subset of $J^k(x|u)$ for some k, i.e., depending on x and a finite number of derivatives of u, will be called a differential function of u. The notation $H[u]$ means that H is a differential function of u. See Ref. 21, for complete definitions.

For the manifold $\mathcal{L}_{(k)}$ to actually represent the system $\mathcal{L}$ of differential equations, the $\mathcal{L}$ have to be *locally solvable* in each point of $\mathcal{L}_{(k)}$. For the application of the Hadamard lemma to differential functions vanishing on the manifold $\mathcal{L}_{(k)}$, we need the system $L_{(k)}$ to be, as a system of algebraic equations defined in the jet space $J^k(x|u)$, *of maximal rank* in each point of $\mathcal{L}_{(k)}$. If for any k the system $\mathcal{L}$ satisfies both these conditions then it is called *totally nondegenerate*. (This definition slightly differs from that given in Ref. 21.)

For certain purposes, e.g., for different potential and pseudopotential frames, it is useful to introduce the more general notion of *weight* of differential variables instead of the order, which

takes into account the structure of the system of differential equations under consideration. Namely, for each variable of the infinite-order jet space $J^\infty(x|u)$ (being the inverse limit of the jet space tower $\{J^k(x|u), k\in\mathbb{N}\cup\{0\}\}$ with respect to the canonical projections $\pi^k: J^k(x|u) \to J^{k-1}(x|u)$, $k\in\mathbb{N}$) we define its weight ϱ by the rule,

$$\varrho(x_i)=0, \quad \varrho(u^a_\alpha)=\varrho_a+|\alpha|.$$

The weights $\varrho(u^a)=\varrho_a$ are defined on the basis of the structure of the system $\mathcal{L}$. (In subsequent sections we will provide concrete examples on how to specify the ϱ_a initially.) In what follows u^a_α stands for the variable in $J^\infty(x|u)$, corresponding to the derivative $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, $\alpha=(\alpha_1,\dots,\alpha_n)$ is an arbitrary multi-index, $\alpha_i\in\mathbb{N}\cup\{0\}$, $|\alpha|:=\alpha_1+\cdots+\alpha_n$. If $\varrho_a=0$ then the weight of u^a_α obviously coincides with the usual derivative order $|\alpha|$. We include in the *weighted jet space* $J^k_\varrho(x|u)$ the variables whose weight is not greater than k. The infinite-order jet space $J^\infty(x|u)$ is the inverse limit of the weighted jet space tower $\{J^k_\varrho(x|u), k\in\mathbb{N}\cup\{0\}\}$ with respect to the canonical projections $\pi^k_\varrho: J^k_\varrho(x|u)\to J^{k-1}_\varrho(x|u)$, $k\in\mathbb{N}$.

The technique of working with weights does not differ from the order technique and so a number of analogous notions can be introduced. Thus, the weight $\varrho(H)$ of any differential function $H[u]$ equals the maximal weight of variables explicitly appearing in H. The weight of the equation $H[u]=0$ equals $\varrho(H)$. A complete set of independent differential consequences of the system $\mathcal{L}$ which have weights not greater than k and the associated manifold in $J^k_\varrho(x|u)$ are denoted by the symbols $L_{[k]}=L_{[k],\varrho}$ and $\mathcal{L}_{[k]}=\mathcal{L}_{[k],\varrho}$, respectively. The system $\mathcal{L}$ is called *totally nondegenerate with respect to the weight* ϱ if for any $k\in\mathbb{N}$ it is locally solvable in each point of $\mathcal{L}_{[k]}$ and the algebraic system $L_{[k]}$ is of maximal rank in each point of $\mathcal{L}_{[k]}$. The Hadamard lemma can be applied, in the conventional way, to differential functions defined in $J^k_\varrho(x|u)$ and vanishing on $\mathcal{L}_{[k]}$.

We will explicitly indicate all places in which the usage of weighted jet spaces is essential. In the other places, the terminology involving orders is used although it can be replaced by that based on weights.

Definition 1: A *conserved vector* of the system $\mathcal{L}$ is an n-tuple $F=(F^1[u],\dots,F^n[u])$ for which the total divergence $\operatorname{Div} F:=D_iF^i$ vanishes for all solutions of $\mathcal{L}$, i.e., $\operatorname{Div} F|_{\mathcal{L}}=0$.

In Definition 1 and below $D_i=D_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i, i.e., $D_i=\partial_{x_i}+u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$, where δ_i is the multi-index whose ith entry equals 1 and whose other entries are zero. We use the summation convention for repeated indices and consider any function as its zero-order derivative. The indices i and j run from 1 to n, the index a runs from 1 to m, and the index s from 1 to p unless otherwise stated. The notation $V|_{\mathcal{L}}$ means that values of V are considered only on solutions of the system $\mathcal{L}$.

Heuristically, a conservation law of the system $\mathcal{L}$ is an expression $\operatorname{Div} F$ vanishing on the solutions of $\mathcal{L}$. The more rigorous definition of conservation laws given below is based on the factorization of the space of conserved vectors with respect to the subspace of trivial conserved vectors. Note that there is also a formalized definition of conservation laws of $\mathcal{L}$ as $(n-1)$-dimensional cohomology classes in the so-called horizontal de Rham complex on the infinite prolongation of the system $\mathcal{L}$.[9,28,29] The formalized definition is appropriate for certain theoretical considerations and reduces to the usual one after local coordinates are fixed.

Definition 2: A conserved vector F is called *trivial* if $F^i=\hat{F}^i+\check{F}^i$ where $\hat{F}^i$ and $\check{F}^i$ are, like F^i, differential functions of u, $\hat{F}^i$ vanishes on the solutions of $\mathcal{L}$ and the n-tuple $\check{F}=(\check{F}^1,\dots,\check{F}^n)$ is a null divergence (i.e., its divergence vanishes identically).

The triviality effected by conserved vectors vanishing on solutions of the system can easily be eliminated by restricting to the manifold of the system, taking into account all its relevant differential consequences. A characterization of all null divergences is given by the following theorem (see, e.g., Ref. 21, Theorem 4.24).

Theorem 2: *The n-tuple* $F=(F^1,\dots,F^n)$, $n\geqslant 2$, *is a null divergence* ($\operatorname{Div} F\equiv 0$) *if and only if there exist differential functions* $v^{ij}[u]$ *such that* $v^{ij}=-v^{ji}$ *and* $F^i=D_jv^{ij}$.

If $n=1$ any null divergence is constant.

Definition 3: Two conserved vectors F and F' are called *equivalent* if the tuple $F'-F$ is a trivial conserved vector.

The above definitions of triviality and equivalence of conserved vectors are natural in view of the usual "empiric" definition of conservation laws of a system of differential equations as divergences of its conserved vectors, i.e., divergence expressions which vanish for all solutions of this system. For example, equivalent conserved vectors correspond to the same conservation law. This allows us to formulate the definition of conservation law in a rigorous style (see, e.g., Ref. 33). Namely, for any system $\mathcal{L}$ of differential equations the set $\mathrm{CV}(\mathcal{L})$ of conserved vectors of its conservation laws is a linear space, and the subset $\mathrm{CV}_0(\mathcal{L})$ of trivial conserved vectors is a linear subspace in $\mathrm{CV}(\mathcal{L})$. The factor space $\mathrm{CL}(\mathcal{L})=\mathrm{CV}(\mathcal{L})/\mathrm{CV}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{CV}(\mathcal{L})$ with respect to the equivalence relation adduced in Definition 3.

Definition 4: The elements of $\mathrm{CL}(\mathcal{L})$ are called *(local) conservation laws* of the system $\mathcal{L}$, and the factor space $\mathrm{CL}(\mathcal{L})$ itself is called *the space of (local) conservation laws* of $\mathcal{L}$.

This is why we view the determination of the set of conservation laws of $\mathcal{L}$ as finding $\mathrm{CL}(\mathcal{L})$, which in turn is equivalent to constructing either a basis if $\dim \mathrm{CL}(\mathcal{L})<\infty$ or a system of generators in the infinite dimensional case. All elements of $\mathrm{CV}(\mathcal{L})$ which belong to the same equivalence class determining a conservation law $\mathcal{F}$ are considered as conserved vectors of this conservation law, and we will additionally identify elements from $\mathrm{CL}(\mathcal{L})$ with their representatives in $\mathrm{CV}(\mathcal{L})$. For $F\in\mathrm{CV}(\mathcal{L})$ and $\mathcal{F}\in\mathrm{CL}(\mathcal{L})$ the notation $F\in\mathcal{F}$ will mean that F is a conserved vector corresponding to the conservation law $\mathcal{F}$. In contrast to the order $\operatorname{ord} F$ of a conserved vector F as the maximal order of derivatives explicitly appearing in F, the *order* $\operatorname{ord}\mathcal{F}$ *of the conservation law* $\mathcal{F}$ is defined as $\min\{\operatorname{ord} F \,|\, F\in\mathcal{F}\}$. The notion of weight of a conservation law is introduced in the same way. By linear dependence of conservation laws we mean linear dependence as elements of $\mathrm{CL}(\mathcal{L})$. Therefore, in the framework of the "representative" approach conservation laws of a system $\mathcal{L}$ are considered *linearly dependent* if there exists a linear combination of their representatives which is a trivial conserved vector.

Substituting any solution u of $\mathcal{L}$ into any conserved vector F results in a null divergence depending only on x. Then the functions v^{ij} of x, introduced according to Theorem 2 and implicitly parametrized by u, are called *potentials* corresponding to the conserved vector F. The equations $D_j v^{ij}=F^i$ determine each potential v^{ij} up to the negligible summand $\breve{v}^{ij}$, where $\breve{v}^{ij}=-\breve{v}^{ji}$ and $D_j\breve{v}^{ij}=0$. Acting on the potentials, the gauge transformation $\tilde{v}^{ij}=v^{ij}+\breve{v}^{ij}$ has no influence on the corresponding tuple F. This gives constant and functional indeterminacies in the potentials if $n=2$ and $n\geqslant 3$, respectively.

Suppose that F and $\widetilde{F}$ are equivalent conserved vectors, i.e., there exist a null divergence $\breve{F}$ and a tuple $\hat{F}$ vanishing on the solutions of $\mathcal{L}$ such that $\widetilde{F}=F+\breve{F}+\hat{F}$. In view of Theorem 2 we can represent $\breve{F}$ in the form $\breve{F}^i=D_j\breve{v}^{ij}$ for some differential functions $\breve{v}^{ij}[u]=-\breve{v}^{ji}[u]$. Then the tuples of potentials (v^{ij}) and $(\tilde{v}^{ij})$, respectively, associated with the conserved vectors F and $\widetilde{F}$ are connected, up to negligible summands $\breve{v}^{ij}$, via the transformation $\tilde{v}^{ij}=v^{ij}+\breve{v}^{ij}[u]$ which allows us to assume that these tuples of potentials are equivalent. Therefore, we can say that the tuple (v^{ij}) (or $(\tilde{v}^{ij})$) of potentials is associated with the conservation law containing the conserved vectors F and $\widetilde{F}$.

III. CHARACTERISTICS OF CONSERVATION LAWS

Let the system $\mathcal{L}$ be totally nondegenerate. Then an application of the Hadamard lemma to the definition of conserved vector and integration by parts imply that the divergence of any conserved vector of $\mathcal{L}$ can always be represented, up to the equivalence relation of conserved vectors, as a linear combination of the left-hand sides of the independent equations from $\mathcal{L}$ with coefficients λ^μ which are functions on a suitable jet space $J^k(x\,|\,u)$,

$$\mathrm{Div}\, F = \lambda^{\mu} L^{\mu}. \tag{1}$$

Here the order k is determined by $\mathcal{L}$ and the order of F, $\mu = \overline{1,l}$. More precisely, the following statement is true.

Proposition 1: *For any conserved vector F of $\mathcal{L}$ there exist a tuple $\hat{F}=(\hat{F}^1[u],\dots,\hat{F}^n[u])$ vanishing on the solutions of $\mathcal{L}$ and differential functions $\lambda^{\mu}[u]$ such that*

$$\mathrm{Div}\, F = \lambda^{\mu} L^{\mu} + \mathrm{Div}\, \hat{F}.$$

If a tuple $F=(F^1[u],\dots,F^n[u])$ satisfies equality (1) for some differential functions $\lambda^{\mu}[u]$ then it obviously is a conserved vector of $\mathcal{L}$.

Definition 5: Formula (1) and the l-tuple $\lambda=(\lambda^1,\dots,\lambda^l)$ are called the *characteristic form* and the *characteristic* of the conservation law containing the conserved vector F, respectively.

The characteristic λ is *trivial* if it vanishes for all solutions of $\mathcal{L}$. Since $\mathcal{L}$ is nondegenerate, the characteristics λ and $\tilde{\lambda}$ satisfy (1) for the same F and, therefore, are called *equivalent* if and only if $\lambda-\tilde{\lambda}$ is a trivial characteristic. Similarly to conserved vectors, the set $\mathrm{Ch}(\mathcal{L})$ of characteristics corresponding to conservation laws of the system $\mathcal{L}$ is a linear space, and the subset $\mathrm{Ch}_0(\mathcal{L})$ of trivial characteristics is a linear subspace in $\mathrm{Ch}(\mathcal{L})$. The factor space $\mathrm{Ch}_{\mathrm{f}}(\mathcal{L})=\mathrm{Ch}(\mathcal{L})/\mathrm{Ch}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{Ch}(\mathcal{L})$ with respect to the above characteristic equivalence relation.

We should like to emphasize that the explicit form of characteristics depends on what set of equations is chosen for the canonical representation of the system $\mathcal{L}$.

The following result[21] forms the cornerstone for the methods of studying conservation laws, which are based on formula (1), including the Noether theorem and the direct method in the version by Anco and Bluman.[2,3]

Theorem 3: *Let $\mathcal{L}$ be a normal, totally nondegenerate system of differential equations. Then the representation of the conservation laws of $\mathcal{L}$ in the characteristic form (1) generates a linear isomorphism between* $\mathrm{CL}(\mathcal{L})$ *and* $\mathrm{Ch}_{\mathrm{f}}(\mathcal{L})$.

Using properties of total divergences, we can eliminate the conserved vector F from (1) and obtain a condition for the characteristic λ only. Namely, a differential function f is a total divergence, i.e., $f=\mathrm{Div}\, F$ for some n-tuple F of differential functions if and only if $\mathsf{E}(f)=0$. Here the Euler operator $\mathsf{E}=(\mathsf{E}_1,\dots,\mathsf{E}_m)$ is the m-tuple of differential operators

$$\mathsf{E}_a = (-D)^{\alpha}\partial_{u^a_{\alpha}}, \quad a=\overline{1,m},$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ runs through the multi-index set ($\alpha_i\in\mathbb{N}\cup\{0\}$), $(-D)^{\alpha}=(-D_1)^{\alpha_1}\cdots(-D_m)^{\alpha_m}$. Therefore, the action of the Euler operator on (1) results in the equation

$$\mathsf{E}(\lambda^{\mu}L^{\mu}) = \mathsf{D}^*_{\lambda}(L) + \mathsf{D}^*_{L}(\lambda) = 0, \tag{2}$$

which is a necessary and sufficient condition on characteristics of conservation laws for the system $\mathcal{L}$. The matrix differential operators D^*_{λ} and D^*_{L} are the adjoints of the Fréchet derivatives D_{λ} and D_{L}, i.e.,

$$\mathsf{D}^*_{\lambda}(L) = \left((-D)^{\alpha}\left(\frac{\partial\lambda^{\mu}}{\partial u^a_{\alpha}}L^{\mu}\right)\right), \quad \mathsf{D}^*_{L}(\lambda) = \left((-D)^{\alpha}\left(\frac{\partial L^{\mu}}{\partial u^a_{\alpha}}\lambda^{\mu}\right)\right).$$

Since $\mathsf{D}^*_{\lambda}(L)=0$ automatically on solutions of $\mathcal{L}$ then Eq. (2) implies a necessary condition for λ to belong to $\mathrm{Ch}(\mathcal{L})$,

$$\mathsf{D}^*_{L}(\lambda)\big|_{\mathcal{L}} = 0. \tag{3}$$

Condition (3) can be considered as adjoint to the criterion $\mathsf{D}_L(\eta)\big|_{\mathcal{L}}=0$ for infinitesimal invariance of $\mathcal{L}$ with respect to an evolutionary vector field with characteristic $\eta=(\eta^1,\dots,\eta^m)$. This is why solutions of (3) are sometimes called *cosymmetries*[25,8] or *adjoint symmetries*.[3]

For the investigation of the connection between characteristics and conserved vectors via formula (1), we need a statement on solutions of the equation $D_iF^i=H$, where $H=H[u]$ is a given differential function and the $F^i=F^i[u]$ are unknown [cf. formula (5.151) and Theorem 5.104 of Ref. 21].

Theorem 4: *Any solution $F=(F^1,\dots,F^n)$ of the equation $D_iF^i[u]=H[u]$ can be represented in the form $F=\check{F}+\widetilde{F}$, where the n-tuple $\check{F}[u]$ is a null divergence ($D_i\check{F}^i=0$) and the n-*tuple *$\widetilde{F}[u]$ is the particular solution of this equation whose components are given by*

$$\widetilde{F}^i=\int_0^1\frac{\alpha_i+1}{|\alpha|+1}D^\alpha(u^a\mathsf{E}_a^{\alpha+\delta_i}(H)[\kappa u])d\kappa+\int_0^1 x^iH(\kappa x,0,\dots,0)d\kappa.$$

Here E_a^α is the higher-order Euler operator acting on an arbitrary differential function $P[u]$ according to

$$\mathsf{E}_a^\alpha(P)=\sum_{\beta\geqslant\alpha}\frac{\beta!}{\alpha!(\beta-\alpha)!}(-D)^{\beta-\alpha}\frac{\partial P}{\partial u_\beta^a}.$$

Recall also that for any multi-index α with components $\alpha_1,\dots,\alpha_n\in\mathbb{N}\cup\{0\}$, we have $\alpha!:=\alpha_1!\cdots\alpha_n!$ and δ_i was introduced after Definition 1. The condition $\beta\geqslant\alpha$ for the multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$ means that $\beta_1\geqslant\alpha_1,\dots,\beta_n\geqslant\alpha_n$.

In fact, we need only a consequence of Theorem 4. It is easy to see that if the function H does not depend on the derivatives of u^a for a fixed value of a then the tuple $\widetilde{F}$ from Theorem 4 possesses the same property with the same value of a.

Corollary 1: *Let F be a conserved vector of a system $\mathcal{L}$, satisfying the equality $D_iF^i=H$, where the differential function $H[u]$ does not depend on the derivatives of $u^{a_1},\dots,u^{a_q}$ for fixed values $a_1,\dots,a_q$. Then the conserved vector F is equivalent to a conserved vector of $\mathcal{L}$ which does not depend on the derivatives of $u^{a_1},\dots,u^{a_q}$.*

IV. A HADAMARD LEMMA FOR FIBER BUNDLES

In this section we derive certain versions of the well-known Hadamard lemma (see, e.g., Ref. 21 Proposition 2.10) which will be needed in our further investigations. To this end we will employ the following notations: let $k,K,\kappa\in\mathbb{N}$. The index s will run from 1 to k, the index S from 1 to K and the index σ from 1 to κ. Let us also recall that the summation convention for summation over repeated indices is in effect.

To begin with we treat a rather elementary special case of the general result below in order to make the underlying ideas transparent and to single out a case of practical relevance. In both cases, we will use unified notations.

Suppose that B and N are manifolds. (Here N can also be a one-element set.) Denote the manifold $B\times N\times\mathbb{R}^\kappa$ by M. Consider the smooth functions $g:B\to\mathbb{R}^k$, $\zeta:B\times N\to\mathbb{R}^\kappa$ and $f:B\to\mathbb{R}$. We associate the function f with the function $\hat{f}:M\to\mathbb{R}$ defined by

$$\hat{f}(y,z',z'')=f(y)\quad\forall\,(y,z',z'')\in M.$$

Lemma 1: *Let $g:B\to\mathbb{R}^k$ be a mapping of maximal rank on the submanifold $B_g=\{y\in B\,|\,g(y)=0\}$. The function $\hat{f}$ vanishes on the submanifold*

$$M_{g,h}=\{(y,z',z'')\in M\mid g(y)=0,\ h(y,z',z'):=z''-\zeta(y,z')=0\}$$

if and only if there exists a smooth function $\lambda:B\to\mathbb{R}^k$ such that

$$f(y)=\lambda^s(y)g^s(y)\quad\forall\,y\in B.$$

Proof: Suppose that the function $\hat{f}$ vanishes on $M_{g,h}$. We fix an arbitrary point y_0 from B_g and some point z_0' from N and put $z_0''=\zeta(y_0,z_0')$. The point (y_0,z_0',z_0'') of M belongs to $M_{g,h}$ and hence

$f(y_0)=\hat{f}(y_0,z_0',z_0'')=0$. In other words, the function f vanishes on the entire submanifold B_g. Then the Hadamard lemma implies the desired result.

The converse statement is obvious. □

Lemma 1, in fact, deals with systems of algebraic equations on trivial fiber bundles, which are partitioned into two subsets of equations. For each appropriate system, the equations of the first subset are pullbacks of equations on the base B of the fiber bundle under consideration. The equations from the second set essentially depend on "fiber variables," i.e., any nonzero combination of them is not an equation of the first kind. Such a system can be called *trivially foliated* since the partition is the same for all points of the fiber bundle. In fact, we can weaken the condition of trivial foliation and demand for systems to have at least local representations as pairs of subsystems with the properties described.

Our next aim is to generalize this result to the general fiber bundle setting. To this end we first introduce some notation (cf., e.g., Ref. 11).

Consider a smooth fiber bundle (M,B,π,F), where M is the total space of the bundle, B the base space, F the fiber, and $\pi:M\to B$ the projection map. We write (U,φ) for the local trivializations (or fiber bundle charts) of the bundle M, $\pi^{-1}(U)\overset{\varphi}{\simeq}U\times F$. Any point $x\in\pi^{-1}(U)$ corresponds to the pair $(y,z)=\varphi(x)\in B\times F$, i.e., $y=\pi(x)=\mathrm{pr}_1(\varphi(x))\in B$ and $z=\mathrm{pr}_2(\varphi(x))\in F$.

Let $H:M\to\mathbb{R}^K$, $g:B\to\mathbb{R}^k$, and $f:B\to\mathbb{R}$ be smooth maps. $B_g=\{y\in B\,|\,g(y)=0\}$ and $M_H=\{x\in M\,|\,H(x)=0\}$ denote the set of solutions of the systems $g(y)=0$ and $H(x)=0$, respectively. We associate the functions f and g with their pullbacks $f\circ\pi:M\to\mathbb{R}$ and $g\circ\pi:M\to\mathbb{R}^k$ under π.

Lemma 2: *Suppose that $\pi(M_H)=B_g$ and $g:B\to\mathbb{R}^k$ has maximal rank on B_g. Then the function $f\circ\pi$ vanishes on M_H if and only if there exists a smooth map $\lambda:B\to\mathbb{R}^k$ such that*

$$f(y)=\lambda^s(y)g^s(y)\qquad\forall\, y\in B. \tag{4}$$

Proof: Suppose that the function $f\circ\pi$ vanishes on M_H. We fix an arbitrary point y_0 from B_g. The condition $\pi(M_H)\supset B_g$ implies that $M_H\cap\pi^{-1}(y_0)\neq\varnothing$. Let $x_0\in M_H\cap\pi^{-1}(y_0)$. Then $f(y_0)=f\circ\pi(x_0)=0$. In other words, the function f vanishes on the entire set B_g. In view of the Hadamard lemma we obtain equality (4).

Conversely, if the function f admits a representation of the form (4), it vanishes on B_g and, therefore, the function $f\circ\pi$ vanishes on $\pi^{-1}(B_g)\supset M_H$. □

Definition 6: Let the smooth maps $H:M\to\mathbb{R}^K$ and $g:B\to\mathbb{R}^k$ have maximal rank on M_H and B_g, respectively. The system $H(x)=0$ is called a *foliated system* over the *base system* $g(y)=0$ if $\pi(M_H)=B_g$.

Definition 6 can be reformulated in terms of a connection between the systems $H(x)=0$ and $g(y)=0$. This reformulation justifies the name foliated system.

Thus, the condition $\pi(M_H)\subset B_g$ is equivalent to the pullback of the system $g(y)=0$ being a consequence of the system $H(x)=0$. Indeed, the condition $\pi(M_H)\subset B_g$ is rewritten as $M_H\subset\pi^{-1}(B_g)$, i.e., the pullback $g\circ\pi$ vanishes on M_H. By the Hadamard lemma, under the condition of maximal rank of H on M_H there exist functions $\Lambda^{sS}:M\to\mathbb{R}$ such that $g^s\circ\pi(x)=\Lambda^{sS}(x)H^S(x)$. This implies that each of the equations $g^s(y)=0$ is a combination of equations of the system $H(x)=0$. Conversely, if the system $g\circ\pi(x)=0$ is a consequence of $H(x)=0$, it is obvious that $\pi(M_H)\subset B_g$.

The condition $\pi(M_H)\supset B_g$ means that for any solution y_0 of $g(y)=0$ there exists a solution x_0 of $H(x)=0$ with $\pi(x_0)=y_0$. Consider a function $f:B\to\mathbb{R}$ whose pullback $f\circ\pi$ vanishes on M_H. Then $f(y_0)=f\circ\pi(x_0)=0$. As a result, f vanishes on B_g and, since the function g is of maximal rank on B_g, in view of the Hadamard lemma we have $f(y)=\lambda^s(y)g^s(y)$ for some smooth functions $\lambda^s:B\to\mathbb{R}$, i.e., the equation $f(y)=0$ is combined from equations of the system $g(y)=0$.

The above arguments are summarized in the following statement.

Proposition 2: *Suppose that $g:B\to\mathbb{R}^k$ and $H:M\to\mathbb{R}^K$ are smooth mappings having maximal rank on the sets B_g and M_H, respectively. Then the system $H(x)=0$ is foliated over the base system $g(y)=0$ if and only if the pullback $g(\pi(x))=0$ of the system $g(y)=0$ (with respect to the projection π) is a consequence of the system $H(x)=0$ and for any solution y_0 of $g(y)=0$ there exists a solution*

x_0 of $H(x)=0$ such that $\pi(x_0)=y_0$. The foliation also implies that the projection of any combination of equations from the system $H(x)=0$, which is the pullback of an equation on B, is a consequence of the system $g(y)=0$.

Let $h:M\to\mathbb{R}^\kappa$ be a smooth map. Then by the *vertical rank* of h in $x\in M$ we mean the rank of the restriction of the tangent map T_xh of h to the vertical subspace of the tangent space T_xM of M at x. (This vertical subspace is just the tangent space of the fiber at x.) If (U,φ) is any trivialization around x and $\varphi(x)=(y,z)$, then the vertical rank of h at x is the rank of $\partial_z(h\circ\varphi^{-1})(y,z)$. After these preparations we may now state the following.

Theorem 5: *Suppose that the system $H(x)=0$ is foliated over the system $g(y)=0$, where $g:B\to\mathbb{R}^k$, $H:M\to\mathbb{R}^K$, $k\leqslant\dim B$, and $K\leqslant\dim M$. Suppose that $x_0\in M_H$, $y_0=\pi(x_0)$, and H is of constant vertical rank (denoted by κ) in a neighborhood of $\pi^{-1}(y_0)\cap M_H$ in M. Then $K=k+\kappa$ and in a neighborhood O_0 of x_0 in M the system $H(x)=0$ is equivalent to the united system of $g(\pi(x))=0$ and $h(x)=0$, where $h:O_0\to\mathbb{R}^\kappa$ is a smooth function with vertical rank κ.*

Proof: We choose a fiber bundle chart (U,φ) around y_0 and set $z_0=\mathrm{pr}_2(\varphi(x_0))$. Let $(y^1,\dots,y^n,z^1,\dots,z^m)$ be local coordinates in a neighborhood of (y_0,z_0) in $U\times F$, where $n=\dim B$ and $m=\dim F$. Then in what follows we may, in fact, suppose that B and F are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We introduce the notation $y''=(y^1,\dots,y^{K-\kappa})$, $y'=(y^{K-\kappa+1},\dots,y^n)$, $z''=(z^1,\dots,z^\kappa)$, and $z'=(z^{\kappa+1},\dots,z^m)$. Up to renumbering of the y- and z-variables we can assume that $\left|\partial(H\circ\varphi^{-1})/\partial(y'',z'')\right|\neq 0$ in the point (y_0,z_0). $H\circ\varphi^{-1}(y_0,z_0)=0$. In view of the implicit function theorem there exist neighborhoods V', V'', W', and W'' of y_0', y_0'', z_0', and z_0'' in the projections of $U\times F$ to the variables y', y'', z', and z'', respectively, and there exist smooth maps $\theta:V'\times W'\to V''$ and $\zeta:V'\times W'\to W''$ such that $H\circ\varphi^{-1}(y,z)=0$ in $\widetilde{O}=V''\times V'\times W''\times W'$ if and only if $y''=\theta(y',z')$ and $z''=\zeta(y',z')$. The derivative $\partial\theta/\partial z'$ identically vanishes since otherwise rank $\partial H/\partial z>\kappa$ for some points in $\widetilde{O}$, i.e., in fact, $\theta:V'\to V''$ and $y''=\theta(y')$. Note that $y_0''=\theta(y_0')$ and $z_0''=\zeta(y_0',z_0')$.

Since for any solution (in $V''\times V'$) of the system $y''=\theta(y')$ there exists a solution of the system $z''=\zeta(y',z')$ from $W''\times W'$ [e.g., $z'=z_0'$ and $z''=\zeta(y',z_0')$], then

$$\pi(M_H\cap O_0)=\{y\in\pi(O_0)|y''=\theta(y')\}\subset B_g\cap\pi(O_0),$$

where $O_0=\varphi^{-1}(\widetilde{O})$ and, therefore, $\pi(O_0)=V''\times V'$. Consequently, the set of projections of tangent vectors to M_H in the points from $\pi^{-1}(y_0)\cap M_H\cap O_0$ coincides with the tangent space to $\pi(M_H\cap O_0)$ in y_0, which has dimension $n-K+\kappa$.

As a result, for any $x\in\pi^{-1}(y_0)\cap M_H$ we can construct a neighborhood O of x in M such that $\pi(M_H\cap O)\subset B_g\cap\pi(O)$ and the set of projections of tangent vectors to M_H in the points from $\pi^{-1}(y_0)\cap M_H\cap O$ is an $n-K+\kappa$-dimensional vector space. It is possible to choose a finite or countable set $\{O_i\}$ of such neighborhoods covering $\pi^{-1}(y_0)\cap M_H$. Hence the set of projections of tangent vectors to M_H in the points from $\pi^{-1}(y_0)\cap M_H$ is at most a countable union of $n-K+\kappa$-dimensional vector spaces. At the same time, it has to coincide with the $n-k$-dimensional tangent space to B_g in y_0 since $\pi(M_H)=B_g$. This implies[1] that $k=K-\kappa$ and, therefore, $\pi(M_H\cap O_0)=B_g\cap\pi(O_0)$, i.e., in view of the Hadamard lemma the systems $y''=\theta(y')$ and $g(y)=0$ are equivalent on $\pi(O_0)$. Finally, the system $H(x)=0$ is equivalent to the combined system of $g(\pi(x))=0$ and $h(x)=0$ on O_0, where $h:O_0\to\mathbb{R}^\kappa$ is the smooth function defined by $h\circ\varphi^{-1}(y,z)=z''-\zeta(y',z')$ and hence having the vertical rank κ. □

Note 1: It follows from the proof of Theorem 5 that any foliated system (under the assumption of constant vertical rank of the associated mapping on the solution submanifold) locally has the structure of a trivially foliated system, as treated in Lemma 1.

[1]If $n-K+\kappa<n-k$ then the Lebesgue measure (in the tangent space $T_{y_0}B_g$) of each of the countably many $n-K+\kappa$-dimensional subspaces would be 0, contradicting the fact that their union is $T_{y_0}B_g$.

V. FOLIATED SYSTEMS OF DIFFERENTIAL EQUATIONS

All the potential frames over systems of differential equations investigated in the subsequent sections are particular cases of the more general notion of foliation of systems of differential equations.

Let $\bar{\mathcal{L}}$ be a system $\bar{L}(x,u_{(\bar{\rho})},v_{(\bar{\rho})})=0$ of $\bar{l}$ differential equations $\bar{L}^1=0,\dots,\bar{L}^{\bar{l}}=0$ for $m+p$ unknown functions $u=(u^1,\dots,u^m)$ and $v=(v^1,\dots,v^p)$ of n independent variables $x=(x_1,\dots,x_n)$. Let $\mathcal{L}$ be a system $L(x,u_{(\rho)})=0$ of l differential equations $L^1=0,\dots,L^l=0$ for only m unknown functions u.

For each $k\in\mathbb{N}\cup\{0\}$ we consider the projection $\varpi_k:J^k(x|u,v)\to J^k(x|u)$: $\varpi_k(x,u_{(k)},v_{(k)})=(x,u_{(k)})$. Any differential function $G=G[u]:J^k(x|u)\to\mathbb{R}$ is naturally associated with its pullback $G[u,v]\circ\varpi_k:J^k(x|u,v)\to\mathbb{R}$ under $\varpi_k:G\circ\varpi_k(x,u_{(k)},v_{(k)})=G(x,u_{(k)})$. It is also possible to consider the projection $\varpi:J^\infty(x|u,v)\to J^\infty(x|u)$ whose restriction to $J^k(x|u,v)$ coincides with ϖ_k and which induces pullbacks of differential functions of u of arbitrary (finite) order. Usually we will notationally suppress the pullback operation in what follows. In order to apply, in particular, the usual and extended characteristic forms of conservation laws and the Hadamard lemma, we suppose that both the systems $\mathcal{L}$ and $\bar{\mathcal{L}}$ are totally nondegenerate.

The definition of foliated systems of differential equations fits well into the general notion of foliation and the geometrical interpretation of systems of differential equations as manifolds in a jet space.

Definition 7: The system $\bar{\mathcal{L}}$ is called a *foliated system over the base system* $\mathcal{L}$ if both the systems $\mathcal{L}$ and $\bar{\mathcal{L}}$ are totally nondegenerate and $\varpi_k(\bar{\mathcal{L}}_{(k)})=\mathcal{L}_{(k)}$ for any $k\in\mathbb{N}$.

It is natural to denote the relation between $\bar{\mathcal{L}}$ and $\mathcal{L}$ by $\varpi\bar{\mathcal{L}}=\mathcal{L}$. Similarly to the algebraic case (cf. the previous section), Definition 7 admits a reformulation in terms of a connection between the systems $\bar{\mathcal{L}}$ and $\mathcal{L}$, which justifies the name foliated system. Namely, the system $\bar{\mathcal{L}}$ is foliated over the system $\mathcal{L}$ if and only if (the pullback of) each equation of $\mathcal{L}$ is a differential consequence of $\bar{\mathcal{L}}$ and for any local solution $u=u^0(x)$ of $\mathcal{L}$ there exist a local solution of the system $\bar{L}|_{u=u^0}=0$ in v. The foliation also implies that any differential consequence of $\bar{\mathcal{L}}$ which does not involve the functions v is (the pullback of) a differential consequence of $\mathcal{L}$. In terms of solution sets, the strip $u=u^0(x)$, where $u^0(x)$ is a fixed solution of $\mathcal{L}$, is the solution set of the system $\bar{L}(x,u^0_{(\bar{\rho})},v_{(\bar{\rho})})=0$.

Definition 8: The system $\bar{\mathcal{L}}$ is called a *strongly foliated system* over the base system $\mathcal{L}$ if $\bar{\mathcal{L}}$ is foliated over $\mathcal{L}$ and each of the equations minimally representing $\mathcal{L}$ can be included in a minimal set of equations forming $\bar{\mathcal{L}}$.

There exist foliated systems which are not strongly foliated. For example, the system $\bar{\mathcal{L}}$ formed by the equations $u^2_x=u^1$, $v_x=u^2$, and $v_t=u^1$ is foliated and not strongly foliated over the system $\mathcal{L}$ consisting of the equations $u^2_x=u^1$ and $u^2_t=u^1_x$. Indeed, the equation $u^2_t=u^1_x$ is a differential consequence of $\bar{\mathcal{L}}$ and cannot be included in the minimal set of equations representing $\bar{\mathcal{L}}$. The cross differentiation of the two last equations of $\bar{\mathcal{L}}$ is the unique way of excluding the derivatives of v from $\bar{\mathcal{L}}$. Therefore, any differential consequence of $\bar{\mathcal{L}}$ which does not involve the function v is a differential consequence of $\mathcal{L}$. This example is directly connected with the main subject of the paper since both the systems are potential systems of the $(1+1)$-linear heat equation, cf. systems (19) and (20) with the value $A=1$.

If $\bar{\mathcal{L}}$ is foliated over $\mathcal{L}$, we will assume that the maximally possible number $\hat{l}$ of equations of $\mathcal{L}$ is included in the minimal equation set forming and canonically representing $\bar{\mathcal{L}}$. Without loss of generality we can additionally assume that these equations are the first $\hat{l}$ equations in both of these systems. Such a representation of $\bar{\mathcal{L}}$ and $\mathcal{L}$ will be called a *canonical foliation* of $\bar{\mathcal{L}}$ over $\mathcal{L}$. The foliation is strong if and only if $\hat{l}=l$.

In the previous example we have $\hat{l}=1$ since the set of equations $u^2_x=u^1$, $v_x=u^2$, and $v_t=u^1$ canonically representing $\bar{\mathcal{L}}$ includes only one equation ($u^2_x=u^1$) from $\mathcal{L}$ and cannot include more equations from $\mathcal{L}$.

The pullback of any conserved vector of $\mathcal{L}$ under ϖ obviously is a conserved vector of $\bar{\mathcal{L}}$ which does not depend on derivatives of v. In view of Lemma 2, the converse statement is also true. Namely, any conserved vector of $\bar{\mathcal{L}}$ which does not depend on derivatives of v is the pullback of a conserved vector of $\mathcal{L}$ under ϖ. This justifies the following definition.

Definition 9: We say that a conservation law $\bar{\mathcal{F}}$ of $\bar{\mathcal{L}}$ is a pullback, with respect to ϖ, of a conservation law $\mathcal{F}$ of $\mathcal{L}$ (i.e., $\bar{\mathcal{F}}=\varpi^*\mathcal{F}$) or, in other words, is *induced* by this conservation law if there exists a conserved vector $\bar{F}\in\bar{\mathcal{F}}$ which is the pullback of a conserved vector $F\in\mathcal{F}$.

Using Definition 9, we can reformulate our results on the pullbacks of conserved vectors.

Proposition 3. *A conservation law $\bar{\mathcal{F}}$ of $\bar{\mathcal{L}}$ is induced by a conservation law $\mathcal{F}$ of $\mathcal{L}$ if and only if the conservation law $\bar{\mathcal{F}}$ contains a conserved vector which does not depend on derivatives of v. This conserved vector necessarily is the pullback of a conserved vector belonging to $\mathcal{F}$.*

Definition 10: Let $\bar{\mathcal{L}}$ be canonically foliated over $\mathcal{L}$. A tuple $\lambda=(\lambda^1[u,v],\dots,\lambda^{l+\bar{l}-\hat{l}}[u,v])$ is called an *extended characteristic* of a conservation law $\bar{\mathcal{F}}$ of $\bar{\mathcal{L}}$ if some conserved vector $\bar{F}\in\bar{\mathcal{F}}$ satisfies the condition

$$D_i\bar{F}^i=\sum_{\mu=1}^{l}\lambda^\mu L^\mu+\sum_{\nu=1}^{\bar{l}-\hat{l}}\lambda^{l+\nu}\bar{L}^{\hat{l}+\nu}. \tag{5}$$

The definition of usual characteristics involves the minimal set of equations canonically representing the system under consideration. In contrast to this, to define extended characteristics of a canonically foliated system, we extend this minimal set by the equations which canonically represent the base system and do not belong to the minimal set of equations of the foliated system.

Definition 11: We say that a usual or extended characteristic of $\bar{\mathcal{L}}$ is *induced* by a characteristic of $\mathcal{L}$ if the tuple of the characteristic components associated with the pullbacks of the equations of $\mathcal{L}$ is the pullback of the characteristic of $\mathcal{L}$ and the other characteristic components vanish.

If the extended characteristic λ is induced by a characteristic of $\mathcal{L}$, the defining equality (5) takes the form $D_i\bar{F}^i=\lambda^\mu[u]L^\mu[u]$, i.e., the total divergence of the associated conserved vector $\bar{F}$ is a function of only x and derivatives of u.

Theorem 6: *Let the system $\bar{\mathcal{L}}$ be canonically foliated with the base system $\mathcal{L}$. A conservation law of $\bar{\mathcal{L}}$ is induced by a conservation law of $\mathcal{L}$ if and only if it has an extended characteristic induced by a characteristic of $\mathcal{L}$.*

Proof: Suppose that $\bar{\mathcal{F}}$ is a conservation law of $\bar{\mathcal{L}}$, induced by a conservation law of $\mathcal{L}$. In view of Proposition 3, it contains a conserved vector $\bar{F}$ which does not depend on derivatives of v. The condition $D_i\bar{F}^i|_{\bar{\mathcal{L}}}=0$ means that the differential function $D_i\bar{F}^i$ [of order $r\leqslant\operatorname{ord}(\bar{F}^1,\dots,\bar{F}^n)+1$] vanishes on the manifold $\bar{\mathcal{L}}_{(r)}$ determined in the jet space $J^r(x|u,v)$ by the system $\bar{\mathcal{L}}$ and its differential consequences. Since $\bar{\mathcal{L}}$ is foliated over $\mathcal{L}$ then $\varpi_r(\bar{\mathcal{L}}_{(r)})=\mathcal{L}_{(r)}$. In view of Lemma 2 there exist functions $\check{\lambda}^{\check{\mu}}$ of only x and derivatives of u up to order r such that $D_i\bar{F}^i=\check{\lambda}^{\check{\mu}}\check{L}^{\check{\mu}}$. Here the equations $\check{L}^{\check{\mu}}=0$, $\check{\mu}=1,\dots,\check{l}$, form a corresponding set of differential consequences of the system $\mathcal{L}$ which have, as differential equations, order not greater than r. Following the conventional way of deriving the characteristic form of conservation laws,[21] we integrate by parts on the right-hand side of the last equality and obtain $D_i\tilde{F}^i=\lambda^\mu L^\mu$, where $\tilde{F}^i$ and λ^μ are functions of x and derivatives of u. The conserved vectors $\bar{F}$ and $\tilde{F}$ are equivalent since their difference vanishes on $\mathcal{L}$. That is why the tuple $(\lambda^1[u],\dots,\lambda^l[u])$ is a characteristic of the system $\mathcal{L}$, associated with the conserved vector $\tilde{F}$ which belongs to the conservation law of $\mathcal{L}$, inducing $\bar{\mathcal{F}}$. Therefore, the tuple

$(\lambda^1[u],\dots,\lambda^l[u],\lambda^{l+1}=0,\dots,\lambda^{l+\bar{l}-\hat{l}}=0)$ is an extended characteristic of the foliated system $\bar{\mathcal{L}}$, associated with the conservation law $\bar{\mathcal{F}}$ and induced by the characteristic $(\lambda^1[u],\dots,\lambda^l[u])$ of the base system $\mathcal{L}$.

Conversely, let the tuple $(\lambda^1,\dots,\lambda^{l+\bar{l}-\hat{l}})$ be an extended characteristic of the foliated system $\bar{\mathcal{L}}$ associated with the conservation law $\bar{\mathcal{F}}$, induced by the characteristic $(\lambda^1,\dots,\lambda^l)$ of the base system $\mathcal{L}$. This means that $\lambda^1=\lambda^1[u],\dots,\lambda^l=\lambda^l[u]$, $\lambda^{l+1}=0,\dots,\lambda^{l+\bar{l}-\hat{l}}=0$, and there exists a conserved vector $\bar{F}=\bar{F}[u,v]\in\bar{\mathcal{F}}$ such that $D_i\bar{F}^i=\lambda^\mu L^\mu$. Since the right-hand side $\lambda^\mu L^\mu$ depends only on x and derivatives of u, the equality $D_i\bar{F}^i=\lambda^\mu L^\mu$ implies in view of Corollary 1 that there exists a conserved vector $\tilde{F}$ of $\bar{\mathcal{L}}$, which depends only on x and derivatives of u, is equivalent to the conserved vectors $\bar{F}$, and, therefore, belongs to $\bar{\mathcal{F}}$. This in turn shows in view of Proposition 3 that the conservation law $\bar{\mathcal{F}}$ is induced by a conservation law of the base system $\mathcal{L}$. □

The proof of Theorem 6 also implies the following statement.

Corollary 2: *An extended characteristic of $\bar{\mathcal{L}}$ is induced by a characteristic of $\mathcal{L}$ if the tuple of the characteristic components associated with the pullbacks of equations of $\mathcal{L}$ does not depend on derivatives of v and the other characteristic components vanish.*

In the general case the equality $D_i\tilde{F}^i=\lambda^\mu L^\mu$ is not a characteristic form of the conservation law of $\bar{\mathcal{L}}$, containing the conserved vector F, since some equations canonically representing $\mathcal{L}$ may lie outside of the canonical foliation $\bar{\mathcal{L}}$. The strong foliation guarantees the inclusion of all the equations $L^1=0,\dots,L^l=0$ in the canonical foliation.

Corollary 3: *A conservation law of the canonically strongly foliated system $\bar{\mathcal{L}}$ is induced by a conservation law of the base system $\mathcal{L}$ if and only if it has a characteristic induced by a characteristic of $\mathcal{L}$.*

VI. THE TWO-DIMENSIONAL CASE

In this section we first derive our results for the case of two independent variables to explain some necessary notions and ideas of the proof clearly. Moreover, this case is special, in particular, with respect to a possible (constant) indeterminacy after the introduction of potentials and due to the high effectiveness of the application of potential symmetries. Only in this case the introduction, according to Theorem 2, of potentials with an arbitrary finite set of conservation laws results in an Abelian covering of the system under consideration, and any Abelian covering can be obtain in this way.

We denote the independent variables by t and x. A conserved vector of the system $\mathcal{L}$ in two independent variables t and x is a pair (F,G) of functions depending on t, x, and a (finite) number of derivatives of u, whose total divergence vanishes for all solutions of $\mathcal{L}$, i.e., $(D_tF+D_xG)|_{\mathcal{L}}=0$. Here D_t and D_x are the operators of total differentiation with respect to t and x, respectively. The components F and G are called the *conserved density* and the *flux* of the conserved vector (F,G). Two conserved vectors (F,G) and (F',G') are *equivalent* and, therefore, associated with the same conservation law if there exist functions $\hat{F}$, $\hat{G}$, and H of t, x and derivatives of u such that $\hat{F}$ and $\hat{G}$ vanish on $\mathcal{L}_{(k)}$ for some k and $F'=F+\hat{F}+D_xH$, $G'=G+\hat{G}-D_tH$.

Any conserved vector (F,G) of $\mathcal{L}$ allows one to introduce the new dependent (potential) variable v by means of the equations

$$v_x=F,\quad v_t=-G. \tag{6}$$

To construct several potentials in one step, we have to use a set of conserved vectors associated with linearly independent conservation laws since otherwise the potentials will be dependent in the following sense: there exists a linear combination of the potentials, which is, up to a negligible constant summand, a differential function of u only (see Proposition 4 below). In the case of two independent variables we can also introduce the more general notion of potential dependence.[22]

Let $v^1,\dots,v^p$ be potentials of the system $\mathcal{L}$. By $\mathcal{L}_{\rm p}$ we denote the combined system of $\mathcal{L}$ and the equations determining the set of potentials $v^1,\dots,v^p$.

Definition 12: The potentials $v^1,\dots,v^p$ are called *dependent on the solution set of the system* $\mathcal{L}$ (or, for short, *dependent*) if there exist $r'\in\mathbb{N}$ and a function Ω of the variables $t,x,u_{(r')},v^1,\dots,v^p$ such that $\Omega_{v^s}\neq 0$ for some s, $1\leqslant s\leqslant p$, and $\Omega(t,x,u_{(r')},v^1,\dots,v^p)=0$ for any solution $(u,v^1,\dots,v^p)$ of $\mathcal{L}_{\rm p}$ (up to gauge transformations, i.e., up to adding constants to the potentials).

A proof of local dependence or independence of potentials seems rather hopeless for general classes of differential equations since it is closely connected with a precise description of the structure of the associated conservation laws. Examples of such proofs for particular classes of differential equations (diffusion-convection equations and linear parabolic equations) were presented in Refs. 22 and 24.

Proposition 4: *If conserved vectors of the system* $\mathcal{L}$ *belong to linearly dependent conservation laws then the associated potentials are locally dependent on the solution set of* $\mathcal{L}$.

Proof: Let (F^s,G^s), $s=1,\dots,p$, be conserved vectors of $\mathcal{L}$ such that the corresponding conservation laws are linearly dependent. This means that $c_sF^s=\hat{F}+D_xH$, $c_sG^s=\hat{G}-D_tH$ for some constants c_s and some functions $\hat{F}$, $\hat{G}$, and H of t, x, and derivatives of u, where $\hat{F}$ and $\hat{G}$ vanish on $\mathcal{L}_{(k)}$ for some k. For each s, the potential v^s associated with the conserved vector (F^s,G^s) satisfies the equations $v^s_x=F^s$ and $v^s_t=-G^s$. Therefore, $c_sv^s_x=D_xH+\hat{F}$ and $c_sv^s_t=D_tH-\hat{G}$, i.e., $c_sv^s-H=c=$const on the solution set of $\mathcal{L}$. As a result, we obtain that the potentials v^s are locally dependent with $\Omega=c_sv^s-H$. (The constant c is negligible up to gauge transformations of the potentials.) □

Proposition 5: *Suppose that two tuples of potentials are associated with tuples of conserved vectors which are equivalent in the following sense: Any conserved vector of each tuple is equivalent to a linear combination of conserved vectors from the other tuple. Then either both these tuples of potentials are locally dependent or both are locally independent on the solution set of the system* $\mathcal{L}$. *Any potential from each of the tuples is a linear combination of potentials from the other tuple with an additional summand which is a differential function of the dependent variables of the initial system.*

It is natural to call tuples of potentials satisfying the conditions of Proposition 5 *equivalent*. Proposition 5 implies that, up to the equivalence of tuples of potentials, any potential system is associated with a subspace of the space of conservation laws of the initial system and does not depend on the choice of a basis in this subspace or of the conserved vectors representing the basis elements.

In the case of a single equation $\mathcal{L}$, pairs of equations of the form (6) combine into the complete potential system if at least one of them is associated with a nonsingular characteristic (since in this case $\mathcal{L}$ is a differential consequence of this pair). As a rule, systems of this kind admit a number of nontrivial symmetries and hence are of great interest. Note that in the case $l=1$ the characteristic $\lambda=\lambda[u]$ is called singular if the differential equation $\lambda[u]=0$ has a solution $u=u(x)$. The importance of distinguishing between singular and nonsingular characteristics was emphasized by Bluman.[5]

Suppose that the system $\mathcal{L}$ has p linearly independent local conservation laws with conserved vectors (F^s,G^s), $s=1,\dots,p$. We introduce the potentials $v^1,\dots,v^p$ associated with this tuple of conserved vectors by the formulas

$$v^s_x=F^s[u],\quad v^s_t=-G^s[u], \tag{7}$$

assuming additionally that these potentials are locally independent on the solution set of the system $\mathcal{L}$. The corresponding potential system $\mathcal{L}_{\rm p}$ is canonically represented by the potential part (7) and a selection of the equations of the system $\mathcal{L}$ excluding a subset of equations which are not differential consequences of (7) and the other equations of $\mathcal{L}$, taken together. This representation is a canonical foliation of the system $\mathcal{L}_{\rm p}$ over the system $\mathcal{L}$. Below the index ν runs through the set $\mathcal{N}$ of the numbers of the equations from $\mathcal{L}$ which are in the canonical representation of $\mathcal{L}_{\rm p}$. The

index ν' runs through the set $\mathcal{N}'=\{1,\dots,l\}\setminus\mathcal{N}$. Note that the total number of such equations is equal to or greater than $l-p$ but is not necessarily equal to $l-p$.

By what was said after Proposition 5, tuples $v=(v^1,\dots,v^p)$ and $\tilde{v}=(\tilde{v}^1,\dots,\tilde{v}^p)$ of potentials associated with the same p-dimensional subspace of the conservation law space $\mathrm{CL}(\mathcal{L})$ of $\mathcal{L}$ are equivalent. In other words, the tuples v and $\tilde{v}$ of potentials are called equivalent if there exist differential functions $\Phi^s[u]$ and constants $c_{s\sigma}$ such that $|c_{s\sigma}|\neq 0$ and the transformation $\tilde{v}^s=c_{s\sigma}v^\sigma+\Phi^s[u]$ (the variables x and derivatives of u are not transformed) maps the system $\mathcal{L}_\mathrm{p}$ associated with v to the system $\tilde{\mathcal{L}}_\mathrm{p}$ associated with $\tilde{v}$. The tuples $(F^s,G^s,s=1,\dots,p)$ and $(\tilde{F}^s,\tilde{G}^s,s=1,\dots,p)$ from the potential parts of these systems are connected by the formulas $(\tilde{F}^s-c_{s\sigma}F^\sigma-D_t\Phi^s)|_{\mathcal{L}}=0$ and $(\tilde{G}^s-c_{s\sigma}G^\sigma+D_x\Phi^s)|_{\mathcal{L}}=0$. We will also say that the systems $\mathcal{L}_\mathrm{p}$ and $\tilde{\mathcal{L}}_\mathrm{p}$ are *equivalent as potential systems* of the system $\mathcal{L}$.

In order to use the characteristic form (1) of conservation laws, we need for the systems $\mathcal{L}$ and $\mathcal{L}_\mathrm{p}$ to be totally nondegenerate in some sense. In the general case, it is difficult to derive the total nondegeneracy of $\mathcal{L}_\mathrm{p}$ in the usual sense[21] from the corresponding property of $\mathcal{L}$. That is why we use the following trick based on the special structure of the potential part (7) of $\mathcal{L}_\mathrm{p}$. For any $k\in\mathbb{N}\cup\{0\}$ we replace the usual jet spaces $J^k(x|u)$ and $J^k(x|u,v)$ by the weighted jet space $J^k_\varrho(x|u)$ with a predefined weight ϱ and the weighted jet space $J^k_\varrho(x|u,v)$ in which the weight ϱ is extended to the derivatives of the potentials v according to the rule

$$\varrho(v^s_\alpha)=\max(0,\varrho(F^s)-1,\varrho(G^s)-1)+|\alpha|.$$

Note that this rule is not the only possible choice. There are a number of different ways for this extension. The main rule for weighting the potentials is that the weights of the left-hand sides of Eq. (7) have to be greater than or equal to the weights of the corresponding right-hand sides. Recall that the weight $\varrho(H)$ of any differential function H equals the maximal weight of the variables explicitly appearing in H. For the extension of the weight ϱ to be canonical (up to permutation of potentials) in the class of potential systems equivalent to $\mathcal{L}_\mathrm{p}$, we have to choose one of the equivalent tuples of potentials which has the minimal value of $\Sigma_s\varrho(v^s)$. The consideration of the preweighted space $J^k_\varrho(x|u)$ is necessary for the investigation of hierarchies of potential systems since the system $\mathcal{L}$ itself may be a potential system of a system with respect to a part of the unknown functions u^a, with the other u's as potentials of the previous level. The first step in this recursive procedure is carried out by assigning the weight 0 to all variables u of any initial system $\mathcal{L}$ in a hierarchy of potential systems.

A complete set $L_{\mathrm{p}[k]}$ of independent differential consequences of the system $\mathcal{L}_\mathrm{p}$ which have extended weights not greater than k is exhausted by the equations

$$\breve{L}^{\breve{\mu}}=0,\quad \breve{\mu}=1,\dots,\breve{l},\quad v^s_{(0,j'+1)}=D_x^{j'}F^s,\quad v^s_{(i+1,j)}=-D_t^iD_x^jG^s.$$

Here the equations $\breve{L}^{\breve{\mu}}=0$, $\breve{\mu}=1,\dots,\breve{l}$, form a complete set $L_{[k]}$ of independent differential consequences of the system $\mathcal{L}$, which have weights not greater than k, and $v^s_{(i,j)}=\partial^{i+j}v^s/\partial t^i\partial x^j$, $i,j\geqslant 0$. For each s the indices j' and (i,j) run through the sets in which $i,j,j'\geqslant 0$, $\varrho(v^s)+j'<k$ and $\varrho(v^s)+i+j<k$.

It is obvious that for any $k\in\mathbb{N}$ the system $L_{\mathrm{p}[k]}$ is of maximal rank on the manifold $\mathcal{L}_{\mathrm{p}[k]}$ in the weighted jet space $J^k_\varrho(x|u,v)$ if and only if the system $L_{[k]}$ is of maximal rank on the manifold $\mathcal{L}_{[k]}$. The local solvability of $\mathcal{L}_\mathrm{p}$ follows from the local solvability of $\mathcal{L}$ and the compatibility conditions for the potential part and implies the local solvability of $\mathcal{L}$ since $\mathcal{L}$ is a subsystem of $\mathcal{L}_\mathrm{p}$.

As a result, we have the following statement.

Lemma 3: *The system $\mathcal{L}$ is totally nondegenerate with respect to a weight if and only if the potential system $\mathcal{L}_\mathrm{p}$ is totally nondegenerate with respect to this weight extended to the derivatives of the potentials.*

Moreover, $\varpi_{[k]}(\mathcal{L}_{\mathrm{p}[k]})=\mathcal{L}_{[k]}$ for any $k\in\mathbb{N}$ since $L_{\mathrm{p}[k]}$ is a trivially foliated system of algebraic equations with the base system $L_{[k]}$. Therefore, two-dimensional potential systems form a particular case of foliated systems of differential equations and all statements of Sec. V are true for conservation laws of such systems. (Only in the proof of Theorem 6 the orders and usual jet spaces have to be replaced by the weights of the same objects and weighted jet spaces, respectively.) At the same time, due to their special structure stronger statements on the connection between conservation laws induced by conservation laws of the corresponding initial systems and the locality of the associated characteristics can be proven.

Lemma 4: *If a characteristic of a two-dimensional potential system depends only on local variables (i.e., independent and nonpotential dependent ones), then the associated conservation law of the potential system has a conserved vector which also does not depend on potentials.*

Proof: Suppose that the potential system $\mathcal{L}_{\mathrm{p}}$ possesses a characteristic

$$(\alpha^s,\ \beta^s,\ \gamma^\nu,\ s=1,\dots,p,\ \nu\in\mathcal{N}),$$

which does not depend on the potentials $v^1,\dots,v^p$. [Due to system (7) the dependence of the characteristic on nonzero-order derivatives of the potentials can be neglected up to the equivalence relation of characteristics.] The components α^s, β^s, and γ^ν are functions of t,x and derivatives of u and correspond to the equations $v^s_t=-G^s$, $v^s_x=F^s$, and $L^\nu=0$, respectively. Therefore, there exists a conserved vector (F,G) of the potential system $\mathcal{L}_{\mathrm{p}}$ such that

$$D_tF+D_xG=\alpha^s(v^s_t+G^s)+\beta^s(v^s_x-F^s)+\gamma^\nu L^\nu=:V. \tag{8}$$

Since the differential function V of t, x, and derivatives of u and v is a total divergence then the value of the extended Euler operator $\mathsf{E}=(\mathsf{E}_{u^1},\dots,\mathsf{E}_{u^m},\mathsf{E}_{v^1},\dots,\mathsf{E}_{v^p})$ on V is the zero $m+p$-tuple. In particular,

$$-\mathsf{E}_{v^s}V=D_t\alpha^s+D_x\beta^s=0,$$

i.e., the tuple $(\alpha^s[u],\beta^s[u])$ is a null divergence. In view of Theorem 2 on null divergences, for each s there exists a differential function $\Phi^s[u]$ such that $\alpha^s=D_x\Phi^s$ and $\beta^s=-D_t\Phi^s$. We set

$$\hat F=F+\Phi^s(v^s_x-F^s),\quad \hat G=G-\Phi^s(v^s_t+G^s).$$

Then Eq. (8) can be rewritten as

$$D_t\hat F+D_x\hat G=\Phi^sD_t(v^s_x-F^s)-\Phi^sD_x(v^s_t+G^s)+\gamma^\nu L^\nu=-\Phi^s(D_tF^s+D_xG^s)+\gamma^\nu L^\nu,$$

and the conserved vector $(\hat F,\hat G)$ is equivalent to the initial conserved vector (F,G). The right-hand side of the last equality is a differential function of u and vanishes on the manifold $\mathcal{L}_{[k]}$ of the jet space $J^k_{\mathcal{Q}}(x|u)$, where k is the highest weight of the variables in this expression. Using the Hadamard lemma and "integration by parts" as in deriving the general characteristic form of conservation laws, we obtain that

$$D_t\check F+D_x\check G=\check\gamma^\mu L^\mu \tag{9}$$

for some differential functions $\check\gamma^\mu[u]$, where the conserved vector $(\check F,\check G)$ is equivalent to $(\hat F,\hat G)$ and, therefore, to (F,G) since it differs from $(\hat F,\hat G)$ on a tuple vanishing on the solution set of $\mathcal{L}$. Since the right-hand side $\check\gamma^\mu L^\mu$ depends only on t,x and derivatives of u, equality (9) implies in view of Corollary 1 that there exists a conserved vector $(\tilde F,\tilde G)$ of $\mathcal{L}_{\mathrm{p}}$, which depends only on t,x and derivatives of u and is equivalent to the conserved vectors $(\check F,\check G)$ and, therefore, (F,G). □

Lemma 5: *If an extended characteristic of a two-dimensional potential system is induced by a characteristic of the corresponding initial system then the associated conservation law of the potential system has a characteristic which does not depend on potentials.*

Proof: Suppose that the potential system $\mathcal{L}_{\mathrm{p}}$ possesses an extended characteristic induced by

a characteristic λ of the initial system $\mathcal{L}$, i.e., there exists a conserved vector (F,G) of $\mathcal{L}_{\rm p}$ such that $D_tF+D_xG=\lambda^\mu[u]L^\mu[u]$. In the general case this equality is not a characteristic form of the conservation law of $\mathcal{L}_{\rm p}$ containing the conserved vector (F,G), since some equations of $\mathcal{L}$ can fall out of the minimal set of equations forming the potential system $\mathcal{L}_{\rm p}$. The indices of such equations form the set $\mathcal{N}'=\{\nu'\}$. If $\mathcal{N}'=\varnothing$, we at once have a characteristic form.

Let $\mathcal{N}'\neq\varnothing$. We represent each $L^{\nu'}$ as a differential consequence of $\mathcal{L}_{\rm p}$. In view of Lemma 1, this representation has the form

$$L^{\nu'}=A^{\nu'\nu}L^\nu+B^{\nu's}(D_tF^s+D_xG^s),$$

where $A^{\nu'\nu}$ and $B^{\nu's}$ are polynomials of the total differentiation operators D_t and D_x with coefficients depending on t, x, and derivatives of u. Note that $D_tF^s+D_xG^s=D_x(v^s_t+G^s)-D_t(v^s_x-F^s)$. Therefore,

$$D_tF+D_xG=\lambda^\nu L^\nu+\lambda^{\nu'}A^{\nu'\nu}L^\nu+\lambda^{\nu'}B^{\nu's}D_x(v^s+G^s)-\lambda^{\nu'}B^{\nu's}D_t(v^s_x-F^s).$$

Integrating by parts on the right-hand side leads to the equality

$$D_t\widetilde{F}+D_x\widetilde{G}=\alpha^s(v^s_t+G^s)+\beta^s(v^s_x-F^s)+\gamma^\nu L^\nu,$$

where $\alpha^s=-D_xB^{s\nu'}{}^*\lambda^{\nu'}$, $\beta^s=D_tB^{s\nu'}{}^*\lambda^{\nu'}$, and $\gamma^\nu=\lambda^\nu+A^{\nu\nu'}{}^*\lambda^{\nu'}$ are functions of t, x and derivatives of u, and $A^{\nu\nu'}{}^*$ and $B^{s\nu'}{}^*$ denote the formally adjoint operators to $A^{\nu'\nu}$ and $B^{\nu's}$, respectively. The conserved vectors (F,G) and $(\widetilde{F},\widetilde{G})$ are equivalent since their difference vanishes on $\mathcal{L}_{\rm p}$.

Finally, we construct the characteristic $(\alpha^s,\beta^s,\gamma^\nu,s=1,\dots,p,\nu\in\mathcal{N})$ of the conservation law with the conserved vector (F,G), which depends only on t, x and derivatives of u. □

Proposition 3, Theorem 6, and Lemmas 4 and 5 can now be combined into the following result.

Theorem 7: *The following statements on a conservation law of a two-dimensional potential system are equivalent.*

(1) The conservation law is induced by a conservation law of the corresponding initial system.
(2) It contains a conserved vector which does not depend on potentials.
(3) Some of its extended characteristics are induced by characteristics of the initial system.
(4) It possesses a characteristic not depending on potentials.

Note 2: If $\mathcal{L}_{\rm p}$ and $\widetilde{\mathcal{L}}_{\rm p}$ are equivalent as potential systems of the system $\mathcal{L}$, then the corresponding equivalence transformation maps any conservation law of $\mathcal{L}_{\rm p}$, possessing the locality properties (1)–(4) of Theorem 7, to a conservation law of $\widetilde{\mathcal{L}}_{\rm p}$ with the same properties. In other words, the locality properties of conservation laws are stable with respect to the equivalence of potential systems.

Note 3: Although the general version of the Hadamard lemma for fiber bundles (Lemma 2) is used in the proof of Theorem 6 involved in deriving Theorem 7, in fact, the simplest version of this lemma (Lemma 1) is sufficient due to the special foliation structure of two-dimensional potential systems to directly prove Theorem 7. The same observation is true for Abelian coverings and standard potential systems without gauges in the multidimensional case.

Consider a *tower* $\{\mathcal{L}^k_{\rm p},k\in K\}$ *of potential systems* over the system $\mathcal{L}=\mathcal{L}^0_{\rm p}$. (Here either $K=\{0,\dots,N\}$ for some $N\in\mathbb{N}$ or $K=\mathbb{N}\cup\{0\}$.) This means that for any $k\in K\setminus\{0\}$ the system $\mathcal{L}^k_{\rm p}$ is a potential system of $\mathcal{L}^{k-1}_{\rm p}$. The system $\mathcal{L}^k_{\rm p}$ will be called a kth *level potential system* associated with $\mathcal{L}$. We will say that the potential system $\mathcal{L}^k_{\rm p}$ is *strictly of* kth *level* if it cannot be included as a potential system of a lower level in another tower of potential systems over $\mathcal{L}$. For any $k,k'\in K$, where $k'<k$, the system $\mathcal{L}^k_{\rm p}$ is foliated over the system $\mathcal{L}^{k'}_{\rm p}$.

A conservation law of a potential system of kth level is a kth level potential conservation law of $\mathcal{L}$. A conservation law of a potential system which is strictly of kth level and is not induced by a conservation law of lower level is called a *potential conservation law which is strictly of* kth

level. A potential in a tower of potential systems is *strictly of k*th *level* if it is introduced with a conservation law which is strictly of $(k-1)$st level. By linearly combining potentials and lowering their levels as far as possible, any finite tower of potential systems over $\mathcal{L}$ can be transformed to a tower in which for any k the dependent variables of $\mathcal{L}_{\rm p}^{k}$, complementary to the dependent variables of $\mathcal{L}_{\rm p}^{k-1}$, are potentials which are strictly of $(k-1)$st level. Another approach to ordering towers of potential systems is to consider only one-dimensional extensions of the spaces of dependent variables for each step between levels (see, e.g., Ref. 20).

An iterative application of Theorem 7 to towers of potential systems implies two statements on potential conservation laws (in terms of a fixed tower and in terms of levels, respectively).

Corollary 4: *Let* $\{\mathcal{L}_{\rm p}^{k},k\in K\}$ *be a tower of potential systems over the system* $\mathcal{L}$ *with two independent variables. For any* $k\in K\backslash\{0\}$ *the following statements on a conservation law of* $\mathcal{L}_{\rm p}^{k}$ *are equivalent*.

(1) *The conservation law is induced by a conservation law of* $\mathcal{L}_{\rm p}^{k'}$ *for some* $k'<k$.
(2) *It contains a conserved vector depending only on variables appearing in* $\mathcal{L}_{\rm p}^{k'}$ *and derivatives involving them.*
(3) *Some of its extended characteristics are induced by characteristics of* $\mathcal{L}_{\rm p}^{k'}$.
(4) *It possesses a characteristic which does not depend on potentials complementary to the dependent variables of* $\mathcal{L}_{\rm p}^{k'}$.

Corollary 5: *The following statements on a k*th *level potential conservation law of a two-dimensional system are equivalent*.

(1) *The conservation law is induced by a conservation law of a lower level.*
(2) *It contains a conserved vector which does not depend on potentials whose strict levels are greater than* $k-1$.
(3) *Some of its extended characteristics are induced by characteristics of a potential system of a lower level.*
(4) *It possesses a characteristic not depending on potentials with strict levels greater than* $k-1$.

VII. ABELIAN COVERINGS

There are two ways to directly generalize the above results for the two-dimensional case to the multidimensional case. One of them deals with so-called Abelian coverings[20] and the other is based on the introduction of potentials according to Theorem 2. In this section we consider Abelian coverings (in the local approach, cf. the remark preceding Definition 13 below).

Suppose that the system $\mathcal{L}$ admits p potentials $v^1,\dots,v^p$ defined by the equation,

$$v_i^s = G^{si}[u], \tag{10}$$

where the differential functions $G^{si}=G^{si}[u]$ satisfy the compatibility conditions $D_jG^{si}=D_iG^{sj}$ on the solution set of the system $\mathcal{L}$. The corresponding potential system $\mathcal{L}_{\rm p}$ is canonically represented by the potential part (10) and a selection of the equations of the system $\mathcal{L}$ excluding a subset of equations which are differential consequences of (10) and other equations of $\mathcal{L}$, taken together. Similarly to Sec. VI, below the index ν runs through the set $\mathcal{N}$ of the numbers of the equations from $\mathcal{L}$ which are in the canonical representation of $\mathcal{L}_{\rm p}$. The index ν' runs through the complementary set $\mathcal{N}'=\{1,\dots,l\}\backslash\mathcal{N}$. The representation described gives a canonical foliation of the system $\mathcal{L}_{\rm p}$ over the system $\mathcal{L}$.

The system $\mathcal{L}_{\rm p}$ defines a (first level) Abelian covering of the system $\mathcal{L}$ since the right-hand sides G^{si} of (10) do not depend on the potentials $v^1,\dots,v^p$. Each of the compatibility conditions $(D_jG^{si}-D_iG^{sj})|_{\mathcal{L}}=0$ can be interpreted as a conservation law of $\mathcal{L}$ with a conserved vector which

has only two nonzero components, namely, the ith component equal to G^{sj} and the jth component equal to $-G^{si}$. Therefore, defining a potential in the framework of Abelian coverings involves $\frac{1}{2}n(n-1)$ conserved vectors of a special form.

Similarly to the two-dimensional case, two tuples $v=(v^1,\dots,v^p)$ and $\tilde{v}=(\tilde{v}^1,\dots,\tilde{v}^p)$ of potentials of Abelian coverings of the same multidimensional system $\mathcal{L}$ are *equivalent* if there exist differential functions $\Phi^s[u]$ and constants $c_{s\sigma}$ such that $|c_{s\sigma}|\neq 0$ and the transformation $\tilde{v}^s=c_{s\sigma}v^\sigma+\Phi^s[u]$ (the variables x and derivatives of u are not transformed) maps the system $\mathcal{L}_{\rm p}$ associated with v to the system $\tilde{\mathcal{L}}_{\rm p}$ associated with $\tilde{v}$. The function tuples $(G^{si}[u])$ and $(\tilde{G}^{si}[u])$ from the potential parts of these systems are connected by the formula $(\tilde{G}^{si}-c_{s\sigma}G^{\sigma i}-D_i\Phi^s)|_{\mathcal{L}}=0$. In fact, in the local-coordinate approach an Abelian covering of $\mathcal{L}$ is an equivalence class of tuples of potentials which are considered along with the corresponding equations of the form (10) and prolongations of the total differentiation operators to the potentials, coinciding on the solution set of $\mathcal{L}$. The equivalence of potential p-tuples agrees with the equivalence of the associated p-element sets of $\frac{1}{2}n(n-1)$-tuples of conserved vectors, involving linear combinations.

Definition 12 is easily generalized to Abelian coverings of arbitrary dimensions.

Definition 13: The potentials $v^1,\dots,v^p$ are called *locally dependent on the solution set of the system* $\mathcal{L}$ (or, briefly speaking, *dependent*) if there exist $r'\in\mathbb{N}$ and a function Ω of the variables x, $u_{(r')}$, $v^1,\dots,v^p$ such that $\Omega_{v^s}\neq 0$ for some s and $\Omega(x,u_{(r')},v^1,\dots,v^p)=0$ for any solution $(u,v^1,\dots,v^p)$ of the system $\mathcal{L}_{\rm p}$, (up to gauge transformations, i.e., adding constants to potentials).

If a linear combination of the tuples $(G^{s1},\dots,G^{sn})$, $s=1,\dots,p$, is a total gradient, i.e., $c_sG^{si}=D_iH$ for certain constants c_s and a differential function $H[u]$, then the potentials $v^1,\dots,v^p$ are dependent since $c_sv^s=H[u]+c_0$ for some negligible constant c_0.

Employing the characteristic form (1) of conservation laws requires the assumption that the systems $\mathcal{L}$ and $\mathcal{L}_{\rm p}$ are totally nondegenerate. We again use the trick of introducing weighted jet spaces and extending the weight to potentials. The procedure is analogous to that in the two-dimensional case. Thus, the rule for extending the weight to the derivatives of the potentials $v^1,\dots,v^p$ is

$$\varrho(v^s_\alpha)=\max(0,\varrho(G^{s1})-1,\dots,\varrho(G^{sn})-1)+|\alpha|.$$

Lemma 6: *The system $\mathcal{L}$ is totally nondegenerate with respect to a weight if and only if the system $\mathcal{L}_{\rm p}$ is totally nondegenerate with respect to this weight extended to the derivatives of the potentials.*

Proof: A complete set $L_{{\rm p}[k]}$ of independent differential consequences of the system $\mathcal{L}_{\rm p}$ which have extended weights not greater than k is exhausted by the equations

$$\breve{L}^{\breve{\mu}}=0,\quad \breve{\mu}=1,\dots,\breve{l},\quad v^s_\alpha=D_i^{\alpha_i-1}D_{i+1}^{\alpha_{i+1}}\cdots D_n^{\alpha_n}G^{si}.$$

Here the equations $\breve{L}^{\breve{\mu}}=0$, $\breve{\mu}=1,\dots,\breve{l}$, form a complete set $L_{[k]}$ of independent differential consequences of the system $\mathcal{L}$, which have weights not greater than k. $v^s_\alpha=\partial^{|\alpha|}v^s/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$. For each i and s the multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ runs through the multi-index set in which $\alpha_1=\cdots=\alpha_{i-1}=0$, $\alpha_i>0$, $\varrho(v^s)+|\alpha|\leqslant k$.

It is obvious that for any $k\in\mathbb{N}$ the system $L_{{\rm p}[k]}$ is of maximal rank on the manifold $\mathcal{L}_{{\rm p}[k]}$ in the weighted jet space $J^k_\varrho(x|u,v)$ if and only if the system $L_{[k]}$ is of maximal rank on the manifold $\mathcal{L}_{[k]}$. The local solvability of $\mathcal{L}_{\rm p}$ follows from the local solvability of $\mathcal{L}$ and the compatibility conditions for the potential part and implies the local solvability of $\mathcal{L}$ since $\mathcal{L}$ is a subsystem of $\mathcal{L}_{\rm p}$. □

Since any potential system representing an Abelian covering is foliated over the corresponding initial system, all statements of Sec. V are applicable to its conservation laws (after the necessary modifications in the proof of Theorem 6, connected with the introduction of weighted jet spaces). Stronger statements on the connection between potential-free characteristics and conservation laws by induced conservation laws of the corresponding initial system can be proven owing to a special structure of the foliation.

Lemma 7: *If a characteristic of the potential system $\mathcal{L}_{\rm p}$ depends only on "local" variables (i.e., it is a function only of x and derivatives of u), then the associated conservation law of $\mathcal{L}_{\rm p}$ has a conserved vector which also does not depend on potentials.*

Proof: Let the potential system $\mathcal{L}_{\rm p}$ possess a characteristic

$$(\alpha^{si},\ \gamma^{\nu},\ s=1,\dots,p,\ i=1,\dots,n,\ \nu\in\mathcal{N}),$$

which does not depend on the potentials $v^1,\dots,v^p$. [By the defining Eq. (10) for the potentials, the dependence of the characteristic on nonzero-order derivatives of the potentials can be neglected up to the equivalence relation of characteristics.] In the above expression, α^{si} and γ^{ν} are differential functions of u corresponding to $v^s_i=G^{si}$ and $L^{\nu}=0$, respectively. From this, we obtain a conserved vector $(F^1,\dots,F^n)$ of $\mathcal{L}_{\rm p}$ with

$$D_iF^i=\alpha^{si}(v^s_i-G^{si})+\gamma^{\nu}L^{\nu}=:V. \tag{11}$$

As the differential function V of x and derivatives of u and v is a total divergence, an application of the extended Euler operator $\mathsf{E}=(\mathsf{E}_{u^1},\dots,\mathsf{E}_{u^m},\mathsf{E}_{v^1},\dots,\mathsf{E}_{v^p})$ on V gives the zero $m+p$-tuple. We conclude that

$$-\mathsf{E}_{v^s}V=D_i\alpha^{si}=0,\quad s=1,\dots,p,$$

so that $(\alpha^{s1}[u],\dots,\alpha^{sn}[u])$ is a null divergence. Thus by Theorem 2 there exist differential functions $\Phi^{sij}[u]$ such that $\alpha^{si}=D_j\Phi^{sij}$ and $\Phi^{sij}=-\Phi^{sji}$. Setting

$$\hat F^i=F^i+\Phi^{sij}(v^s_j-G^{sj}),$$

the tuple $\hat F=(\hat F^1,\dots,\hat F^n)$ is a conserved vector equivalent to the initial conserved vector F. In terms of $\hat F$ Eq. (11) can be rewritten as

$$D_i\hat F^i=\alpha^{si}(v^s_i-G^{si})+\gamma^{\nu}L^{\nu}+(D_i\Phi^{sij})(v^s_j-G^{sj})+\Phi^{sij}(v^s_{ij}-D_iG^{sj})=\sum_{i<j}\Phi^{sij}(D_jG^{si}-D_iG^{sj})+\gamma^{\nu}L^{\nu}.$$

The right-hand side of the last equality vanishes on the solution set of $\mathcal{L}$. The standard way of deriving the characteristic form of conservation laws implies that

$$D_i\check F^i=\check\gamma^{\mu}L^{\mu} \tag{12}$$

for some differential functions $\check\gamma^{\mu}[u]$ and some conserved vector $\check F$ equivalent to $\hat F$ and, therefore, F. (The conserved vector $\check F$ differs from $\hat F$ by a tuple vanishing on the solution set of $\mathcal{L}$.) As $\check\gamma^{\mu}L^{\mu}$ depends only on x and derivatives of u, by (9) and Corollary 1 we obtain that there exist a conserved vector $\tilde F$ of $\mathcal{L}_{\rm p}$ which depends only on x and derivatives of u and is equivalent to the conserved vector $\check F$ and, consequently, to F. □

Lemma 8: *If an extended characteristic of a potential system $\mathcal{L}_{\rm p}$ is induced by a characteristic of the initial system $\mathcal{L}$, then the associated conservation law of $\mathcal{L}_{\rm p}$ has a characteristic which does not depend on potentials.*

Proof: Let the system $\mathcal{L}_{\rm p}$ define an Abelian covering of the system $\mathcal{L}$ and suppose that $\mathcal{L}_{\rm p}$ possesses an extended characteristic induced by a characteristic λ of $\mathcal{L}$. Equivalently, there exists a conserved vector F of $\mathcal{L}_{\rm p}$ with $D_iF^i=\lambda^{\mu}[u]L^{\mu}[u]$. In general, this equation need not be a characteristic form of the conservation law of $\mathcal{L}_{\rm p}$, containing the conserved vector F, since some equations of $\mathcal{L}$ may fail to be contained in the minimal set of equations forming the potential system $\mathcal{L}_{\rm p}$. We collect the indices of such equations in the set $\mathcal{N}'=\{\nu'\}$ and suppose that $\mathcal{N}'\neq\varnothing$ (as otherwise we already have a characteristic form).

By Lemma 1, the representation of any $L^{\nu'}$ as a differential consequence of $\mathcal{L}_{\rm p}$ is of the form

$$L^{\nu'} = A^{\nu'\nu}L^{\nu} + \sum_{i<j} B^{\nu' sij}(D_i G^{sj} - D_j G^{si}),$$

where $A^{\nu'\nu}$ and $B^{\nu' sij}$ are polynomials of the total differentiation operators D_i with smooth coefficients depending on x and derivatives of u, $B^{\nu' sij} = -B^{\nu' sji}$. Since $D_i G^{sj} - D_j G^{si} = D_j(v_i^s - G^{si}) - D_i(v_j^s - G^{sj})$ we get

$$D_i F^i = \lambda^{\nu} L^{\nu} + \lambda^{\nu'} A^{\nu'\nu} L^{\nu} + \lambda^{\nu'} B^{\nu' sij} D_j(v_i^s - G^{si}),$$

which, by integrating by parts, entails

$$D_i \widetilde{F}^i = \gamma^{\nu} L^{\nu} + \alpha^{si}(v_i^s - G^{si}).$$

Here $\alpha^{si} = -D_j B^{jis\nu'} * \lambda^{\nu'}$ and $\gamma^{\nu} = \lambda^{\nu} + A^{\nu\nu'} * \lambda^{\nu'}$ are functions of x and derivatives of u. By $A^{\nu\nu'} *$ and $B^{jis\nu'} *$ we denote the formally adjoint operators to $A^{\nu'\nu}$ and $B^{\nu' sij}$, respectively. F and $\widetilde{F}$ are equivalent conserved vectors as their difference vanishes on $\mathcal{L}_{\mathrm{p}}$.

Thus we obtain the characteristic $(\alpha^{si}, \gamma^{\nu}, s = 1, \ldots, p, i = 1, \ldots, n, \nu \in \mathcal{N})$ of the conservation law with conserved vector F, depending exclusively on x and derivatives of u. □

Thus we may combine Proposition 3, Theorem 6, and Lemmas 7 and 8 into the following result.

Theorem 8: *The following statements on a conservation law of a system determining an Abelian covering are equivalent.*

(1) The conservation law is induced by a conservation law of the corresponding initial system.
(2) It contains a conserved vector which does not depend on potentials.
(3) Some of its extended characteristics are induced by characteristics of the initial system.
(4) It possesses a characteristic not depending on potentials.

Note 4: The locality properties of conservation laws, listed in Theorem 8, are preserved under equivalence transformations of potential systems. More precisely, if the systems $\mathcal{L}_{\mathrm{p}}$ and $\widetilde{\mathcal{L}}_{\mathrm{p}}$ belong to the same Abelian covering of the system $\mathcal{L}$ then the corresponding equivalence transformation maps any conservation law of $\mathcal{L}_{\mathrm{p}}$ with these locality properties to a conservation law of $\widetilde{\mathcal{L}}_{\mathrm{p}}$ with the same properties. Therefore the statement on locality properties of conservation laws of potential systems can be reformulated as an analogous statement for Abelian coverings.

VIII. STANDARD POTENTIALS

Consider potential systems obtained via introducing potentials according to Theorem 2 in the case $n > 2$. Suppose that the system $\mathcal{L}$ has p linearly independent local conservation laws with conserved vectors $G^s = (G^{s1}, \ldots, G^{sn})$, $s = 1, \ldots, p$. We introduce the potentials $v^{sij} = -v^{sji}$ associated with this set of conserved vectors by the equations

$$v_j^{sij} = G^{si}, \tag{13}$$

assuming additionally that these potentials are locally independent on the solution set of the system $\mathcal{L}$. The canonical representation of the corresponding *standard potential system* $\mathcal{L}_{\mathrm{p}}$ consists of the potential part (13) and a selection of the equations of the system $\mathcal{L}$ excluding a subset of equations which are differential consequences of (13) and other equations of $\mathcal{L}$, taken together. Below the index ν runs through the set $\mathcal{N}$ of the numbers of the equations from $\mathcal{L}$ which are in the canonical representation of $\mathcal{L}_{\mathrm{p}}$. The index ν' runs through the set $\mathcal{N}' = \{1, \ldots, l\} \setminus \mathcal{N}$. (Note that the total number of elements in $\mathcal{N}$ is equal to or greater than $l - p$ but is not necessarily equal to $l - p$.) The above representation is a canonical foliation of the system $\mathcal{L}_{\mathrm{p}}$ over the system $\mathcal{L}$.

Tuples $v = (v^{sij})$ and $\tilde{v} = (\tilde{v}^{sij})$ of potentials associated with the same p-dimensional subspace of the conservation law space $\mathrm{CL}(\mathcal{L})$ of $\mathcal{L}$ are considered *equivalent*. In other words, the tuples of potentials v and $\tilde{v}$ are equivalent if there exist differential functions $\Phi^{sij}[u]$ and constants $c_{s\sigma}$ such that $\Phi^{sij} = -\Phi^{sji}$, $|c_{s\sigma}| \neq 0$, and the transformation $\tilde{v}^{sij} = c_{s\sigma} v^{\sigma ij} + \Phi^{sij}[u]$ (the variables x and deriva-

tives of u are not transformed) maps the system $\mathcal{L}_{\rm p}$ associated with v to the system $\widetilde{\mathcal{L}}_{\rm p}$ associated with $\tilde{v}$. The tuples of the corresponding conserved vectors G^s and $\widetilde{G}^s$ are connected by the formula $(\widetilde{G}^{si}-c_{s\sigma}G^{\sigma i}-D_i\Phi^{sij})|_{\mathcal{L}}=0$. We will also say that the systems $\mathcal{L}_{\rm p}$ and $\widetilde{\mathcal{L}}_{\rm p}$ are equivalent as potential systems of the system $\mathcal{L}$.

The procedure of grading the jet space with respect to potentials in the case $n>2$ is analogous to the one in the two-dimensional case (see Sec. VI). The difference is that the weights of the potentials arising from the same conservation law (i.e., having the same value of the index s) are assumed equal, i.e.,

$$\varrho(v^{sij}_{\alpha}) = \max(0, \varrho(G^{s1}) - 1, \dots, \varrho(G^{sn}) - 1) + |\alpha|.$$

Lemma 9: *The system $\mathcal{L}$ is totally nondegenerate with respect to a weight if and only if the system $\mathcal{L}_{\rm p}$ is totally nondegenerate with respect to this weight extended to the derivatives of the potentials.*

Proof: A complete set $L_{{\rm p}[k]}$ of independent differential consequences of the system $\mathcal{L}_{\rm p}$ which have extended weights not greater than k is exhausted by the equations

$$\breve{L}^{\breve{\mu}}=0, \quad \breve{\mu}=1,\dots,\breve{l}, \quad v^{sij}_{\alpha+\delta_j}=D_1^{\alpha_1}\cdots D_n^{\alpha_n}G^{si}.$$

Here the equations $\breve{L}^{\breve{\mu}}=0$, $\breve{\mu}=1,\dots,\breve{l}$ form a complete set $L_{[k]}$ of independent differential consequences of the system $\mathcal{L}$, which have weights not greater than k and $v^s_{\alpha}=\partial^{|\alpha|}v^s/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$. For each i and s the multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ runs through the multi-index set in which $\varrho(v^s)+|\alpha|<k$ and additionally $\alpha_1=0$ if $i=1$. The symbol δ_i was introduced after Definition 1.

It is obvious that for any $k\in\mathbb{N}$ the system $L_{{\rm p}[k]}$ is of maximal rank on the manifold $\mathcal{L}_{{\rm p}[k]}$ in the weighted jet space $J^k_{\varrho}(x|u,v)$ if and only if the system $L_{[k]}$ is of maximal rank on the manifold $\mathcal{L}_{[k]}$. The local solvability of $\mathcal{L}_{\rm p}$ follows from the local solvability of $\mathcal{L}$ and the compatibility conditions for the potential part and implies the local solvability of $\mathcal{L}$ since $\mathcal{L}$ is a subsystem of $\mathcal{L}_{\rm p}$. □

Similarly to two-dimensional potential systems and systems representing Abelian coverings, multidimensional potential systems are foliated over the corresponding initial systems in a special way. In addition to using all statements of Sec. V, this allows us to prove stronger statements on their conservation laws induced by conservation laws of the initial systems.

Lemma 10: *If a characteristic of the potential system $\mathcal{L}_{\rm p}$ depends only on local variables (i.e., independent and nonpotential dependent ones), then the associated conservation law of $\mathcal{L}_{\rm p}$ has a conserved vector which also does not depend on potentials.*

Proof: By assumption, the potential system $\mathcal{L}_{\rm p}$ has a characteristic

$$(\alpha^{si},\ \gamma^{\nu},\ s=1,\dots,p,\ i=1,\dots,n,\ \nu\in\mathcal{N}),$$

which does not depend on the potentials $v^1,\dots,v^p$. [Here the dependence of the characteristic on nonzero-order derivatives of the potentials can be neglected up to the equivalence relation of characteristics by (13).] Since the α^{si} and γ^{ν} are functions of x and derivatives of u corresponding to the equations $D_jv^{sij}=G^{si}$ and $L^{\nu}=0$, respectively, there exists a conserved vector F of the potential system $\mathcal{L}_{\rm p}$ with

$$D_iF^i=\alpha^{si}(v^{sij}_j-G^{si})+\gamma^{\nu}L^{\nu}=:V. \tag{14}$$

It follows that the differential function $V=V[u,v]$ is a total divergence, so the extended Euler operator $\mathsf{E}=(\mathsf{E}_{u^1},\dots,\mathsf{E}_{u^m},\mathsf{E}_{v^1ij},\dots,\mathsf{E}_{v^pij},1\leqslant i<j\leqslant n)$ annihilates V. Thus,

$$-\mathsf{E}_{v^{sij}}V=D_j\alpha^{si}-D_i\alpha^{sj}=0.$$

These conditions mean that for each s the "horizontal" differential 1-form $\omega^s=\alpha^{si}[u]dx_i$ is closed with respect to the total differential D since

$$\mathsf{D}\omega^s = D_j\alpha^{si}dx_j\wedge dx_i = \sum_{i<j}(D_j\alpha^{si} - D_i\alpha^{sj})dx_j\wedge dx_i = 0.$$

The horizontal de Rahm complex[9] (also called D-complex[21]) over a totally star-shaped domain of the independent variable x and the dependent variable u is exact (see, e.g., Theorem 5.59 of Ref. 21). Therefore, the form ω^s is D-exact, i.e., there exists a horizontal differential 0-form (in other words, a differential function) $\Phi^s=\Phi^s[u]$ such that $\omega^s=\mathsf{D}\Phi^s$. Writing the last equality in components, we obtain $\alpha^{si}=D_i\Phi^s$.

Consider the conserved vector $\hat{F}$ with the components

$$\hat{F}^i = F^i - \Phi^s(v_j^{sij} - G^{si}),$$

which is equivalent to the initial conserved vector F. Then Eq. (14) can be rewritten as

$$D_i\hat{F}^i = -\Phi^s(v_{ij}^{sij} - D_iG^{si}) + \gamma^\nu L^\nu = \Phi^s D_iG^{si} + \gamma^\nu L^\nu.$$

The right-hand side of this equation is a function of x and derivatives of u and vanishes on the manifold $\mathcal{L}_{(k)}$ in the jet space $J^k(x|u)$, where k is the highest order of derivatives in this expression. Using the Hadamard lemma and integration by parts as in the derivation of the general characteristic form of conservation laws, we obtain that

$$D_i\check{F}^i = \check{\gamma}^\mu L^\mu \tag{15}$$

for some differential functions $\check{\gamma}^\mu[u]$, where the conserved vector $\check{F}$ is equivalent to $\hat{F}$ and, therefore, F since it differs from $\hat{F}$ by a tuple vanishing on the solution set of $\mathcal{L}$. Since the right-hand side $\check{\gamma}^\mu L^\mu$ depends only on x and derivatives of u, equality (15) implies in view of Corollary 1 that there exists a conserved vector $\widetilde{F}$ of $\mathcal{L}_{\rm p}$, which depends only on x and derivatives of u and is equivalent to the conserved vector $\check{F}$ and, therefore, F. □

Lemma 11: *If an extended characteristic of the potential system $\mathcal{L}_{\rm p}$ is induced by a characteristic of the system $\mathcal{L}$, then the associated conservation law of $\mathcal{L}_{\rm p}$ has a characteristic which does not depend on potentials.*

Proof: Assume that the multidimensional potential system $\mathcal{L}_{\rm p}$ possesses an extended characteristic which is induced by a characteristic λ of the initial system $\mathcal{L}$. This means that there exists a conserved vector $F=(F^1,\dots,F^n)$ of $\mathcal{L}_{\rm p}$ such that $D_iF^i=\lambda^\mu[u]L^\mu[u]$. Again this equation need not be a characteristic form of the conservation law of $\mathcal{L}_{\rm p}$ which contains the conserved vector F, since some equations of $\mathcal{L}$ may not be contained in the minimal set of equations forming the potential system $\mathcal{L}_{\rm p}$. We form the set $\mathcal{N}'=\{\nu'\}$ of indices of such equations and may suppose that $\mathcal{N}'\neq\varnothing$. Then by Lemma 1, $L^{\nu'}$, being a differential consequence of $\mathcal{L}_{\rm p}$, can be represented as

$$L^{\nu'} = A^{\nu'\nu}L^\nu + B^{\nu's}D_iG^{si},$$

where $A^{\nu'\nu}$ and $B^{\nu's}$ are polynomials of the total differentiation operators D_i with coefficients depending on x and derivatives of u. Since $D_iG^{si}=D_i(v_j^{sij}-G^{si})$, it follows that

$$D_iF^i = \lambda^\nu L^\nu + \lambda^{\nu'}A^{\nu'\nu}L^\nu + \lambda^{\nu'}B^{\nu's}D_i(v_j^{sij} - G^{si}),$$

and integrating by parts on the right-hand side leads to

$$D_i\widetilde{F}^i = \alpha^{si}(v_j^{sij} - G^{si}) + \gamma^\nu L^\nu.$$

In this expression, $\alpha^{si}=-D_iB^{s\nu'}{}^*\lambda^{\nu'}$ and $\gamma^\nu=\lambda^\nu+A^{\nu\nu'}{}^*\lambda^{\nu'}$ are differential functions of u and $A^{\nu\nu'}{}^*$ and $B^{s\nu'}{}^*$ are the formally adjoint operators to $A^{\nu'\nu}$ and $B^{\nu's}$. Also, F and $\widetilde{F}$ are equivalent as conserved vectors as their difference vanishes on $\mathcal{L}_{\rm p}$.

This gives the characteristic $(\alpha^{si},\ \gamma^\nu,\ s=1,\dots,p,\ i=1,\dots,n,\ \nu\in\mathcal{N})$ of the conservation law with conserved vector F, which depend only on x and derivatives of u, as claimed. □

Summarizing Proposition 3, Theorem 6, and Lemmas 10 and 11, we arrive at the following.

Theorem 9: *The following statements on a conservation law of a standard potential system (without gauges) are equivalent.*

(1) The conservation law is induced by a conservation law of the corresponding initial system.
(2) It contains a conserved vector which does not depend on potentials.
(3) Some of its extended characteristics are induced by characteristics of the initial system.
(4) It possesses a characteristic not depending on potentials.

Note 5: The locality properties of conservation laws, listed in Theorem 7, are stable with respect to the equivalence of potential systems. In other words, if potential systems $\mathcal{L}_{\rm p}$ and $\widetilde{\mathcal{L}}_{\rm p}$ of the system $\mathcal{L}$ are equivalent, then the corresponding equivalence transformation maps any conservation law of $\mathcal{L}_{\rm p}$ possessing the above locality properties to a conservation law of $\widetilde{\mathcal{L}}_{\rm p}$ with the same properties.

If $n>2$, Eq. (13) associated with a fixed solution $u=u(x)$ of the system $\mathcal{L}$ forms an underdetermined system with respect to the potentials v^{sij}. Therefore, we can add gauge conditions on the potentials to $\mathcal{L}_{\rm p}$. In fact, such additional conditions are absolutely necessary in the case $n>2$ for the potential system to have nontrivial symmetries and conservation laws. It is stated in Theorem 2.7 of Ref. 1 for a quite general situation that every local symmetry of a potential system with unconstrained potentials is projectable to a local symmetry of the initial system, i.e., such a potential system gives no nontrivial potential symmetries. Moreover, each conservation law of such a system is invariant with respect to gauge transformations of the potentials.[4]

Definition 14: A system $\mathcal{L}_{\rm g}$ of differential equations with the independent variables x and the dependent variables u and v is called a *gauge* on the potentials v^{sij} defined by Eq. (13) if any differential consequence of the coupled system $\mathcal{L}_{\rm gp}=\mathcal{L}_{\rm p}\cap\mathcal{L}_{\rm g}$, which does not involve the potentials v^{sij}, is a differential consequence of the initial system $\mathcal{L}$. The coupled system $\mathcal{L}_{\rm gp}$ is called a *gauged potential system*. The gauge $\mathcal{L}_{\rm g}$ is called *weak* if a minimal set of equations generating all the differential consequences of $\mathcal{L}_{\rm p}$ is contained in a minimal set representing the coupled system $\mathcal{L}_{\rm gp}$ called a *weakly gauged potential system*.

The gauged potential system $\mathcal{L}_{\rm gp}$ is a foliated system over the base system $\mathcal{L}$. Therefore, the statements of Sec. V are true for conservation laws of such systems and can be sharpened in the following way.

Proposition 6: *A conservation law of a gauged potential system contains a conserved vector which does not depend on potentials if and only if it is induced by the conservation law of the corresponding initial system with the same conserved vector and if and only if some of its extended characteristics are induced by characteristics of the initial system.*

A weakened version of Theorem 9 on potential systems without gauges can be extended to weakly gauged potential systems. The proof is analogous to those already presented. Only the general version of the Hadamard lemma for fiber bundles (Lemma 2) has to be applied instead of the simplest one (Lemma 1).

Theorem 10: *A conservation law of a weakly gauged potential system contains a conserved vector which does not depend on potentials if and only if it has a characteristic which also does not depend on potentials and whose components corresponding to the gauge equations vanish.*

IX. GENERAL COVERINGS

The idea of general coverings arose in the well-known paper by Wahlquist and Estabrook[31] in the form of prolongation structures involving *pseudopotentials*. Later this idea was rigorously formulated and developed in geometrical terms.[9,17,18,30] Here we treat coverings in the framework of the local approach by introducing local coordinates.

The statement on the simultaneous locality of conserved vectors and characteristics is not true for conservation laws of general coverings.

Suppose that the system $\mathcal{L}$ admits p pseudopotentials $v^1,\dots,v^p$ defined by the equation

$$v_i^s = G^{si}[u|v], \tag{16}$$

where the differential functions $G^{si}=G^{si}[u|v]$ satisfy the compatibility conditions $\hat{D}_j G^{si}=\hat{D}_i G^{sj}$ on the solution set of the system $\mathcal{L}$. The notation $G[u|v]$ means that G is a differential function of u and v, depending on x, v, and derivatives of u (there are no derivatives of v of orders greater than 0!). We will briefly call $G[u|v]$ a differential function of $(u|v)$. $\hat{D}_i$ is the operator of total differentiation, acting on differential functions of the above type according to system (16), i.e., $\hat{D}_i=\partial_{x_i}+u^a_{\alpha,i}\partial_{u^a_\alpha}+G^{si}[u|v]\partial_{v^s}$.

The canonical representation of the corresponding potential system $\mathcal{L}_{\mathrm{p}}$ consists of the pseudopotential part (16) and a selection of the equations of the system $\mathcal{L}$ excluding a subset of equations which are differential consequences of (16) and the other equations of $\mathcal{L}$, taken together. The system $\mathcal{L}_{\mathrm{p}}$ defines a *covering* of the system $\mathcal{L}$. It is an example of a foliated system, where $\mathcal{L}$ is the base system.

Two tuples of pseudopotentials $v=(v^1,\dots,v^p)$ and $\tilde{v}=(\tilde{v}^1,\dots,\tilde{v}^p)$ of the same system $\mathcal{L}$ are considered *equivalent* if there exist differential functions $\Omega^s[u|v]$ such that $|\Omega^s_{v^\sigma}|\neq 0$ and if the transformation Ω: $\tilde{v}^s=\Omega^s[u|v]$ (the variables x and derivatives of u are not transformed) maps the system $\mathcal{L}_{\mathrm{p}}$ associated with v to the system $\tilde{\mathcal{L}}_{\mathrm{p}}$ associated with $\tilde{v}$. The functions $G^{si}[u|v]$ and $\tilde{G}^{si}[u|\tilde{v}]$ from the pseudopotential parts of these systems are connected by the formula $(\tilde{G}^{si}-\hat{D}_i\Omega^s)|_{\mathcal{L}}=0$. Hence the prolongations of the total differentiation operators to equivalent tuples of pseudopotentials coincide on the solution set of $\mathcal{L}$. In fact, in the local-coordinate approach a covering of $\mathcal{L}$ is an equivalence class of tuples of pseudopotentials which are considered along with the corresponding equations of the form (16) and prolongations of the total differentiation operators coinciding on the solution set of $\mathcal{L}$.

Since two conserved vectors of $\mathcal{L}_{\mathrm{p}}$, whose difference vanishes identically in view of subsystem (16) are equivalent, any conservation law of $\mathcal{L}_{\mathrm{p}}$ contains a conserved vector $F[u|v]$ whose components $F^i[u|v]$ do not depend on nonzero-order derivatives of the pseudopotentials. In view of Lemma 1, the defining formula $D_iF^i|_{\mathcal{L}_{\mathrm{p}}}=0$ for conserved vectors of this kind can be rewritten in the form $\hat{D}_iF^i|_{\mathcal{L}}=0$. The same is true for characteristics and extended characteristics of the system $\mathcal{L}_{\mathrm{p}}$. Namely, up to equivalence determined by the subsystem (16), the components of any (extended) characteristic of $\mathcal{L}_{\mathrm{p}}$ can be assumed to be differential functions of $(u|v)$. Conserved vectors (characteristics and extended characteristics) whose components do not depend on the nonzero-order derivatives of the pseudopotentials will be called *reduced*.

Due to the structure of Eq. (16) defining the pseudopotentials, any weight defined for the variables x and u^a_α is extendable to the derivatives of pseudopotentials. To extend the weight, we use the following rule. We will assume that all the pseudopotentials v have the same weight equal, e.g., to

$$\varrho_v=\max(0,\ \varrho(G^{si})-1,\ s=1,\dots,p,\ i=1,\dots,n).$$

Therefore, $\varrho(v^s_\alpha)=\varrho_v+|\alpha|$. This equation reflects the fact that pseudopotentials appear on the right-hand sides of Eq. (16).

Lemma 12: *The system $\mathcal{L}$ is totally nondegenerate with respect to a weight if and only if the system $\mathcal{L}_{\mathrm{p}}$ is totally nondegenerate with respect to this weight extended to the derivatives of the pseudopotentials.*

The proof of Lemma 12 is analogous to that of Lemma 6. Only the total differentiation operators $\hat{D}_i$ have to be used instead of the standard ones. Thus only the total nondegeneracy of the system $\mathcal{L}$ has to be assumed for working with the usual and extended characteristics of conservation laws of both the system $\mathcal{L}$ and the system $\mathcal{L}_{\mathrm{p}}$. Since any potential system determining a covering of the system $\mathcal{L}$ is a foliated system with base system $\mathcal{L}$, the statements of Sec. V remain true for conservation laws of such systems (after the necessary replacements in the proof of Theorem 6, taking into account the grading of the jet spaces). Let us combine these statements and formulate them in a specific way.

Proposition 7: *A conservation law of a system determining a covering contains a conserved vector which does not depend on potentials if and only if it is induced by the conservation law of the corresponding initial system which has the same conserved vector and if and only if some of its extended characteristics are induced by characteristics of the initial system.*

Unfortunately, the property of characteristic locality cannot be included in the chain of equivalent statements of Proposition 7 and, moreover, this property is not preserved under the equivalence transformations of tuples of pseudopotentials. In fact, if the potential systems $\mathcal{L}_{\rm p}$ and $\widetilde{\mathcal{L}}_{\rm p}$ of the system $\mathcal{L}$ are equivalent with respect to an equivalence transformation Ω and the system $\mathcal{L}_{\rm p}$ possesses a conservation law $\mathcal{F}$ with a local characteristic, then there is no guarantee that the conservation law $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{L}}_{\rm p}$, equivalent to $\mathcal{F}$ with respect to Ω, also has a local characteristic.

A partial locality property of extended characteristics of covering systems is connected with the linearity of associated conserved vectors with respect to pseudopotentials.

Theorem 11: *A conservation law of a system determining a covering contains a reduced conserved vector which linearly depends on pseudopotentials if and only if it has a reduced extended characteristic whose components corresponding to the pseudopotential part of the system do not depend on pseudopotentials.*

Proof: Suppose that a conservation law $\mathcal{F}$ of the system $\mathcal{L}_{\rm p}$ contains a reduced conserved vector $F[u|v]$ which linearly depends on pseudopotentials, i.e., $F^i=F^{is}[u]v^s+F^{i0}[u]$. The defining formula $\hat{D}_iF^i|_{\mathcal{L}}=0$ for reduced conserved vectors implies that

$$((D_iF^{is})v^s + F^{is}G^{si} + D_iF^{i0})|_{\mathcal{L}} = 0.$$

Following the conventional way of deriving the characteristic form of conservation laws, we apply the Hadamard lemma, integrate by parts on the right-hand side of the derived equality, and finally obtain that

$$(D_iF^{is})v^s + F^{is}G^{si} + D_iF^{i0} = \gamma^\mu L^\mu + D_i\hat{F}^i$$

for some differential functions $\gamma^\mu=\gamma^\mu[u|v]$ and $\hat{F}^i=\hat{F}^i[u|v]$, and the functions $\hat{F}^i$ vanish on the solutions of $\mathcal{L}$ identically with respect to v. Therefore, the tuple $\hat{F}=(\hat{F}^1,\dots,\hat{F}^n)$ is a trivial conserved vector of $\mathcal{L}_{\rm p}$. The conserved vector $\widetilde{F}=F-\hat{F}$ belongs to $\mathcal{F}$ (since it is equivalent to F) and satisfies the equality

$$D_i\widetilde{F}^i = F^{is}(v_i^s - G^{si}) + \gamma^\mu L^\mu.$$

This means that the tuple $(F^{is}[u],\ i=1,\dots,n,\ s=1,\dots,p,\ \gamma^\mu[u|v],\ \mu=1,\dots,l)$ is a reduced extended characteristic of the system $\mathcal{L}_{\rm p}$, which is associated with the conservation law $\mathcal{F}$ and obviously has the necessary property.

Conversely, let the tuple $(F^{is}[u],\ i=1,\dots,n,\ s=1,\dots,p,\ \gamma^\mu[u|v],\ \mu=1,\dots,l)$ be a reduced extended characteristic associated with the conservation law $\mathcal{F}$ of the system $\mathcal{L}_{\rm p}$. Then there exists a conserved vector F belonging to $\mathcal{F}$ such that

$$D_iF^i = F^{is}(v_i^s - G^{si}) + \gamma^\mu L^\mu. \tag{17}$$

Acting by the extended Euler operator $\mathsf{E}=(\mathsf{E}_{u^1},\dots,\mathsf{E}_{u^m},\mathsf{E}_{v^1},\dots,\mathsf{E}_{v^p})$ on both the sides of the last equality, we have, in particular, that

$$0 = \mathsf{E}_{v^s}D_iF^i = -D_iF^{is} - F^{is}G^{si}_{v^s} + \gamma^\mu_{v^s}L^\mu.$$

Simultaneously integrating these equations, we obtain that

$$-F^{is}G^{si} + \gamma^\mu L^\mu = (D_iF^{is})v^s + H[u]$$

for some differential function $H=H[u]$. The substitution of the last expression into Eq. (17) results in the equality $D_iF^i=F^{is}v_i^s+(D_iF^{is})v^s+H$, i.e., $D_i(F^i-F^{is}v^s)=H[u]$. This immediately implies in

view of Corollary 1 that there exist an n-tuple $\breve{F}=\breve{F}[u]$ and a null divergence $\breve{F}=\breve{F}[u,v]$ such that $F^i-F^{is}v^s=\breve{F}^i+\breve{F}^i$. Finally, the tuple $\widetilde{F}=F-\breve{F}$ differs from F by the null divergence $\breve{F}$ and, therefore, also is a conserved vector of $\mathcal{L}_{\rm p}$, belonging to the conservation law $\mathcal{F}$. Its components $\widetilde{F}^i=F^{is}[u]v^s+\breve{F}^i[u]$ are linear with respect to the pseudopotentials. □

X. A CRITERION FOR PURELY POTENTIAL CONSERVATION LAWS

The main applications of the results collected in Theorem 1 are connected with the construction of (nonlocal) potential conservation laws and hierarchies of potential systems. At first sight it appears that they are important mostly for those approaches to finding conservation laws which involve the characteristic form (1) of conservation laws or its consequences (2) and (3), including the Noether symmetry approach.[2,3,9,21] (A detailed comparative analysis of different methods of finding conservation laws and their realizations is given in Ref. 32.) A more careful consideration reveals that these results are also important for the direct method based on the definitions of conserved vectors and conservation laws.[22] Given a conserved vector depending on derivatives of potentials, usually it is difficult to test whether this conserved vector is equivalent to a conserved vector which does not depend on potentials. The reason of the difficulty is the duplicity of the equivalence relation of conserved vectors, which is generated by summands of two kinds—null divergences and tuples of differential functions identically vanishing on the solution set of the corresponding system of differential equations. That is why it seems impossible to formulate, directly in terms of conserved vectors, an effective criterion for testing whether a conservation law of a potential system is induced by a conservation law of the corresponding initial system. At the same time, such a criterion is easily formulated in terms of characteristics.

Proposition 8: *Let a system $\mathcal{L}$ be totally nondegenerate with respect to a weight, $\mathcal{L}_{\rm p}$ be a system determining an Abelian covering of $\mathcal{L}$ (a potential system of $\mathcal{L}$ in the two-dimensional case). Moreover, let a characteristic λ of $\mathcal{L}_{\rm p}$ be completely reduced, i.e., the derivatives of potentials of orders greater than 0 are excluded from λ due to differential consequences of the potential part of $\mathcal{L}_{\rm p}$ and then the constrained derivatives of u are excluded from λ due to differential consequences of $\mathcal{L}$. Then the characteristic λ is associated with a conservation law of $\mathcal{L}_{\rm p}$, which is not induced by a conservation law of $\mathcal{L}$, if and only if it depends on potentials.*

Proof. If a characteristic λ of $\mathcal{L}_{\rm p}$ is completely reduced and depends on potentials, then it is unconditionally inequivalent to any characteristic free from all derivatives of potentials. That is why the necessary statement directly follows from Theorem 8 (Theorem 7). □

Let us consider the two-dimensional case in some more detail, employing the notations of Sec. VI. Suppose that a conserved vector (F,G) of a potential system $\mathcal{L}_{\rm p}$ is associated with a characteristic

$$\lambda=(\alpha^s[u],\ \beta^s[u],\ \gamma^\nu[u],\ s=1,\dots,p,\ \nu\in\mathcal{N}),$$

which does not depend on derivatives of potentials. Then we can algorithmically find a conserved vector $(\widetilde{F},\widetilde{G})$ which is equivalent to (F,G) and also does not depend on derivatives of potentials, avoiding the direct application of the complicated formula from Theorem 4. The algorithm is based on the proof of Lemma 4. Since each tuple (α^s,β^s) is a null divergence, there exist differential functions $\Phi^s[u]$ such that $D_x\Phi^s=\alpha^s$ and $D_t\Phi^s=-\beta^s$. Then the conserved vector with the components

$$\hat{F}=F+\Phi^s(v^s_x-F^s),\quad \hat{G}=G-\Phi^s(v^s_t+G^s)$$

is equivalent to the initial conserved vector (F,G) since the difference of (F,G) and $(\hat{F},\hat{G})$ vanishes on the solution set of $\mathcal{L}_{\rm p}$, and the total divergence of $(\hat{F},\hat{G})$ is a differential function of u. This means that the conserved vector $(\widetilde{F},\widetilde{G})$ differs from $(\hat{F},\hat{G})$ by a null divergence whose components are, in general, differential functions of u and v. See the next section for examples on the application of this procedure.

Suppose that the potential system $\mathcal{L}_{\rm p}$ has q linearly independent conservation laws induced by conservation laws of the initial system $\mathcal{L}$. Let the tuples $(\widetilde{F}^{\varsigma},\widetilde{G}^{\varsigma})$, $\varsigma=1,\dots,q$, be conserved vectors of these conservation laws which do not depend on derivatives of potentials. The second-level potential system (see Ref. 22, for definitions) constructed from $\mathcal{L}_{\rm p}$ with the conserved vectors $(\widetilde{F}^{\varsigma},\widetilde{G}^{\varsigma})$, $\varsigma=1,\dots,q$, is equivalent, with respect to a local transformation changing only potentials, to the first-level potential system $\mathcal{L}'_{\rm p}$ obtained from $\mathcal{L}$ with the conserved vectors (F^s,G^s), $s=1,\dots,p$, and $(\widetilde{F}^{\varsigma},\widetilde{G}^{\varsigma})$, $\varsigma=1,\dots,q$ (cf. the end of Sec. II). The potential part of $\mathcal{L}'_{\rm p}$ differs from the potential part of $\mathcal{L}_{\rm p}$ in the equations $v^{p+\varsigma}_x=\widetilde{F}^{\varsigma}$, $v^{p+\varsigma}_t=-\widetilde{G}^{\varsigma}$, $\varsigma=1,\dots,q$. An analogous argument holds for potential systems of an arbitrary level.

XI. AN EXAMPLE

To present an illustrative example, we give a new detailed interpretation of results from Ref. 22 on hierarchies of conservation laws and potential systems of diffusion-convection equations, involving tools developed in this paper. See also Refs. 16, 22, and 24, for the method of classification of potential conservation laws for a class of differential equations with respect to the equivalence group of this class.

The class of diffusion-convection equations of the general form

$$u_t=(A(u)u_x)_x+B(u)u_x, \tag{18}$$

where $A=A(u)$ and $B=B(u)$ are arbitrary smooth functions of u, $A\neq 0$, possesses the equivalence group $G^{\sim}$ formed by the transformations

$$\widetilde{t}=\varepsilon_4 t+\varepsilon_1,\quad \widetilde{x}=\varepsilon_5 x+\varepsilon_7 t+\varepsilon_2,\quad \widetilde{u}=\varepsilon_6 u+\varepsilon_3,\quad \widetilde{A}=\varepsilon_4^{-1}\varepsilon_5^2 A,\quad \widetilde{B}=\varepsilon_4^{-1}\varepsilon_5 B-\varepsilon_7,$$

where $\varepsilon_1,\dots,\varepsilon_7$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6\neq 0$. The kernel (intersection) $G^{\cap}$ of the maximal Lie invariance groups of equations from class (18) consists of the transformations $\widetilde{t}=t+\varepsilon_1$, $\widetilde{x}=x+\varepsilon_2$, $\widetilde{u}=u$.

Any equation from class (18) has the conservation law $\mathcal{F}^0$ whose density, flux, and characteristic are

$$\mathcal{F}^0=\mathcal{F}^0(A,B):\quad F=u,\quad G=-Au_x-\int B,\quad \lambda=1.$$

A complete list of $G^{\sim}$-inequivalent equations (18) having additional (i.e., linearly independent of $\mathcal{F}^0$) conservation laws is exhausted by the following ones:

$$B=0,\quad \mathcal{F}^1=\mathcal{F}^1(A):\quad F=xu,\quad G=\int A-xAu_x,\quad \lambda=x,$$

$$B=A,\quad \mathcal{F}^2=\mathcal{F}^2(A):\quad F=e^x u,\quad G=-e^x Au_x,\quad \lambda=e^x,$$

$$A=1,\quad B=0,\quad \mathcal{F}^3_h:\quad F=hu,\quad G=h_x u-hu_x,\quad \lambda=h.$$

where $\int A=\int A(u)du$, $\int B=\int B(u)du$, $h=h(t,x)$ is an arbitrary solution of the backward linear heat equation $h_t+h_{xx}=0$. (Along with constrains for A and B the above table also contains complete lists of densities, fluxes, and characteristics of additional conservation laws.)

General case: In the general case Eq. (18) has the unique linearly independent local conservation law $\mathcal{F}^0(A,B)$. The corresponding potential system

$$v^1_x = u, \quad v^1_t = Au_x + \int B$$

possesses only the zero conservation law, i.e., Eq. (18) of the general form admits no purely potential conservation laws.

B=0: Any equation with $B=0$ and a general value of A admits exactly two linearly independent local conservation laws $\mathcal{F}^0=\mathcal{F}^0(A,0)$ and $\mathcal{F}^1=\mathcal{F}^1(A)$, and up to linear dependence any conservation law is $G^\cap$-equivalent to one of them. Using these conservation laws, we introduce the potentials v^1 and v^2, where

$$v^1_x = u, \quad v^1_t = Au_x, \tag{19}$$

$$v^2_x = xu, \quad v^2_t = xAu_x - \int A. \tag{20}$$

The pairs of equations (19) and (20), considered separately, form two potential systems for Eq. (18) (with vanishing B) in the unknown functions (u,v^1) and (u,v^2), respectively. The third potential system is formed by (19) and (20) simultaneously, and the three functions u, v^1, and v^2 are assumed unknown. Since the characteristics $\lambda=1$ and $\lambda=x$ are nonsingular, the initial equation is a differential consequence of both the potential parts (19) and (20) and is not included in the minimal sets of equations representing the potential systems. Therefore, the characteristics of systems (19) and (20) have two components. The components β and α correspond to the first and second equations of these systems, respectively.

System (19) has only one linearly independent local conservation law $\mathcal{F}$ whose conserved vector $(F,G)=(v^1,-\int A)$ is associated with the characteristic $(\alpha,\beta)=(1,0)$. In view of Theorem 7, this conservation law is induced by a conservation law of the initial equation. Let us find a conserved vector $(\tilde{F},\tilde{G})$ which is equivalent to (F,G) and additionally does not depend on derivatives of potentials. The function Φ (see Sec. X) satisfies the equations $D_x\Phi=\alpha=1$ and $D_t\Phi=-\beta=0$. We choose the value $\Phi=x$ and consider the conserved vector $(\hat{F},\hat{G})$ equivalent to (F,G) with the components

$$\hat{F} = F + \Phi(v^1_x - u) = v^1 + x(v^1_x - u) = (xv^1)_x - xu,$$

$$\hat{G} = G - \Phi(v^1_t - Au_x) = -\int A - x(v^1_t - Au_x) = -(xv^1)_t - \int A + xAu_x.$$

Up to the summand $((xv^1)_x,-(xv^1)_t)$ which obviously is a null divergence, the conserved vector $(\hat{F},\hat{G})$ is equivalent to the conserved vector $(\tilde{F},\tilde{G})=(-xu,xAu_x-\int A)$ belonging to the conservation law $-\mathcal{F}^1$. That is why the "second-level" potential system,

$$v^1_x = u, \quad w^1_x = v^1, \quad w^1_t = \int A, \tag{21}$$

obtained from (19) by introducing the second-level potential w^1 with the conservation law $\mathcal{F}$ is, in fact, equivalent, with respect to the point transformation $w^1=xv^1-v^2$, to the "first-level" united potential systems (19) and (20). Although system (21) formally belongs to the second level, it is the most convenient one for further investigation since it has the simplest form among the potential systems constructed with two conservation laws from Eq. (18) with $B=0$.

Analogously, system (20) possesses only one linearly independent local conservation law $\mathcal{F}$ with the conserved vector $(F,G)=(x^{-2}v^2,-x^{-1}\int A)$ and the characteristic $(\alpha,\beta)=(x^{-2},0)$. Theorem 7 implies that this conservation law is induced by a conservation law of the initial equation. As a solution of the equations $D_x\Phi=\alpha=x^{-2}$ and $D_t\Phi=-\beta=0$, we choose the value $\Phi=-x^{-1}$. Then

$$\hat{F} = x^{-2}v^2 - x^{-1}(v_x^2 - xu) = -(x^{-1}v^2)_x + u,$$

$$\hat{G} = -x^{-1}\int A + x^{-1}\left(v_t^2 - xAu_x + \int A\right) = -(x^{-1}v^2)_t - Au_x.$$

The conserved vector $(\hat{F},\hat{G})$ is equivalent, by construction, to (F,G) on the solution set of (21). Up to the null divergence $((x^{-1}v^2)_x,-(x^{-1}v^2)_t)$, it is also equivalent to the conserved vector $(\tilde{F},\tilde{G})=(u,-Au_x)$ which depends only on derivatives of u and belongs to the conservation law $\mathcal{F}^0$. Therefore the second-level potential system,

$$v_x^2 = xu, \quad w_x^2 = x^{-2}v^2, \quad w_t^2 = x^{-1}\int A,$$

obtained from (20) by introducing the second-level potential w^2 with the conservation law $\mathcal{F}$ is also equivalent, with respect to the point transformation $w^2=v^1-x^{-1}v^2$ to the united systems (19) and (20).

The space of conservation laws of the united systems (19) and (20) is zero dimensional. Therefore, for any equation (18) with $B=0$ all potential conservation laws are induced by local ones and all inequivalent potential systems are exhausted by systems (19)–(21).

B=A: This case is analyzed in a way similar to the previous one. Any equation with $B=A$ and a general value of A has a two-dimensional space of local conservation laws generated by $\mathcal{F}^0=\mathcal{F}^0(A,A)$ and $\mathcal{F}^2=\mathcal{F}^2(A)$, and up to linear dependence any conservation law is $G^{\cap}$-equivalent to either $\mathcal{F}^0$ or $\mathcal{F}^2+\varepsilon\mathcal{F}^0$, where $\varepsilon\in\{0,\pm1\}\mathrm{mod}\,G^{\cap}$. Using the conservation laws $\mathcal{F}^0$ and $\mathcal{F}^2+\varepsilon\mathcal{F}^0$, we can introduce the independent potentials v^1 and v^3, satisfying the conditions

$$v_x^1 = u, \quad v_t^1 = Au_x + \int A, \tag{22}$$

$$v_x^3 = (e^x+\varepsilon)u, \quad v_t^3 = (e^x+\varepsilon)Au_x + \varepsilon\int A. \tag{23}$$

The pairs of equations (22) and (23) considered separately form two potential systems for Eq. (18) with $B=A$ in the unknown functions (u,v^1) and (u,v^3), respectively. The third potential system is formed by Eqs. (22) and (23) simultaneously, and the three functions u, v^1, and v^3 are assumed as unknown. Since the characteristics $\lambda=1$ and $\lambda=e^x+\varepsilon$ are nonsingular, the initial equation is a differential consequence of both the potential parts (22) and (23) and is not included in the minimal sets of equations representing the potential systems. Therefore, characteristics of systems (22) and (23) have two components. The components β and α correspond to the first and second equations of these systems, respectively.

System (22) has only one linearly independent local conservation law $\mathcal{F}$ whose conserved vector $(F,G)=(e^xv^1,-e^x\int A)$ is associated with the characteristic $(\alpha,\beta)=(e^x,0)$. We choose the solution $\Phi=e^x$ of the equations $D_x\Phi=\alpha=1$ and $D_t\Phi=-\beta=0$ and put

$$\hat{F} = e^xv^1 + e^x(v_x^1 - u) = (e^xv^1)_x - e^xu,$$

$$\hat{G} = -e^x\int A - e^x\left(v_t^1 - Au_x - \int A\right) = -(e^xv^1)_t + e^xAu_x.$$

The conserved vector $(\hat{F},\hat{G})$ is equivalent to (F,G) by construction and, up to the null divergence $((e^xv^1)_x,-(e^xv^1)_t)$, is equivalent to the conserved vector $(\tilde{F},\tilde{G})=(-e^xu,e^xAu_x)$. This vector does not depend on the potential v^1 and belongs to the conservation law $-\mathcal{F}^2$. Hence the conservation

law $\mathcal{F}$ of the potential system (23) is induced by the conservation law $-\mathcal{F}^2$ of the initial equation. Therefore, the second-level potential system,

$$v^1_x = u, \quad w^1_x = e^x v^1, \quad w^1_t = e^x \int A, \tag{24}$$

obtained from (23) by introducing the second-level potential w^1 with the conservation law $\mathcal{F}$ is equivalent, with respect to the point transformation $w^1 = e^x v^1 - v^3$, to the united systems (22) and (23), where $\varepsilon = 0$. Although system (24) formally belongs to the second level, it is most convenient for our further investigation among the potential systems constructed with two conservation laws from Eq. (18) with $B = A$ since it has the simplest form.

System (23) also admits only one linearly independent local conservation law $\mathcal{F}$ which contains the conserved vector $(F, G) = (e^x(e^x+\varepsilon)^{-2}v^3, -e^x(e^x+\varepsilon)^{-1}\int A)$ associated with the characteristic $(\alpha, \beta) = (e^x(e^x+\varepsilon)^{-2}, 0)$ and, hence, is induced by a conservation law of the initial equation in view of Theorem 7. We choose the solution $\Phi = -(e^x+\varepsilon)^{-1}$ of the equations $D_x\Phi = \alpha$ and $D_t\Phi = -\beta$ and put

$$\hat{F} = \frac{e^x v^3}{(e^x+\varepsilon)^2} - \frac{v^3_x - (e^x+\varepsilon)u}{e^x+\varepsilon} = -\left(\frac{v^3}{e^x+\varepsilon}\right)_x + u,$$

$$\hat{G} = \frac{e^x \int A}{e^x+\varepsilon} + \frac{v^3_t - (e^x+\varepsilon)Au_x - \varepsilon \int A}{e^x+\varepsilon} = \left(\frac{v^3}{e^x+\varepsilon}\right)_t - Au_x - \int A.$$

Again the conserved vector $(\hat{F}, \hat{G})$ is equivalent to (F, G) and up to a null divergence is also equivalent to the conserved vector $(\tilde{F}, \tilde{G}) = (u, -Au_x - \int A)$ which depends only on derivatives of u and belongs to the conservation law $\mathcal{F}^0$. Therefore the second-level potential system,

$$v^3_x = (e^x+\varepsilon)u, \quad w^3_x = \frac{e^x}{(e^x+\varepsilon)^2}v^3, \quad w^3_t = \frac{e^x}{e^x+\varepsilon}\int A,$$

obtained from (23) by introducing the second-level potential w^3 with the conservation law $\mathcal{F}$ is also equivalent, with respect to the point transformation $w^3 = v^1 - (e^x+\varepsilon)^{-1}v^3$, to the united systems (22) and (23).

The space of conservation laws of the united systems (22) and (23) is zero dimensional. Therefore, for any equation (18) with $B = A$, all potential conservation laws are induced by local ones and all inequivalent potential systems are exhausted by systems (22)–(24).

$B = \int A + uA$: From the point of view of local conservation laws, this case does not differ from the general one. Any equation from class (18) with such a value of B and an arbitrary value of A has the unique linearly independent local conservation law $\mathcal{F}^0 = \mathcal{F}^0(A, \int A + uA)$. At the same time, the corresponding potential system

$$v^1_x = u, \quad v^1_t = Au_x + u\int A \tag{25}$$

also admits the unique linearly independent local conservation law $\mathcal{F}^4 = \mathcal{F}^4(A)$ with the conserved vector $(F, G) = (e^{v^1}, -e^{v^1}\int A)$ and the characteristic $(\alpha, \beta) = (e^{v^1}, -e^{v^1}\int A)$. Since the characteristic is completely reduced and depends on the potential v^1, in view of Proposition 8 the conservation law $\mathcal{F}^4$ is not induced by a local conservation law of the initial equation, i.e., it is a purely potential conservation law. The potential system (25) is reduced to the potential system (22) by means of the potential hodograph transformation,

$$\tilde{t}=t,\quad \tilde{x}=v^1,\quad \tilde{v}^1=x,\quad \tilde{u}=u^{-1},\quad \tilde{A}=u^2A, \tag{26}$$

and the conservation law $\mathcal{F}^4$ is mapped to the one induced by $\mathcal{F}^2$. The same transformation extended by the formula $\tilde{w}=-w+v^1e^x$ to the second-level potential w introduced with $\mathcal{F}^4$ also reduces the second-level potential system $v_x=u$, $w_x=e^v$, $w_t=e^v\int A$ to system (24). As a result, although any equation from class (18) with $B=\int A+uA$ admits a nontrivial potential conservation law, this case does not give principally new potential systems.

Linear heat equation: The space of local conservation laws of the linear heat equation $u_t=u_{xx}$ is infinite dimensional and formed by $\mathcal{F}^4_h$, where $h=h(t,x)$ runs through solutions of the backward linear heat equation $h_t+h_{xx}=0$.[12] Fixing an arbitrary $p\in\mathbb{N}$ and choosing p linearly independent solutions $h^1,\dots,h^p$ of the backward linear heat equation, we obtain p linearly independent conservation laws $\mathcal{F}^4_{h^1},\dots,\mathcal{F}^4_{h^p}$. In view of Theorem 5 of Ref. 22 (see also Lemma 6 of Ref. 24), the potentials $v^1,\dots,v^p$ introduced for these conservation laws by

$$v^s_x=h^su,\quad v^s_t=h^su_x-h^s_xu,\quad s=1,\dots,p \tag{27}$$

are independent in the sense of Definition 12. According to Theorem 8 of Ref. 22 or Theorem 5 of Ref. 24, any local conservation law of system (27) is induced by a local conservation law of the linear heat equation. As a result, the systems of the form (27) exhaust all possible potential systems of the linear heat equation and all potential conservation laws of this equations are induced by local ones.

Linearizable equations: Up to $G^{\sim}$-equivalence, class (18) contains three linearizable equations. These are the u^{-2}-diffusion equation $u_t=(u^{-2}u_x)_x$,[7,26] the related equation $u_t=(u^{-2}u_x)_x+u^{-2}u_x$,[13,27] and the Burgers equation $u_t=u_{xx}+2uu_x$.[14,15,10] These equations are well known to be linearized by nonlocal transformations [the so-called potential equivalence transformations in the class (18) (Refs. 23 and 19)] to the linear heat equation. While possessing the usual properties concerning local conservation laws, they are distinguished from the other diffusion-convection equations of the form (18) by possessing an infinite number of linearly independent purely potential conservation laws.

The u^{-2}-diffusion equation $\mathbf{u_t=(u^{-2}u_x)_x}$ admits, as a subcase of the case $B=0$, two linearly independent local conservation laws $\mathcal{F}^0=\mathcal{F}^0(u^{-2},0)$ and $\mathcal{F}^1=\mathcal{F}^1(u^{-2})$. The potential system constructed by $\mathcal{F}^1$ has the form (20) with $A=u^{-2}$ and possesses the same properties as for general A (see the case $B=0$). The conservation law $\mathcal{F}^0$ gives a potential system of the form (19) with $A=u^{-2}$, whose space of local conservation laws, in contrast to the general value of A, is infinite dimensional and consists of the conservation laws $\mathcal{F}^5_\sigma$ with the conserved vectors $(F,G)=(\sigma,\sigma_vu^{-1})$ and the characteristics $(\alpha,\beta)=(\sigma_v,-\sigma_tu^{-1})$. Here the parameter function $\sigma=\sigma(t,v)$ runs through the solution set of the backward linear heat equation $\sigma_t+\sigma_{vv}=0$ and the potential v^1 is redenoted by v. Since any of the above characteristics is completely reduced and depends on the potential v in the case of $\sigma_{vv}\neq0$, then in view of Proposition 8 each conservation law $\mathcal{F}^5_\sigma$ with $\sigma_{vv}\neq0$ is not induced by a local conservation law of the initial equation, i.e., it is a purely potential conservation law. The conservation law $\mathcal{F}^5_v$ is induced by $\mathcal{F}^2=\mathcal{F}^2(u^{-2})$ and $\mathcal{F}^5_1$ is the zero conservation law.

The u^{-2}-diffusion equation is reduced to the linear heat equation[7] by the potential hodograph transformation (26). More precisely, the transformation (26) is a local transformation between the corresponding potential systems $v_x=u$, $v_t=u^{-2}u_x$, and $v_x=u$, $v_t=u_x$, constructed by means of the conservation laws $\mathcal{F}^0(u^{-2},0)$ and $\mathcal{F}^0(1,0)=\mathcal{F}^4_1$, respectively. Hence the action of (26) maps each of these conservation laws to zero of the target system. Moreover, the transformation (26) provides the correspondence between the conservation laws $\mathcal{F}^5_\sigma$ and $\mathcal{F}^4_h$ with the same values of the parameter functions $\sigma(t,v)=h(\tilde{t},\tilde{x})$.

After fixing an arbitrary $p\in\mathbb{N}$ and choosing p solutions $\sigma^1,\dots,\sigma^p$ of the backward linear heat equation any of whose linear combinations is not a constant, we construct the second-level potential system $\mathcal{S}$ from system (19) with $A=u^{-2}$ using the p linearly independent conservation laws $\mathcal{F}^5_{\sigma^1},\dots,\mathcal{F}^5_{\sigma^p}$. The system $\mathcal{S}$ is pointwise equivalent to the potential system of the linear heat

equation, associated with the conservation laws $\mathcal{F}^4_1$, $\mathcal{F}^4_{\sigma^1},\dots,\mathcal{F}^4_{\sigma^p}$. The above results on conservation laws of the linear heat equation imply that any conservation law of $\mathcal{S}$ is induced by a conservation law of system (19) with $A=u^{-2}$. Consequently, this case does not give principally new potential systems although the u^{-2}-diffusion equation admits an infinite-dimensional space of first-level potential conservation laws connected with system (19).

Since the equation $\mathbf{u_t}=\mathbf{(u^{-2}u_x)_x}+\mathbf{u^{-2}u_x}$ is reduced to the u^{-2}-diffusion equation by the point transformation $\tilde t=t$, $\tilde x=e^x$, $\tilde u=e^{-x}u$, its conservation laws are connected with ones of the linear heat equation in a way similar to the previous case. Thus, the space of local conservation laws of the equation $u_t=(u^{-2}u_x)_x+u^{-2}u_x$ is the usual one for the case $B=A$. It is generated by two linearly independent conservation laws $\mathcal{F}^0=\mathcal{F}^0(u^{-2},u^{-2})$ and $\mathcal{F}^2=\mathcal{F}^2(u^{-2})$. The potential system associated with $\mathcal{F}^2+\varepsilon\mathcal{F}^0$ is of the form (23) with $A=u^{-2}$. Its properties are as usual for the case $B=A$. At the same time, the other inequivalent potential system which is associated with $\mathcal{F}^0$ possesses an infinite-dimensional space of local conservation laws, equal to $\{\mathcal{F}^6_\sigma\}$. Here $\mathcal{F}^6_\sigma$ is a conservation law with the conserved vector $(\sigma e^x,\sigma_v u^{-1}e^x)$ and the characteristic $(\sigma_v e^x,-\sigma_t u^{-1}e^x)$. The parameter function $\sigma=\sigma(t,v)$ again runs through the solution set of the backward linear heat equation $\sigma_t+\sigma_{vv}=0$ and the potential v^1 is redenoted by v. Since any of the above characteristics is completely reduced and depends on the potential v in the case of $\sigma_{vv}\neq0$, then in view of Proposition 8 each conservation law $\mathcal{F}^6_\sigma$ with $\sigma_{vv}\neq0$ is not induced by a local conservation law of the initial equation, i.e., it is a purely potential conservation law. At the same time, these conservation laws lead to potential systems which are equivalent to potential systems of the linear heat equation, which have form (27).

The Burgers equation $\mathbf{u_t}=\mathbf{u_{xx}}+\mathbf{2uu_x}$ is distinguished from the equations of the form (18) with $B=\int A+uA$ through its potential conservation laws. As any equation with $B=\int A+uA$, it possesses the unique linearly independent local conservation law $\mathcal{F}^0=\mathcal{F}^0(1,2u)$. The associated potential system $v_x=u$, $v_t=u_x+u^2$ has the infinite-dimensional space of conservation laws $\mathcal{F}^7_h$ with the conserved vectors $(he^v,h_xe^v-hue^v)$ and the characteristics $(he^v,h_xe^v-hue^v)$. Here the parameter function $h=h(t,x)$ runs through the solution set of the backward linear heat equation $h_t+h_{xx}=0$. Any of the above characteristics is completely reduced and depends on the potential v if $h\neq0$. Hence in view of Proposition 8 each conservation law $\mathcal{F}^7_h$ with $h\neq0$ is not induced by a local conservation law of the initial equation, i.e., it is a purely potential conservation law.

The potential system $v_x=u$, $v_t=u_x+u^2$ of the Burgers equation $u_t=u_{xx}+2uu_x$ is mapped to the potential system $\tilde v_{\tilde x}=\tilde u$, $\tilde v_{\tilde t}=\tilde u_{\tilde x}$ (constructed from the linear heat equation $\tilde u_{\tilde t}=\tilde u_{\tilde x\tilde x}$ with the "common" conservation law $\mathcal{F}^0(1,0)=\mathcal{F}^4_1$) by the point transformation,

$$\tilde t=t,\quad \tilde x=x,\quad \tilde u=ue^v,\quad \tilde v=e^v.$$

This transformation establishes the correspondence between the conservation law $\mathcal{F}^7_{h_x}$ and the conservation law of the potential system $\tilde v_{\tilde x}=\tilde u$, $\tilde v_{\tilde t}=\tilde u_{\tilde x}$ induced by $\mathcal{F}^4_h$. Note that if the parameter function $h=h(t,x)$ is a solution of the backward linear heat equation, then its derivative h_x also is a solution of the same equation. The famous Cole–Hopf transformation[10,15] (first found in Ref. 14) is a consequence of the above transformation and, in fact, linearizes the Burgers equation to the linear heat equation with respect to the potential $\tilde v$.[19,23]

For some $p\in\mathbb{N}$ we choose p solutions $h^1,\dots,h^p$ of the backward linear heat equation such that any of their linear combinations is not a constant. The second-level potential system $\mathcal{S}$ constructed from the potential system $v_x=u$, $v_t=u_x+u^2$ using the p linearly independent conservation laws $\mathcal{F}^7_{h^1_x},\dots,\mathcal{F}^7_{h^p_x}$ is pointwise equivalent to the potential system of the linear heat equation, associated with the conservation laws $\mathcal{F}^4_1$, $\mathcal{F}^4_{h^1},\dots,\mathcal{F}^4_{h^p}$. The above results on conservation laws of the linear heat equation imply that any conservation law of $\mathcal{S}$ is induced by a conservation law of the potential system $v_x=u$, $v_t=u_x+u^2$. Therefore this case gives only potential systems which are pointwise equivalent to systems of the form (27).

XII. POTENTIAL INDETERMINACY AND POTENTIAL CONSERVATION LAWS

Suppose that $\mathcal{L}_{\rm p}$ is a system determining an Abelian covering of $\mathcal{L}$ (a potential system of $\mathcal{L}$ in the two-dimensional case). The potential part of $\mathcal{L}_{\rm p}$ consisting of Eq. (10) defines the potentials $v^1,\dots,v^p$ up to arbitrary constant summands. This means that the system $\mathcal{L}_{\rm p}$ is invariant with respect to the gauge transformations of the form $\tilde{x}_i=x_i$, $\tilde{u}^a=u^a$, and $\tilde{v}^s=v^s+c_s$, where $c_s=$const, i.e., the operators ∂_{v^s} belong to the maximal Lie invariance group of $\mathcal{L}_{\rm p}$. It is well known that, acting by an appropriately prolonged generalized symmetry operator of a system of differential equations on a conserved vector of the same system, one obtains a conserved vector of this system (cf. Ref. 21 Proposition 5.64). Due to the special structure of $\mathcal{L}_{\rm p}$, the statement on the action by the operators ∂_{v^s} to conserved vectors of $\mathcal{L}_{\rm p}$ can be formulated more precisely.

Proposition 9: *Any derivative of any conserved vector of $\mathcal{L}_{\rm p}$ with respect to potentials is a conserved vector of $\mathcal{L}_{\rm p}$. The same derivative of a characteristic of the conservation law containing the initial conserved vector represents a characteristic associated with the differentiated conserved vector.*

Proof: Let $F\in{\rm CV}(\mathcal{L}_{\rm p})$. In view of Proposition 1, there exist differential functions $\bar{\lambda}^{si}[u,v]$ and $\lambda^{\nu}[u,v]$ and an n-tuple $\hat{F}$ vanishing on the solutions of $\mathcal{L}_{\rm p}$ such that

$$D_iF^i=\bar{\lambda}^{si}(v^s_i-G^{si})+\lambda^{\nu}L^{\nu}+D_i\hat{F}^i.$$

The functions $\bar{\lambda}^{si}$ and λ^{ν} are the components of a characteristic of the conservation law containing F. For a fixed value of s, we act on the latter equality with the infinite prolongation of the operator ∂_{v^s}, which formally coincides with ∂_{v^s}, and use the property of commutation of any infinitely prolonged operator with each total differentiation operator,

$$D_iF^i_{v^s}=\bar{\lambda}^{si}_{v^s}(v^s_i-G^{si})+\lambda^{\nu}_{v^s}L^{\nu}+D_i\hat{F}^i_{v^s}.$$

Since ∂_{v^s} is a symmetry operator of $\mathcal{L}_{\rm p}$ then $\hat{F}^i_{v^s}$ vanishes on the solutions of $\mathcal{L}_{\rm p}$. Therefore, F_{v^s} is a conserved vector of $\mathcal{L}_{\rm p}$ and $(\bar{\lambda}^{si}_{v^s},\lambda^{\nu}_{v^s})$ is a characteristic of the conservation law containing this conserved vector. □

Moreover, there exists an interesting connection between conserved vectors and characteristics of the potential systems determining Abelian coverings.

Proposition 10: *For any fixed value of s, the components of an arbitrary characteristic λ of a conservation law of the system $\mathcal{L}_{\rm p}$, which corresponds to the equations defining the potential v^s, form a conserved vector of $\mathcal{L}_{\rm p}$ belonging to the conservation law with the characteristic $-\lambda_{v^s}$.*

Proof: Since $\lambda\in{\rm Ch}(\mathcal{L}_{\rm p})$, there exists a conserved vector F of $\mathcal{L}_{\rm p}$ such that

$$\bar{\lambda}^{si}(v^s_i-G^{si})+\lambda^{\nu}L^{\nu}=D_iF^i. \tag{28}$$

Applying the component E_{v^s} of the extended Euler operator to Eq. (28), we obtain

$$0=D_i\bar{\lambda}^{si}+D^{\alpha}(\bar{\lambda}^{\sigma i}_{v^{\sigma}_{\alpha}}(v^{\sigma}_i-G^{\sigma i}))+D^{\alpha}(\lambda^{\nu}_{v^{\sigma}_{\alpha}}L^{\nu}),$$

where α runs through the multi-index set. The derived equality implies that $(\bar{\lambda}^{s1},\dots,\bar{\lambda}^{sn})$ is a conserved vector of $\mathcal{L}_{\rm p}$ since all the summands excluding $D_i\bar{\lambda}^{si}$ obviously vanish on the solutions of $\mathcal{L}_{\rm p}$. This equality can be represented as a characteristic form of a conservation law of $\mathcal{L}_{\rm p}$,

$$D_i\bar{\lambda}^{si}=-\bar{\lambda}^{\sigma i}_{v^{\sigma}}(v^{\sigma}_i-G^{\sigma i})-\lambda^{\nu}_{v^{\sigma}}L^{\nu}-\sum_{|\alpha|>0}D^{\alpha}(\bar{\lambda}^{\sigma i}_{v^{\sigma}_{\alpha}}(v^{\sigma}_i-G^{\sigma i}))-\sum_{|\alpha|>0}D^{\alpha}(\lambda^{\nu}_{v^{\sigma}_{\alpha}}L^{\nu}),$$

which associates the conserved vector $(\bar{\lambda}^{s1},\dots,\bar{\lambda}^{sn})$ with the characteristic $(-\bar{\lambda}^{\sigma i}_{v^s},-\lambda^{\nu}_{v^s})$. Therefore, the conserved vector $(\bar{\lambda}^{s1},\dots,\bar{\lambda}^{sn})$ is equivalent to the conserved vector $-F_{v^s}$. □

Good illustrative examples for the above statements are given by linearizable diffusion-convection equations (cf. the previous section). Thus, the potential system $v_x=u$, $v_t=u^{-2}u_x$ of the

u^{-2}-diffusion equation $u_t=(u^{-2}u_x)_x$ possesses the infinite-dimensional space of the local conservation laws $\mathcal{F}^5_\sigma$ with the conserved vectors $(\sigma,\sigma_v u^{-1})$ and the characteristics $(\sigma_v,-\sigma_t u^{-1})$. Here the parameter function $\sigma=\sigma(t,v)$ runs through the solution set of the backward linear heat equation $\sigma_t+\sigma_{vv}=0$. Recall that the potential v^1 is re-denoted by v. Since the derivative σ_v also is a solution of the backward linear heat equation, the image of the conserved vector $(\sigma,\sigma_v u^{-1})$ under the action of the operator ∂_v is a conserved vector belonging to the conservation law $\mathcal{F}^5_{\sigma_v}$. The characteristic $(\sigma_v,-\sigma_t u^{-1})$ of $\mathcal{F}^5_\sigma$ coincides with this conserved vector from $\mathcal{F}^5_{\sigma_v}$. Analogously, the local conservation laws of the potential system $v_x=u$, $v_t=u^{-2}u_x-u^{-1}$ of the equation $u_t=(u^{-2}u_x)_x+u^{-2}u_x$ is exhausted by the conservation laws $\mathcal{F}^6_\sigma$ with the conserved vectors $(\sigma e^x,\sigma_v u^{-1}e^x)$ and the characteristics $(\sigma_v e^x,-\sigma_t u^{-1}e^x)$. The action of the operator ∂_v maps the conserved vector $(\sigma e^x,\sigma_v u^{-1}e^x)$ to the conserved vector $(\sigma_v e^x,\sigma_{vv}u^{-1}e^x)$ which belongs to the conservation law $\mathcal{F}^6_{\sigma_v}$ and coincides with the characteristic $(\sigma_v e^x,-\sigma_t u^{-1}e^x)$ of $\mathcal{F}^6_\sigma$. Any local conservation law of the potential system $v_x=u$, $v_t=u_x+u^2$ associated with the Burgers equation $u_t=u_{xx}+2uu_x$ has a conserved vector and a characteristic of the same form $(he^v,h_x e^v-hue^v)$, where the parameter function $h=h(t,x)$ runs through the solution set of the backward linear heat equation $h_t+h_{xx}=0$. The action of ∂_v does not change such conserved vectors and characteristics. This explains in view of Proposition 10 why they have the same form. A similar observation is true for diffusion-convection equations of the form (18) with $B=\int A+uA$. Each of these equations has the unique linearly independent potential conservation law $\mathcal{F}^4=\mathcal{F}^4(A)$ with the coinciding conserved vector $(F,G)=(e^v,-e^v\int A)$ and characteristic $(\alpha,\beta)=(e^v,-e^v\int A)$. The action of the operator ∂_v maps $\mathcal{F}^4$ to itself. The other non-linearizable diffusion-convection equations admit only potential conservation laws induced by local ones, which are mapped by differentiations with respect to potentials to the zero conservation laws of these equations.

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Paper 5

Symmetry preserving parameterization schemes

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Symmetry preserving parameterization schemes

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Methods for the design of physical parameterization schemes that possess certain invariance properties are discussed. These methods are based on different techniques of group classification and provide means to determine expressions for unclosed terms arising in the course of averaging of nonlinear differential equations. The demand that the averaged equation is invariant with respect to a subalgebra of the maximal Lie invariance algebra of the unaveraged equation leads to a problem of inverse group classification which is solved by the description of differential invariants of the selected subalgebra. Given no prescribed symmetry group, the direct group classification problem is relevant. Within this framework, the algebraic method or direct integration of determining equations for Lie symmetries can be applied. For cumbersome parameterizations, a preliminary group classification can be carried out. The methods presented are exemplified by parameterizing the eddy vorticity flux in the averaged vorticity equation. In particular, differential invariants of (infinite-dimensional) subalgebras of the maximal Lie invariance algebra of the unaveraged vorticity equation are computed. A hierarchy of normalized subclasses of generalized vorticity equations is constructed. Invariant parameterizations possessing minimal symmetry extensions are described and a restricted class of invariant parameterization is exhaustively classified. The physical importance of the parameterizations designed is discussed.
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I. INTRODUCTION

The problem of parameterization is one of the most important issues in modern dynamic meteorology and climate research.[20,46] As even the most accurate present days numerical models are not capable to resolve all small scale features of the atmosphere, there is a necessity for finding ways to incorporate these unresolved processes in terms of the resolved ones. This technique is referred to as parameterization. The physical processes being parameterized in numerical weather and climate prediction models can be quite different, including, e.g., cumulus convection, momentum, heat and moisture fluxes, gravity wave drag, and vegetation effects. The general problem of parameterization is intimately linked to the design of closure schemes for averaged (or filtered) nonlinear equations. By averaging, a nonlinear differential equation becomes unclosed, that is, there arise additional terms for which no prognostic or diagnostic equation exist. These terms must hence be re-expressed in a physically reasonable way to be included in the averaged equations.

It has been noted in Ref. 47 that every parameterization scheme ought to retain some basic properties of the unresolved terms, which must be expressed by the resolved quantities. These properties include, just to mention a few, correct dimensionality, tensorial properties, invariance under changes of the coordinate system and invariance with respect to Galilean transformations. While the formulation of a parameterization scheme with correct dimensions is in general a straightforward

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task, not all parameterization schemes that have been used in practice are indeed Galilean invariant. An example for this finding is given by the classical Kuo convection scheme.[23,24] In this scheme, it is assumed that the vertically integrated time-change of the water vapor at a point locally balances a fraction of the observed precipitation rate [p. 528 of Ref. 12]. This also implies that the moisture convergence is proportional to the precipitation rate. However, while the precipitation rate is clearly a Galilean invariant quantity, the moisture convergence depends on the motion of the observer.[21] That is, the Kuo scheme does not properly account for pure symmetry constraints, which is a potential source of unphysical effects in the results of a numerical model integration.

The latter finding is the main motivation for the present investigations. Galilean invariance is an important example for a Lie symmetry, but it is by no means the only invariance characteristic that might be of importance in the course of the parameterization process. This is why it is reasonable to focus on parameterization schemes that also preserve other symmetries. This is not an academic task. Almost all real-world processes exhibit miscellaneous symmetry characteristics. These characteristics are reflected in the symmetry properties of differential equations and correspondingly should also be reflected in case where these processes cannot be explicitly modeled by differential equations, i.e., in the course of parameterizations. What is hence desirable is a constructive method for the design of symmetry-preserving parameterization schemes. It is the aim of this paper to demonstrate that techniques from group analysis do provide such constructive methods. In particular, we state the following proposition:

Any problem of finding invariant parameterizations is a group classification problem.

Implications following from the above proposition form the core of the present study. It appears that this issue was first opened in Ref. 33 dealing with the problem of turbulence closure of the averaged Navier–Stokes equations. We aim to build on this approach and extend it in several directions. As the equations of hydrodynamics and geophysical fluid dynamics usually possess wide symmetry groups,[2,5,8,14,16] the design of symmetry-preserving parameterizations will in general lead to a great variety of different classes of invariant schemes.

Needless to say that the parameterization problem is too comprehensive both in theory and applications to be treated exhaustively in a single paper. Therefore, it is crucial to restrict to a setting that allows to demonstrate the basic ideas of invariant parameterizations without overly complicating the presentation by physical or technical details. This is the reason for illustrating the invariant parameterization procedure with the rather elementary barotropic vorticity equation. For the sake of simplicity, we moreover solely focus on local closure schemes in the present study. That is, the quantities to be parameterized at each point are substituted with known quantities defined at the same respective point.[47] This renders it possible to thoroughly use differential equations and hence it will not be necessary to pass to integro-differential equations, as would be the case for nonlocal closure schemes. On the other hand, this restriction at once excludes a number of processes with essential nonlocal nature, such as atmospheric convection. Nevertheless, there are several processes that can be adequately described within the framework of the present paper, most notably different kinds of turbulent transport phenomena.

The organization of this paper is the following: Section II discusses different possibilities for the usage of symmetry methods in the parameterization procedure, most noteworthy the application of techniques of direct and inverse group classifications. We restate some basic results from the theory of group classification and relate them to the parameterization problem. Section III is devoted to the construction of several parameterization schemes for the eddy vorticity flux of the vorticity equation using the methods introduced in Sec. II. Generating sets of differential invariants and operators of invariant differentiation for subalgebras of the maximal Lie invariance algebra of the vorticity equation are computed and used in the framework of invariant parameterization (Sec. III A). It should emphasized that up to now only very few examples on exhaustive descriptions of differential invariants for infinite-dimensional Lie algebras exist in the literature.[10,15] A hierarchy of nested normalized subclasses of a class of generalized vorticity equations is constructed in Sec. III C. Additionally, in Sec. III B the equivalence algebras of some subclasses are directly found within the framework of the infinitesimal approach. The algebraic method of group classification is

used to determine inequivalent invariant parameterization schemes. For a restricted class of generalized vorticity equations, it is proved in Sec. III D that the algebraic method provides an exhaustive description of all inequivalent parameterizations of the eddy vorticity flux. For a wider class of generalized vorticity equations, in Sec. III E we study the problem of invariant parameterization within the framework of preliminary group classification. Namely, inequivalent invariant parameterizations possessing at least one-dimensional symmetry extensions are listed. A short discussion of the results of the paper is presented in Sec. IV, together with an outlook on forthcoming works in the field of invariant parameterization theory. In the Appendix A, details on the classification of inequivalent one-dimensional subalgebras of the equivalence algebra from Theorem 1, which is used in Sec. III E, can be found.

II. THE GENERAL IDEA

Throughout the paper, the notation we adopt follows closely that presented in the textbook.[36] Let there be given a system of differential equations

$$\Delta^l(x, u_{(n)}) = 0, \qquad l = 1, \dots, m, \tag{1}$$

where $x = (x^1, \dots, x^p)$ denote the independent variables and the tuple $u_{(n)}$ includes all dependent variables $u = (u^1, \dots, u^q)$ as well as all derivatives of u with respect to x up to order n. Hereafter, subscripts of functions denote differentiation with respect to the corresponding variables.

Both numerical representations of (1) as well as real-time measurements are not able to capture the instantaneous value of u, but rather only provide some mean values. That is, to employ (1) in practice usually requires an averaging or filter procedure. For this purpose, u is separated according to

$$u = \bar{u} + u',$$

where $\bar{u}$ and u' refer to the averaged and the deviation quantities, respectively. The precise form of the averaging or filter method used determines additional calculation rules, e.g., $\overline{ab} = \bar{a}\bar{b} + \overline{a'b'}$ for the classical Reynolds averaging. At the present stage it is not essential to already commit oneself to a definite averaging method. For nonlinear system (1) averaging usually gives expressions

$$\tilde{\Delta}^l(x, \bar{u}_{(n)}, w) = 0, \qquad l = 1, \dots, m, \tag{2}$$

where $\tilde{\Delta}^l$ are smooth functions of their arguments whose explicit form is precisely determined by the form of Δ^l and the chosen averaging rule. The tuple $w = (w^1, \dots, w^k)$ includes all averaged nonlinear combinations of terms, which cannot be obtained by means of the quantities $\bar{u}_{(n)}$. These combinations typically include such expressions as $\overline{u'u'}$, $\overline{u'\bar{u}}$, $\overline{u'u'_x}$, etc., referred to as subgrid scale terms. Stated in another way, system (2) contains more unknown quantities than equations. To solve system (2), suitable assumptions on w have to be made. An adequate choice for these assumptions is the problem of parameterization.

The most straightforward way to tackle this issue is to directly express the unclosed terms w as functions of the variables x and $\bar{u}_{(r)}$ for some r which can be greater than n. In other words, system (2) is closed via

$$\tilde{\Delta}^l(x, \bar{u}_{(n)}, f(x, \bar{u}_{(r)})) = 0, \qquad l = 1, \dots, m, \tag{3}$$

using the relation $w^s = f^s(x, \bar{u}_{(r)})$, $s = 1, \dots, k$. The purpose of this paper is to discuss different paradigms for the choice of the functions $f = (f^1, \dots, f^k)$ within the symmetry approach, where k is the number of unclosed terms which are necessary to be parameterized. In other words, we should carry out, in different ways, group analysis of the class (3) with the arbitrary elements running through a set of differential functions. To simplify notation, we will omit bars over the dependent variables in systems where parameterization of w is already applied.

Remark: In the theory of group classification, any class of differential equations is considered in a jet space of a fixed order. That is, both the explicit part of the expression of the general equation from the class and the arbitrary elements can be assumed to depend on derivatives up to the same

order. In contrast to this, for the construction of parameterization schemes it is beneficial to allow for varying the orders of arbitrary elements while the order of the explicitly resolved terms is fixed. This is why we preserve different notations for the orders of derivatives in the explicit part of the expression of the general equation and in the arbitrary elements of the class (3).

A. Parameterization via inverse group classification

Parameterizations based on Lie symmetries appear to have been first investigated for the Navier–Stokes equations. It was gradually realized that the consideration of symmetries plays a key role in the construction of subgrid scale models for the Navier–Stokes equations to allow for realistic simulations of flow evolution. See Refs. 33 and 34 for a further discussions on this subject. The approach involving symmetries for the design of local closure schemes, was later extended in Refs. 42, 43, and 44 in order to incorporate also the second law of thermodynamics into the consideration.

For an arbitrary system of differential equations, this approach can be sketched as follows: First, determine the Lie symmetry group G (resp. the corresponding Lie invariance algebra $\mathfrak{g}$) of the model to be investigated. For common models of hydro-thermodynamics these computations were already carried out and results can be found in collections like Ref. 16. Subsequently, determine the differential invariants of the group G. If the left hand side of system (2) is formulated in terms of these invariants by an adequate choice of the function f, it is guaranteed that the parameterized system will admit the same group of point symmetries as the unfiltered system. Usually this leads to classes of differential equations rather than to a single model. That is, among all models constructed this way it is be possible to select those which also satisfy other desired physical and mathematical properties.

The procedure outlined above can be viewed as a special application of techniques of *inverse group classification*. Inverse group classification starts with a prescribed symmetry group and aims to determine the entire class of differential equations admitting the given group as a symmetry group.[38] Thus, in Refs. 33, 34, and 42–44 it is assumed that the closure scheme for the subgrid scale terms leads to classes of differential equations admitting the complete Lie symmetry group of the Navier–Stokes equations. From the mathematical point of view, this assumption is justified as filtering (or averaging) of the Navier–Stokes equations introduces a turbulent friction term among the viscous friction term that already appears in the unfiltered equations. That is, filtering does not principally perturb the structure of the Navier–Stokes equations. However, this assumption may not be as well justified if a model is chosen, where filtering leads to terms of forms not already included in the unfiltered model. In such cases, it may be more straightforward to solve the parameterization problem by inverse group classification only with respect to particular subgroups of the Lie symmetry group G of the initial system $\mathcal{S}$ of differential equations. The selection of proper subgroups of G can be realized involving physical arguments.

Another possible way for such a selection may be related to boundary-value problems. One can choose a subgroup of G consisting of either symmetries of a particular boundary-value problem for $\mathcal{S}$ or equivalence transformations of a relevant class $\mathcal{B}$ of similar boundary-value problems for $\mathcal{S}$. The re-interpretation of symmetries of $\mathcal{S}$ as equivalence transformations for $\mathcal{B}$ is natural because they often have a clear physical significance, such as the rescaling of a domain (e.g., when conducting numerical tests), shifts of space and time variables or the transformation from a resting reference frame to reference frames moving with constant velocity, and all of these fundamental symmetries are usually broken when considering a fixed boundary-value problem. A scaling symmetry of $\mathcal{S}$ is restored as an equivalence transformation for $\mathcal{B}$ if the class $\mathcal{B}$ consists of boundary-value problems of all possible domain sizes. The same argument holds for shifts and Galilean boosts. This re-interpretation does not change the general algorithm for the construction of parameterization schemes using inverse group classification. It rather requires an analysis of the parameterization problem to be treated using group classification methods. The argument is to determine which symmetries map any particular boundary-value problem from $\mathcal{B}$ to another problem from $\mathcal{B}$. The symmetries fulfilling this requirement are to be interpreted as equivalence transformations for the

given class $\mathcal{B}$ of boundary-value problems. The symmetries not compatible with $\mathcal{B}$ could therefore be excluded from the consideration.

The approach of inverse group classification usually relies on the notion of differential invariants.[35,38] Differential invariants are defined as the invariants of the prolonged action of a given symmetry group. They can be determined either with the infinitesimal method[15,38] or with the technique of moving frames.[10,13,37] In the present paper we will use the former method which is briefly described here for this reason.

Let X be the p-dimensional space of independent and U be the q-dimensional space of dependent variables. The connected Lie group G acts locally as a point transformation group on the space $J^0 = X \times U$, with $\mathfrak{g}$ denoting the associated Lie algebra of infinitesimal generators. (The whole consideration is assumed local.) Each element of $\mathfrak{g}$ is of the form $Q = \xi^i(x,u)\partial_{x^i} + \varphi^a(x,u)\partial_{u^a}$. In this section the indices i and j run from 1 to p while the indices a and b run from 1 to q, and the summation convention over repeated indices is used. The space $J^r = X \times U_{(r)}$ is the rth prolongation of the space $X \times U$ (the rth order jet space), which is the space endowed with coordinates x_i and u^a_α, $|\alpha| := \alpha_1 + \cdots + \alpha_p < r$, where u^a_α stands for the variable corresponding to the derivative $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\dots\partial x_p^{\alpha_p}$, and $\alpha = (\alpha_1, \dots, \alpha_p)$ is an arbitrary multi-index, $\alpha_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The action of G can be extended to an action on J^r and so the elements of $\mathfrak{g}$ can be prolonged via

$$Q_{(r)} = Q + \sum_{\alpha>0} \varphi^{a\alpha}\partial_{u^a_\alpha}, \quad \varphi^{a\alpha} := \mathrm{D}_1^{\alpha_1}\cdots\mathrm{D}_p^{\alpha_p}(\varphi^a - \xi^i u^a_i) + \xi^i u^a_{\alpha+\delta_i}. \tag{4}$$

Here, $\mathrm{D}_i = \mathrm{D}_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i, i.e., $\mathrm{D}_i = \partial_{x_i} + u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$, where δ_i is the multi-index whose ith entry equals 1 and whose other entries are zero. More details can be found in the textbooks.[35,36,38]

A differential function f (i.e., a smooth function from J^r to $\mathbb{R}$ for some r) is called an (rth order) *differential invariant* of the group G if for any transformation $g : (x,u) \mapsto (\tilde{x},\tilde{u})$ from G we have that $f(\tilde{x},\tilde{u}_{(r)}) = f(x,u_{(r)})$. The function f is a differential invariant of G if and only if the equality $Q_{(r)}f = 0$ holds for any $Q \in \mathfrak{g}$. A vector field $\mathfrak{d}$ defined in the infinite jet space J^∞, which is the inverse limit of the sequence of natural projections from J^{p+1} to J^p for $p \in \mathbb{N}_0$, is called an *operator of invariant differentiation* for the group G if the result $\mathfrak{d}f$ of its action to any differential invariant f of G also is a differential invariant of G.

The *fundamental basis theorem* states that any finite-dimensional Lie group (or, more generally, any Lie pseudo-group satisfying certain condition) acting on J^0 possesses exactly p operators of invariant differentiation, which are independent up to linear combining with coefficients depending on differential invariants, and a finite basis of differential invariants, i.e., a finite set of differential invariants such that any differential invariant of the group can be obtained from basis invariants by a finite number of functional operations and actions by the chosen independent operators of invariant differentiation.

For a vector field $\mathfrak{d}$ in J^∞ to be an operator of invariant differentiation of G, it is sufficient that it commutes with every infinitely prolonged operator from the corresponding Lie algebra $\mathfrak{g}$, i.e., $[\mathfrak{d}, Q_{(\infty)}] = 0$ for any $Q \in \mathfrak{g}$. If the group G is finite-dimensional, a set of p independent operators of invariant differentiation can be found in the form $\mathfrak{d} = h^i\mathrm{D}_i$ by solving, with respect to the differential functions $h^i = h^i(x, u_{(r)})$, the system of first order quasi-linear partial differential equations

$$Q_{(r)}h^i = h^j\mathrm{D}_j\xi^i,$$

where Q runs through a basis of the corresponding Lie algebra $\mathfrak{g}$ and r equals the minimum order for which the rank of the prolonged basis operators of $\mathfrak{g}$ coincides with its dimension. Eventually, it may be convenient to determine h^i in the implicit form $\Omega^j(h^1,\dots,h^p,x,u_{(r)}) = 0$, where $\det(\Omega^j_{h_i}) \neq 0$ and Ω^j satisfy the associated system of homogeneous equations

$$\left(Q_{(r)} + (h^{i'}\mathrm{D}_{i'}\xi^i)\partial_{\lambda^i}\right)\Omega^j = 0.$$

In the infinite-dimensional case, the construction of invariant differentiation operators is analogous though more sophisticated.

A systematic approach to parameterization via inverse group classification hence consists of determining the basis differential invariants of a group together with the list of operators of invariant differentiation. Subsequently, there are infinitely many parameterizations that can be constructed, which admit the given group as a symmetry group.

B. Parameterization via direct group classification

The main assumption in the approach presented in Refs. 33 and 44 is that a realistic subgrid scale model for the Navier–Stokes equations should admit the symmetry group of the original equations. However, this assumption is rather restrictive in more general situations. While it is true that a filtered model should be a realistic approximation of the unfiltered equations, parameterization schemes also have to take into account physical processes for which we may not have a precise understanding yet. That is, one eventually has to face the problem to deal with processes for which we may not even have a differential equation. This particularly means that a fixed set of symmetries (as for the Navier–Stokes equations) may not be obtainable.

On the other hand, symmetries do provide a useful guiding principle for the selection of physical models. As nature tends to prefer states with a high degree of symmetry, a general procedure for the derivation of symmetry-preserving parameterization schemes seems reasonable. The only crucial remark is, that we may not know in advance, which symmetries are most essential for capturing the characteristics of the underlying physical processes. For such problems, application of inverse group classification techniques is at once limited. Rather, it may be beneficial to derive parameterization schemes admitting different symmetry groups and subsequently test these various schemes to select among them those which best describe the processes under consideration. That is, instead of expressing the tuple w in system (2) using differential invariants of a symmetry group of the unfiltered equations (or another convenient symmetry group) from the beginning, we investigate symmetries of system (3) for different realizations of the functions f which are eventually required to satisfy some prescribed conditions. This way, we could be interested in special classes of parameterizations, such as time- or spatially independent ones. This naturally leads back to the usual problem of *direct group classification*: Let there be given a class of differential equations, parameterized by arbitrary functions. First determine the symmetries admitted for all choices of these functions, leading to the kernel of symmetry groups of the class under consideration. Subsequently, investigate for which special values of these parameter-functions there are extensions of the kernel group.[38,40]

To systematically carry out direct group classification, it is necessary to determine the equivalence group of the class, i.e., the group of transformations mapping an equation from the class (3) to an equation from the same class. Classification of extensions of the kernel group is then done up to equivalence imposed by the equivalence group of the class (3).

The continuous part of the equivalence group can be found using infinitesimal methods in much the same way as Lie symmetries can be found using the infinitesimal invariance criterion. This firstly yields the equivalence algebra, the elements of which can then be integrated to give the continuous equivalence group. See Refs. 38 and 40 for more details on this subject.

We now formalize the method reviewed in the previous paragraphs. Let there be given a class of differential equations of the form (3), $\tilde{\Delta}^l(x, \bar{u}_{(n)}, f(x, \bar{u}_{(r)})) = 0$, $l = 1, \dots, m$. The arbitrary elements f usually satisfy an auxiliary system of equations $S(x, u_{(r)}, f_{(\rho)}(x, u_{(r)})) = 0$, $S = (S^1, \dots, S^s)$, and an inequality $\Sigma(x, u_{(r)}, f_{(\rho)}(x, u_{(r)})) \neq 0$, where $f_{(\rho)}$ denotes the collection of f and all derivatives of f with respect to the variables x and $u_{(r)}$ up to order ρ. The conditions $S = 0$ and $\Sigma \neq 0$ restrict the generality of f and hence allow the design of specialized parameterizations. We denote the solution set of the auxiliary system by $\mathcal{S}$, the system of form (3) corresponding to an $f \in \mathcal{S}$ by $\mathcal{L}_f$ and the entire class of such system by $\mathcal{L}|_{\mathcal{S}}$.

The set of all (nondegenerate) point transformations that map a system $\mathcal{L}_f$ to a system $\mathcal{L}_{\tilde{f}}$, where both $f, \tilde{f} \in \mathcal{S}$ is denoted by $\mathrm{T}(f, \tilde{f})$ and is referred to as the set of admissible transformations from the system $\mathcal{L}_f$ to the system $\mathcal{L}_{\tilde{f}}$. The collection of all point transformations relating at least two systems from the class $\mathcal{L}|_S$ gives rise to the set of admissible transformations of $\mathcal{L}|_S$.

Definition 1: The set of admissible transformations of the class $\mathcal{L}|_{\mathcal{S}}$ is the set $\mathrm{T}(\mathcal{L}|_{\mathcal{S}}) = \{(f, \tilde{f}, \varphi) \mid f, \tilde{f} \in \mathcal{S}, \varphi \in \mathrm{T}(f, \tilde{f})\}$.

That is, an admissible transformation is a triple, consisting of the initial system (with arbitrary elements f), the target system (with arbitrary elements $\tilde{f}$), and a mapping φ between these two systems. It is obvious that the usual composition of mappings defines the groupoid structure on the set $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ and the general point equivalence between equations from the class $\mathcal{L}|_{\mathcal{S}}$ coincides with that generated by elements of $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$. This is why we can also call $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ the *equivalence groupoid* of the class $\mathcal{L}|_{\mathcal{S}}$.

The *usual equivalence group* $G^{\sim} = G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ of the class $\mathcal{L}|_{\mathcal{S}}$ is defined in a rigorous way in terms of admissible transformations. Namely, any element Φ from $G^{\sim}$ is a point transformation in the space of $(x, u_{(r)}, f)$, which is projectable on the space of $(x, u_{(r')})$ for any $0 \leqslant r' \leqslant r$, so that the projection is the r'th order prolongation of $\Phi|_{(x,u)}$, the projection of Φ on the variables (x, u), and for any arbitrary elements $f \in \mathcal{S}$ we have that $\Phi f \in \mathcal{S}$ and $\Phi|_{(x,u)} \in \mathrm{T}(f, \Phi f)$. The admissible transformations of the form $(f, \Phi f, \Phi|_{(x,u)})$, where $f \in \mathcal{S}$ and $\Phi \in G^{\sim}$, are called induced by transformations from the equivalence group $G^{\sim}$. Needless to say, that in general not all admissible transformations are induced by elements from the equivalence group. Different generalizations of the notion of usual equivalence groups exist in the literature.[30,40] By $\mathfrak{g}^{\sim}$ we denote the algebra associated with the equivalence group $G^{\sim}$ and call it the equivalence algebra of the class $\mathcal{L}|_{\mathcal{S}}$.

After clarifying the notion of admissible transformations and equivalence groups, we move on with the description of a common technique in the course of group analysis of differential equations, namely, the algebraic method. Within this method one at first should classify inequivalent subalgebras of the corresponding equivalence algebra and then solve the inverse group classification problem for each of the subalgebras obtained. This procedure usually yields most of the cases of extensions and therefore leads to preliminary group classification (see, e.g., Refs. 17, 18, and 48 for applications of this technique to various classes of differential equations).

The algebraic method rests on the following two propositions:[11]

Proposition 1: Let $\mathfrak{a}$ *be a subalgebra of the equivalence algebra* $\mathfrak{g}^{\sim}$ *of the class* $\mathcal{L}|_{\mathcal{S}}$, $\mathfrak{a} \subset \mathfrak{g}^{\sim}$, *and let* $f^0(x, u_{(r)}) \in \mathcal{S}$ *be a value of the tuple of arbitrary elements* f *for which the algebraic equation* $f = f^0(x, u_{(r)})$ *is invariant with respect to* $\mathfrak{a}$. *Then the differential equation* $\mathcal{L}|_{f^0}$ *is invariant with respect to the projection of* $\mathfrak{a}$ *to the space of variables* (x, u).

Proposition 2: Let $\mathcal{S}_i$ *be the subset of* $\mathcal{S}$ *that consists of all arbitrary elements for which the corresponding algebraic equations are invariant with respect to the same subalgebra of the equivalence algebra* $\mathfrak{g}^{\sim}$ *and let* $\mathfrak{a}_i$ *be the maximal subalgebra of* $\mathfrak{g}^{\sim}$ *for which* $\mathcal{S}_i$ *satisfies this property,* $i = 1, 2$. *Then the subalgebras* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *are equivalent with respect to the adjoint action of* $G^{\sim}$ *if and only if the subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are mapped to each other by transformations from* $G^{\sim}$.

The result of preliminary group classification is a list of inequivalent (with respect to the equivalence group) members $\mathcal{L}_f$ of the class $\mathcal{L}|_{\mathcal{S}}$, admitting symmetry extension of the kernel of symmetry algebras using subalgebras of the equivalence algebra.

Although the algebraic method is a straightforward tool to derive cases of symmetry extensions for classes of differential equations with arbitrary elements, there remains the important question when it gives complete group classification, i.e., preliminary and complete group classification coincide. This question is of importance also for the problem of parameterization, as only complete group classification will lead to an exhaustive description of all possible parameterization schemes feasible for some class of differential equations. The answer is that the class under consideration should be *weakly normalized in infinitesimal sense*, i.e., it should satisfy the following property: The span of maximal Lie invariance algebras of all equations from the class is contained in the projection of the corresponding equivalence algebra to the space of independent and dependent variables,

$$\langle \mathfrak{g}_f \mid f \in \mathcal{S} \rangle \subset \mathrm{P}\mathfrak{g}^{\sim}.$$

At the same time, it is better to use a stronger notion of normalization introduced in Ref. 40.

Definition 2: The class $\mathcal{L}|_{\mathcal{S}}$ is *normalized* if its equivalence groupoid is generated by its equivalence group, i.e., $\forall\,(f,\tilde f,\varphi)\in \mathrm{T}(\mathcal{L}|_{\mathcal{S}})\ \exists\Phi\in G^{\sim}\colon \tilde f=\Phi f$ and $\varphi=\Phi|_{(x,u)}$.

The normalization of $\mathcal{L}|_{\mathcal{S}}$ in the sense of Definition 2 additionally implies that the group classification of equations from this class up to $G^{\sim}$-equivalence coincides with the group classification using the general point transformation equivalence. Due to this fact we have no additional equivalences between cases obtained under the classification up to $G^{\sim}$-equivalence. As a result, solving the group classification problem for normalized classes of differential equations is especially convenient and effective.

In turn, depending on normalization properties of the given class (or their lacking), different strategies of group classification should be applied.[40] For a normalized class, the group classification problem is reduced, within the infinitesimal approach, to classification of subalgebras of its equivalence algebra.[3,25,40,51] A class that is not normalized can eventually be embedded into a normalized class which is not necessarily minimal among the normalized superclasses.[40,41] One more way to treat a non-normalized class is to partition it into a family of normalized subclasses and to subsequently classify each subclass separately.[40,50] If a partition into normalized subclasses is difficult to construct due to the complicated structure of the set of admissible transformations, conditional equivalence groups and additional equivalence transformations may be involved in the group classification.[19,39,50] In the case when the class is parameterized by constant arbitrary elements or arbitrary elements depending only on one or two arguments, one can apply the direct method of group classification based on compatibility analysis and integration of the determining equations for Lie symmetries up to $G^{\sim}$-equivalence.[1,32,38] Recall that these determining equations involve both coefficients of a Lie symmetry operator of a system $\mathcal{L}_f$ and the corresponding tuple of arbitrary elements f and follow from the infinitesimal invariance criterion,[36,38]

$$Q_{(r')}\tilde\Delta^l(x,\bar u_{(n)},f(x,\bar u_{(r)}))|_{\mathcal{L}_f}=0,\quad l=1,\dots,m.$$

Here $r'=\max\{n,r\}$, the prolongation $Q_{(r')}$ of Q is defined by (4) and the symbol $|_{\mathcal{L}_f}$ means that above relation holds on solutions of the system $\mathcal{L}_f$.

III. SYMMETRY-PRESERVING PARAMETERIZATIONS FOR VORTICITY EQUATION

The inviscid barotropic vorticity equations in Cartesian coordinates reads

$$\zeta_t+\{\psi,\zeta\}=0, \tag{5}$$

where $\{a,b\}=a_xb_y-a_yb_x$ denotes the usual Poisson bracket with respect the variables x and y. The vorticity ζ and the stream function ψ are related through the Laplacian, i.e., $\zeta=\nabla^2\psi$. The two-dimensional wind field $\mathbf{v}=(u,v,0)^{\mathrm{T}}$ is reconstructed from the stream function via the relation $\mathbf{v}=\mathbf{k}\times\nabla\psi$, where $\mathbf{k}$ is the vertical unit vector.

The maximal Lie invariance algebra $\mathfrak{g}_0$ of the Eq. (5) is generated by the operators

$$\begin{aligned}
&\mathcal{D}_1=t\partial_t-\psi\partial_\psi,\quad \partial_t,\quad \mathcal{D}_2=x\partial_x+y\partial_y+2\psi\partial_\psi,\\
&\mathcal{J}=-y\partial_x+x\partial_y,\quad \mathcal{J}^t=-ty\partial_x+tx\partial_y+\tfrac12(x^2+y^2)\partial_\psi,\\
&\mathcal{X}(\gamma^1)=\gamma^1(t)\partial_x-\gamma^1_t(t)y\partial_\psi,\quad \mathcal{Y}(\gamma^2)=\gamma^2(t)\partial_y+\gamma^2_t(t)x\partial_\psi,\\
&\mathcal{Z}(\chi)=\chi(t)\partial_\psi,
\end{aligned} \tag{6}$$

where γ^1, γ^2, and χ run through the set of smooth functions of t. See, e.g., Refs. 2 and 5 for further discussions.

Reynolds averaging the above equation leads to

$$\bar\zeta_t+\{\bar\psi,\bar\zeta\}=\nabla\cdot(\overline{\mathbf{v}'\zeta'}). \tag{7}$$

The term $\overline{\mathbf{v}'\zeta'}=(\overline{u'\zeta'},\overline{v'\zeta'},0)^{\mathrm{T}}$ is the horizontal eddy vorticity flux. Its divergence provides a source term for the averaged vorticity equation. The presence of this source term destroys several of the properties of (5), such as possessing conservation laws. In this paper we aim to find parameterizations of this flux term, which admit certain symmetries.

A simple choice for a parameterization of the eddy vorticity flux is given by the *down-gradient ansatz*

$$\overline{\mathbf{v}'\zeta'} = -K\nabla\overline{\zeta},$$

where the eddy viscosity coefficient K still needs to be specified. Physically, this ansatz accounts for the necessity of the vorticity flux to be directed down-scale, as enstrophy (integrated squared vorticity) is continuously dissipated at small scales. Moreover, this ansatz will lead to a uniform distribution of the mean vorticity field, provided there is no external forcing that counteracts this tendency.[27] The simplest form of the parameter K is apparently $K = K(x, y)$, i.e., the eddy viscosity coefficient is only a function of space. More advanced ansatzes for K assume dependence on $\overline{\zeta'^2}$, which is the eddy enstrophy[27] (see also the discussion in the recent paper[28]). This way, the strength of the eddy vorticity flux depends on the intensity of two-dimensional turbulence, which gives a more realistic model for the behavior of the fluid. There also exist a number of other parameterization schemes that can be applied to the vorticity equation, such as methods based on statistical mechanics[22] or the anticipated potential vorticity method.[45,49]

In the present framework, we exclusively focus on first order closure schemes. This is why we are only able to parameterize the eddy vorticity flux using the independent and dependent variables, as well as all derivatives of the dependent variables. This obviously excludes the more sophisticated and recent parameterization ansatzes of geophysical fluid dynamics from the present study. On the other hand, the basic method of invariant parameterization can already be demonstrated for this rather simple model. Indeed, symmetries of the vorticity equation employing the down-gradient ansatz or related parameterizations are investigated below using both inverse and direct group classification. Physically more advanced examples for parameterizations can be constructed following the methods outlined in Sec. II and exemplified subsequently.

A. Parameterization via inverse group classification

This is the technique by Refs. 33 and 44 applied to the inviscid vorticity equation. In view of the description of Sec. II A this approach consists of singling out subgroups (subalgebras) of the maximal Lie invariance group (algebra) of the vorticity equation and computation of the associated differential invariants (via a basis of differential invariants and operators of invariant differentiation). These differential invariants can then be used to construct different parameterizations of the eddy vorticity flux.

It is important to note that singling out subgroups of the maximal Lie invariance group of the vorticity equation is a meteorological way of group classification. This is why it is necessary to have a basic understanding of the processes to be parameterized before the selection of a particular group is done (otherwise, we would have to face the problem of how to combine these invariants to physically meaningful parameterizations). For the vorticity equation, we demonstrate the basic mechanisms of parameterizations via inverse group classification by singling out subgroups that allow to include the above down-gradient ansatz. This choice is of course not unique as there exist various other possibilities for parameterizations of the eddy vorticity flux. However, this choice allows us to demonstrate several of the issues of parameterization via inverse group classification.

1. Invariance under the whole Lie symmetry group

To present differential invariants of the whole Lie invariance algebra $\mathfrak{g}_0$, we use the notation

$$\zeta = \psi_{xx} + \psi_{yy}, \quad \theta = \psi_{xx} - \psi_{yy}, \quad \eta = 2\psi_{xy},$$
$$\sigma = \psi_{xxx} - 3\psi_{xyy}, \quad \varsigma = 3\psi_{xxy} - \psi_{yyy}, \quad V = \mathrm{D}_t + \psi_x\mathrm{D}_y - \psi_y\mathrm{D}_x.$$

The algebra $\mathfrak{g}_0$ possesses no differential invariants up to order two. At the same time, it has the singular second order manifold determined by the equations $\psi_{xx} = \psi_{yy}$ and $\psi_{xy} = 0$, which is not essential for our consideration. A generating set $\mathcal{I}_0$ of functionally independent differential invariants

of $\mathfrak{g}_0$ consists of the third order differential invariants

$$\frac{V\zeta}{\theta^2+\eta^2},\quad \frac{\theta V\theta+\eta V\eta}{(\theta^2+\eta^2)^{3/2}},\quad \frac{(V\theta+2\eta\zeta)^2+(V\eta-2\theta\zeta)^2}{(\theta^2+\eta^2)^2},$$
$$\frac{\sigma^2+\varsigma^2}{\zeta_x{}^2+\zeta_y{}^2},\quad \frac{\theta(\zeta_x{}^2-\zeta_y{}^2)+2\eta\zeta_x\zeta_y}{(\theta^2+\eta^2)^{1/2}(\zeta_x{}^2+\zeta_y{}^2)},\quad \frac{\sigma\zeta_x(\zeta_x{}^2-3\zeta_y{}^2)+\varsigma\zeta_y(3\zeta_x{}^2-\zeta_y{}^2)}{(\zeta_x{}^2+\zeta_y{}^2)^{3/2}}.$$

A complete set $\mathcal{O}_0$ of independent operators of invariant differentiation for this algebra is formed by the operators

$$(\theta^2+\eta^2)^{-1/2}V,\quad (\zeta_x{}^2+\zeta_y{}^2)^{-1/2}(\zeta_x\mathrm{D}_x+\zeta_y\mathrm{D}_y),\quad (\zeta_x{}^2+\zeta_y{}^2)^{-1/2}(\zeta_x\mathrm{D}_y-\zeta_y\mathrm{D}_x).$$

The computation of $\mathcal{I}_0$ and $\mathcal{O}_0$ is cumbersome and will be presented elsewhere, jointly with the selection of a basis (i.e., minimal generating set) of differential invariants. At the same time, the result of the computation can be checked in a rather direct and simple way. Indeed, the cardinality of $\mathcal{O}_0$ equals three. The elements of $\mathcal{O}_0$ are linearly independent over the ring of differential invariants of $\mathfrak{g}_0$ and commute with the infinite prolongations of all vector fields from the generating set (6) of $\mathfrak{g}_0$. Since each element I of $\mathcal{I}_0$ satisfies the condition $Q_{(3)}I=0$, where the operator Q runs through the operators (6), it is a differential invariant of $\mathfrak{g}_0$. The invariants belonging to $\mathcal{I}_0$ are functionally independent. Moreover, for any fixed order r an rth order universal basis of differential invariants of $\mathfrak{g}_0$ can be constructed via acting by operators from $\mathcal{O}_0$ on invariants from $\mathcal{I}_0$. We only sketch the proof of the last assertion. The cardinality of any rth order universal basis of differential invariants of $\mathfrak{g}_0$, where $r\geqslant 4$, equals the difference between the dimension of the jet space J^r and the rank of the rth prolongation of $\mathfrak{g}_0$,

$$N=3+\binom{r+3}{r}-(3r+8).$$

Acting on elements of $\mathcal{I}_0$ by operators from $\mathcal{O}_0$ $k-3$ times, $3\leqslant k\leqslant r$, we obtain a set of k-order differential invariants which is of maximal rank with respect to the k-order derivatives involving at least two differentiations with respect to space variables. Choosing, for each k, a subset of invariants associated with a nonzero k-order minor in the corresponding Jacobi matrix and uniting such subsets for $k\leqslant r$, we construct exactly N functionally independent differential invariants of order not greater than r, which hence form an rth order universal basis of differential invariants of $\mathfrak{g}_0$.

The above case where invariance of the parameterization under the whole symmetry group of the vorticity equation is desired can be neglected for physical reasons. This is since it is impossible to realized, e.g., the down-gradient ansatz within this framework. It can easily be checked that the corresponding vorticity equation with parameterized eddy vorticity flux only admits one scaling operator for any physically meaningful ansatz for K. In contrast to the example of the Navier–Stokes equations discussed in Ref. 33, the vorticity equation hence does not allow physical parameterizations leading to a closed model invariant under the same symmetry group as the original vorticity equation. This is why it is beneficial to single out several subgroups of the maximal Lie invariance group and consider the invariant parameterization problem only with respect to these subgroups.

2. Explicit spatial dependency

If the two-dimensional fluid is anisotropic and inhomogeneous the only subalgebra of (6) that can be admitted is spanned by the operators

$$\partial_t,\quad \mathcal{Z}(\chi)=\chi(t)\partial_\psi.$$

For this subalgebra, a basis of invariants is formed by x, y, ψ_x, and ψ_y. Independent operators of invariant differentiation are exhausted by D_t, D_x, and D_y. If we express the right hand side of (7) in terms of differential invariants of the above subalgebra, a possible representation reads

$$\zeta_t+\{\psi,\zeta\}=K(x,y)\nabla^2\zeta.$$

Hence, we assembled our parameterization using the (differential) invariants x, y, $\mathrm{D}_x^3\psi_x = \psi_{xxxx}$, $\mathrm{D}_y^2\mathrm{D}_x\psi_x = \psi_{xxyy}$, and $\mathrm{D}_y^3\psi_y = \psi_{yyyy}$. This boils down to the usual gradient ansatz for the eddy flux term, where the eddy viscosity K explicitly depends on the position in the space. Note, however, that this ansatz is only one possibility which is feasible within this class of models.

3. Rotationally invariant fluid

In case the two-dimensional fluid is isotropic, the resulting parameterized system should also admit rotations. Hence, we seek for differential invariants of the subalgebra j spanned by the operators

$$\partial_t, \quad \mathcal{J} = x\partial_y - y\partial_x, \quad \mathcal{J}^t = tx\partial_y - ty\partial_x + \tfrac{1}{2}(x^2+y^2)\partial_\psi, \quad \mathcal{Z}(\chi) = \chi(t)\partial_\psi.$$

A complete set of independent operators of invariant differentiation for j consists of

$$\mathrm{D}_t + \psi_x\mathrm{D}_y - \psi_y\mathrm{D}_x, \quad x\mathrm{D}_x + y\mathrm{D}_y, \quad -y\mathrm{D}_x + x\mathrm{D}_y$$

and a generating set of functionally independent differential invariants is formed by

$$\rho = \tfrac{1}{2}(x^2+y^2), \quad x\psi_y - y\psi_x, \quad (x^2+y^2)(\psi_{xx}+\psi_{yy}) - 2(x\psi_x + y\psi_y),$$
$$(x^2+y^2)(x\psi_{tx} + y\psi_{ty}) + (x\psi_x + y\psi_y)(xy(\psi_{yy}-\psi_{xx}) + (x^2-y^2)\psi_{xy} + x\psi_y - y\psi_x).$$

In the modified polar coordinates (ρ, φ) with $\varphi = \arctan y/x$, these sets have, after an additional rearrangement, simpler representations $\mathcal{O} = \{\tilde{\mathrm{D}}_t, \mathrm{D}_\rho, \mathrm{D}_\varphi\}$ and $\mathcal{I} = \{I^\alpha,\ \alpha = 0, \dots, 3\}$, respectively, where $\tilde{\mathrm{D}}_t = \mathrm{D}_t + \psi_\rho\mathrm{D}_\varphi$ and

$$I^0 = \rho, \quad I^1 = \psi_\varphi, \quad I^2 = \psi_{\rho\rho}, \quad I^3 = \psi_{t\rho} + \psi_\rho\psi_{\rho\varphi}.$$

Any element of $\mathcal{O}$ indeed is an invariant differentiation operator for j since it commutes with the infinite prolongation of every vector field from j. The fact that I^α, $\alpha = 0, \dots, 3$, are differential invariants of j is also checked in a rather direct way, by the substitution to the condition $Q_{(2)}I = 0$, where the operator Q runs through j. These invariants obviously are functionally independent.

The most difficult part is to prove that for any fixed order r we can construct an rth order universal basis of differential invariants of j by invariant differentiations of I^α, $\alpha = 0, \dots, 3$. The number of elements in any rth order universal basis of differential invariants of j, where $r \geqslant 1$, equals

$$3 + \binom{r+3}{r} - (r+4) = \frac{r}{6}(r+1)(r+5)$$

(the dimension of the jet space J^r minus the rank of the rth prolongation of j). The commutation relations between the operators of invariant differentiation are

$$[\mathrm{D}_\rho, \mathrm{D}_\varphi] = 0, \quad [\mathrm{D}_\rho, \tilde{\mathrm{D}}_t] = \psi_{\rho\rho}\mathrm{D}_\varphi, \quad [\mathrm{D}_\varphi, \tilde{\mathrm{D}}_t] = \psi_{\rho\varphi}\mathrm{D}_\varphi.$$

The elements I^1, I^2, and I^3 of $\mathcal{I}$ can be represented in the form $I^1 = \mathrm{D}_\varphi\psi$, $I^2 = \mathrm{D}_\rho^2\psi$, and $I^3 = \tilde{\mathrm{D}}_t\mathrm{D}_\rho\psi$. Hence, acting by the operators of invariant differentiation on elements of $\mathcal{I}$, we can construct

$$1 + \binom{r-1+3}{r-1} + \binom{r-2+2}{r-2} + \binom{r-2+1}{r-2} = \frac{r}{6}(r+1)(r+5)$$

functionally independent invariants of order not greater than r (the zeroth order invariant ρ plus acting on I^1 by the operators $\tilde{\mathrm{D}}_t^{\alpha_1}\mathrm{D}_\rho^{\alpha_2}\mathrm{D}_\varphi^{\alpha_3}$, where $\alpha_1+\alpha_2+\alpha_3 \leqslant r-1$, plus acting on I^2 by D_ρ and then $\tilde{\mathrm{D}}_t$ at most $r-2$ times in total and plus acting on I^3 by D_ρ at most $r-2$ times). As the above numbers coincide, the proof is completed.

For the set $\mathcal{O}$ of operators of invariant differentiation, the generating set $\mathcal{I}$ of differential invariants is not minimal. On the domain singled out in the corresponding infinite jet space by the condition $\psi_{\varphi\varphi} \neq 0$ we have

$$I^2 = \frac{[\mathrm{D}_\rho, \tilde{\mathrm{D}}_t]I^1}{\mathrm{D}_\varphi I^1}$$

and hence the invariant I^2 can be excluded from the generating set of invariants. At the same time, the remaining invariant I^0, I^1, and I^3 form a basis (i.e., minimal generating set) of invariants with respect to the set $\mathcal{O}$ of operators of invariant differentiation. Indeed, any function of ρ and invariants obtained from I^1 by invariant differentiations is represented as a function of ρ, $\tilde{\mathrm{D}}_t^{\alpha_1}\mathrm{D}_\rho^{\alpha_2}\mathrm{D}_\varphi^{\alpha_3}\psi_\varphi$, and $\tilde{\mathrm{D}}_t^{\beta_1}\mathrm{D}_\rho^{\beta_2}\psi_{\rho\rho}$, where $(\alpha_1, \alpha_2, \alpha_3)$ and (β_1, β_2) run through certain subsets of $\mathbb{N}_0^3$ and $\mathbb{N}_0^2$, respectively, and hence this function cannot coincide with I^3. Analogously, any function of ρ and invariants constructed from I^3 by invariant differentiations is represented as a function of ρ and $\tilde{\mathrm{D}}_t^{\alpha_1}\mathrm{D}_\rho^{\alpha_2}\mathrm{D}_\varphi^{\alpha_3}\psi_\rho$, where $(\alpha_1, \alpha_2, \alpha_3)$ runs through certain subset of $\mathbb{N}_0^3$, and hence this function cannot coincide with I^1.

As an example, the parameterizations of the form

$$\zeta_t + \{\psi, \zeta\} = K(\sqrt{x^2 + y^2})\nabla^2\zeta$$

are invariant with respect to j because ρ, $\zeta_t + \{\psi, \zeta\}$ and $\nabla^2\zeta$ are differential invariants of j.

In the same fashion it would be possible to derive classes of parameterizations that preserve other subalgebras of $\mathfrak{g}_0$, e.g., including (generalized) Galilean symmetry or a scaling symmetry, but we do not derive them in this paper.

B. Equivalence algebras of classes of generalized vorticity equations

In order to demonstrate different possible techniques, we present the details of the calculation of the usual equivalence algebra $\mathfrak{g}_1^{\sim}$ for the class of equations

$$\zeta_t + \{\psi, \zeta\} = \mathrm{D}_i f^i(t, x, y, \zeta_x, \zeta_y) = f^i_i + f^i_{\zeta_j}\zeta_{ij}, \quad \zeta := \psi_{ii}, \tag{8}$$

where for convenience we introduce another notation for the independent variables, $t = z_0$, $x = z_1$, and $y = z_2$, and omit bars over the dependent variables. Throughout the section the indices i, j, and k range from 1 to 2, while the indices κ, λ, μ, and ν run from 0 to 2. The summation over repeated indices is understood. A numerical subscript of a function denotes the differentiation with respect to the corresponding variable z_μ.

In fact, the equivalence algebra of class (8) can be easily obtained from the much more general results on admissible transformations, presented in Sec. III C. At the same time, calculations using the direct method applied for finding admissible transformations are too complicated and lead to solving nonlinear overdetermined systems of partial differential equations. This is why the infinitesimal approach is wider applied and realized within symbolic calculation systems. The usage of the infinitesimal approach for the construction of the equivalence algebra of (8) has specific features richly deserving to be demonstrated here.

Theorem 1: *The equivalence algebra* $\mathfrak{g}_1^{\sim}$ *of class* (8) *is generated by the operators*

$$\begin{aligned}
&\tilde{\mathcal{D}}_1 = t\partial_t - \psi\partial_\psi - \zeta_x\partial_{\zeta_x} - \zeta_y\partial_{\zeta_y} - 2f^1\partial_{f^1} - 2f^2\partial_{f^2}, \qquad \partial_t,\\
&\tilde{\mathcal{D}}_2 = x\partial_x + y\partial_y + 2\psi\partial_\psi - \zeta_x\partial_{\zeta_x} - \zeta_y\partial_{\zeta_y} + f^1\partial_{f^1} + f^2\partial_{f^2},\\
&\tilde{\mathcal{J}}(\beta) = \beta x\partial_y - \beta y\partial_x + \frac{\beta_t}{2}(x^2 + y^2)\partial_\psi + \beta(\zeta_x\partial_{\zeta_y} - \zeta_y\partial_{\zeta_x})\\
&\qquad + (\beta_{tt}x - \beta f^2)\partial_{f^1} + (\beta_{tt}y + \beta f^1)\partial_{f^2},\\
&\tilde{\mathcal{X}}(\gamma^1) = \gamma^1\partial_x - \gamma^1_t y\partial_\psi, \qquad \tilde{\mathcal{Y}}(\gamma^2) = \gamma^2\partial_y + \gamma^2_t x\partial_\psi,\\
&\tilde{\mathcal{R}}(\sigma) = \frac{\sigma}{2}(x^2 + y^2)(\partial_\psi + \zeta_y\partial_{f^1} - \zeta_x\partial_{f^2}) + \sigma_t x\partial_{f^1} + \sigma_t y\partial_{f^2},\\
&\tilde{\mathcal{H}}(\delta) = \delta(\partial_\psi + \zeta_y\partial_{f^1} - \zeta_x\partial_{f^2}), \qquad \tilde{\mathcal{G}}(\rho) = \rho_x\partial_{f^2} - \rho_y\partial_{f^1}, \qquad \tilde{\mathcal{Z}}(\chi) = \chi\partial_\psi,
\end{aligned} \tag{9}$$

where β, γ^i, σ, and χ are arbitrary smooth functions of t solely, $\delta = \delta(t, x, y)$ is an arbitrary solution of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$ and $\rho = \rho(t, x, y)$ is an arbitrary smooth function of its arguments.

Remark: Although the coefficients of ∂_{ζ_x} and ∂_{ζ_y} can be obtained by standard prolongation from the coefficients associated with the equation variables, it is necessary to include the corresponding terms in the representation of the basis elements (9) in order to guarantee that they commute in a proper way.

Remark: The operators $\tilde{\mathcal{G}}(\rho)$ and $\tilde{\mathcal{H}}(\chi) - \tilde{\mathcal{Z}}(\chi)$ arise due to the total divergence expression of the right hand side of the first equation in (8), leading to the gauge freedom in rewriting the right hand side of the class (8). They do not generate transformations of the independent and dependent variables and hence form the gauge equivalence subalgebra of the equivalence algebra (9).[40] The parameter-function ρ is defined up to summand depending on t.

Proof: As coordinates in the underlying fourth-order jet space $\mathfrak{J}^{(4)}$, we choose the variables

$$z_\mu, \quad \psi, \quad \psi_\mu, \quad \psi_{\mu\nu},\ \mu \leqslant \nu, \quad \psi_{\lambda\mu\nu},\ \lambda \leqslant \mu \leqslant \nu,\ (\mu, \nu) \neq (2, 2), \quad \zeta_\mu,$$
$$\psi_{\kappa\lambda\mu\nu},\ \kappa \leqslant \lambda \leqslant \mu \leqslant \nu,\ (\mu, \nu) \neq (2, 2), \quad \zeta_{\mu\nu},\ \mu \leqslant \nu.$$

(Variables of the jet space and related values are defined by their notation up to permutation of indices.) The variable ζ_0 of the jet space is assumed principal, i.e., it is expressed via the other coordinate variables (called the parametric ones) in view of Eq. (8). Under calculation we also carry out the substitutions $\psi_{22\mu} = \zeta_\mu - \psi_{11\mu}$. To avoid repetition of the above conditions for indices, in what follows we assume that the index tuples (μ, ν), (λ, μ, ν), and $(\kappa, \lambda, \mu, \nu)$ satisfy these conditions by default.

Due to the special form of the arbitrary elements f^i, we have to augment Eq. (8) with the following auxiliary system for f^i:

$$f^i_\psi = f^i_{\psi_\mu} = f^i_{\psi_{\mu\nu}} = f^i_{\psi_{\lambda\mu\nu}} = f^i_{\zeta_0} = f^i_{\psi_{\kappa\lambda\mu\nu}} = f^i_{\zeta_{\mu\nu}} = 0. \tag{10}$$

As we compute the usual equivalence algebra rather than the generalized one[30] and the arbitrary elements f^i do not depend on fourth-order derivatives of ψ, the elements of the algebra are assumed to be vector fields in the joint space of the variables of $\mathfrak{J}^{(3)}$ and the arbitrary elements f^i, which are projectable to both the spaces (t, x, y, ψ) and $\mathfrak{J}^{(3)}$. In other words, the algebra consists of vector fields of the general form

$$Q = \xi^\mu \partial_\mu + \eta \partial_\psi + \eta^\mu \partial_{\psi_\mu} + \eta^{\mu\nu} \partial_{\psi_{\mu\nu}} + \eta^{\lambda\mu\nu} \partial_{\psi_{\lambda\mu\nu}} + \theta^\mu \partial_{\zeta_\mu} + \varphi^i \partial_{f^i},$$

where $\xi^\mu = \xi^\mu(t, x, y, \psi)$, $\eta = \eta(t, x, y, \psi)$, the coefficients corresponding to derivatives of ψ are obtained by the standard prolongation (4) from ξ^μ and η, the coefficients θ^ν are obtained by the standard prolongation from ξ^μ and $\theta = \eta^{ii}$, and the coefficients φ^i depends on all the variables of $\mathfrak{J}^{(3)}$ and the arbitrary elements f^j. As a result, each element from the equivalence algebra is determined by its coefficients ξ^μ, η, and φ^i. To act on the Eqs. (8) and (10) by the operator Q, we should additionally prolong it to the variables $\psi_{\kappa\lambda\mu\nu}$ and $\zeta_{\mu\nu}$ in the conventional way and to the derivatives of f, assuming all the variables of $\mathfrak{J}^{(3)}$ as usual ones:

$$\bar{Q} = Q + \eta^{\kappa\lambda\mu\nu} \partial_{\psi_{\kappa\lambda\mu\nu}} + \theta^{\mu\nu} \partial_{\zeta_{\mu\nu}} + \varphi^{i\mu} \partial_{f^i_\mu} + \varphi^{i\psi} \partial_{f^i_\psi} + \varphi^{i\psi_\mu} \partial_{f^i_{\psi_\mu}} + \varphi^{i\psi_{\mu\nu}} \partial_{f^i_{\psi_{\mu\nu}}} + \varphi^{i\psi_{\lambda\mu\nu}} \partial_{f^i_{\psi_{\lambda\mu\nu}}} + \varphi^{i\zeta_\mu} \partial_{f^i_{\zeta_\mu}}.$$

First we consider the infinitesimal invariance conditions associated with Eqs. (10). The invariance condition for the equation $f^i_\psi = 0$ is

$$\varphi^{i\psi}\big|_{\text{Eq. (10)}} = \varphi^i_\psi - \xi^\mu_\psi f^i_\mu - \theta^k_\psi f^i_{\zeta_k} = 0.$$

Splitting with respect to derivatives of f^i in the latter equation implies that $\varphi^i_{\psi}=0$, $\xi^\mu_\psi=0$, $\theta^i_\psi=0$. As $\theta^i=\mathrm{D}_j\mathrm{D}_j\mathrm{D}_i(\eta-\xi^\mu\psi_\mu)+\xi^\mu\psi_{\mu jji}$, we additionally derive the simple determining equation $\eta_{\psi\psi}=0$.

In a similar way, the invariance conditions for the equations $f^i_{\psi_\mu}=0$, $f^i_{\psi_{\mu\nu}}=0$, $f^i_{\psi_{\lambda\mu\nu}}=0$ and $f^i_{\zeta_0}=0$ can be presented in the form

$$\begin{aligned}
&\varphi^{i\psi_\mu}\big|_{\text{Eq. (10)}}=\varphi^i_{\psi_\mu}-\theta^k_{\psi_\mu}f^i_{\zeta_k}=0,\\
&\varphi^{i\psi_{\mu\nu}}\big|_{\text{Eq. (10)}}=\varphi^i_{\psi_{\mu\nu}}-\theta^k_{\psi_{\mu\nu}}f^i_{\zeta_k}=0,\\
&\varphi^{i\psi_{\lambda\mu\nu}}\big|_{\text{Eq. (10)}}=\varphi^i_{\psi_{\lambda\mu\nu}}-\theta^k_{\psi_{\lambda\mu\nu}}f^i_{\zeta_k}=0,\\
&\varphi^{i\zeta_0}\big|_{\text{Eq. (10)}}=\varphi^i_{\zeta_0}-\theta^k_{\zeta_0}f^i_{\zeta_k}=0,
\end{aligned}$$

which is split into $\varphi^i_{\psi_\mu}=0$, $\theta^k_{\psi_\mu}=0$; $\varphi^i_{\psi_{\mu\nu}}=0$, $\theta^k_{\psi_{\mu\nu}}=0$; $\varphi^i_{\psi_{\lambda\mu\nu}}=0$, $\theta^k_{\psi_{\lambda\mu\nu}}=0$; and $\varphi^i_{\zeta_0}=0$, $\theta^k_{\zeta_0}=0$, respectively. The equations $\theta^k_{\psi_\mu}=0$, $\theta^k_{\psi_{\lambda\mu\nu}}=0$, and $\theta^k_{\zeta_0}=0$ provide no essential restrictions on the coefficients ξ^μ, η, and φ^i. From the equation $\theta^k_{\psi_{\lambda\mu\nu}}=0$ we derive that $\xi^0_j=0$, $\xi^1_2+\xi^2_1=0$ and $\xi^1_1-\xi^2_2=0$. Hence,

$$\theta=\eta^{jj}=\eta_{jj}+2\eta_{j\psi}\psi_j+\eta_\psi\psi_{jj}-2\xi^i_j\psi_{ij}=\eta_{jj}+2\eta_{j\psi}\psi_j+(\eta_\psi-2\xi^1_1)\zeta.$$

It remains to solve the determining equations following from the invariance condition for Eq. (8). The invariance condition reads

$$\theta^0+\eta^1\zeta_2+\psi_1\theta^2-\eta^2\zeta_1+\psi_2\theta^1=\varphi^{ii}+\varphi^{i\zeta_j}\zeta_{ji}+f^i_{\zeta_j}\theta^{ji},$$

or explicitly

$$\begin{aligned}
&\eta_{jjt}+\eta_{jj\psi}\psi_t+2\eta_{tj\psi}\psi_j+2\eta_{j\psi}\psi_{tj}+(\eta_{t\psi}-2\xi^1_{t1})\zeta\\
&+(\eta_\psi-2\xi^1_1-\xi^0_t)(f^i_i+f^i_{\zeta_j}\zeta_{ij}-\psi_1\zeta_2+\psi_2\zeta_1)-\xi^i_t\zeta_i\\
&+(\eta_1+\eta_\psi\psi_1-\xi^i_1\psi_i)\zeta_2+\psi_1(\eta_{jj2}+\eta_{jj\psi}\psi_2+2\eta_{2j\psi}\psi_j+2\eta_{j\psi}\psi_{2i}+(\eta_\psi-2\xi^1_1)\zeta_2-\xi^i_2\zeta_i)\\
&-(\eta_2+\eta_\psi\psi_2-\xi^i_2\psi_i)\zeta_1-\psi_2(\eta_{jj1}+\eta_{jj\psi}\psi_1+2\eta_{1j\psi}\psi_j+2\eta_{j\psi}\psi_{1j}+(\eta_\psi-2\xi^1_1)\zeta_1-\xi^i_1\zeta_i)\\
&=\varphi^i_i+\varphi^i_{f^j}f^j_i-\xi^j_if^i_j-\theta^i_kf^i_{\zeta_k}+\zeta_{ij}(\varphi^i_{\zeta_j}+\varphi^i_{f^k}f^k_{\zeta_j}-\theta^k_{\zeta_j}f^i_{\zeta_k})+f^i_{\zeta_j}\theta^{ij}.
\end{aligned}$$

Collecting the coefficients of ψ_{tj} gives $\eta_{j\psi}=0$. This implies that $\theta_\psi=0$. Similarly, the coefficients of $\psi_i\zeta_j$ lead to the equation $\eta_{ijj}=0$ and $\eta_\psi-2\xi^1_1+\xi^0_t=0$. As $\xi^0_i=0$ and $\eta_{i\psi}=0$, the second equation together with the relations $\xi^1_1=\xi^2_2$ and $\xi^1_2+\xi^2_1=0$ implies that $\xi^i_{jk}=0$. Then, the coefficient of ζ gives $\xi^0_{tt}=0$ and the coefficients of f^i_j lead to $\varphi^1_{f^2}=\xi^1_2$, $\varphi^2_{f^1}=\xi^2_1$ and $\varphi^1_{f^1}=\varphi^2_{f^2}=\xi^1_1-2\xi^0_t$. In view of the determining equations that we have already derived, the terms involving $f^i_{\zeta_j}$ are identically canceled. Note that the coefficients of $f^i_{\zeta_j}\zeta_{kl}$ simultaneously lead to the same set of equations as the coefficients of f^i_j.

The remaining part of the invariance condition is $\eta_{jjt}-\xi^i_t\zeta_i+\eta_1\zeta_2-\eta_2\zeta_1=\varphi^i_i+\zeta_{ij}\varphi^i_{\zeta_j}$. Splitting with respect to ζ_{ij} in this relation gives $\varphi^1_{\zeta_1}=\varphi^2_{\zeta_2}=0$, $\varphi^1_{\zeta_2}+\varphi^2_{\zeta_1}=0$ and

$$\varphi^i_i=\eta_{jjt}-\xi^i_t\zeta_i+\eta_1\zeta_2-\eta_2\zeta_1.$$

Acting on the last equation by the operator $\partial_j\partial_{\zeta_j}$, we obtain $\xi^i_{it}=0$. Further splitting with respect to ζ_1 and ζ_2 is not possible since φ^j may depend on them.

Finally, the reduced system of determining equations reads

$$\xi^0_\psi=\xi^0_i=\xi^0_{tt}=0,\quad \xi^i_\psi=\xi^i_{jk}=0,\quad \xi^i_{it}=0,\quad \xi^1_1=\xi^2_2,\quad \xi^1_2+\xi^2_1=0,$$
$$\eta_{\psi\psi}=0,\quad \eta_\psi=2\xi^1_1-\xi^0_t,\quad \eta_{ijj}=0,$$
$$\varphi^i_\psi=0,\quad \varphi^i_{\psi_\mu}=0,\quad \varphi^i_{\psi_{\mu\nu}}=0,\quad \varphi^i_{\psi_{\lambda\mu\nu}}=0,\quad \varphi^i_{\zeta_0}=0,$$
$$\varphi^1_{f^2}=\xi^1_2,\quad \varphi^2_{f^1}=\xi^2_1,\quad \varphi^1_{f^1}=\varphi^2_{f^2}=\xi^1_1-2\xi^0_t,$$
$$\varphi^1_{\zeta_1}=\varphi^2_{\zeta_2}=0,\quad \varphi^1_{\zeta_2}+\varphi^2_{\zeta_1}=0,\quad \varphi^i_i=\eta_{jjt}-\xi^i_t\zeta_i+\eta_1\zeta_2-\eta_2\zeta_1.$$

The solution of this system provides the principal coefficients of the operators from the equivalence algebra of the class (8),

$$\begin{aligned}
&\xi^0=c_1t+c_0,\qquad \xi^1=c_2x-\beta y+\gamma^1,\qquad \xi^2=\beta x+c_2y+\gamma^2,\\
&\eta=(2c_2-c_1)\psi+\delta-\gamma^1_t y+\gamma^2_t x+\frac{\beta_t}{2}(x^2+y^2)+\frac{\sigma}{2}(x^2+y^2)+\chi,\\
&\varphi^1=(c_2-2c_1)f^1-\beta f^2+\delta\zeta_y+\frac{\sigma}{2}(x^2+y^2)\zeta_y+\beta_{tt}x+\sigma_t x-\rho_y,\\
&\varphi^2=\beta f^1+(c_2-2c_1)f^2-\delta\zeta_x-\frac{\sigma}{2}(x^2+y^2)\zeta_x+\beta_{tt}y+\sigma_t y+\rho_x,
\end{aligned}\tag{11}$$

where β, γ^i, σ, and χ are real-valued smooth functions of t only, c_0, c_1, and c_2 are arbitrary constants, ρ is an arbitrary function of t, x, and y and $\delta=\delta(t,x,y)$ is an arbitrary solution of the Laplace equation $\delta_{jj}=0$.

Splitting with respect to parametric values in (11), we obtain the coefficients of the basis operators (9) of the algebra $\mathfrak{g}^{\sim}_1$. Recall that the coefficients η^μ, $\eta^{\mu\nu}$, $\eta^{\lambda\mu\nu}$, and θ^ν are calculated from ξ^μ and η via the standard procedure of prolongation and the coefficients φ^i do not depend on ψ_μ, $\psi_{\mu\nu}$, $\psi_{\lambda\mu\nu}$, and ζ_0. Therefore, both the operators from $\mathfrak{g}^{\sim}_1$ and their commutators are completely determined by the coefficients of ∂_μ, ∂_ψ, ∂_{ζ_i}, and ∂_{f^j}. This is why in (9) and similar formulas we omit the other terms for sake of brevity. □

Remark: The auxiliary system for the arbitrary elements is an important component of the definition of a class of differential equations. Its choice is usually guided by some prior knowledge about the processes to be parameterized. We have decided to assume that the arbitrary elements f^1 and f^2 depend also on t, keeping in mind two more, purely mathematical, reasons. The first reason is that the projection of the corresponding equivalence algebra on the space (t,x,y,ψ) contains the maximal Lie invariance algebra $\mathfrak{g}_0$ of the vorticity Eq. (5) which is the initial point of the entire consideration. The basis operators (6) of $\mathfrak{g}_0$ are obtained from (9) by

$$\mathcal{D}_1=\mathrm{P}\tilde{\mathcal{D}}_1,\qquad \partial_t=\mathrm{P}\partial_t,\qquad \mathcal{D}_2=\mathrm{P}\tilde{\mathcal{D}}_2,\qquad \mathcal{J}=\mathrm{P}\tilde{\mathcal{J}}(1),\qquad \mathcal{J}^t=\mathrm{P}\tilde{\mathcal{J}}(t),$$
$$\mathcal{X}(\gamma^1)=\mathrm{P}\tilde{\mathcal{X}}(\gamma^1),\qquad \mathcal{Y}(\gamma^2)=\mathrm{P}\tilde{\mathcal{Y}}(\gamma^2),\qquad \mathcal{Z}(\chi)=\mathrm{P}\tilde{\mathcal{Z}}(\chi),$$

where P denotes the projection operator on the space (t,x,y,ψ). (Though the expressions for the operator ∂_t (resp. $\tilde{\mathcal{X}}(\gamma^1)$, $\tilde{\mathcal{Y}}(\gamma^2)$ or $\tilde{\mathcal{Z}}(\chi)$) and its projection formally coincide, they in fact determine vector fields on different spaces.) The second reason is that the class (8) is normalized, cf. Sec. III C. This in particular implies that the maximal Lie invariance algebra of any equation from the class (8) is contained in the projection of the equivalence algebra $\mathfrak{g}^{\sim}_1$ of this class.

We also calculate the equivalence algebras of two subclasses of the class (8).

The first subclass corresponds to parameterizations not depending on time explicitly and, therefore, is singled out from the class (8) by the further auxiliary equation

$$f^i_t=0,$$

which has no influence on splitting of the invariance conditions for the Eqs. (8) and (10) and gives the additional determining equations $\varphi^i_t=\xi^i_t=\theta^i_t=0$. These determining equations imply that β, γ^i, and σ are constant, δ is a function only of x and y and ρ can be assumed as a function only of x

and y. Therefore, the equivalence algebra of this subclass is

$$\langle \tilde{\mathcal{D}}_1,\ \partial_t,\ \tilde{\mathcal{D}}_2,\ \tilde{\mathcal{J}}(1),\ \tilde{\mathcal{X}}(1),\ \tilde{\mathcal{Y}}(1),\ \tilde{\mathcal{R}}(1),\ \tilde{\mathcal{H}}(\delta),\ \tilde{\mathcal{G}}(\rho),\ \tilde{\mathcal{Z}}(\chi)\rangle,$$

where the parameter-function $\delta = \delta(x, y)$ runs through the set of solutions of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$ and $\rho = \rho(x, y)$ is an arbitrary function of its arguments.

The second subclass is associated with spatially independent parameterizations. Hence, we additionally set

$$f^i_j = 0.$$

It has to be noted that after attaching this condition we cannot split with respect to f^i_j as we did in the course of solving the determining equations. However, precisely the same conditions obtained from splitting with respect to f^i_j can also be obtained from splitting with respect to $f^i_{\zeta_j}$. Hence, the condition $f^i_j = 0$ only leads to the additional restriction $\varphi^i_j = 0$ and, therefore, we find that $\delta_i = 0$, $\sigma = 0$, $\beta_{tt} = 0$, and $\rho_{ij} = 0$. Without loss of generality we can set $\rho = \rho^i(t)z_i$, where ρ^i are arbitrary smooth functions of t. As a result, the equivalence algebra $\mathfrak{g}^{\sim}_2$ of the second subclass is generated by the operators

$$\tilde{\mathcal{D}}_1,\ \partial_t,\ \tilde{\mathcal{D}}_2,\ \tilde{\mathcal{J}}(1),\ \tilde{\mathcal{J}}(t),\ \tilde{\mathcal{X}}(\gamma^1),\ \tilde{\mathcal{Y}}(\gamma^2),\ \tilde{\mathcal{H}}(\delta),\ \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y),\ \tilde{\mathcal{Z}}(\chi),$$

where γ^i, ρ^i, δ, and χ are arbitrary smooth functions of t.

The intersection of the above subclasses corresponds to the set of parameterizations independent of both t and (x, y) and is singled out from the class (8) by the joint auxiliary system

$$f^i_t = f^i_j = 0.$$

Its equivalence algebra is the intersection of the equivalence algebras of the above subclasses and, therefore, equals

$$\langle \tilde{\mathcal{D}}_1,\ \partial_t,\ \tilde{\mathcal{D}}_2,\ \tilde{\mathcal{J}}(1),\ \tilde{\mathcal{X}}(1),\ \tilde{\mathcal{Y}}(1),\ \tilde{\mathcal{H}}(1),\ \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y),\ \tilde{\mathcal{Z}}(\chi)\rangle,$$

where ρ^1, ρ^2, and χ are arbitrary smooth functions of t.

C. Normalized classes of generalized vorticity equations

In the course of computing the set of admissible transformations of a class of differential equations, it is often convenient to construct a hierarchy of normalized superclasses for this class.[40,41] This is why here we also start with the quite general class of differential equations

$$\zeta_t - F(t, x, y, \psi, \psi_x, \psi_y, \zeta, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{xy}, \zeta_{yy}) = 0, \quad \zeta := \psi_{ii}, \tag{12}$$

where $(F_{\zeta_x}, F_{\zeta_y}, F_{\zeta_{xx}}, F_{\zeta_{xy}}, F_{\zeta_{yy}}) \neq (0, 0, 0, 0, 0)$, to assure that the generalized vorticity equations of the form (8) belong to this class. We use notations and agreements from Sec. III B. In particular, $z = (z_0, z_1, z_2) = (t, x, y)$, the indices i, j, and k again run through $\{1, 2\}$, while the indices κ, λ, μ, and ν range from 0 to 2.

Admissible transformations are determined using the direct method in terms of finite transformations. Namely, we aim to exhaustively describe point transformations of the form

$$\mathcal{T}: \quad \tilde{z}_\mu = Z^\mu(z, \psi), \quad \tilde{\psi} = \Psi(z, \psi), \quad \text{where} \quad J = \frac{\partial(Z^0, Z^1, Z^2, \Psi)}{\partial(z_0, z_1, z_2, \psi)} \neq 0,$$

which map an equation from class (12) to an equation from the same class. We express derivatives of the "old" dependent variable ψ with respect to the "old" independent variables z via derivatives of the "new" dependent variable $\tilde{\psi}$ with respect to the "new" independent variables $\tilde{z}$. The latter derivatives will be marked by tilde over ψ. Thus, the derivative of $\tilde{\psi}$ with respect to $\tilde{z}_\mu$ is briefly denoted by $\tilde{\psi}_\mu$, etc. Then we substitute the expressions for derivatives into the equation $\zeta_t - F = 0$, exclude the new principal derivative $\tilde{\psi}_{022}$ using the transformed equation $\tilde{\psi}_{022} = -\tilde{\psi}_{011} + \tilde{F}$, split with respect to parametric variables whenever this is possible and solve the obtained determining equations for Z^μ and Ψ supplemented with the inequality $J \neq 0$, considering all arising cases for values of the arbitrary element F and simultaneously finding the expression for $\tilde{F}$ via F, Z^μ, and Ψ.

The first order derivatives ψ_μ are expressed in the following manner:

$$\psi_\mu = -\frac{\Psi_\mu - \tilde\psi_\nu Z^\nu_\mu}{\Psi_\psi - \tilde\psi_\nu Z^\nu_\psi} = -\frac{V_\mu}{V_\psi},$$

where we have introduced the notation $V = V(z, \psi, \tilde z) := \Psi(z, \psi) - \tilde\psi_\nu(\tilde z)Z^\nu(z, \psi)$ which is assumed as a function of the old dependent and independent variables and the new independent variables, so that $V_\mu = \Psi_\mu - \tilde\psi_\nu Z^\nu_\mu$ and $V_\psi = \Psi_\psi - \tilde\psi_\nu Z^\nu_\psi$. We will not try to express the old variables via the new variables by inverting the transformation. This is a conventional trick within the direct method, which essentially simplifies the whole consideration. In what follows we will also use three more abbreviations similar to V_μ:

$$U^{\mu\nu} := Z^\mu_\nu V_\psi - Z^\mu_\psi V_\nu, \quad W^{\mu\nu} := U^{\mu i}U^{\nu j}F_{\zeta_{ij}}, \quad P^\mu := U^{\mu 0} - U^{\mu i}F_{\zeta_i}.$$

Higher order derivatives are expressible in an analogous way. The Laplacian of ψ, e.g., reads

$$\psi_{ii} = V_\psi^{-3}(U^{\mu i}U^{\nu i}\tilde\psi_{\mu\nu} - V_\psi^2 V_{ii} + 2V_i V_\psi V_{i\psi} - V_i^2 V_{\psi\psi}).$$

For the class (12) considered here, we need the derivatives of the Laplacian up to second order. The highest derivatives required are of the form

$$\psi_{iijk} = V_\psi^{-5}U^{\mu i}U^{\nu i}U^{\kappa j}U^{\lambda k}\tilde\psi_{\mu\nu\kappa\lambda} + \cdots,$$

where the tail contains only derivatives of $\tilde\psi$ up to order three.

Denote by G the left hand side of the equation obtained by substituting all the expressions for derivatives into (12). For the transformation $\mathcal{T}$ to be admissible, the condition $G_{\tilde\psi_{\mu\nu\kappa\lambda}} = 0$ has to be satisfied for any tuple of the subscripts $(\mu, \nu, \kappa, \lambda)$ in which at least one of the subscripts equals 0. Under varying the subscripts, this condition leads to the following system:

$$\begin{aligned}
&G_{\tilde\psi_{0000}} = 0: \quad U^{0k}U^{0k}W^{00} = 0,\\
&G_{\tilde\psi_{000i}} = 0: \quad U^{0k}U^{0k}W^{0i} + U^{0k}U^{ik}W^{00} = 0,\\
&G_{\tilde\psi_{00ij}} = 0: \quad U^{0k}U^{0k}W^{ij} + 2U^{0k}U^{ik}W^{0j} + 2U^{0k}U^{jk}W^{0i} + U^{ik}U^{jk}W^{00} = 0.
\end{aligned}$$

Suppose that $U^{0k}U^{0k} \neq 0$. Then the above equations imply that $W^{\mu\nu} := U^{\mu i}U^{\nu j}F_{\zeta_{ij}} = 0$. If $\operatorname{rank}(U^{\mu i}) < 2$ then for any μ and ν

$$U^{\mu 1}U^{\nu 2} - U^{\mu 2}U^{\nu 1} = \left(\frac{\partial(Z^\mu, Z^\nu, \Psi)}{\partial(z_1, z_2, \psi)} - \tilde\psi_\kappa \frac{\partial(Z^\mu, Z^\nu, Z^\kappa)}{\partial(z_1, z_2, \psi)}\right) V_\psi = 0$$

and after splitting with respect to $\tilde\psi_\lambda$ we obtain that

$$\frac{\partial(Z^\mu, Z^\nu, \Psi)}{\partial(z_1, z_2, \psi)} = \frac{\partial(Z^\mu, Z^\nu, Z^\kappa)}{\partial(z_1, z_2, \psi)} = 0 \quad \text{or} \quad Z^\kappa_\psi = \Psi_\psi = 0,$$

but this contradicts the transformation nondegeneracy condition $J \neq 0$. Hence, $\operatorname{rank}(U^{\mu i}) = 2$ and, therefore, the equation $U^{\mu i}U^{\nu j}F_{\zeta_{ij}} = 0$ sequentially implies that $U^{\nu j}F_{\zeta_{ij}} = 0$ and $F_{\zeta_{ij}} = 0$. Then, the necessary conditions $G_{\tilde\psi_{000}} = 0$ and $G_{\tilde\psi_{00i}} = 0$ for admissible transformations are, respectively, equivalent to the equations $U^{0k}U^{0k}P^0 = 0$ and $U^{0k}U^{0k}P^i + 2U^{0k}U^{ik}P^0 = 0$ which jointly gives in view of the condition $U^{0k}U^{0k} \neq 0$ that $P^\mu = 0$. Thus, we should have $\det(U^{\mu\nu}) = 0$. At the same time,

$$\begin{aligned}
\det(U^{\mu\nu}) &= V_\psi{}^2\big(|Z^0_\nu, Z^1_\nu, Z^2_\nu|V_\psi - |V_\nu, Z^1_\nu, Z^2_\nu|Z^0_\psi - |Z^0_\nu, V_\nu, Z^2_\nu|Z^1_\psi - |Z^0_\nu, Z^1_\nu, V_\nu|Z^2_\psi\big)\\
&= V_\psi{}^2\frac{\partial(Z^0, Z^1, Z^2, V)}{\partial(z_0, z_1, z_2, \psi)} = V_\psi{}^2 J \neq 0
\end{aligned}$$

that leads to a contradiction. Therefore, the supposition $U^{0k}U^{0k} \neq 0$ is not true, i.e., $U^{0k}U^{0k} = 0$ and hence $U^{0k} = 0$. Substituting the expressions for U^{0k} and V into the last equation and splitting with respect to $\tilde\psi_\mu$, we derive the equations

$$Z^0_k Z^\mu_\psi = Z^0_\psi Z^\mu_k, \quad Z^0_k \Psi_\psi = Z^0_\psi \Psi_k.$$

The tuples (Z_1^μ, Ψ_1), (Z_2^μ, Ψ_2), and (Z_ψ^μ, Ψ_ψ) are not proportional since $J \neq 0$. This is why we finally obtain the first subset of determining equations $Z_k^0 = Z_\psi^0 = 0$. It follows from them that $Z_0^0 \neq 0$ (otherwise $J = 0$) and expressions for "old" derivatives with respect to only x and y contain "new" derivatives only of the same type. In other words, derivatives of $\tilde\psi$ involving differentiation with respect to $\tilde t$ appear only in the expressions for ψ_{0aa} and we can simply split with respect to them via collecting their coefficients.

Equating the coefficients of $\tilde\psi_{012}$ leads, in view of the condition $Z_0^0 \neq 0$, to the equation $U^{1k}U^{2k} = 0$, i.e.,

$$\left(Z_k^1\Psi_\psi - Z_\psi^1\Psi_k + (Z_\psi^1 Z_k^2 - Z_k^1 Z_\psi^2)\tilde\psi_2\right)\left(Z_k^2\Psi_\psi - Z_\psi^2\Psi_k - (Z_\psi^1 Z_k^2 - Z_k^1 Z_\psi^2)\tilde\psi_1\right) = 0. \tag{13}$$

We split Eq. (13) with respect to $\tilde\psi_1$ and $\tilde\psi_2$. Collecting the coefficients of $\tilde\psi_1\tilde\psi_2$ gives the equation $(Z_\psi^1 Z_k^2 - Z_k^1 Z_\psi^2)(Z_\psi^1 Z_k^2 - Z_k^1 Z_\psi^2) = 0$, or equivalently $Z_\psi^1 Z_k^2 - Z_k^1 Z_\psi^2 = 0$. As $\operatorname{rank}(Z_1^i, Z_2^i, Z_\psi^i) = 2$, this implies that $Z_\psi^i = 0$ and, therefore, $\Psi_\psi \neq 0$. Consequently, Eq. (13) is reduced to $Z_k^1 Z_k^2 = 0$.

The derivative $\tilde\psi_{022}$ is assumed principal, $\tilde\psi_{022} = -\tilde\psi_{011} + \tilde F$. Hence, another third order derivative of the above type appropriate for splitting is only $\tilde\psi_{011}$. The corresponding equation $Z_k^1 Z_k^1 = Z_k^2 Z_k^2 := L$ joint with the equation $Z_k^1 Z_k^2 = 0$ implies that the functions Z^1 and Z^2 satisfy the Cauchy–Riemann system $Z_1^1 = \varepsilon Z_2^2$, $Z_2^1 = -\varepsilon Z_1^2$, where $\varepsilon = \pm 1$, and hence $Z_{kk}^i = 0$. Note that $L \neq 0$ since $J \neq 0$.

Analogously, collecting the coefficients of $\tilde\psi_{0i}$ and further splitting with respect to $\tilde\psi_j$ lead to the equations $Z_k^i Z_k^j \Psi_{\psi\psi} = 0$ and $Z_k^i \Psi_{k\psi} = 0$. Therefore, $\Psi_{\psi\psi} = 0$ and $\Psi_{k\psi} = 0$. Here, we take into account the inequalities $L \neq 0$ and $\det(Z_k^i) \neq 0$.

We do not have more possibilities for splitting. The derived system of determining equations consists of the equations

$$Z_k^0 = Z_\psi^0 = 0, \quad Z_\psi^i = 0, \quad Z_k^1 Z_k^2 = 0, \quad Z_k^1 Z_k^1 = Z_k^2 Z_k^2, \quad \Psi_{\psi\psi} = \Psi_{k\psi} = 0.$$

The remaining terms determine the transformation rule for the arbitrary element F. This is why any point transformation satisfying the above determining equations maps every equation from class (12) to an equation from the same class and, therefore, belongs to the equivalence group $G_1^{\sim}$ of class (12). In other words, any admissible point transformation of class (12) is induced by a transformation from $G_1^{\sim}$, i.e., class (12) is normalized. As a result, we have the following theorem.

Theorem 2: *Class* (12) *is normalized. Its equivalence group* $G_1^{\sim}$ *consists of the transformations*

$$\tilde t = T(t), \quad \tilde x = Z^1(t,x,y), \quad \tilde y = Z^2(t,x,y), \quad \tilde\psi = \Upsilon(t)\psi + \Phi(t,x,y),$$
$$\tilde F = \frac{1}{T_t}\left(\frac{\Upsilon}{L}F + \left(\frac{\Upsilon}{L}\right)_0\zeta + \left(\frac{\Phi_{ii}}{L}\right)_0 - \frac{Z_t^i Z_j^i}{L}\left(\frac{\Upsilon}{L}\zeta_j + \left(\frac{\Upsilon}{L}\right)_j\zeta + \left(\frac{\Phi_{ii}}{L}\right)_j\right)\right),$$

where T, Z^i, Υ, *and* Φ *are arbitrary smooth functions of their arguments, satisfying the conditions* $Z_k^1 Z_k^2 = 0$, $Z_k^1 Z_k^1 = Z_k^2 Z_k^2 := L$, $T_t \Upsilon L \neq 0$, *and the subscripts 1 and 2 denote differentiation with respect to x and y, respectively.*

The expression for the transformed vorticity is also simple: $\tilde\zeta = L^{-1}(\Upsilon\zeta + \Phi_{ii})$.

Remark: The continuous component of unity of the group $G_1^{\sim}$ consists of the transformations from $G_1^{\sim}$ with $T_t > 0$, $\varepsilon = 1$, and $\Upsilon > 0$. Therefore, a complete set of independent discrete transformations in $G_1^{\sim}$ is exhausted by the uncoupled changes of the signs of t, y, and ψ. In particular, the value $\varepsilon = -1$ corresponds to alternating the sign of y.

Consider the subclass of class (12), singled out by the constraints $F_\psi = 0$, $F_{\psi_x} = -\zeta_y$, and $F_{\psi_y} = \zeta_x$, i.e., the class consisting of the equations of the form

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x = H(t,x,y,\zeta,\zeta_x,\zeta_y,\zeta_{xx},\zeta_{xy},\zeta_{yy}), \quad \zeta := \psi_{ii}, \tag{14}$$

where H is an arbitrary smooth function of its arguments, which is assumed as an arbitrary element instead of $F = H - \psi_x\zeta_y + \psi_y\zeta_x$. The class (14) is still a superclass of the class (8).

Theorem 3: *Class (14) is normalized. The equivalence group $G_2^{\sim}$ of this class is formed by the transformations*

$$\begin{aligned}
&\tilde t = \tau, \quad \tilde x = \lambda(x\mathfrak{c} - y\mathfrak{s}) + \gamma^1, \quad \varepsilon\tilde y = \lambda(x\mathfrak{s} + y\mathfrak{c}) + \gamma^2,\\
&\tilde\psi = \varepsilon\frac{\lambda}{\tau_t}\left(\lambda\psi + \frac{\lambda}{2}\beta_t(x^2+y^2) - \gamma_t^1(x\mathfrak{s} + y\mathfrak{c}) + \gamma_t^2(x\mathfrak{c} - y\mathfrak{s})\right) + \delta + \frac{\sigma}{2}(x^2+y^2),\\
&\tilde H = \frac{\varepsilon}{\tau_t^2}\left(H - \frac{\tau_{tt}}{\tau_t}\zeta - \frac{\lambda_t}{\lambda}(x\zeta_x + y\zeta_y) + 2\beta_{tt} - 2\frac{\tau_{tt}}{\tau_t}\beta_t\right) - \frac{\delta_y + \sigma y}{\tau_t\lambda^2}\zeta_x + \frac{\delta_x + \sigma x}{\tau_t\lambda^2}\zeta_y\\
&\qquad + \frac{2}{\tau_t}\left(\frac{\sigma}{\lambda^2}\right)_t,
\end{aligned} \tag{15}$$

where $\varepsilon = \pm 1$, $\mathfrak{c} = \cos\beta$, $\mathfrak{s} = \sin\beta$; $\tau, \lambda, \beta, \gamma^i$, and σ are arbitrary smooth functions of t satisfying the conditions $\lambda > 0$ and $\tau_t \neq 0$; $\delta = \delta(t, x, y)$ runs through the set of solutions of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$.

Proof: The class (14) is a subclass of the class (12) and the class (12) is normalized. Therefore, any admissible transformation of the class (14) is generated by a transformation from the equivalence group $G_1^{\sim}$ of the superclass. It is only necessary to derive the additional restrictions on transformation parameters caused by narrowing the class.

The group $G_1^{\sim}$ is a usual equivalence group,[38] i.e., in contrast to different generalizations of equivalence groups,[29,40] it consists of point transformations of the joint space of the equation variables and arbitrary elements, and the components of transformations for the variables do not depend on the arbitrary elements. Any transformation from $G_1^{\sim}$ is additionally projectable to the space of the independent variables and the space of the single variable t. This is why it already becomes convenient, in contrast to the proof of Theorem 2, to express the new derivatives via old ones. Then we substitute the expressions for new derivatives into the transformed equation $\tilde\zeta_{\tilde t} + \tilde\psi_{\tilde x}\tilde\zeta_{\tilde y} - \tilde\psi_{\tilde y}\tilde\zeta_{\tilde x} = \tilde H$, exclude the principal derivative ψ_{tyy} using the equation

$$\psi_{tyy} = -\psi_{txx} - \psi_x\zeta_y + \psi_y\zeta_x + H,$$

split with respect to parametric variables whenever this is possible and solve the obtained determining equations. As equations from the class (14) involve derivatives ψ_x and ψ_y in an explicitly defined (linear) manner, we can split with respect to these derivatives, simply collecting their coefficients. Since these coefficients do not involve the arbitrary element H, we can further split them with respect to other derivatives. As a result, we obtain the equations

$$\Upsilon = \varepsilon\frac{L}{T_t}, \quad L_i = 0, \quad \Phi_{jji} = 0,$$

where $\varepsilon = \pm 1$ and other notations are defined in the proof of Theorem 2. Therefore, L and Φ_{jj} are functions of t only. As $L > 0$, we can introduce the function $\lambda = \sqrt{L}$ of t. Acting by the Laplace operator ∂_{jj} on the conditions $Z^1_k Z^1_k = \lambda^2$ and $Z^2_k Z^2_k = \lambda^2$ and taking into account that Z^i are solutions of the Laplace equation, $Z^i_{kk} = 0$, we derive the important differential consequences $Z^i_{jk} = 0$, which imply that the functions Z^i are affine in (x, y). Hence, there exists a function $\beta = \beta(t)$ such that $Z^1_1 = \lambda\mathfrak{c}$ and $Z^1_2 = -\lambda\mathfrak{s}$, where $\mathfrak{c} = \cos\beta$ and $\mathfrak{s} = \sin\beta$, and, therefore, $Z^1_1 = \varepsilon\lambda\mathfrak{s}$ and $Z^1_2 = \varepsilon\lambda\mathfrak{c}$. We re-denote T by τ for the sake of notation consistency and represent Φ in the following form:

$$\Phi = \delta(t, x, y) + \frac{\sigma}{2}(x^2+y^2) + \varepsilon\frac{\lambda}{\tau_t}\left(\frac{\lambda}{2}\beta_t(x^2+y^2) - \gamma_t^1(x\mathfrak{s} + y\mathfrak{c}) + \gamma_t^2(x\mathfrak{c} - y\mathfrak{s})\right),$$

where σ is a function of t and $\delta = \delta(t, x, y)$ is a solution of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$.

Note that there is an ambiguity in representations of Z^i and Φ. For example, the last summand in the representation of Φ can be omitted. The usage of the above complicated representations is motivated by a few reasons: the consistency with the notation of basis operators of the equivalence

algebra $\mathfrak{g}_1^{\sim}$ from Theorem 1, the simplification of the expression for the transformed arbitrary element $\tilde{H}$ and the convenience of studying admissible transformations within subclasses of the class (14).

Collecting the terms without ψ_x and ψ_y gives the transformation for the arbitrary element H.

Similarly to the proof of Theorem 2, any transformation from $G_1^{\sim}$ satisfying the above additional constraints maps every equation from the class (14) to an equation from the same class and, therefore, belongs to the equivalence group $G_2^{\sim}$ of the class (14). In other words, any admissible point transformation of the class (14) is induced by a transformation from $G_2^{\sim}$, i.e., the class (14) is normalized. □

Remark: The transformations from the equivalence group $G_2^{\sim}$, which are associated with the parameter-function δ depending only on t, and only such transformations identically act on the arbitrary element H and, therefore, their projections to the space of independent and dependent variables form the kernel (intersection) of point symmetry groups of the class (14).

Corollary 1: The subclass of the class (14) *singled out by the constraint* $H_\zeta = 0$ *is normalized. Its equivalence group* $G_3^{\sim}$ *consists of the elements of* $G_2^{\sim}$ *with* $\tau_{tt} = 0$.

Proof: As the vorticity and its derivatives are transformed by elements of $G_2^{\sim}$ according to the formulas

$$\tilde{\zeta} = \frac{\varepsilon}{\tau_t}(\zeta + \beta_t) + 2\frac{\sigma}{\lambda^2}, \quad \tilde{\zeta}_i = \frac{\varepsilon Z_j^i}{\tau_t \lambda^2}\zeta_j, \tag{16}$$

it follows from (15) under the constraints $H_\zeta = 0$ and $\tilde{H}_{\tilde{\zeta}} = 0$ that $\tau_{tt} = 0$. The rest of the proof is similar to the end of the proof of Theorem 3. □

Corollary 2: The subclass of the class (14) *singled out by the constraints* $H_i = 0$ *is normalized. Its equivalence group* $G_4^{\sim}$ *consists of the elements of* $G_2^{\sim}$ *with* $\lambda_t = 0$, $\sigma = 0$, *and* $\delta_{ij} = 0$.

Proof: As any admissible transformation of the class (14) has the form (15) and, therefore, the vorticity and its derivatives are transformed according to (16), the system $\tilde{H}_{\tilde{x}} = 0$, $\tilde{H}_{\tilde{y}} = 0$ is equivalent to the system $\tilde{H}_x = 0$, $\tilde{H}_y = 0$. After differentiating the last equation in (15) with respect to x and y and splitting with respect to ζ_x and ζ_y, we derive all the above additional constraints on transformation parameters. The rest of the proof is similar to the end of the proof of Theorem 3. □

Corollary 3: The subclass of the class (14) *singled out by the constraints* $H_\zeta = 0$ *and* $H_i = 0$ *is normalized. Its equivalence group* $G_5^{\sim}$ *consists of the elements of* $G_2^{\sim}$ *with* $\tau_{tt} = 0$, $\lambda_t = 0$, $\sigma = 0$, *and* $\delta_{ij} = 0$.

Proof: The subclass under consideration is normalized as it is the intersection of the normalized subclasses from Corollaries 1 and 2. Therefore, we also have $G_5^{\sim} = G_3^{\sim} \cap G_4^{\sim}$. □

Remark: For the subclass from Corollary 3, the kernel of point symmetry groups is essentially extended in comparison with the whole class (14). It is formed by the projections of elements of the equivalence group $G_2^{\sim}$, associated with the parameter-functions γ^1 and γ^2 and the parameter-function δ depending only on t, to the space of independent and dependent variables, cf. Sec. III D.

A further narrowing is given by the condition that the arbitrary element H with $H_\zeta = 0$ is a total divergence with respect to the space variables, i.e., $H = \mathrm{D}_i f^i$ for some differential functions $f^i = f^i(t, x, y, \zeta_x, \zeta_y)$. The corresponding subclass rewritten in the terms of f^i coincides with the class (8) and is singled out from the class (14) by the constraints $H_\zeta = 0$ and $\mathsf{E}H = 0$, where $\mathsf{E} = \partial_\zeta - \mathrm{D}_i\partial_{\zeta_i} + \sum_{i \leqslant j} \mathrm{D}_i\mathrm{D}_j\partial_{\zeta_{ij}} + \dots$ is the associated Euler operator. In this Euler operator, the role of independent and dependent variables is played by (x, y) and ζ, respectively, and the variable t is assumed as a parameter. The vorticity ζ can be considered in E as the dependent variable

instead of ψ since the arbitrary element H depends only on combinations of derivatives of ψ being derivatives of ζ.

Remark: It is obvious that the arbitrary element H satisfies the constraints $H_\zeta = 0$ and $\mathsf{E}H = 0$ if it is represented in the form $H = \mathrm{D}_i f^i$ for some differential functions $f^i = f^i(t, x, y, \zeta_x, \zeta_y)$. The converse claim should be proved. Thus, the constraint $\mathsf{E}H = 0$ implies the representation $H = \mathrm{D}_i f^i$ for some differential functions $f^i(t, x, y, \zeta, \zeta_x, \zeta_y)$, which may depend on ζ. Substituting this representation into the constraint $H_\zeta = 0$ and splitting the resulting equations with respect to the second derivatives of ζ, we obtain the following system of partial differential equations for the functions f^i: $f^i_{\zeta i} + f^i_{\zeta\zeta}\zeta_i = 0$, $f^1_{\zeta\zeta_1} = 0$, $f^2_{\zeta\zeta_2} = 0$, $f^1_{\zeta\zeta_2} + f^2_{\zeta\zeta_1} = 0$. Its general solution has the form $f^1 = D_2\Psi + \tilde f^1$ and $f^2 = -D_1\Psi + \tilde f^2$ for some smooth functions $\Psi = \Psi(t, x, y, \zeta)$ and $\tilde f^i = \tilde f^i(t, x, y, \zeta_x, \zeta_y)$. The first summands in the expressions for f^i can be neglected due to the gauge equivalence in the set of arbitrary elements (f^1, f^2). As a result, we construct the necessary representation for the arbitrary element H.

Corollary 4: The class (8) is normalized. The equivalence group $G_6^{\sim}$ of this class represented in terms of the arbitrary element H consists of the elements of $G_2^{\sim}$ with $\tau_{tt} = 0$ and $\lambda_t = 0$. The arbitrary elements f^i are transformed in the following way:

$$
\begin{aligned}
\tilde f^1 &= \varepsilon\lambda\frac{f^1\mathfrak{c} - f^2\mathfrak{s}}{\tau_t^2} + \left(\frac{\delta}{\tau_t\lambda} + \frac{\sigma}{2\tau_t\lambda}(x^2+y^2) - \frac{\varepsilon\chi}{\lambda^2}\right)(\zeta_x\mathfrak{s} + \zeta_y\mathfrak{c})\\
&\quad + (\varepsilon\lambda^2\beta_{tt} + \tau_t\sigma_t)\frac{x\mathfrak{c} - y\mathfrak{s}}{{\tau_t}^2\lambda} - \varepsilon\frac{\rho_x\mathfrak{s} + \rho_y\mathfrak{c}}{\lambda^2},\\
\tilde f^2 &= \lambda\frac{f^1\mathfrak{s} + f^2\mathfrak{c}}{{\tau_t}^2} - \varepsilon\left(\frac{\delta}{\tau_t\lambda} + \frac{\sigma}{2\tau_t\lambda}(x^2+y^2) - \frac{\varepsilon\chi}{\lambda^2}\right)(\zeta_x\mathfrak{c} - \zeta_y\mathfrak{s})\\
&\quad + \varepsilon(\varepsilon\lambda^2\beta_{tt} + \tau_t\sigma_t)\frac{x\mathfrak{s} + y\mathfrak{c}}{{\tau_t}^2\lambda} + \frac{\rho_x\mathfrak{c} - \rho_y\mathfrak{s}}{\lambda^2},
\end{aligned}
\tag{17}
$$

where $\chi = \chi(t)$ and $\rho = \rho(t, x, y)$ are arbitrary functions of their arguments.

Proof: The class (8) is contained in the normalized subclass of the class (14) singled out by the constraint $H_\zeta = 0$. Therefore, any admissible transformation of the class (8) is generated by an element of $G_2^{\sim}$ with $\tau_{tt} = 0$, and the corresponding transformations of the space variables are affine with respect to these variables, $Z^i_{jk} = 0$. Then $\tilde{\mathrm{D}}_j\tilde f^j = \mathrm{D}_i(\lambda^{-2}Z^j_i\tilde f^j)$, i.e., the differential function $\tilde H$ is a total divergence with respect to the new space variables if and only if it is a total divergence with respect to the old space variables. Applying the Euler operator E to the last equality in (15) under the conditions $H_\zeta = 0$, $\tilde H_{\tilde\zeta} = 0$, $\mathsf{E}H = 0$, $\tilde{\mathsf{E}}\tilde H = 0$, and $\tau_{tt} = 0$, we derive the additional constraint $\lambda_t = 0$. The remaining part of the proof of normalization of the class (8) and its equivalence group is analogous to the end of the proof of Theorem 3.

In order to construct the transformations of the arbitrary elements f^i, we represent the right hand side of the last equality in (15) as a total divergence: $\tilde H = \mathrm{D}_i h^i$, where

$$
\begin{aligned}
h^1 &= \frac{\varepsilon}{\tau_t^2}(f^1 + \beta_{tt}x) + \frac{\sigma_t}{\tau_t\lambda^2}x + \left(\delta + \frac{\sigma}{2}(x^2+y^2)\right)\frac{\zeta_y}{\tau_t\lambda^2},\\
h^2 &= \frac{\varepsilon}{\tau_t^2}(f^2 + \beta_{tt}y) + \frac{\sigma_t}{\tau_t\lambda^2}y - \left(\delta + \frac{\sigma}{2}(x^2+y^2)\right)\frac{\zeta_x}{\tau_t\lambda^2}.
\end{aligned}
$$

As $\tilde H = \tilde{\mathrm{D}}_j\tilde f^j = \mathrm{D}_i h^i = \tilde{\mathrm{D}}_j Z^j_i h^i$, the pair of the differential functions $\tilde f^j - Z^j_i h^i$ is a null divergence, $\tilde{\mathrm{D}}_i(\tilde f^j - Z^j_i h^i) = 0$. In view of Theorem 4.24 from Ref. 36 there exists a differential function Q depending on t, x, y, and derivatives of ζ such that $\tilde f^1 - Z^1_i h^i = -\tilde{\mathrm{D}}_2 Q$ and $\tilde f^2 - Z^2_i h^i = \tilde{\mathrm{D}}_2 Q$. As $\tilde{\mathrm{D}}_i Q$ and, therefore, $\mathrm{D}_i Q$ should be functions of t, x, y, ζ_x, and ζ_y, the function Q is represented in the form $Q = \chi(t)\zeta + \rho(t, x, y)$ for some smooth functions $\chi = \chi(t)$ and $\rho = \rho(t, x, y)$. □

Remark: The equivalence transformations associated with the parameter-functions χ and ρ are identical with respect to both the independent and dependent variables, i.e., they transform only arbitrary elements with no effect on the corresponding equation and, therefore, are *trivial* [p. 53 of Ref. 26] or *gauge* [Sec. 2.5 of Ref. 40] equivalence transformations. These transformations arise due to the special representation of the arbitrary element H as a total divergence and form a normal subgroup of the entire equivalence group considered in terms of the arbitrary elements f^1 and f^2, called the *gauge equivalence group* of the class (8).

Remark: The continuous component of unity of the group $G_6^{\sim}$ is singled out from $G_6^{\sim}$ by the conditions $\tau_t > 0$ and $\varepsilon = 1$. Therefore, a complete set of independent discrete transformations in $G_6^{\sim}$ is exhausted by alternating signs either in the tuple (t, ψ) or in the tuple (y, ψ, f^1).

Consider the subclass of the class (8), singled out by the further auxiliary equation $f^i_j = 0$, i.e., the class of equations

$$\zeta_t + \{\psi, \zeta\} = \mathrm{D}_i f^i(t, \zeta_x, \zeta_y), \quad \zeta := \psi_{ii}, \tag{18}$$

with the arbitrary elements $f^i = f^i(t, \zeta_x, \zeta_y)$.

Remark: Rewritten in the terms of H, the class (18) is a well-defined subclass of (14). It is singled out from the class (14) by the constraints $\mathsf{E}H = 0$, $H_\zeta = 0$, $H_i = 0$, and $\zeta_{ij} H_{\zeta_{ij}} = H$. Indeed, the representation $H = \mathrm{D}_i f^i(t, \zeta_x, \zeta_y)$ obviously implies that the arbitrary element H does not depend on x, y, and ζ, is annulated by the Euler operator E and is a (homogenous) linear function in the totality of the derivatives ζ_{ij}. Hence all the above constraints are necessary. Conversely, the constraint $\mathsf{E}H = 0$ implies that the arbitrary element H is affine in the totality of ζ_{ij} and, therefore, in view of the constraint $\zeta_{ij} H_{\zeta_{ij}} = H$ it is a (homogenous) linear function in these derivatives of ζ. As a result, we have the representation $H = h^{ij}\zeta_{ij}$, where the coefficients h^{ij}, $h^{12} = h^{21}$, depend solely on t, ζ_x, and ζ_y since $H_\zeta = 0$ and $H_i = 0$. Then the constraint $\mathsf{E}H = 0$ is equivalent to the single equation

$$2h^{12}_{\zeta_1\zeta_2} = h^{11}_{\zeta_2\zeta_2} + h^{22}_{\zeta_1\zeta_1}$$

whose general solutions is represented in the form $h^{11} = f^1_{\zeta_1}$, $h^{12} = f^1_{\zeta_2} + f^2_{\zeta_1}$, and $h^{22} = f^2_{\zeta_2}$ for some differential functions $f^i = f^i(t, \zeta_x, \zeta_y)$. This finally gives the necessary representation for H.

Remark: In view of the previous remark, the subclass of the class (14), singled out by the constraints $\mathsf{E}H = 0$, $H_\zeta = 0$, and $H_i = 0$ is a proper superclass for the class (18) rewritten in the terms of H. This superclass of (18) is normalized since it is the intersection of the normalized class from Corollary 3 and the normalized class (8). Its equivalence group coincides with the group $G_5^{\sim}$ described in Corollary 3.

In a way analogous to the above proofs, the normalization of the superclass and formulas (15) and (17) imply the following assertion.

Corollary 5: The class (18) *is normalized. The equivalence group* $G_7^{\sim}$ *of this class represented in terms of the arbitrary element* H *consists of the elements of* $G_2^{\sim}$ *with* $\tau_{tt} = 0$, $\lambda_t = 0$, $\beta_{tt} = 0$, $\sigma = 0$, *and* $\delta_i = 0$. *The arbitrary elements* f^i *are transformed according to* (17), *where additionally* $\rho_{ij} = 0$.

Remark: The above consideration of normalized classes is intended for the description of invariant parameterizations of the forms (8) and (18). The hierarchy of normalized classes constructed is, in some sense, minimal and optimal for this purpose. It can be easily extended with related normalized classes. For instance, the subclass singled out from the class (14) by the constraints $\mathsf{E}H = 0$ is normalized. Other hierarchies of normalized classes, which are related to the vorticity equation (5) and different from the hierarchy presented, can be constructed.

Remark: In fact, all subclasses of generalized vorticity equations studied in this section are strongly normalized.[40]

D. Parameterization via direct group classification

As proved in Sec. III C (see Corollary 5), the class (18) is normalized. Hence, the complete group classification for this class can be obtained within the algebraic method. Another way to justify the sufficiency of the algebraic method is to check the weak normalization of the class (18) in infinitesimal sense, i.e., the condition that the linear span the maximal Lie invariance algebras of equations from the class (18) is contained in the projection of its equivalence algebra $\mathfrak{g}^{\sim}_2$ (cf. Sec. III B) to the space of independent and dependent variables. A vector field Q in the space of the variables (t, x, y, ψ) has the form $Q = \xi^\mu \partial_\mu + \eta\partial_\psi$, where the coefficients ξ^μ and η smoothly depend on (t, x, y, ψ). For Q to be a Lie symmetry operator of an equation from the class (18), its coefficients should satisfy the following system of determining equations that do not involve the arbitrary elements (f^1, f^2):

$$\xi^\mu_\psi = 0, \quad \xi^0_i = 0, \quad \xi^0_{tt} = 0, \quad \xi^i_{jk} = 0, \quad \xi^1_{1t} = 0, \quad \xi^1_1 = \xi^2_2, \quad \xi^1_2 = -\xi^2_1,$$
$$\eta_{\psi\psi} = 0, \quad \eta_{\psi t} = 0, \quad \eta_{\psi 1} = \xi^1_t, \quad \eta_{\psi 2} = -\xi^2_t, \quad \eta_\psi - 2\xi^1_1 + \xi^0_t = 0,$$

The integration of the above system immediately implies that $Q \in \mathrm{P}\mathfrak{g}^{\sim}_2$.

The equivalence algebra $\mathfrak{g}^{\sim}_2$ can be represented as a semidirect sum $\mathfrak{g}^{\sim}_2 = \tilde{\mathfrak{i}} \in \tilde{\mathfrak{a}}$, where $\tilde{\mathfrak{i}} = \langle \tilde{\mathcal{X}}(\gamma^1), \tilde{\mathcal{Y}}(\gamma^2), \tilde{\mathcal{Z}}(\chi)\rangle$ and $\tilde{\mathfrak{a}} = \langle \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \partial_t, \tilde{\mathcal{J}}^1, \tilde{\mathcal{J}}^t, \tilde{\mathcal{K}}(\delta), \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y)\rangle$ are an ideal and a subalgebra of $\mathfrak{g}^{\sim}_2$, respectively. Here, $\gamma^1, \gamma^2, \rho^1, \rho^2, \delta$, and χ run through the set of smooth functions of the variable t and we use the notation $\tilde{\mathcal{J}}^1 = \tilde{\mathcal{J}}(1)$, $\tilde{\mathcal{J}}^t = \tilde{\mathcal{J}}(t)$ and $\tilde{\mathcal{K}}(\delta) = \tilde{\mathcal{H}}(\delta) - \tilde{\mathcal{Z}}(\delta)$. The intersection (*kernel*) of the maximal Lie invariance algebras of equations from class (18) is

$$\mathfrak{g}^{\cap}_2 = \langle \mathcal{X}(\gamma^1), \mathcal{Y}(\gamma^2), \mathcal{Z}(\chi)\rangle = \mathrm{P}\tilde{\mathfrak{i}}.$$

In other words, the complete infinite-dimensional part $\mathrm{P}\tilde{\mathfrak{i}}$ of the projection of the equivalence algebra $\mathfrak{g}^{\sim}_2$ to the space of variables (t, x, y, ψ) is already a Lie invariance algebra for any equation from the class (18). Therefore, any Lie symmetry extension is only feasible via (finite-dimensional) subalgebras of the five-dimensional solvable algebra

$$\mathfrak{a} = \langle \mathcal{D}_1, \partial_t, \mathcal{D}_2, \mathcal{J}, \mathcal{J}^t\rangle = \mathrm{P}\tilde{\mathfrak{a}}.$$

In other words, for any values of the arbitrary elements $f^i = f^i(t, \zeta_x, \zeta_y)$ the maximal Lie invariance algebra $\mathfrak{g}^{\max}_f$ of the corresponding equation $\mathcal{L}_f$ from the class (18) is represented in the form $\mathfrak{g}^{\max}_f = \mathfrak{g}^{\mathrm{ext}}_f \notin \mathfrak{g}^{\cap}_2$, where $\mathfrak{g}^{\mathrm{ext}}_f$ is a subalgebra of $\mathfrak{a}$. A nonzero linear combination of the operators $\mathcal{J}$ and $\mathcal{J}^t$ is a Lie symmetry operator of the equation $\mathcal{L}_f$ if and only if this equation is invariant with respect to the algebra $\langle \mathcal{J}, \mathcal{J}^t\rangle$. Therefore, for any extension within the class (18) we have that either $\mathfrak{g}^{\mathrm{ext}}_f \cap \langle \mathcal{J}, \mathcal{J}^t\rangle = \{0\}$ or $\mathfrak{g}^{\mathrm{ext}}_f \supset \langle \mathcal{J}, \mathcal{J}^t\rangle$, i.e.,

$$\dim(\mathfrak{g}^{\mathrm{ext}}_f \cap \langle \mathcal{J}, \mathcal{J}^t\rangle) \in \{0, 2\}. \tag{19}$$

Moreover, as $\mathrm{P}\mathfrak{g}^{\sim}_2 = \mathfrak{g}_0$, the maximal Lie invariance algebra of the inviscid barotropic vorticity equation (5), the normalization of class (18) means that only subalgebras of $\mathfrak{g}_0$ can be used to construct spatially independent parameterization schemes within the class (18). That is, for such parameterizations, the approach from Ref. 34 based on inverse group classification is quite natural and gives the same exhaustive result as direct group classification. Due to the normalization, the complete realization of preliminary group classification of equations from the class (18) is also equivalent to its direct group classification which can be carried out for this class with the algebraic method.

Note that the class (18) possesses the nontrivial gauge equivalence algebra

$$\mathfrak{g}^{\mathrm{gauge}} = \langle \tilde{\mathcal{K}}(\delta), \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y)\rangle,$$

cf., the second remark after Theorem 1. As we have $\mathrm{P}\mathfrak{g}^{\mathrm{gauge}} = \{0\}$, the projections of operators from $\mathfrak{g}^{\mathrm{gauge}}$ obviously do not appear in $\mathfrak{g}^{\mathrm{ext}}_f$ for any value of f. At the same time, they are essential for finding all possible parameterizations that admit symmetry extensions.

Therefore, two equivalent ways for the further use of the algebraic method in this problem depending on subalgebras of what algebra will be classified.

As a first impression, the optimal way is to construct a complete list of inequivalent subalgebras of the Lie algebra $\mathfrak{a}$ and then substitute basis operators of each obtained subalgebra to the infinitesimal invariance criterion in order to derive the associated system of equations for f^i that should be integrated. The algebra $\mathfrak{a}$ is finite-dimensional and has the structure of a direct sum, $\mathfrak{a} = \langle \mathcal{D}_1,\ \partial_t,\ \mathcal{J},\ \mathcal{J}^t\rangle \oplus \langle \mathcal{D}_2\rangle$. The first summand is the four-dimensional Lie algebra $\mathfrak{g}^{-1}_{4.8}$ in accordance with Mubarakzyanovx's classification of low-dimensional Lie algebras[31] whose nilradical is isomorphic to the Weyl (Bianchi II) algebra $\mathfrak{g}_{3.1}$. The classification of inequivalent subalgebra up to the equivalence relation generated by the adjoint action of the corresponding Lie group on $\mathfrak{a}$ is a quite simple problem. Moreover, the set of subalgebras to be used is reduced after taking into account the condition (19). At the same time, the derived systems for f^i consist of second order partial differential equations and have to be integrated up to $G^{\sim}_7$-equivalence.

This is why another way is optimal. It is based on the fact that $\mathfrak{g}^{\rm ext}_f$ coincides with a subalgebra $\mathfrak{b}$ of $\mathfrak{a}$ if and only if there exists a subalgebra $\tilde{\mathfrak{b}}$ of $\tilde{\mathfrak{a}}$ such that $\mathrm{P}\tilde{\mathfrak{b}} = \mathfrak{b}$ and the arbitrary elements f^i satisfy the equations

$$\xi^0 f^i_t + \theta^j f^i_{\zeta_j} = \varphi^i \tag{20}$$

for any operator $\tilde{\mathcal{Q}}$ from $\tilde{\mathfrak{b}}$, where ξ^0, θ^j, and φ^i are coefficients of ∂_t, ∂_{ζ_j}, and ∂_{f^i} in $\tilde{\mathcal{Q}}$, respectively. In fact, the system (20) is the invariant surface condition for the operator $\tilde{\mathcal{Q}}$ and the functions f^i depending only on t and ζ^j. This system is not compatible for any operator from $\tilde{\mathfrak{a}}$ of the form $\tilde{\mathcal{Q}} = \tilde{\mathcal{K}}(\delta) + \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y)$, where at least one of the parameter-functions δ, ρ^1, or ρ^2 does not vanish. In other words, each operator from $\tilde{\mathfrak{b}}$ should have a nonzero part belonging to $\langle \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \partial_t, \tilde{\mathcal{J}}^1, \tilde{\mathcal{J}}^t\rangle$ and hence $\dim \mathrm{P}\tilde{\mathfrak{b}} = \dim \tilde{\mathfrak{b}} \leqslant 5$. Taking into account also the condition (19), we obtain the following algorithm for classification of possible Lie symmetry extensions within the class (18):

1. We classify $G^{\sim}_7$-inequivalent subalgebras of the algebra $\tilde{\mathfrak{a}}$ each of which satisfies the conditions $\dim \mathrm{P}\tilde{\mathfrak{b}} = \dim \tilde{\mathfrak{b}}$ and $\dim(\tilde{\mathfrak{b}} \cap \langle \mathcal{J}, \mathcal{J}^t\rangle) \in \{0, 2\}$. Adjoint actions corresponding to operators from $\tilde{\mathfrak{i}}$ can be neglected.
2. We fix a subalgebra $\tilde{\mathfrak{b}}$ from the list constructed in the first step. This algebra is necessarily finite-dimensional, $\dim \tilde{\mathfrak{b}} \leqslant 5$. We solve the system consisting of equations of the form (20), where the operator $\tilde{\mathcal{Q}}$ runs through a basis of $\tilde{\mathfrak{b}}$. For every solution of this system we have $\mathfrak{g}^{\rm ext}_f = \mathrm{P}\tilde{\mathfrak{b}}$.
3. Varying $\tilde{\mathfrak{b}}$, we get the required list of values of the arbitrary elements (f^1, f^2) and the corresponding Lie symmetry extensions.

In order to realize the first step of the algorithm, we list the nonidentical adjoint actions related to basis elements of $\tilde{\mathfrak{a}}$,

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D}_1 = \mathcal{D}_1 - \varepsilon\partial_t, && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\partial_t = e^{\varepsilon}\partial_t,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{J}^t})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{J}^t, && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{J}^t = e^{-\varepsilon}\mathcal{J}^t,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{K}(\delta)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{K}(t\delta_t + \delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{K}(\delta) = \mathcal{K}(e^{-\varepsilon}\delta(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{G}(t\rho_t + 2\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{G}(\rho) = \mathcal{G}(e^{-2\varepsilon}\rho(e^{-\varepsilon}t, x, y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{K}(\delta)})\partial_t = \partial_t + \varepsilon\mathcal{K}(\delta_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{J}^t = \mathcal{J}^t - \varepsilon\mathcal{J}^1,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\partial_t = \partial_t + \varepsilon\mathcal{G}(\rho_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{K}(\delta) = \mathcal{K}(\delta(t - \varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{J}^t})\partial_t = \partial_t + \varepsilon\mathcal{J}, && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{G}(\rho) = \mathcal{G}(\rho(t - \varepsilon, x, y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{K}(\delta)})\mathcal{D}_2 = \mathcal{D}_2 + \varepsilon\mathcal{K}(2\delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{K}(\delta) = \mathcal{K}(e^{2\varepsilon}\delta(t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_2 = \mathcal{D}_2 + \varepsilon\mathcal{G}(2\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{G}(\rho) = \mathcal{G}(e^{-\varepsilon}\rho(t, e^{-\varepsilon}x, e^{-\varepsilon}y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{J}^1 = \mathcal{J}^1 + \varepsilon\mathcal{G}(\rho^2 x - \rho^1 y), && \mathrm{Ad}(e^{\varepsilon\mathcal{J}^1})\mathcal{G}(\rho) = \mathcal{G}(\hat{\rho}^{\varepsilon}),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{J}^t = \mathcal{J}^t + \varepsilon\mathcal{G}(t\rho^2 x - t\rho^1 y), && \mathrm{Ad}(e^{\varepsilon\mathcal{J}^t})\mathcal{G}(\rho) = \mathcal{G}(\check{\rho}^{\varepsilon}),
\end{aligned}$$

where we omit tildes in the notation of operators and also omit arguments of parameter-functions if these arguments are not changed under the corresponding adjoint action, $\rho = \rho^1 x + \rho^2 y$, $\hat\rho^\varepsilon = (\rho^1 x + \rho^2 y)\cos\varepsilon + (\rho^1 y - \rho^2 x)\sin\varepsilon$, $\check\rho^\varepsilon = (\rho^1 x + \rho^2 y)\cos\varepsilon t + (\rho^1 y - \rho^2 x)\sin\varepsilon t$.

Based upon these adjoint actions, we derive the following list of $G_7^{\sim}$-inequivalent subalgebras of $\tilde{\mathfrak{a}}$ satisfying the above restrictions (we again omit tildes in the notation of operators):

one-dimensional subalgebras:

$$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\mathcal{J}^1\rangle, \quad \langle \partial_t + c\mathcal{D}_2 + \hat c\mathcal{J}^t\rangle, \quad \langle \mathcal{D}_2 + \mathcal{J}^t\rangle, \quad \langle \mathcal{D}_2 + a\mathcal{J}^1\rangle;$$

two-dimensional subalgebras:

$$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\mathcal{J} + \mathcal{K}(c) + \mathcal{G}(\tilde c x), \partial_t\rangle, \quad \langle \mathcal{D}_1 + a\mathcal{J}^1, \mathcal{D}_2 + \hat a\mathcal{J}^1\rangle,$$
$$\langle \partial_t + c\mathcal{J}^t, \mathcal{D}_2 + \hat a\mathcal{J}^1\rangle, \quad \langle \mathcal{J}^1 + \mathcal{K}(\delta^1(t)), \mathcal{J}^t + \mathcal{K}(\delta^2(t))\rangle;$$

three-dimensional subalgebras:

$$\langle \mathcal{D}_1 + a\mathcal{J}^1, \partial_t, \mathcal{D}_2 + \hat a\mathcal{J}^1\rangle, \quad \langle \mathcal{D}_1 + b\mathcal{D}_2, \mathcal{J}^1 + \mathcal{K}(c|t|^{2b-1}), \mathcal{J}^t + \mathcal{K}(\hat c|t|^{2b})\rangle,$$
$$\langle \partial_t + \tilde c\mathcal{D}_2, \mathcal{J}^1 + \mathcal{K}(ce^{2\tilde c t}), \mathcal{J}^t + \mathcal{K}((ct + \hat c)e^{2\tilde c t})\rangle, \quad \langle \mathcal{D}_2, \mathcal{J}^1, \mathcal{J}^t\rangle;$$

four-dimensional subalgebras:

$$\langle \mathcal{D}_1 + b\mathcal{D}_2 + \mathcal{K}(\nu_2), \partial_t, \mathcal{J}^1 + \mathcal{K}(\nu_1), \mathcal{J}^t + \mathcal{K}(\nu_1 t + \nu_0)\rangle,\ (2b-1)\nu_1 = 0,\ b\nu_0 = 0,$$
$$\langle \mathcal{D}_1, \mathcal{D}_2, \mathcal{J}^1, \mathcal{J}^t\rangle, \quad \langle \partial_t, \mathcal{D}_2, \mathcal{J}^1, \mathcal{J}^t\rangle;$$

five-dimensional subalgebra:

$$\langle \mathcal{D}_1, \partial_t, \mathcal{D}_2, \mathcal{J}^1, \mathcal{J}^t\rangle.$$

In the above subalgebras, due to adjoint actions we can put the following restrictions on the algebra parameters: $a \geqslant 0$, $c, \tilde c \in \{0, 1\}$, $\hat a \geq 0$ if $a = 0$ (resp. $c = 0$), $\hat c \in \{0, 1\}$ if $c = 0$; additionally, in the first two-dimensional subalgebra we can set $(1 + 2b)c = 0$ and $((1 + b)^2 + a^2)\tilde c = 0$; in the first four-dimensional subalgebra one non-zero parameter among ν_0, ν_1, ν_2 can be set to 1. In the last two-dimensional subalgebra, the parameters δ^1 and δ^2 are arbitrary smooth functions of t. The subalgebras with parameter tuples (δ^1, δ^2) and $(\tilde\delta^1, \tilde\delta^2)$ are equivalent if and only if there exist constants ε_0, ε_1, and ε_2 such that $\tilde\delta^1 = e^{\varepsilon_2 - \varepsilon_1}\delta^1(e^{-\varepsilon_1}t + \varepsilon_0)$ and $\tilde\delta^2 = e^{\varepsilon_2}\delta^2(e^{-\varepsilon_1}t + \varepsilon_0)$.

Concerning the realization of the second step of the algorithm, we note that the system corresponding to the last two-dimensional subalgebra is compatible if and only if $\delta^2(t) = t\delta^1(t)$. We re-denote δ^1 by δ. As the general solution of the system is parameterized by functions of two arguments, we put the associated two-dimensional symmetry extension into Table I, where the other extensions are one-dimensional. A similar remark is true for the three last three-dimensional subalgebras, which is why we list them in Table II containing symmetry extensions parameterized by functions of a single argument.

TABLE I. Symmetry extensions parameterized by functions of two arguments.

$\mathfrak{g}_f^{\rm ext}$	Arguments of I_1, I_2		f^1, f^2
$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\mathcal{J}\rangle$	$\lvert t\rvert^{b+1}(\zeta_x\cos\tau + \zeta_y\sin\tau)$,	$\tau := a\ln\lvert t\rvert$	$\lvert t\rvert^{b-2}(I_1\cos\tau - I_2\sin\tau)$,
	$\lvert t\rvert^{b+1}(\zeta_y\cos\tau - \zeta_x\sin\tau)$,		$\lvert t\rvert^{b-2}(I_1\sin\tau + I_1\cos\tau)$
$\langle \partial_t + c\mathcal{D}_2 + \hat c\mathcal{J}^t\rangle$	$e^{ct}(\zeta_x\cos\tau + \zeta_y\sin\tau)$,	$\tau := \frac{\hat c}{2}t^2$	$e^{ct}(I_1\cos\tau - I_2\sin\tau)$,
	$e^{ct}(\zeta_y\cos\tau - \zeta_x\sin\tau)$,		$e^{ct}(I_1\sin\tau + I_1\cos\tau)$
$\langle \mathcal{D}_2 + \mathcal{J}^t\rangle$	$t, Re^{\Phi/t}$		P_1, P_2
$\langle \mathcal{D}_2 + a\mathcal{J}\rangle$	$t, R^a e^{\Phi}$		P_1, P_2
$\langle \mathcal{J}, \mathcal{J}^t\rangle$	t, R		$\zeta_x I^1 - \zeta_y I^2 + \delta(t)\zeta_y\Phi$,
			$\zeta_y I^1 + \zeta_x I^2 - \delta(t)\zeta_x\Phi$

TABLE II. Symmetry extensions parameterized by functions of a single argument.

$\mathfrak{g}^{\rm ext}_f$	Argument of I_1, I_2	f^1, f^2
$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\mathcal{J}, \partial_t\rangle, b \neq -1, \frac{1}{2}$	$R^a e^{(1+b)\Phi}$	$R^{\alpha_1} P_1 - \mu\zeta_y$, $R^{\alpha_1} P_2 + \mu\zeta_x$
$\langle \mathcal{D}_1 + \frac{1}{2}\mathcal{D}_2 + a\mathcal{J}, \partial_t\rangle$	$R^a e^{3\Phi/2}$	$R^2 P_1 - \mu\zeta_y \ln R$, $R^2 P_2 + \mu\zeta_x \ln R$
$\langle \mathcal{D}_1 - \mathcal{D}_2 + a\mathcal{J}, \partial_t\rangle, a \neq 0$	R	$e^{\alpha_2\Phi} P_1 - \mu\zeta_y$, $e^{\alpha_2\Phi} P_2 + \mu\zeta_x$
$\langle \mathcal{D}_1 + a\mathcal{J}, \mathcal{D}_2 + \hat{a}\mathcal{J}\rangle$	$\lvert t\rvert^{\hat{a}-a} R^{\hat{a}} e^{\Phi}$	$t^{-3} P_1$, $t^{-3} P_2$
$\langle \partial_t + c\mathcal{J}^t, \mathcal{D}_2 + \hat{a}\mathcal{J}\rangle$	$R^{\hat{a}} e^{\Phi - ct^2/2}$	P_1, P_2
$\langle \mathcal{D}_1 + b\mathcal{D}_2, \mathcal{J}, \mathcal{J}^t\rangle$	$\lvert t\rvert^{b+1} R$	$\lvert t\rvert^{2b-1}(\zeta_x I^1 - \zeta_y I^2 + c\zeta_y\Phi)$, $\lvert t\rvert^{2b-1}(\zeta_y I^1 + \zeta_x I^2 - c\zeta_x\Phi)$
$\langle \partial_t + \tilde{c}\mathcal{D}_2, \mathcal{J}, \mathcal{J}^t\rangle$	$e^{\tilde{c}t} R$	$e^{2\tilde{c}t}(\zeta_x I^1 - \zeta_y I^2 + c\zeta_y\Phi)$, $e^{2\tilde{c}t}(\zeta_y I^1 + \zeta_x I^2 - c\zeta_x\Phi)$
$\langle \mathcal{D}_2, \mathcal{J}, \mathcal{J}^t\rangle$	t	P_1, P_2

The system associated with the first two-dimensional subalgebra is compatible if and only if $(a, b) \neq (0, -1)$. The solution of the system is split into three cases, (i) $b \neq -1, 1/2$, (ii) $b = 1/2$, and (iii) $b = -1$ and $a \neq 0$. We will use the notation $\mu = c/(2b - 1)$ for $b \neq 1/2$ and $\mu = 2c/3$ in case of $b = 1/2$.

For the second and third three-dimensional subalgebras, the corresponding systems are compatible if and only if $c = \hat{c}$ and $\hat{c} = 0$, respectively.

For the reason of compatibility, in the first four-dimensional subalgebra we have $\nu_0 = 0$ and $b \neq -1$. Due to the condition $(2b - 1)\nu_1 = 0$, the solution of the corresponding system should be split into the two cases $b \neq 1/2$ and $b = 1/2$. For simplicity of the representation of the results in Table III we introduce the notation $\mu = \nu_2/(2b - 1)$ if $b \neq 1/2$ and $\tilde{\nu}_2 = 2\nu_2/3$ for $b = 1/2$.

In Tables I–III, I_1 and I_2 are arbitrary functions of two indicated arguments, arbitrary functions of one indicated argument or arbitrary constants, respectively,

$$R = \sqrt{\zeta_x^2 + \zeta_y^2}, \quad \Phi = \arctan\frac{\zeta_y}{\zeta_x}, \quad P_1 = \frac{\zeta_x I_1 - \zeta_y I_2}{\zeta_x^2 + \zeta_y^2}, \quad P_2 = \frac{\zeta_y I_1 + \zeta_x I_2}{\zeta_x^2 + \zeta_y^2}.$$

Moreover, $\alpha_1 = 3/(b + 1)$ (for $b \neq -1$), $\alpha_2 = 3/a$ (for $b = -1$ and $a \neq 0$), and $\alpha_3 = 3/(\hat{a} - a)$ (for $\hat{a} \neq a$). In Table I, δ is an arbitrary function of t.

Up to gauge equivalence, the single parameterization admitting five-dimensional symmetry extension within the class (18) is the trivial parameterization, $f^1 = f^2 = 0$, in which we neglect the eddy vorticity flux. This shows the limits of applicability of the method proposed in Ref. 33, cf. Sec. III A.

E. Parameterization via preliminary group classification

The technique of preliminary group classification is based on classifications of extensions of the kernel Lie invariance algebra by operators obtained via projection of elements of the corresponding equivalence algebra to the space of independent and dependent variables.[18] It is illustrated here with the class (8) whose equivalence algebra $\mathfrak{g}^{\sim}_1$ is calculated in Sec. III B.

TABLE III. Symmetry extensions parameterized by constants.

$\mathfrak{g}^{\rm ext}_f$	f^1, f^2
$\langle \mathcal{D}_1 + a\mathcal{J}^1, \partial_t, \mathcal{D}_2 + \hat{a}\mathcal{J}^1\rangle, \hat{a} \neq a$	$R^{\alpha_3\hat{a}} e^{\alpha_3\Phi} P_1$, $R^{\alpha_3\hat{a}} e^{\alpha_3\Phi} P_2$
$\langle \mathcal{D}_1 + b\mathcal{D}_2, \partial_t, \mathcal{J}, \mathcal{J}^t\rangle, b \neq -1, \frac{1}{2}$	$R^{\alpha_1} P_1 - \mu\zeta_y$, $R^{\alpha_1} P_2 + \mu\zeta_x$
$\langle \mathcal{D}_1 + \frac{1}{2}\mathcal{D}_2, \partial_t, \mathcal{J}, \mathcal{J}^t\rangle$	$R^2 P_1 + (\tilde{\nu}_2 \ln R + \nu_1\Phi)\zeta_y$, $R^2 P_2 - (\tilde{\nu}_2 \ln R + \nu_1\Phi)\zeta_x$
$\langle \mathcal{D}_1, \mathcal{D}_2, \mathcal{J}, \mathcal{J}^t\rangle$	$t^{-3} P_1, t^{-3} P_2$
$\langle \partial_t, \mathcal{D}_2, \mathcal{J}, \mathcal{J}^t\rangle$	P_1, P_2

The kernel Lie invariance algebra $\mathfrak{g}_1^{\cap}$ of the class (8) (i.e., the intersection of the maximal Lie invariance algebras of equations from the class) is $\langle \mathcal{Z}(\chi)\rangle$. Denote by $\tilde{\mathfrak{g}}_1^{\cap}$ the ideal of $\mathfrak{g}_1^{\sim}$ corresponding to $\mathfrak{g}_1^{\cap}$, $\tilde{\mathfrak{g}}_1^{\cap} = \langle \tilde{\mathcal{Z}}(\chi)\rangle$. In view of the classification of one-dimensional subalgebras of the equivalence algebra in Appendix A (list (A1)) and since for preliminary group classification we are only concerned with extensions of the complement of $\tilde{\mathfrak{g}}_1^{\cap}$ in $\mathfrak{g}_1^{\sim}$, we essentially have to consider the inequivalent subalgebras

$$\langle \tilde{\mathcal{D}}_1 + a\tilde{\mathcal{D}}_2\rangle, \quad \langle \partial_t + b\tilde{\mathcal{D}}_2\rangle, \quad \langle \tilde{\mathcal{D}}_2 + \tilde{\mathcal{J}}(\beta) + \tilde{\mathcal{R}}(\sigma)\rangle, \quad \langle \tilde{\mathcal{J}}(\beta) + \tilde{\mathcal{R}}(\sigma)\rangle,$$
$$\langle \tilde{\mathcal{X}}(\gamma^1) + \tilde{\mathcal{R}}(\sigma)\rangle, \quad \langle \tilde{\mathcal{R}}(\sigma) + \tilde{\mathcal{H}}(\delta) + \tilde{\mathcal{G}}(\rho)\rangle.$$

Here, $a \in \mathbb{R}$, $b \in \{-1, 0, 1\}$, $\beta = \beta(t)$, $\sigma = \sigma(t)$, $\gamma^1 = \gamma^1(t)$, and $\rho = \rho(t, x, y)$ are smooth functions of their arguments and $\delta = \delta(t, x, y)$ is a solution of the Laplace equation, $\delta_{xx} + \delta_{yy} = 0$. All parameters are arbitrary but fixed for a particular subalgebra. For each of the subalgebras, the corresponding arbitrary elements f^i satisfy the equations

$$\xi^\mu f^i_\mu + \theta^j f^i_{\zeta_j} = \varphi^i, \tag{21}$$

where ξ^μ, θ^j, and φ^i, respectively, are coefficients of ∂_μ, ∂_{ζ_j}, and ∂_{f^i} in the basis element of the subalgebra. It now remains to present the parameterization schemes constructed, which can be found in Table IV.

In this table, I_1 and I_2 are arbitrary functions of four indicated arguments, $r = \sqrt{x^2 + y^2}$, $\varphi = \arctan y/x$ and $\mathcal{R}(\sigma) = \mathrm{P}\tilde{\mathcal{R}}(\sigma) = \frac{1}{2}\sigma r^2\partial_\psi$.

In the last class of subalgebras no ansatz can be constructed due to the special form of functions f^i. Namely, as the variable ψ is not included in the list of arguments of f^i, any nonzero operator of the form $\tilde{\mathcal{R}}(\sigma) + \tilde{\mathcal{H}}(\delta) + \tilde{\mathcal{G}}(\rho)$ gives an incompatible system of the form (21) and hence its projection does not belong to Lie invariance algebras of equations from the class (8).

Note that some of the extensions presented are not maximal even for the general values of the invariant functions I^1 and I^2. In particular, if an equation from the class (8) possesses a Lie symmetry operator of the form $\mathcal{X}(\gamma^1)$ with a fixed function γ^1, it possesses all the operators of this form.

As the class (8) is normalized (see Corollary 4), its complete group classification also can be obtained by the algebraic method. For this it is enough to classify only special subalgebras of the equivalence algebra $\mathfrak{g}_1^{\sim}$, cf. a similar classification in Sec. III D which is also used here. The restrictions for appropriate subalgebras are mentioned above under the classification of (at least) one-dimensional Lie symmetry extensions. Now we precisely formulate them,

TABLE IV. One-dimensional symmetry algebra extensions for the case $f^i = f^i(t, x, y, \zeta_x, \zeta_y)$.

Extension	Arguments of I_1, I_2	f^1, f^2
$\langle \mathcal{D}_1 + a\mathcal{D}_2\rangle$	$\lvert t\rvert^{-a}x$, $\lvert t\rvert^{-a}y$, $tx\zeta_x$, $ty\zeta_y$	$t^{-2}xI_1$, $t^{-2}yI_2$
$\langle \partial_t + a\mathcal{D}_2\rangle$	$e^{-at}x$, $e^{-at}y$, $x\zeta_x$, $y\zeta_y$	xI_1, yI_2
$\langle \mathcal{D}_2 + \mathcal{J}(\beta) + \mathcal{R}(\sigma)\rangle$	$t, \varphi - \beta\ln r$,	$xI^1 - yI^2 + \frac{\sigma}{2}r^2\zeta_y \ln r + (\beta_{tt} + \sigma_t)x\ln r$,
	$x\zeta_x + y\zeta_y, y\zeta_x - x\zeta_y$	$yI^1 + xI^2 - \frac{\sigma}{2}r^2\zeta_x \ln r + (\beta_{tt} + \sigma_t)y\ln r$
$\langle \mathcal{J}(\beta) + \mathcal{R}(\sigma)\rangle$,	t, r,	$xI^1 - yI^2 + \frac{\sigma}{2\beta}r^2\zeta_y\varphi + \frac{\beta_{tt} + \sigma_t}{\beta}x\varphi$,
$\beta \neq 0$	$x\zeta_x + y\zeta_y, y\zeta_x - x\zeta_y$	$yI^1 + xI^2 - \frac{\sigma}{2\beta}r^2\zeta_x\varphi + \frac{\beta_{tt} + \sigma_t}{\beta}y\varphi$
$\langle \mathcal{X}(\gamma^1) + \mathcal{R}(\sigma)\rangle$,	t, y, ζ_x, ζ_y	$I_1 + \frac{\sigma_t}{\gamma^1}\frac{x^2}{2} + \frac{\sigma\zeta_y}{6\gamma^1}(x^3 + 3xy^2)$,
$\gamma^1 \neq 0$		$I_2 + \frac{\sigma_t}{\gamma^1}xy - \frac{\sigma\zeta_x}{6\gamma^1}(x^3 + 3xy^2)$

- The projection of a subalgebra $\mathfrak{s}$ of $\mathfrak{g}_1^{\sim}$ to the space of variables (t, x, y, ψ) is a Lie invariance algebra of an equation from the class (8) if and only if the corresponding system of equations of the form (21) for the arbitrary elements f^1 and f^2 is compatible.
- Only subalgebras of $\mathfrak{g}_1^{\sim}$ should be classified whose projections to the space of variables (t, x, y, ψ) are the maximal Lie invariance algebra of certain equations from the class (8).

As a result, the classification is split into several cases. For each of the cases we have a common part of Lie symmetry extensions, which may be infinite-dimensional. All additional extensions are finite-dimensional and can be classified with reasonable efforts. We briefly describe only the main cases arising under the classification. The complete classification will be presented elsewhere.

Let $\mathfrak{s}$ be an appropriate subalgebra of $\mathfrak{g}_1^{\sim}$. As remarked above, any appropriate subalgebra does not contain nonzero operators of the form $\tilde{\mathcal{R}}(\sigma) + \tilde{\mathcal{H}}(\delta) + \tilde{\mathcal{G}}(\rho)$ and includes $\tilde{\mathfrak{g}}_1^{\cap} = \langle \tilde{\mathcal{Z}}(\chi) \rangle$ as a proper ideal. Denote by $\mathfrak{j}$ the subspace of $\mathfrak{g}_1^{\sim}$ spanned by the operators $\tilde{\mathcal{X}}(\gamma^1)$, $\tilde{\mathcal{Y}}(\gamma^2)$, $\tilde{\mathcal{R}}(\sigma)$, $\tilde{\mathcal{H}}(\delta)$, and $\tilde{\mathcal{G}}(\rho)$, where the parameters runs through the corresponding sets of functions, cf. Theorem 1. Then denote by r_0 the rank of the set of tuples of functional parameters (γ^1, γ^2) appearing in operators from $\mathfrak{s} \cap \mathfrak{j}$. It is obvious that $r_0 \in \{0, 1, 2\}$. We consider each of the possible values of r_0 separately.

If $r_0 = 0$, any nonzero operator from the complement of $\tilde{\mathfrak{g}}_1^{\cap}$ in $\mathfrak{s}$ has a nonzero projection to the subalgebra $\langle \tilde{D}_1, \tilde{D}_2, \partial_t, \tilde{\mathcal{J}}(\beta) \rangle$, where the parameter-function β runs through the set of smooth functions of t. Suppose that the operators $Q^a = \tilde{\mathcal{J}}(\beta^a) + T^a$ with fixed linearly independent functions β^a and tails $T^a = \tilde{\mathcal{X}}(\gamma^{a1}) + \tilde{\mathcal{Y}}(\gamma^{a2}) + \tilde{\mathcal{R}}(\sigma^a) + \tilde{\mathcal{H}}(\delta^a) + \tilde{\mathcal{G}}(\rho^a) \in \mathfrak{j}$, $a = 1, \dots, n$, where $n \geqslant 2$, belong to $\mathfrak{s}$. Up to $G_6^{\sim}$-equivalence we can assume that $T^1 = \tilde{\mathcal{R}}(\sigma^1)$, i.e., $\gamma^{11} = 0$, $\gamma^{12} = 0$, $\delta^1 = 0$, and $\rho^1 = 0$. As $r_0 = 0$, we have that the commutator of any pair of operators Q's should be a linear combinations of certain Q's and $\tilde{\mathcal{Z}}(\chi)$. This condition taken for Q^1 and the other Q's implies that $\gamma^{a1} = \gamma^{a2} = 0$. Denote by $\mathcal{E}^a$ the equation of the form (21), associated with the operator Q^a. For each $a \neq 1$, we subtract the equation $\mathcal{E}^1$ multiplied by β^a from the equation $\mathcal{E}^a$ multiplied by β^1. This results in the equation that does not involves f^i and, therefore, is an identity. Splitting it with respect to ζ_x and ζ_y, we obtain the system

$$
\begin{aligned}
&(\beta^1\sigma^a - \beta^a\sigma^1)(x^2 + y^2) + 2\beta^1\delta^a = 0,\\
&(\beta^1\beta^a_{tt} - \beta^a\beta^1_{tt} + \beta^1\sigma^a_t - \beta^a\sigma^1_t)x + \beta^1\rho^a_y = 0,\\
&(\beta^1\beta^a_{tt} - \beta^a\beta^1_{tt} + \beta^1\sigma^a_t - \beta^a\sigma^1_t)y - \beta^1\rho^a_x = 0.
\end{aligned}
$$

Taking into account that $\delta^a_{xx} + \delta^a_{yy} = 0$ and cross differentiating the two last equations of the system, we then derive that $\beta^1\sigma^a - \beta^a\sigma^1 = 0$, $\delta^a = 0$, $\rho^a_x = 0$, $\rho^a_y = 0$, and

$$
\beta^1\beta^a_{tt} - \beta^a\beta^1_{tt} + \beta^1\sigma^a_t - \beta^a\sigma^1_t = 0. \tag{22}
$$

Since the parameter-functions ρ^a are defined up to summands being arbitrary smooth functions of t, we can assume, in view of the equations $\rho^a_x = 0$ and $\rho^a_y = 0$, that $\rho^a = 0$. The equations $\beta^1\sigma^a - \beta^a\sigma^1 = 0$ mean that the tuples of β's and σ's are proportional for each t, i.e., there exists a smooth function $\alpha = \alpha(t)$ such that $(\sigma^1, \dots, \sigma^n) = \alpha(\beta^1, \dots, \beta^n)$. We combine the last condition with Eqs. (22) and solve the resulting equations

$$
(\beta^1\beta^a_t - \beta^a\beta^1_t)_t + \alpha(\beta^1\beta^a_t - \beta^a\beta^1_t) = 0
$$

with respect to β^a. The solutions are $\beta^a = c_{1a}\beta^1 \int (\beta^1)^{-2}\tilde{\alpha}\, dt + c_{2a}\beta^1$, where $\tilde{\alpha} = e^{-\int \alpha\, dt}$ and c_{1a} and c_{2a} are arbitrary constants. Therefore, the number n of linearly independent functions β^a cannot be greater than 2. Summing up the above consideration, we conclude that basis elements of $\mathfrak{s}$ belonging to the complement of $\tilde{\mathfrak{g}}_1^{\cap}$ can be assumed to have the following form:

$$
S^b + \tilde{\mathcal{J}}(\hat{\beta}^b) + \hat{T}^b, \quad b = 1, \dots, m, \quad \tilde{\mathcal{J}}(\beta^a) + \tilde{\mathcal{R}}(\sigma^a), \quad a = 1, \dots, n,
$$

where $\langle S^b, b = 1, \dots, m \rangle$ is an m-dimensional subalgebra of $\langle \tilde{D}_1, \tilde{D}_2, \partial_t \rangle$ and hence $0 \leqslant m \leqslant 3$, $\hat{T}^b \in \mathfrak{j}$, $0 \leqslant n \leqslant 2$, the functions β^a are linearly independent and $\sigma^a = -(\ln|\beta^1\beta^2_t - \beta^2\beta^1_t|)_t\beta^a$ if $n = 2$. The total dimension of extension in this case equals $m + n$ and is not greater than 5.

The condition $r_0 = 1$ implies that the subalgebra $\mathfrak{s}$ contains no operators of the form $\tilde{\mathcal{J}}(\beta) + T$, where $\beta \neq 0$ and $T \in \mathfrak{j}$. Suppose that operators $T^s = \tilde{\mathcal{X}}(\gamma^{s1}) + \tilde{\mathcal{Y}}(\gamma^{s2}) + \tilde{\mathcal{R}}(\sigma^s) + \tilde{\mathcal{H}}(\delta^s) + \tilde{\mathcal{G}}(\rho^s)$ from $\mathfrak{j}$, $s = 1, \dots, p$, where $p \geqslant 2$ and $(\gamma^{s1}, \gamma^{s2})$ are linearly independent pairs of functions, belong to $\mathfrak{s}$. Up to $G_6^{\sim}$-equivalence we can assume that $\gamma^{12} = 0$, $\delta^1 = 0$ and $\rho^1 = 0$. As $r_0 = 1$, this also means that $\gamma^{s2} = 0$, $s = 2, \dots, p$, and the parameter-functions $\gamma^{s1} = 0$, $s = 1, \dots, p$, are linearly independent. Analogously to the previous case, denote by $\mathcal{E}^s$ the equation of the form (21), associated with the operator T^s. For each $s \neq 1$, we subtract the equation $\mathcal{E}^1$ multiplied by γ^{s1} from the equation $\mathcal{E}^s$ multiplied by γ^{11}. This results in the equation that does not involves f^i and, therefore, is an identity. Making the same manipulations with the identity as those in the previous case, we obtain $\delta^s = 0$, $\rho^s = 0$, $\gamma^{11}\sigma^s_t = \gamma^{s1}\sigma^1_t$, $\gamma^{11}\sigma^s = \gamma^{s1}\sigma^1$ and, therefore, $\gamma^{11}_t\sigma^s = \gamma^{s1}_t\sigma^1$. In view of the linear independence of γ^{s1} and γ^{11}, the last two conditions form a well-determined homogenous system of linear algebraic equations with respect to σ^1 and σ^s and hence imply that $\sigma^s = \sigma^1 = 0$. At the same time, if an equation from the class (8) possesses a Lie symmetry operator $\mathcal{X}(\gamma^1)$ with a fixed function γ^1, it possesses all the operators of this form. This means that there are only two $G_6^{\sim}$-inequivalent possibility for $\mathfrak{s} \cap \mathfrak{j}$ in this case, namely, $\mathfrak{s} \cap \mathfrak{j}$ is either spanned by a single operator $\tilde{\mathcal{X}}(\gamma^{01}) + \tilde{\mathcal{R}}(\sigma^0)$, where γ^{01} and σ^0 are fixed smooth nonvanishing functions of t, or equal to the entire set of operators of the form $\tilde{\mathcal{X}}(\gamma^1)$, where γ^1 runs through the set of smooth functions of t. Additional extensions are realized only by tuple of operators of the form $S^b + \tilde{\mathcal{J}}(\hat{\beta}^b) + \hat{T}^b$, $b = 1, \dots, m$, where $\hat{T}^b \in \mathfrak{j}$, $\langle S^b, b = 1, \dots, m\rangle$ is an m-dimensional subalgebra of $\langle \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \partial_t\rangle$ and hence $0 \leqslant m \leqslant 3$.

Let $r_0 = 2$. We use notations of the previous case and assume summation for the repeated index i. Suppose that operators T^s, $s = 1, \dots, p$, where $p \geqslant 3$ and $(\gamma^{s1}, \gamma^{s2})$ are linearly independent pairs of functions, belong to $\mathfrak{s}$. In view of the condition $r_0 = 2$, up to permutation of the operators T^s we can assume without loss of generality that $\gamma^{11}\gamma^{22} - \gamma^{12}\gamma^{21} \neq 0$. Then for each $s > 2$ there exist smooth functions α^{si} of t, $i = 1, 2$, such that $(\gamma^{s1}, \gamma^{s2}) = \alpha^{si}(\gamma^{i1}, \gamma^{i2})$. Subtracting the equation $\mathcal{E}^i$ multiplied by α^{si} from the equation $\mathcal{E}^s$, we derive the equation which should identically satisfied and, therefore, implies after certain manipulations that $\delta^s = \alpha^{si}\delta^i$, $\rho^s = \varrho^i$, $\sigma^s = \alpha^{si}\sigma^i$, $\sigma^s_t = \alpha^{si}\sigma^i_t$ and hence $\alpha^{si}_t\sigma^i = 0$. We should separately consider two subcases depending on either vanishing or nonvanishing $\sigma^i\sigma^i$.

If $\sigma^i\sigma^i \neq 0$ then $\mathfrak{s} \cap \mathfrak{j}$ coincides with the set of operators of the general form

$$\tilde{\mathcal{X}}(\alpha^i\gamma^{i1}) + \tilde{\mathcal{Y}}(\alpha^i\gamma^{i2}) + \tilde{\mathcal{R}}(\alpha^i\sigma^i) + \tilde{\mathcal{H}}(\alpha^i\delta^i) + \tilde{\mathcal{G}}(\alpha^i\rho^i),$$

where (α^1, α^2) runs through the set of pairs of smooth functions of t satisfying the condition $\alpha^i_t\sigma^i = 0$. In view of commutation relations between $\tilde{\mathcal{J}}(\beta)$ and operators from $\mathfrak{j}$, no operator of the form $\tilde{\mathcal{J}}(\beta) + T$, where $\beta \neq 0$ and $T \in \mathfrak{j}$, belongs to $\mathfrak{s}$. Additional extensions are realized only by tuple of operators of the form $S^b + \tilde{\mathcal{J}}(\hat{\beta}^b) + \hat{T}^b$, $b = 1, \dots, m$, where $\hat{T}^b \in \mathfrak{j}$, $\langle S^b, b = 1, \dots, m\rangle$ is an m-dimensional subalgebra of $\langle \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \partial_t\rangle$ and hence $0 \leqslant m \leqslant 3$.

Suppose that $\sigma^1 = \sigma^2 = 0$. The condition $[T^1, T^2] \in \mathfrak{s}$ implies that

$$\gamma^{11}\delta^2_x + \gamma^{12}\delta^2_y = \gamma^{21}\delta^1_x + \gamma^{22}\delta^1_y, \quad \gamma^{11}\rho^2_x + \gamma^{12}\rho^2_y = \gamma^{21}\rho^1_x + \gamma^{22}\rho^1_y.$$

Therefore, using the push-forwards of transformations from $G_6^{\sim}$, we can set $\delta^i = 0$, $\rho^i = 0$. In other words, we can assume that the subalgebra $\mathfrak{s}$ contains the operators $T^i = \tilde{\mathcal{X}}(\gamma^{i1}) + \tilde{\mathcal{Y}}(\gamma^{i2})$, where $\gamma^{11}\gamma^{22} - \gamma^{12}\gamma^{21} \neq 0$. The system of equations of the form (21), associated with these operators, is equivalent to the system $f^i_x = f^i_y = 0$, $i = 1, 2$, which singles out the subclass (18) from the class (8). The complete group classification of this subclass has been carried out in Sec. III D.

IV. CONCLUSION

In this paper, we have addressed the question of symmetry-preserving parameterization schemes. It was demonstrated that the problem of finding invariant parameterization schemes can be treated as a group classification problem. In particular, the interpretation of parameterizations as particular elements of classes of differential equations renders it possible to use well-established methods of symmetry analysis for the design of general classes of closure schemes with prescribed symmetry properties. For parameterizations to admit selected subgroups of the maximal Lie invariance group

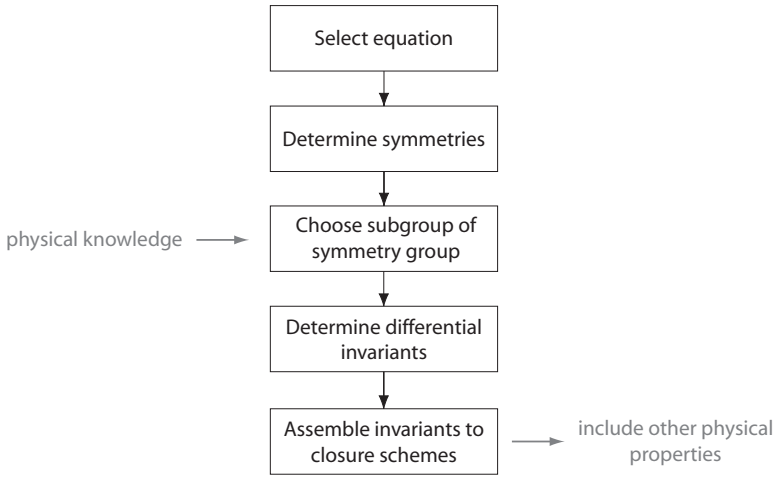

FIG. 1. Schematic overview of the construction of invariant parameterization schemes based on methods of inverse group classification.

of the unaveraged differential equation, they should be expressed in terms of related differential invariants. The general outline of this approach is depicted in Figure 1. Differential invariants can be computed either using infinitesimal methods or the method of moving frames, cf. Sec. III A.

It should be stressed that the selection of subgroups with respect to which a parameterization scheme should be invariant can be naturally justified when considering boundary-value problems. It is usually the case that explicitly taking into account particular initial and/or boundary conditions strongly decreases the number of admitted symmetries, see, e.g., Refs. 6, 7, and 9 for further discussions and particular examples related to geophysical fluid dynamics. For selected subgroups not to be trivial, one can consider a class of similar boundary-value problems instead of a fixed problem and selected those symmetries that are extended to equivalence transformations of this class of boundary-value problems. Hence, symmetry-subgroup admitting parameterization schemes can be especially useful when a parameterization scheme is constructed for particular boundary-value problems.

For parameterization ansatzes with prescribed functional dependence on the resolved quantities and no prescribed symmetry group, the direct group classification problem should be solved. In the case where the given class of differential equations is normalized (which can be checked by the computation of the set of admissible transformations), it is possible and convenient to carry out the classification using the algebraic method.[40] In the case where the class fails to be normalized (or in the case where it is impossible to compute the set of admissible transformations), an exhaustive investigation of parameterizations might be possible due to applying compatibility analysis of the corresponding determining equations or by combining the algebraic and compatibility methods. For more involved classes of differential equations at least symmetry extensions induced by subalgebras of the equivalence algebra can be found, i.e., preliminary group classification can be carried out. The framework of invariant parameterization involving methods of direct group classification is depicted in Figure 2.

Irrespectively of whether one uses direct or inverse group classification techniques, the procedure of invariant parameterization in fact yield classes of parameterization ansatzes rather than a particular fixed parameterization. This gives a certain degree of freedom which allows one to include other desirable physical or structural features into the parameterization scheme. For example, the specification of the parameterizations in Tables I–IV can be done by prescribing a particular form of the functions I_1 and I_2. In the case of inverse group classification, one has to formulate a precise functional relation among related differential invariants. From the point of view of application the freedom in tuning a parameterization is extremely important as the preservation of symmetries is only one feature that might be required when parameterizing a given subgrid-scale process.

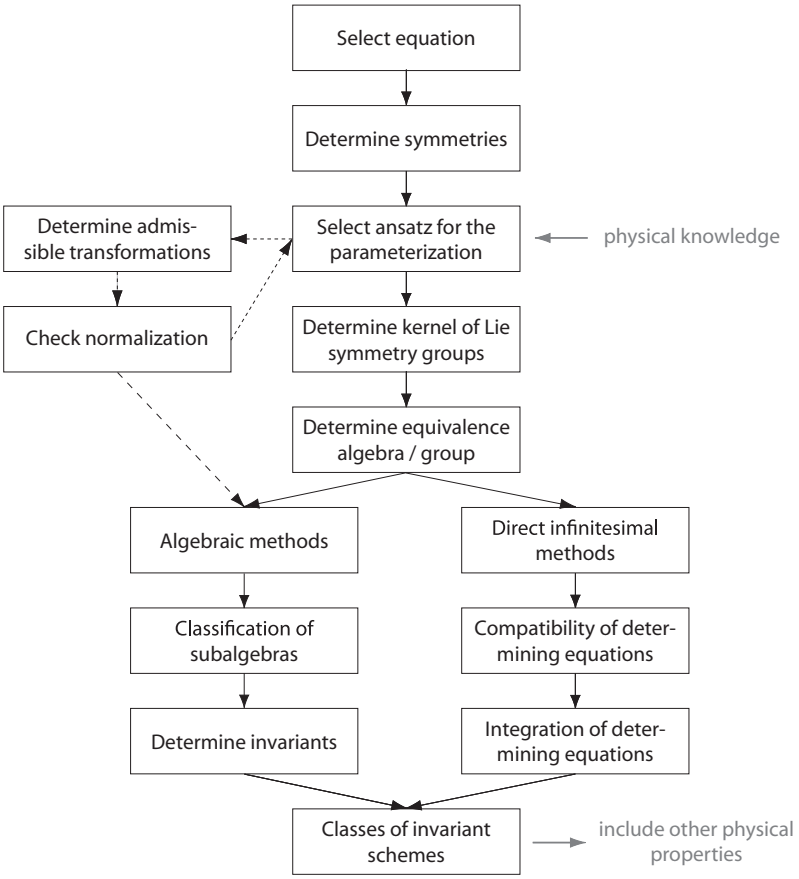

FIG. 2. Schematic overview of the construction of invariant parameterization schemes based on methods of direct group classification.

Since the primary aim of this paper is a clear presentation of the variety of invariant parameterization methods, we focused on rather simple first order local closure schemes for the classical barotropic vorticity equation, cf. the introduction of Sec. III. That is, we parameterized already the eddy vorticity flux $\overline{\mathbf{v}'\zeta'}$ using $\bar{\zeta}$ and its derivatives. Admittedly, this is a quite simple ansatz for one of the simplest physically relevant models in geophysical fluid dynamics. On the other hand, it can be seen that already for this particular simple example the computations involved were rather elaborate. This is in particular true for the computation of the set of admissible transformations for the various classes of vorticity equations considered in Sec. III C. Needless to say that irrespectively of practical computational problems the same technique would be applicable to higher order closure schemes as well. In designing such schemes it is necessary to explicitly include differential equations for the first or higher order correlation terms. In the case of the vorticity equations, a second order closure schemes is obtainable upon retaining the equations governing the evolution of $\overline{\mathbf{v}'\zeta'}$ and parameterize the higher order correlation terms arising in these equations. In practice, however, it becomes increasingly difficult to acquire real atmospheric data for such higher order correlation quantities, which therefore makes it difficult to propose parameterization schemes based solely on physical considerations.[47] We argue that especially in such cases symmetries could provide a useful guiding principle to determine general classes of relevant parameterizations.

Up to now, we have restricted ourselves to the problem of invariant local closure schemes. Nonlocal schemes constructed using symmetry arguments should be investigated in a subsequent work. This extension to nonlocal parameterization schemes is crucial in order to make general methods available that can be used in the development of parameterization schemes for other types of physical processes in atmosphere-ocean dynamics, including, e.g., convection. A further perspective

for generalization of the present work is the design of parameterization schemes that preserve conservation laws. This is another aspect that is of major importance in practical applications. For parameterizations of conservative processes, it is crucial that the corresponding closed differential equation preserves energy conservation. This is by no means self-evident. In fact, energy conservation is violated by various classes of down-gradient ansatzes,[49] which is straightforward to check also for parameterizations constructed in this paper. The construction of parameterization schemes that retain conservation laws will call for the classification of conservation laws in the way similar as the usual group classification. A main complication is that there is no restriction on the order of conservation laws for general systems of partial differential equations (so far, such restrictions are only known for $(1+1)$-dimensional evolution equations of even order and some similar classes of equations). The combination of invariant and conservative parameterization schemes is also conceivable. As shown in Ref. 4, it works for the barotropic vorticity equation on the beta-plane.

It is beyond the scope of the present paper to explicitly test the various parameterization schemes proposed though it was indicated above that some of them might have a physical importance whereas other schemes are obviously flawed. An example on the application of invariant parameterization schemes for the barotropic vorticity equation on the beta-plane to the problem of two-dimensional freely decaying turbulence has been presented in Ref. 4.

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APPENDIX: INEQUIVALENT ONE-DIMENSIONAL SUBALGEBRAS OF THE EQUIVALENCE ALGEBRA OF CLASS (8)

In this appendix, we classify one-dimensional subalgebras of the equivalence algebra $\mathfrak{g}_1^{\sim}$ with basis elements (9). For this means, we subsequently present the commutator table of $\mathfrak{g}_1^{\sim}$. In what follows we omit tildes in the notation of operators.

Based on Table V, it is straightforward to recover the following nontrivial adjoint actions:

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D}_1 = \mathcal{D}_1 - \varepsilon\partial_t, && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\partial_t = e^{\varepsilon}\mathcal{D}_1,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{J}(t\beta_t), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{J}(\beta) = \mathcal{J}(\beta(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{X}(t\gamma_t^1), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{X}(\gamma^1) = \mathcal{X}(\gamma^1(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{Y}(t\gamma_t^2), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{Y}(\gamma^2) = \mathcal{Y}(\gamma^2(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{R}(t\sigma_t + \sigma), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{R}(\sigma) = \mathcal{R}(e^{-\varepsilon}\sigma(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{H}(t\delta_t + \delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{H}(\delta) = \mathcal{H}(e^{-\varepsilon}\delta(e^{-\varepsilon}t, x, y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{G}(t\rho_t + 2\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{G}(\rho) = \mathcal{G}(e^{-2\varepsilon}\rho(e^{-\varepsilon}t, x, y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(\chi)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{Z}(t\chi_t + \chi), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{Z}(\chi) = \mathcal{Z}(e^{-\varepsilon}\chi(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})\partial_t = \partial_t + \varepsilon\mathcal{J}(\beta_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{J}(\beta) = \mathcal{J}(\beta(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\partial_t = \partial_t + \varepsilon\mathcal{X}(\gamma_t^1), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{X}(\gamma^1) = \mathcal{X}(\gamma^1(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\partial_t = \partial_t + \varepsilon\mathcal{Y}(\gamma_t^2), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Y}(\gamma^2) = \mathcal{Y}(\gamma^2(t-\varepsilon)),
\end{aligned}$$

TABLE V. Commutation relations for the algebra $\mathfrak{g}_1^{\sim}$.

	$\mathcal{D}_1$	$\mathcal{D}_2$	∂_t	$\mathcal{J}(\beta)$	$\mathcal{X}(\gamma^1)$
$\mathcal{D}_1$	0	0	$-\partial_t$	$\mathcal{J}(t\beta_t)$	$\mathcal{X}(t\gamma_t^1)$
$\mathcal{D}_2$	0	0	0	0	$-\mathcal{X}(\gamma^1)$
∂_t	∂_t	0	0	$\mathcal{J}(\beta_t)$	$\mathcal{X}(\gamma_t^1)$
$\mathcal{J}(\tilde\beta)$	$-\mathcal{J}(t\tilde\beta_t)$	0	$-\mathcal{J}(\tilde\beta_t)$	0	$-\mathcal{Y}(\tilde\beta\gamma^1)+\mathcal{G}(\gamma^1\tilde\beta_{tt}y)$
$\mathcal{X}(\tilde\gamma^1)$	$-\mathcal{X}(t\tilde\gamma_t^1)$	$\mathcal{X}(\tilde\gamma^1)$	$-\mathcal{X}(\tilde\gamma_t^1)$	$\mathcal{Y}(\beta\tilde\gamma^1)-\mathcal{G}(\tilde\gamma^1\beta_{tt}y)$	0
$\mathcal{Y}(\tilde\gamma^2)$	$-\mathcal{Y}(t\tilde\gamma_t^2)$	$\mathcal{Y}(\tilde\gamma^2)$	$-\mathcal{Y}(\tilde\gamma_t^2)$	$-\mathcal{X}(\beta\tilde\gamma^2)+\mathcal{G}(\tilde\gamma^2\beta_{tt}x)$	$-\mathcal{Z}((\gamma^1\tilde\gamma^2)_t)$
$\mathcal{R}(\tilde\sigma)$	$-\mathcal{R}(t\tilde\sigma_t+\tilde\sigma)$	0	$-\mathcal{R}(\tilde\sigma_t)$	0	$-\mathcal{H}(\gamma^1\tilde\sigma x)+\mathcal{G}(\gamma^1\tilde\sigma_t y)$
$\mathcal{H}(\tilde\delta)$	$-\mathcal{H}(t\tilde\delta_t+\tilde\delta)$	$-\mathcal{H}(x\tilde\delta_x+y\tilde\delta_y-2\tilde\delta)$	$-\mathcal{H}(\tilde\delta_t)$	$-\mathcal{H}(\beta x\tilde\delta_y-\beta y\tilde\delta_x)$	$-\mathcal{H}(\gamma^1\tilde\delta_x)$
$\mathcal{G}(\tilde\rho)$	$-\mathcal{G}(t\tilde\rho_t+2\tilde\rho)$	$-\mathcal{G}(x\tilde\rho_x+y\tilde\rho_y+\tilde\rho)$	$-\mathcal{G}(\tilde\rho_t)$	$-\mathcal{G}(\beta x\tilde\rho_y-\beta y\tilde\rho_x)$	$\mathcal{G}(\gamma^1\tilde\rho_x)$
$\mathcal{Z}(\tilde\chi)$	$-\mathcal{Z}(t\tilde\chi_t+\tilde\chi)$	$2\mathcal{Z}(\tilde\chi)$	$-\mathcal{Z}(\tilde\chi_t)$	0	0
	$\mathcal{Y}(\gamma^2)$	$\mathcal{R}(\sigma)$	$\mathcal{H}(\delta)$	$\mathcal{G}(\rho)$	$\mathcal{Z}(\chi)$
$\mathcal{D}_1$	$\mathcal{Y}(t\gamma_t^2)$	$\mathcal{R}(t\sigma_t+\sigma)$	$\mathcal{H}(t\delta_t+\delta)$	$\mathcal{G}(t\rho_t+2\rho)$	$\mathcal{Z}(t\chi_t+\chi)$
$\mathcal{D}_2$	$-\mathcal{Y}(\gamma^2)$	0	$H(x\delta_x+y\delta_y-2\delta)$	$\mathcal{G}(x\rho_x+y\rho_y+\rho)$	$-2\mathcal{Z}(\chi)$
∂_t	$\mathcal{Y}(\gamma_t^2)$	$\mathcal{R}(\sigma_t)$	$\mathcal{H}(\delta_t)$	$\mathcal{G}(\rho_t)$	$\mathcal{Z}(\chi_t)$
$\mathcal{J}(\tilde\beta)$	$\mathcal{X}(\tilde\beta\gamma^2)-\mathcal{G}(\gamma^2\tilde\beta_{tt}x)$	0	$\mathcal{H}(\tilde\beta x\delta_y-\tilde\beta y\delta_x)$	$\mathcal{G}(\tilde\beta x\rho_y-\tilde\beta y\rho_x)$	0
$\mathcal{X}(\tilde\gamma^1)$	$\mathcal{Z}((\tilde\gamma^1\gamma^2)_t)$	$\mathcal{H}(\tilde\gamma^1\sigma x)-\mathcal{G}(\tilde\gamma^1\sigma_t y)$	$\mathcal{H}(\tilde\gamma^1\delta_x)$	$\mathcal{G}(\tilde\gamma^1\rho_x)$	0
$\mathcal{Y}(\tilde\gamma^2)$	0	$\mathcal{H}(\tilde\gamma^2\sigma y)+\mathcal{G}(\tilde\gamma^2\sigma_t x)$	$\mathcal{H}(\tilde\gamma^2\delta_y)$	$\mathcal{G}(\tilde\gamma^2\rho_y)$	0
$\mathcal{R}(\tilde\sigma)$	$-\mathcal{H}(\gamma^2\tilde\sigma y)-\mathcal{G}(\gamma^2\tilde\sigma_t x)$	0	0	0	0
$\mathcal{H}(\tilde\delta)$	$-\mathcal{H}(\gamma^2\tilde\delta_y)$	0	0	0	0
$\mathcal{G}(\tilde\rho)$	$-\mathcal{G}(\gamma^2\tilde\rho_y)$	0	0	0	0
$\mathcal{Z}(\tilde\chi)$	0	0	0	0	0

$$\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\partial_t=\partial_t+\varepsilon\mathcal{R}(\sigma_t),\qquad \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{R}(\sigma)=\mathcal{R}(\sigma(t-\varepsilon)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\partial_t=\partial_t+\varepsilon\mathcal{H}(\delta_t),\qquad \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{H}(\delta)=\mathcal{H}(\delta(t-\varepsilon,x,y)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\partial_t=\partial_t+\varepsilon\mathcal{G}(\rho_t),\qquad \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{G}(\rho)=\mathcal{G}(\rho(t-\varepsilon,x,y)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(\chi)})\partial_t=\partial_t+\varepsilon\mathcal{Z}(\chi_t),\qquad \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Z}(\chi)=\mathcal{Z}(\chi(t-\varepsilon)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{D}_2=\mathcal{D}_2-\varepsilon\mathcal{X}(\gamma^1),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{X}(\gamma^1)=\mathcal{X}(e^{\varepsilon}\gamma^1),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{D}_2=\mathcal{D}_2-\varepsilon\mathcal{Y}(\gamma^2),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{Y}(\gamma^2)=\mathcal{Y}(e^{\varepsilon}\gamma^2),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{D}_2=\mathcal{D}_2+\varepsilon\mathcal{H}(x\delta_x+y\delta_y-2\delta),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{H}(\delta)=\mathcal{H}(e^{2\varepsilon}\delta(t,e^{-\varepsilon}x,e^{-\varepsilon}y)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_2=\mathcal{D}_2+\varepsilon\mathcal{G}(x\rho_x+y\rho_y+\rho),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{G}(\rho)=\mathcal{G}(e^{-\varepsilon}\rho(t,e^{-\varepsilon}x,e^{-\varepsilon}y)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(\chi)})\mathcal{D}_2=\mathcal{D}_2-2\varepsilon\mathcal{Z}(\chi),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{Z}(\chi)=\mathcal{Z}(e^{2\varepsilon}\chi),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{J}(\beta)=A_1,\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})\mathcal{X}(\gamma^1)=A_3,$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{J}(\beta)=A_2,\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})\mathcal{Y}(\gamma^2)=A_4,$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{J}(\beta)=\mathcal{J}(\beta)+\varepsilon\mathcal{H}(\beta x\delta_y-\beta y\delta_x),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})\mathcal{H}(\delta)=A_5,$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{J}(\beta)=\mathcal{J}(\beta)+\varepsilon\mathcal{G}(\beta x\rho_y-\beta y\rho_x),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})\mathcal{G}(\rho)=A_6,$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1)+\varepsilon\mathcal{Z}((\gamma^1\gamma^2)_t),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2)-\varepsilon\mathcal{Z}((\gamma^1\gamma^2)_t),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\mathcal{X}(\gamma^1)=A_7,\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{R}(\sigma)=A_8,$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1)+\varepsilon\mathcal{H}(\gamma^1\delta_x),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{H}(\delta)=\mathcal{H}(\delta(t,x-\varepsilon\gamma^1,y)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1)+\varepsilon\mathcal{G}(\gamma^1\rho_x),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{G}(\rho)=\mathcal{G}(\rho(t,x-\varepsilon\gamma^1,y)),$$
$$\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\mathcal{Y}(\gamma^2)=A_9,\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{R}(\sigma)=A_{10},$$
$$\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2)+\varepsilon\mathcal{H}(\gamma^2\delta_y),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{H}(\delta)=\mathcal{H}(\delta(t,x,y-\varepsilon\gamma^2)),$$
$$\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2)+\varepsilon\mathcal{G}(\gamma^2\rho_y),\qquad \mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{G}(\rho)=\mathcal{G}(\rho(t,x,y-\varepsilon\gamma^2)),$$

where

$$A_1:=\mathcal{J}(\beta)-\varepsilon\big(\mathcal{Y}(\beta\gamma^1)-\mathcal{G}(\beta_{tt}\gamma^1 y)\big)+\tfrac12\varepsilon^2\mathcal{Z}\big((\beta(\gamma^1)^2)_t\big),$$
$$A_2:=\mathcal{J}(\beta)+\varepsilon\big(\mathcal{X}(\beta\gamma^2)-\mathcal{G}(\beta_{tt}\gamma^2 x)\big)+\tfrac12\varepsilon^2\mathcal{Z}\big((\beta(\gamma^2)^2)_t\big),$$
$$A_3:=\mathcal{X}(\gamma^1\cos\beta\varepsilon)+\mathcal{Y}(\gamma^1\sin\beta\varepsilon)-\varepsilon\mathcal{G}\big(\gamma^1\beta_{tt}(-x\sin\beta\varepsilon+y\cos\beta\varepsilon)\big),$$
$$A_4:=-\mathcal{X}(\gamma^2\sin\beta\varepsilon)+\mathcal{Y}(\gamma^2\cos\beta\varepsilon)+\varepsilon\mathcal{G}\big(\gamma^1\beta_{tt}(x\cos\beta\varepsilon+y\sin\beta\varepsilon)\big),$$
$$A_5:=\mathcal{H}(\delta(t,x\cos\beta\varepsilon+y\sin\beta\varepsilon,-x\sin\beta\varepsilon+y\cos\beta\varepsilon)),$$
$$A_6:=\mathcal{G}(\rho(t,x\cos\beta\varepsilon+y\sin\beta\varepsilon,-x\sin\beta\varepsilon+y\cos\beta\varepsilon)),$$
$$A_7:=\mathcal{X}(\gamma^1)+\varepsilon\big(\mathcal{H}(\gamma^1\sigma x)-\mathcal{G}(\gamma^1\sigma_t y)\big),$$
$$A_8:=\mathcal{R}(\sigma)-\varepsilon\left(\mathcal{H}(\gamma^1\sigma x)-\mathcal{G}(\gamma^1\sigma_t y)\right)+\tfrac12\varepsilon^2\mathcal{H}\left((\gamma^1)^2\sigma\right),$$
$$A_9:=\mathcal{Y}(\gamma^2)+\varepsilon\left(\mathcal{H}\left(\gamma^2\sigma y\right)+\mathcal{G}(\gamma^2\sigma_t x)\right),$$
$$A_{10}:=\mathcal{R}(\sigma)-\varepsilon\left(\mathcal{H}(\gamma^2\sigma y)+\mathcal{G}(\gamma^2\sigma_t x)\right)+\tfrac12\varepsilon^2\mathcal{H}\left((\gamma^2)^2\sigma\right).$$

Using the above adjoint actions, we construct the following optimal list of inequivalent one-dimensional subalgebras of $\mathfrak{g}_1^{\sim}$:

$$\begin{aligned}&\langle\mathcal{D}_1+a\mathcal{D}_2\rangle,\quad \langle\partial_t+b\mathcal{D}_2\rangle,\quad \langle\mathcal{D}_2+\mathcal{J}(\beta)+\mathcal{R}(\sigma)\rangle,\quad \langle\mathcal{J}(\beta)+\mathcal{R}(\sigma)+\mathcal{Z}(\chi)\rangle,\\ &\langle\mathcal{X}(\gamma^1)+\mathcal{R}(\sigma)\rangle,\quad \langle\mathcal{R}(\sigma)+\mathcal{H}(\delta)+\mathcal{G}(\rho)+\mathcal{Z}(\chi)\rangle,\end{aligned}\tag{A1}$$

where $a\in\mathbb{R}$, $b\in\{-1,0,1\}$. In fact, each element of the above list represents a parameterized class of subalgebras rather than a single subalgebra. Particular subalgebras correspond to arbitrary but fixed values of parameters. Subalgebras within each of the four last classes can be equivalent. Thus, in the third class we can use adjoint action $\mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})$ to rescale σ as well as the argument t of β and σ. Using $\mathrm{Ad}(e^{\varepsilon\partial_t})$ allows us to shift t in the functions β and σ. In the fourth class, equivalence is understood up to actions of $\mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})$ and $\mathrm{Ad}(e^{\varepsilon\partial_t})$, which permit rescaling of σ, χ and their argument t, scaling of χ as well as shifts of t in β, σ and χ. Similar equivalence is also included in the fifth class. The last class comprises equivalence with respect to actions of $\mathrm{Ad}(e^{\varepsilon\mathcal{J}(\beta)})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})$. In the three last classes we can also rescale the entire basis elements.

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