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Contents

1. Preface	3
2. Prerequisites	4
2.1. Jets and equations	5
2.2. Linearization and adjoint	5
2.3. Symmetries	6
2.4. Cosymmetries and conservation laws	7
2.5. Differential coverings	7
2.6. Nonlocal symmetries	8
2.7. Bäcklund transformations and recursion operators	9
2.8. Zero curvature representations	9
3. Infinitely many commuting nonlocal symmetries for modified Martínez Alonso–Shabat Equation [I]	10
3.1. The recursion operator	10
3.2. Nonlocal symmetries	10
4. On the four 3-dimensional Lax integrable equations [II], [III], [V]	11
4.1. Symmetries and Lie algebra structure of 4E equations	12
4.2. Lax pairs and differential coverings	14
4.3. Nonlocal symmetries, Lie algebra structure, recursion operators	16
5. 4E Symmetry reductions and its integrability properties	18
5.1. The complete list of 2-dimensional reductions [V]	18
5.2. Integrability properties of some reductions [IV]	21
6. Integrable Weingarten surfaces	24
6.1. Weingarten surfaces	26
6.2. Constant astigmatism equation [VII]	26
6.3. The classification [VI]	27
References	28
Publications concerning the thesis	31

1. Preface

Different approaches to integrability of partial differential equations (PDEs) are based on their diverse but related properties such as existence of infinite hierarchies of (local or nonlocal) symmetries and/or conservation laws, zero-curvature representations, Lax integrability, recursion operators etc.

This thesis consists of papers

- [I] Baran, H. *Infinitely many commuting nonlocal symmetries for modified Martinez Alonso–Shabat equation.* Communications in Nonlinear Science and Numerical Simulation **96** (2021), 105692.
- [II] Baran, H., Krasil'shchik, I.S., Morozov, O.I., and Vojčák, P. *Nonlocal Symmetries of Integrable Linearly Degenerate Equations: A Comparative Study.* Theoretical and Mathematical Physics **196** (2) (2018), 1089–1110.
- [III] Baran, H., Krasil'shchik, I.S., Morozov, O.I., and Vojčák, P. *Coverings over Lax integrable equations and their nonlocal symmetries.* Theoretical and Mathematical Physics **188** (3) (2016), 1273–1295.
- [IV] Baran, H., Krasil'shchik, I.S., Morozov, O.I., and Vojčák, P. *Integrability properties of some equations obtained by symmetry reductions.* Journal of Nonlinear Mathematical Physics **22** (2) (2015), 210–232.
- [V] Baran, H., Krasil'shchik, I.S., Morozov, O.I., and Vojčák, P. *Symmetry reductions and exact solutions of Lax integrable 3-dimensional systems.* Journal of Nonlinear Mathematical Physics **21** (4) (2014), 643–671.
- [VI] Baran, H. and Marvan, M. *Classification of integrable Weingarten surfaces possessing an sl(2)-valued zero curvature representation.* Nonlinearity **23** (10) (2010), 2577–2597.
- [VII] Baran, H. and Marvan, M. *On integrability of Weingarten surfaces: A forgotten class.* Journal of Physics A: Mathematical and Theoretical **42** (40) (2009), 404007.

All of them study integrability properties of some or several nonlinear PDE.

Section 2 is a brief review of basic definitions from geometry of PDEs and fixes some notation.

Section 3 reviews the results of the paper [I] on the 4-dimensional *modified Martínez Alonso–Shabat* equation

$$u_y u_{xz} + \alpha u_x u_{ty} - (u_z + \alpha u_t) u_{xy} = 0$$

and presents its recursion operator and an infinite commuting hierarchy of nonlocal symmetries. Discovering explicit form of the infinite-dimensional nonlocal symmetry algebras for multidimensional integrable PDEs, rather than just finding shadows of nonlocal symmetries, appears to be quite difficult and hence was only done for a very small number of examples, especially in the case of four or more independent variables. On the other hand, the situation seems to be different in 3D, where infinite-dimensional noncommutative algebras of nonlocal symmetries for a number of dispersionless integrable systems were found by direct computations, as we can see in [II, III].

In Section 4, we focus on the papers [II, III] where we considered the four 3-dimensional Lax-integrable equations: The *universal hierarchy* equation

$$u_{yy} = u_t u_{xy} - u_y u_{tx},$$

the *rdDym* equation

$$u_{ty} = u_x u_{xy} - u_y u_{xx},$$

the *modified Veronese web equation*

$$u_{ty} = u_t u_{xy} - u_y u_{tx}$$

and the *3D Pavlov* equation

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}.$$

All of the above four equations (we denote them as 4E) can be obtained as reductions of five-dimensional equation

$$u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs}$$

studied in [3]. In the papers [II, III], for all the 4E equations, a Lie algebra of local symmetries is described, two infinite-dimensional differential coverings are constructed, a complete description of nonlocal symmetry algebras associated to these coverings is given and actions of recursion operators on shadows of nonlocal symmetries are discussed.

In Section 5 we study 2-dimensional reductions of 4E equations following the papers [IV, V]. The paper [V] presents a complete description of 2-dimensional equations that arise as symmetry reductions of 4E equations. In the paper [IV], we study the behavior of the integrability features of 4E equations under the reduction procedure. We show that the zero-curvature representations are transformed to nonlinear differential coverings of the resulting 2-dimensional systems similar to the one found for the Gibbons-Tsarev equation. Using these coverings we construct infinite series of (nonlocal) conservation laws and prove their nontriviality. We also show that the recursion operators are not preserved under reductions.

The Section 6 follows the papers [VI, VII], where we study classes of surfaces immersed in the Euclidean space whose Gauss–Mainardi–Codazzi equations are integrable in the sense of soliton theory. The paper [VII] reveals an integrable class, consisting of surfaces with a constant difference between the principal radii of curvature, which we called *surfaces of constant astigmatism* and are described by the integrable nonlinear PDE

$$z_{yy} + (1/z)_{xx} + 2 = 0,$$

the *constant astigmatism equation*. In the paper [VI] we classify integrable Weingarten surfaces, where the criterion of integrability is that the associated Gauss equation possesses an $\mathfrak{sl}(2)$-valued zero curvature representation with a nonremovable parameter. Under certain restrictions on the jet order, the answer is given by a third order ordinary differential equation to govern the functional dependence of the principal curvatures. We give a general solution of the governing equation in terms of elliptic integrals. We show that the instances when the elliptic integrals degenerate to elementary functions were known to nineteenth century geometers.

Note that all the symbolic computations in papers constituting this thesis were performed using the software `Jets` [4].

2. Prerequisites

We give here (in a simplified, local coordinate form) the basics of the geometrical approach to differential equations and differential coverings following [2] and [22].

2.1. **Jets and equations.** Consider $\mathbb{R}^n$ with coordinates $x^1, \dots, x^n$ and $\mathbb{R}^m$ coordinated by $u^1, \dots, u^m$. The space of *k-jets* $J^k(n,m)$, $k = 0, 1, \dots, \infty$, carries the coordinates $x^1, \dots, x^n$ and u^j_σ, where $j = 1, \dots, m$ and σ is a symmetrical multi-index of length $|\sigma| \le k$, $u^j_\varnothing = u^j$. If $u^j = f(x^1, \dots, x^n)$ is a vector-function then the collection

$$u^j_\sigma = \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}, \qquad j = 1, \dots, m, \quad |\sigma| \le k,$$

is called its *k-jet*.

At a fixed point $\theta \in J^k(n,m)$ tangent planes to the graphs of k-jets passing through this point span the *Cartan plane* $\mathscr{C}_\theta$ and the correspondence $\mathscr{C} \colon \theta \mapsto \mathscr{C}_\theta$ is called the *Cartan distribution*. For $k = \infty$, a basis of $\mathscr{C}$ consists of the vector fields

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \qquad i = 1, \dots, n,$$

called the *total derivatives*. The total derivatives commute which amounts to the formal integrability of the Cartan distribution on $J^\infty(n,m)$. We put

$$D_\sigma = D_{x^{i_1}} \circ \dots \circ D_{x^{i_k}}$$

for $\sigma = i_1 \dots i_k$.

The *differential equation of order k* is a submanifold in $J^k(n,m)$ given by the relations

$$F^1(x^i, u^j_\sigma) = \dots = F^r(x^i, u^j_\sigma) = 0; \tag{2.1}$$

for the sake of simplicity we speak of differential equations even if we in fact deal with systems of those.

The *infinite prolongation* $\mathscr{E} \subset J^\infty(n,m)$ of (2.1) is given by

$$D_\sigma(F^j) = 0, \qquad j = 1, \dots, r, \quad |\sigma| \ge 0.$$

Everywhere below we deal with infinite prolongations only and identify them with differential equations under study.

The total derivatives, as well as all differential operators expressed in terms of total derivatives, are restrictable to the infinite prolongations defined above, and we preserve the same notation for these restrictions. Total derivatives then span the Cartan distribution on $\mathscr{E}$. Maximal integral manifolds of this distribution are solutions of $\mathscr{E}$.

Given an $\mathscr{E}$, we for a subsequent computations always choose *internal coordinates* in it, which are local coordinates on the infinite prolongation $\mathscr{E}$. The choice of internal coordinates is not unique. To restrict an operator to $\mathscr{E}$ essentially amounts to expressing this operator in terms of internal coordinates.

2.2. **Linearization and adjoint.** The *linearization* $\ell_{\mathscr{E}}$ of $\mathscr{E}$ is defined as the restriction of the matrix operator

$$\ell_F = \left(\sum_\sigma \frac{\partial F^\alpha}{\partial u^\beta_\sigma} D_\sigma \right)^{\alpha = 1, \dots, r}_{\beta = 1, \dots, m} \tag{2.2}$$

to $\mathscr{E}$.

Let Δ be a differential operator in the matrix form $\Delta = (\Delta^{\alpha\beta})$, $\Delta^{\alpha\beta} = \sum_\sigma \Delta^{\alpha\beta}_\sigma D_\sigma$. Its *adjoint* is the matrix operator

$$\Delta^* = \left(\Delta^{*\alpha\beta}\right), \quad \Delta^{*\alpha\beta} = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ \Delta^{\alpha\beta}_\sigma.$$

In particular, the adjoint to ℓ_F is given by

$$\ell^*_F = \left(\sum_\sigma (-1)^{|\sigma|} D_\sigma \circ \frac{\partial F^\alpha}{\partial u^\beta_\sigma}\right)^T.$$

If $\mathscr{E}$ is the equation defined by F, we use the notation

$$\ell_{\mathscr{E}} = \ell_F|_{\mathscr{E}}, \quad \ell^*_{\mathscr{E}} = \ell^*_F|_{\mathscr{E}}.$$

2.3. **Symmetries.** Consider an equation $\mathscr{E} \subset J^\infty(n,m)$. We shall assume below that the natural projection $\mathscr{E} \to J^0(n,m) = \mathbb{R}^n \times \mathbb{R}^m$ is a surjective map *onto* its target. This means that the differential consequences of (2.1) do not contain 0-order functions.. Consequently, the algebra $C^\infty(J^0(n,m))$ is embedded into the algebra $C^\infty(\mathscr{E})$.

A vector field $X\colon C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ is called *vertical* if $X|_{C^\infty(J^0(n,m))} = 0$, i.e., X does not contain components of the form $\partial/\partial x^i$. A vertical field X is a (*higher*, or *generalized*) *symmetry* of $\mathscr{E}$ if it preserves the Cartan distribution, i.e., $[X, \mathscr{C}] \subset \mathscr{C}$. Symmetries of $\mathscr{E}$ form a Lie algebra denoted by $\mathrm{sym}(\mathscr{E})$.

A vector field is a symmetry if and only if it has the *evolutionary* form

$$\mathbf{E}_\varphi = \sum_\sigma D_\sigma(\varphi^j)\frac{\partial}{\partial u^j_\sigma}, \tag{2.3}$$

where summation is taken over the internal coordinates on $\mathscr{E}$ and $\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector-function on $\mathscr{E}$ called the *generating section* (or *characteristic*) of the symmetry that satisfies the equation

$$\ell_{\mathscr{E}}(\varphi) = 0.$$

Generating sections are (vector) functions that form a Lie algebra with respect to the *Jacobi bracket*

$$\{\varphi, \psi\}^j = \sum_\sigma \left(D_\sigma(\varphi^l)\frac{\partial \psi^j}{\partial u^l_\sigma} - D_\sigma(\psi^l)\frac{\partial \varphi^j}{\partial u^l_\sigma}\right),$$

which can be defined in the coordinate-free fashion as

$$\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi).$$

A solution u of the equation (2.1) is said to be *invariant* with respect to a symmetry $\varphi \in \mathrm{sym}\,\mathscr{E}$ if it enjoys the equation

$$\varphi\Big(x, \dots, \frac{\partial^{|\sigma|} u}{\partial x^\sigma}, \dots\Big) = 0. \tag{2.4}$$

The *reduction* of $\mathscr{E}$ with respect to φ is equation (2.1) rewritten in terms of first integrals of the equation (2.4).

2.4. **Cosymmetries and conservation laws.** A *cosymmetry* of the equation $\mathscr{E}$ is a solution of the equation

$$\ell^*_{\mathscr{E}}(\psi) = 0.$$

The space of cosymmetries is denoted by $\operatorname{cosym}\mathscr{E}$.

A horizontal $(n-1)$-form on $\mathscr{E}$

$$\omega = \omega_1\, dx^2\wedge dx^3\wedge\dots\wedge dx^n + \omega_2\, dx^1\wedge dx^3\wedge\dots\wedge dx^n + \dots + \omega_n\, dx^1\wedge dx^2\wedge\dots\wedge dx^{n-1}$$

defines a *conservation law* of $\mathscr{E}$ if

$$\sum_{i=1}^{n}(-1)^{i+1}D_i(\omega_i) = 0,$$

i. e. when is closed with respect to the horizontal de Rham differential

$$d_h = \sum_{i=1}^{n} dx^i \wedge D_{x^i}.$$

A conservation law is *trivial* if it is d_h-exact, i.e. $\omega = d_h\rho$ for some horizontal $(n-2)$-form ρ. We are interested in nontrivial conservation laws. Two conservation laws are *equivalent* if their difference is a trivial one.

Let ω be a conservation law and let us extend the form ω on $\mathscr{E}$ to $\tilde{\omega}$ on $J^\infty(n,m)$ in an arbitrary way. Then

$$\sum_{i=1}^{n}(-1)^i D_{x^i}(\tilde{\omega}_i) = \Delta(F) \tag{2.5}$$

for some differential operator Δ. Function $\psi_\omega = \Delta^*(1)|_{\mathscr{E}}$ is called the *generating function* of the conservation law ω. Generating function ψ_ω of a given conservation law ω is a cosymmetry of $\mathscr{E}$.

To compute conservation laws, their generating sections are used: Integrating by parts eq. (2.5) order of Δ can be reduced to zero which gives a relation

$$\sum_{i=1}^{n}(-1)^i D_{x^i}(\omega_i) = \psi^1 F^1 + \dots + \psi^r F^r, \tag{2.6}$$

where $\psi = (\psi^1, \dots, \psi^r)$ is a vector function. Thus, to find conservation laws corresponding to a cosymmetry ψ the eq. (2.6) must be solved w.r.t. the unknown functions a_i. All the solutions, if any, differ by a trivial conservation law.

2.5. **Differential coverings.** Consider the space $\tilde{\mathscr{E}} = \mathbb{R}^s \times \mathscr{E}$, $s \le \infty$, and the natural projection $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$. We say that τ is an s-dimensional (*differential*) *covering* over $\mathscr{E}$ if $\tilde{\mathscr{E}}$ is endowed with vector fields $\tilde{D}_{x^1}, \dots, \tilde{D}_{x^n}$ such that

$$[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0, \quad \tau_*(\tilde{D}_{x^i}) = D_{x^i}, \qquad i,j = 1,\dots,n.$$

Let $\{w^\alpha\}$ be coordinates in $\mathbb{R}^s$ (they are called *nonlocal variables*). Then the covering structure is given by

$$\tilde{D}_{x^i} = D_{x^i} + X_i$$

such that

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0,$$

where

$$X_i = \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}$$

are τ-vertical vector fields.

There exists a distinguished class of coverings that are associated with two-component conservation laws of $\mathscr{E}$. Fix two integers i and j, $1 \le i < j \le n$, and consider a differential form

$$\omega = X_i\,dx^1\wedge\dots\wedge dx^{i-1}\wedge dx^{i+1}\wedge\dots\wedge dx^n + X_j\,dx^1\wedge\dots\wedge dx^{j-1}\wedge dx^{j+1}\wedge\dots\wedge dx^n,$$

such that

$$D_{x^i}(X_i) = (-1)^{i+j-1}D_{x^j}(X_j).$$

Consider the Euclidean space V with the coordinates w^σ, where σ is symmetric multi-index whose entries are any integers $1,\dots,n$ except for i and j. Thus, $\dim V = 1$ if $n=2$ and $\dim V = \infty$ otherwise. Then the system of vector fields

$$\begin{aligned}
\tilde{D}_{x^k} &= D_{x^k} + \sum_\sigma w^{\sigma k}\frac{\partial}{\partial w^\sigma}, \qquad k \ne i,j,\\
\tilde{D}_{x^i} &= D_{x^i} + \sum_\sigma \tilde{D}_\sigma(X_j)\frac{\partial}{\partial w^\sigma},\\
\tilde{D}_{x^j} &= D_{x^j} + (-1)^{i+j-1}\sum_\sigma \tilde{D}_\sigma(X_i)\frac{\partial}{\partial w^\sigma}
\end{aligned}$$

defines a covering structure on $\tilde{\mathscr{E}}_\omega = V\times\mathscr{E}$. The coverings of this type are called *Abelian*.

2.6. **Nonlocal symmetries.** Denote by $\mathscr{C}$ the distribution on $\tilde{\mathscr{E}}$ spanned by the fields $\tilde{D}_{x^1},\dots,\tilde{D}_{x^n}$ and let X be a field vertical with respect to the composition $\tilde{\mathscr{E}} \to \mathscr{E} \to \mathbb{R}^n$. Such a field is called a *nonlocal symmetry* if it preserves $\tilde{\mathscr{C}}$. These symmetries form a Lie algebra denoted by $\operatorname{sym}_\tau(\mathscr{E})$. The restriction $X|_{C^\infty(\mathscr{E})}: C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ is called a nonlocal τ-*shadow*. A nonlocal symmetry is said to be *invisible* if its shadow vanishes.

In local coordinates, any $X \in \operatorname{sym}_\tau(\mathscr{E})$ is of the form

$$X = \tilde{\mathbf{E}}_\varphi + \sum_\alpha \psi^\alpha\frac{\partial}{\partial w^\alpha},$$

where $\varphi = (\varphi^1,\dots,\varphi^m)$, ψ^α are functions on $\tilde{\mathscr{E}}$ satisfying the equations

$$\begin{aligned}
&\tilde{\ell}_{\mathscr{E}}(\varphi) = 0,\\
&\tilde{D}_{x^i}(\psi^\alpha) = \sum_{j,\sigma}\frac{\partial X_i^\alpha}{\partial u_\sigma^j}\tilde{D}_\sigma(\varphi^j) + \sum_\beta\frac{\partial X_i^\alpha}{\partial w^\beta}\psi^\beta,
\end{aligned}$$

where $\tilde{\mathbf{E}}_\varphi$ ans $\tilde{\ell}_{\mathscr{E}}$ are obtained from the expressions (2.3) and (2.2), respectfully, by changing D_{x^i} to $\tilde{D}_{x^i}$. Nonlocal shadows are the operators $\tilde{\mathbf{E}}_\varphi$ while invisible symmetries are obtained from general ones by setting $\varphi = 0$.

In particular, for coverings of the form $\tilde{\mathscr{E}}_\omega$, where ω is a 2-component conservation law, the symmetries acquire the form

$$X = \tilde{\mathbf{E}}_\varphi + \sum_\sigma D_\sigma(\psi)\frac{\partial}{\partial w^\sigma},$$

where φ and ψ satisfy

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0,$$

$$\tilde{D}_{x^i}(\psi) = \sum_{\sigma,k} \frac{\partial X_j}{\partial u^k_\sigma} \tilde{D}_\sigma(\varphi^k) + \sum_\sigma \frac{\partial X_j}{\partial w^\sigma} \tilde{D}_\sigma(\psi),$$

$$\tilde{D}_{x^j}(\psi) = (-1)^{i+j-1} \left(\sum_{\sigma,k} \frac{\partial X_i}{\partial u^k_\sigma} \tilde{D}_\sigma(\varphi^k) + \sum_\sigma \frac{\partial X_i}{\partial w^\sigma} \tilde{D}_\sigma(\psi) \right).$$

2.7. **Bäcklund transformations and recursion operators.** Let $\mathscr{E}_1$, $\mathscr{E}_2$ be equations. A *Bäcklund transformation* between $\mathscr{E}_1$ and $\mathscr{E}_2$ is the diagram

$$\begin{array}{ccccc} & & \tilde{\mathscr{E}} & & \\ & \swarrow^{\tau_1} & & \searrow^{\tau_2} & \\ \mathscr{E}_1 & & & & \mathscr{E}_2, \end{array}$$

where τ_1, τ_2 are coverings. When $\mathscr{E}_1 = \mathscr{E}_2$, it is called a *Bäcklund auto-transformation*. If τ_1 is finite-dimensional and $\gamma \subset \mathscr{E}_1$ is a graph of solution then, generically, $\tau_2\left(\tau_1^{-1}(\gamma)\right)$ is a finite-dimensional manifold endowed with an integrable n-dimensional distribution whose integral manifolds are solutions of $\mathscr{E}_2$.

Consider now an equation $\mathscr{E}$ given by (2.1) and the system

$$F(x^i, u^j_\sigma) = 0, \qquad \ell_F(q) = 0,$$

where $F = (F^1, \dots, F^r)$. This system is called the *tangent* equation to $\mathscr{E}$ and denoted by $\mathscr{T}\mathscr{E}$, while the projection $\mathrm{t}\colon \mathscr{T}\mathscr{E} \to \mathscr{E}$ is called the *tangent covering*. Sections of this covering that preserve the Cartan distribution are identified with generating sections of symmetries.

Let $\mathscr{R}$ be a Bäcklund transformation between $\mathscr{T}\mathscr{E}_1$ and $\mathscr{T}\mathscr{E}_2$. Then it follows from the above said that it accomplishes a correspondence between symmetries of the two equations. If $\mathscr{E}_1 = \mathscr{E}_2$ then such a correspondence is called a *recursion operator*, [30].

2.8. **Zero curvature representations.** Given a system $\mathscr{E}$ of PDEs in independent variables x, y and a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-valued *zero curvature representation* for $\mathscr{E}$ is a form $\alpha = A\,dx + B\,dy$ with $A, B \in \mathfrak{g}$ such that

$$D_y A - D_x B + [A, B] = 0$$

as a consequence of the system $\mathscr{E}$.

Zero curvature representations have many applications in the field of integrability theory of PDEs: there is a connection with inverse scattering method (Zakharov–Shabat formulation), Bäcklund transformations, nonlocal symmetries, pseudosymmetries (factorizations of PDE), recursion operators and hierarchies.

Zero curvature representations come in huge families (gauge equivalence classes): Let $\sum_i A_i\,dx^i$ be a $\mathfrak{g}$-valued zero curvature representation, G the Lie group corresponding to the Lie algebra $\mathfrak{g}$. The left action

$$S(A_i) = D_i S S^{-1} + S A_i S^{-1}$$

by a G-valued function S is called the *gauge transformation*.

Two zero curvature representation are called *gauge equivalent* if one can be obtained from the other by gauge transformation. A zero curvature representation is called *trivial* if it is gauge equivalent to zero.

We say that a PDE is *Lax-integrable* if it admits a Lax pair with a non-removable parameter.

3. Infinitely many commuting nonlocal symmetries for modified Martínez Alonso–Shabat Equation [I]

In the paper [I], we study the 4-dimensional modified Martínez Alonso–Shabat equation

$$u_y u_{xz} + \alpha u_x u_{ty} - (u_z + \alpha u_t) u_{xy} = 0 \tag{3.1}$$

involving a nonzero real parameter α, found in [34], and present its recursion operator and an infinite commuting hierarchy of full-fledged nonlocal symmetries (rather than mere shadows). To date such hierarchies were found only for very few integrable systems in more than three independent variables. Equation (3.1) is an integrable PDE as it has a known Lax pair involving the spectral parameter $\lambda \neq \alpha$ [24, 42, 43]

$$r_y = \frac{\lambda}{\alpha}\frac{u_y}{u_x} r_x, \quad r_z = \frac{\lambda}{\alpha}\frac{u_z + \alpha u_t}{u_x} r_x - \lambda r_t. \tag{3.2}$$

3.1. The recursion operator.

Starting with (3.2) and using the deformation procedure described in [44] we readily find that (3.1) admits, in addition to (3.2), a Lax pair

$$\begin{aligned} q_y &= \frac{\lambda u_y q_x + (\alpha - \lambda) q u_{xy}}{\alpha u_x}, \\ q_z &= \frac{\lambda\left((\alpha u_t + u_z) q_x - \alpha u_x q_t - q u_{xz}\right) + \alpha q u_{xz}}{\alpha u_x}. \end{aligned} \tag{3.3}$$

In particular, for any given λ equations (3.3) define a covering, which we denote by $\mathscr{Q}_\lambda$, over (3.1). Unlike r, if q satisfies (3.3) then it is a nonlocal symmetry shadow for (3.1) in the covering $\mathscr{Q}_\lambda$. Using the techniques from [42, 44] we obtain

Proposition 3.1. *Equation (3.1) admits a recursion operator $\mathcal{R}$ defined by the relations*

$$\begin{aligned} \psi_y &= \frac{u_y \varphi_x - u_{xy}\varphi + \alpha u_{xy}\psi}{\alpha u_x}, \\ \psi_z &= \frac{(\alpha u_t + u_z)\varphi_x - \alpha u_x \varphi_t - u_{xz}\varphi + \alpha u_{xz}\psi}{\alpha u_x}, \end{aligned} \tag{3.4}$$

meaning that for any nonlocal symmetry shadow φ for (3.1) $\mathcal{R}$ produces another nonlocal symmetry shadow $\mathcal{R}(\varphi) \stackrel{\text{def}}{=} \psi$ for (3.1).

3.2. Nonlocal symmetries.

While, as we have seen in the preceding section, q is a nonlocal symmetry shadow in the covering $\mathscr{Q}_\lambda$, this shadow cannot be lifted to a full-fledged nonlocal symmetry in the covering under study.

To circumvent this difficulty, consider a formal expansion $q = \sum_{i=0}^{\infty} q_i \lambda^i$. Substituting this expansion into (3.3) shows that $q_0 = F u_x$, where $F(x, t)$ is an arbitrary function, while the remaining q_i are defined by the equations

$$(q_1)_y = \frac{\alpha u_{xy} q_1 + (u_{xx} u_y - u_{xy} u_x) F + u_x u_y F_x}{\alpha u_x},$$

$$(q_1)_z = \frac{\alpha u_{xz} q_1 + (\alpha (u_t u_x)_x + u_{xx} u_z - u_{xz} u_x)\, F + (\alpha u_t + u_z) u_x F_x - \alpha u_x^2 F_t}{\alpha u_x},$$

$$(q_i)_y = \frac{\alpha u_{xy} q_i - u_{xy} (q_{i-1}) + u_y (q_{i-1})_x}{\alpha u_x},$$

$$(q_i)_z = \frac{\alpha u_{xz} q_i - u_{xz} (q_{i-1}) - \alpha u_x (q_{i-1})_t + \alpha u_t (q_{i-1})_x + u_z (q_{i-1})_x}{\alpha u_x},$$

$i = 2, 3, \ldots$, that define an infinite-dimensional covering, which we denote by $\mathscr{Q}_\infty$, over (3.1).

Theorem 3.1. *Infinite prolongations of the vector fields*

$$Q_i = q_i \frac{\partial}{\partial u} + \sum_{j=1}^{\infty} B_i^j \frac{\partial}{\partial q_j}, \quad i = 1, 2, \ldots, \tag{3.5}$$

form an infinite hierarchy of commuting nonlocal symmetries for (3.1) in the covering $\mathscr{Q}_\infty$.

Here

$$B_i^j = \frac{\left([u_x, q_{i+j-1}]_x - \alpha\, [u_x, q_{i+j}]_x\right) F - (\alpha\, q_{i+j} - q_{i+j-1})\, u_x\, F_x}{\alpha\, u_x} + \frac{\left(q_{i+j-s(i,j)-1}\right)_x q_{s(i,j)+1}}{u_x} + \sum_{k=1}^{s(i,j)} \frac{\alpha\, [q_{i+j-k}, q_k]_x - [q_{i+j-k-1}, q_k]_x}{\alpha u_x}, \tag{3.6}$$

where $s(i,j) = \min(i-1, j-1)$ and $[A, B]_x = A_x B - A B_x$.

Finding explicit form of the symmetries Lie algebra generators and providing rigorous proofs of commutation relations for infinite-dimensional algebras of nonlocal symmetries for multidimensional integrable PDEs, rather than merely finding shadows of nonlocal symmetries, appears to be quite rare, especially in the case of four (or more) independent variables. The situation seems to be quite different in 3D, where infinite-dimensional noncommutative algebras of nonlocal symmetries for a number of dispersionless integrable systems were found by direct computations, see e.g. [II], [III] or [17, 21].

4. On the four 3-dimensional Lax integrable equations [II],[III], [V]

In the series of papers [II],[III], [V] we consider the four 3-dimensional Lax-integrable equations

- the *universal hierarchy equation* [29]

$$u_{yy} = u_z u_{xy} - u_y u_{xz}, \tag{4.1}$$

- the *3D rdDym equation* [7, 35, 37]

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \tag{4.2}$$

- the *Veronese web equation* [1, 10, 14, 49]

$$u_{ty} = u_t u_{xy} - u_y u_{tx}, \tag{4.3}$$

- the *Pavlov equation* [11, 36]

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}. \tag{4.4}$$

All the four above listed equations (denote them as 4E) may be obtained as the symmetry reductions of the following Lax-integrable 4-dimensional systems

$$u_{yz} = u_{tx} + u_x\, u_{xy} - u_y\, u_{xx}, \tag{4.5}$$

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{4.6}$$

introduced in [15] and [29], respectively, while the latter two, in turn, are the reductions of

$$u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs} \tag{4.7}$$

with five independent variables t, x, y, z, s, studied in [3], which is a particular case of Manakov–Santini equation [27, 28] and is related to the five-dimensional equation considered in [29]. Some of 4E equations arise also in [15] as integrable reductions of multi-dimensional dispersionless PDEs.

Reductions of (4.7) to (4.5) and (4.6) and consequently to 4E are described in [3] and visualized in Figure 1. Integrability properties of the equation (4.7) were

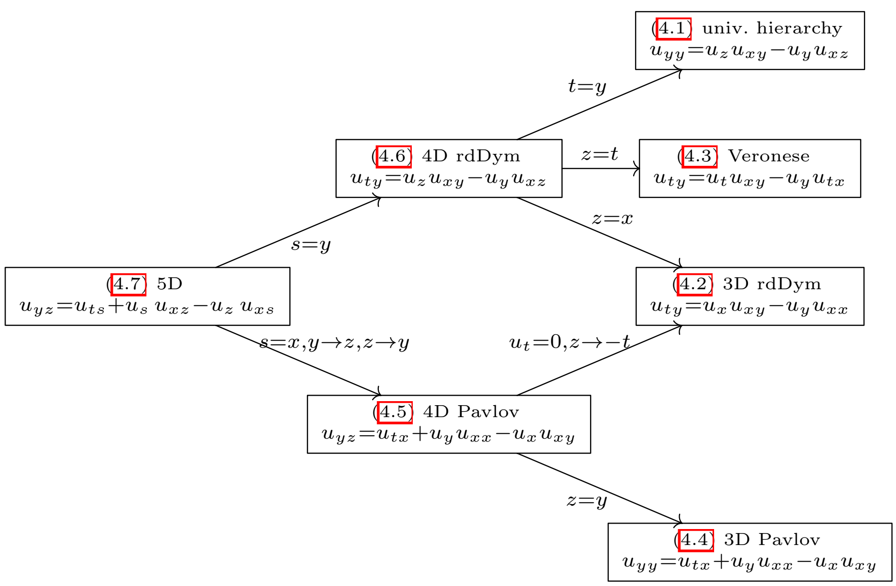

FIGURE 1. 4E reduction diagram

studied in [3]: We found the Lie algebra of symmetries, conservation laws, differential coverings with non-removable parameter (Lax-integrability) and the recursion operator together with its action on symmetries for (4.7).

The notation used within the papers [II],[III], [V] here and there slightly differs, so we fix here and bellow the notation to the form used in the last paper [II].

4.1. Symmetries and Lie algebra structure of 4E equations.

In [V] we found symmetries and corresponding Lia algebra structure for the 4E equations (4.1)–(4.4).

4.1.1. *The universal hierarchy equation* (4.1). The space of local symmetries is spanned by the functions

$$\theta_0(X) = Xu_x - X'u, \qquad \theta_1(X) = X,$$
$$\varphi_0(T) = Tu_t + T'yu_y, \qquad \varphi_1(T) = Tu_y, \qquad \upsilon = yu_y + u,$$

where X is a function of x and T is a function of t and here and everywhere bellow we using the notation $[R, \bar{R}] = R\bar{R}' - \bar{R}R'$ for functions R and $\bar{R}$, while 'prime' denotes the corresponding derivative. The commutators of local symmetries are presented in Table 1.

	υ	$\theta_0(\bar{X})$	$\theta_1(\bar{X})$	$\varphi_0(\bar{T})$	$\varphi_1(\bar{T})$
υ	0	0	$-\theta_1(\bar{X})$	0	$\varphi_1(\bar{T})$
$\theta_0(X)$	...	$\theta_0([\bar{X}, X])$	$\theta_1([\bar{X}, X])$	0	0
$\theta_1(X)$	...	...	0	0	0
$\varphi_0(T)$	...	...	...	$\varphi_0([\bar{T}, T])$	$\varphi_1([\bar{T}, T])$
$\varphi_1(T)$	...	...	...	...	0

Table 1: The UHE: commutators of local symmetries.

4.1.2. *rdDym equation* (4.2). The space of local symmetries is spanned by the functions

$$\psi_0 = -x\,u_x + 2\,u, \qquad \upsilon_0(Y) = Y u_y,$$
$$\theta_0(T) = Tu_t + T'(xu_x - u) + \frac{1}{2}T''x^2,$$
$$\theta_{-1}(T) = Tu_x + T'x, \qquad \theta_{-2}(T) = T,$$

where $T = T(t)$, $Y = Y(y)$ are arbitrary functions of their arguments and 'prime' denotes the corresponding derivative. Commutators of symmetries are presented in Table 2.

	ψ_0	$\upsilon_0(\bar{Y})$	$\theta_0(\bar{T})$	$\theta_{-1}(\bar{T})$	$\theta_{-2}(\bar{T})$
ψ_0	0	0	0	$\theta_{-1}(\bar{T})$	$2\theta_{-2}(\bar{T})$
$\upsilon_0(Y)$	...	$\upsilon_0([Y, \bar{Y}])$	0	0	0
$\theta_0(T)$	...	...	$\theta_0([T, \bar{T}])$	$\theta_{-1}([\bar{T}, T])$	$\theta_{-2}([\bar{T}, T])$
$\theta_{-1}(T)$	...	...	...	$\theta_{-2}([\bar{T}, T])$	0
$\theta_{-2}(T)$	...	...	...	...	0

Table 2: The rdDym equation: commutators of local symmetries.

4.1.3. *Veronese web equation* (4.3). The modified Veronese web equation (mVWE) was studied in [1] and is related to the Veronese web equation, [10, 49], by the Bäcklund transformation (4.9).

The space of local symmetries is generated by the functions

$$\varphi(T) = Tu_t, \quad \upsilon(Y) = Yu_y, \quad \theta_0(X) = Xu_x - X'u, \quad \theta_1(X) = X,$$

where $X = X(x)$, $Y = Y(y)$, and $T = T(t)$ are arbitrary functions of their arguments. The commutators of the symmetries are presented in Table 3.

	$\varphi(\bar{T})$	$\theta_0(\bar{X})$	$\theta_1(\bar{X})$	$\upsilon(\bar{Y})$
$\varphi(T)$	$\varphi([\bar{T},T])$	0	0	0
$\theta_0(X)$	$\dots$	$\theta_0([\bar{X},X])$	$\theta_1([\bar{X},X])$	0
$\theta_1(X)$	$\dots$	$\dots$	0	0
$\upsilon(Y)$	$\dots$	$\dots$	$\dots$	$\upsilon([\bar{Y},Y])$

Table 3: The mVwe: commutators of local symmetries.

4.1.4. *The Pavlov equation* (4.4). The space of local symmetries is spanned by the functions

$$\varphi_1 = 2x - yu_x, \qquad \varphi_2 = 3u - 2xu_x - yu_y,$$

$$\theta_0(T) = Tu_t + T'(xu_x + yu_y - u) + \frac{1}{2}T''(y^2u_x - 2xy) - \frac{1}{6}T'''y^3,$$

$$\theta_1(T) = Tu_y + T'(yu_x - x) - \frac{1}{2}T''y^2, \qquad \theta_2(T) = Tu_x - T'y, \qquad \theta_3(T) = T,$$

where T is a function of t and 'prime' denotes the t-derivatives. Commutators of these symmetries are presented in Table 4.

	φ_1	φ_2	$\theta_0(\bar{T})$	$\theta_1(\bar{T})$	$\theta_2(\bar{T})$	$\theta_3(\bar{T})$
φ_1	0	φ_1	0	$-2\theta_2(\bar{T})$	$2\theta_3(\bar{T})$	0
φ_2	$\dots$	0	0	$-\theta_1(\bar{T})$	$-2\theta_2(\bar{T})$	$-3\theta_3(\bar{T})$
$\theta_0(T)$	$\dots$	$\dots$	$\theta_0([\bar{T},T])$	$\theta_1([\bar{T},T])$	$\theta_2([\bar{T},T])$	$\theta_3([\bar{T},T])$
$\theta_1(T)$	$\dots$	$\dots$	$\dots$	$\theta_2([\bar{T},T])$	$\theta_3([\bar{T},T])$	0
$\theta_2(T)$	$\dots$	$\dots$	$\dots$	$\dots$	0	0
$\theta_3(T)$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	0

Table 4: The Pavlov equation: commutators of local symmetries.

4.2. **Lax pairs and differential coverings.** The results presented in this section were obtained in the paper [III] for the rdDym equation and in [II] for the remaining equations.

For each equation, we construct two infinite hierarchies of two-component nonlocal conservation laws (corresponding to non-negative and non-positive powers of the spectral parameter). To these hierarchies there correspond two infinite-dimensional coverings τ^+, τ^-(in the sense of [22]) which we call *positive* and *negative*.

4.2.1. *The universal hierarchy equation* (4.1). The UHE admits the Lax representation

$$q_t = \lambda^{-2}(\lambda u_t - u_y)q_x, \quad q_y = \lambda^{-1}u_yq_x.$$

Expansion in powers of λ leads to the system

$$q_{i,t} = u_tq_{i+1,x} - u_yq_{i+2,x}, \quad q_{i,y} = u_yq_{i+1,x}.$$

The corresponding positive covering is of the form

$$q_{1,y} = \frac{u_t}{u_y}, \qquad q_{1,x} = \frac{1}{u_y};$$

$$q_{i,y} = \frac{u_t}{u_y} q_{i-1,y} - q_{i-1,t}, \qquad\qquad q_{i,x} = \frac{q_{i-1,y}}{u_y},$$

$i > 1$, with the additional variables $q_i^{(j)}$ that satisfy the relations

$$q_i^{(0)} = q_i, \quad q_i^{(j+1)} = q_{i,t}^{(j)}.$$

The equations defining the negative covering are

$$\begin{aligned} r_{1,y} &= u_x u_y, & r_{1,t} &= u_x u_t - u_y; \\ r_{i,y} &= u_y r_{i-1,x}, & r_{i,t} &= u_t r_{i-1,x} - r_{i-1,y}, \end{aligned}$$

$i > 1$, with $r_i^{(j)}$ defined by relations $r_i^{(j+1)} = r_{i,x}^{(j)}$.

4.2.2. *rdDym equation* (4.2). The system

$$w_t = (u_x - \lambda) w_x \qquad w_y = \lambda^{-1} u_y w_x, \tag{4.8}$$

is a Lax pair for (4.2). Setting $w = \sum_{i=-\infty}^{+\infty} \lambda^i w_i$ and inserting this expansion into (4.8), we obtain

$$w_{i,t} = u_x w_{i,x} - w_{i-1,x}, \quad w_{i,y} = u_y w_{i+1,x}.$$

The corresponding positive covering is defined by the system

$$\begin{aligned} q_{1,t} &= \frac{u_x}{u_y}, & q_{1,x} &= \frac{1}{u_y}, \\ q_{i,t} &= \frac{u_x}{u_y} q_{i-1,y} - q_{i-1,x}, & q_{i,x} &= \frac{q_{i-1,y}}{u_y}, \end{aligned}$$

where $i \geq 2$, with the additional nonlocal variables $q_i^{(j)}$ defined by relations

$$q_i^{(0)} = q_i, \quad q_i^{(j+1)} = \left(q_i^{(j)}\right)_y.$$

The negative covering is defined by the system

$$\begin{aligned} r_{1,x} &= u_x^2 - u_t, & r_{1,y} &= u_x u_y, \\ r_{i,x} &= u_x r_{i-1,x} - r_{i-1,t}, & r_{i,y} &= u_y r_{i-1,x} \end{aligned}$$

enriched by additional nonlocal variables $r_i^{(j)}$ defined by relations

$$r_i^{(0)} = r_i, \quad r_i^{(j+1)} = \left(r_i^{(j)}\right)_t.$$

4.2.3. *Veronese web equation* (4.3). The mVwe admits the Lax pair

$$q_t = (\lambda + 1)^{-1} u_t q_x, \qquad q_y = \lambda^{-1} u_y q_x. \tag{4.9}$$

Expanding in powers of λ, one obtains

$$q_{i-1,t} + q_{i,t} = u_t q_{i,x}, \quad q_{i-1,y} = u_y q_{i,x}.$$

Then the positive covering acquires the form

$$\begin{aligned} q_{1,t} &= \frac{u_t}{u_y}, & q_{1,x} &= \frac{1}{u_y}, \\ q_{i,x} &= \frac{q_{i-1,y}}{u_y}, & q_{i,t} &= \frac{u_t}{u_y} q_{i-1,y} - q_{i-1,t}, \end{aligned}$$

$i > 1$, the additional variables being $q_i^{(j)}$ defined by relations

$$q_i^{(0)} = q_i, \quad q_i^{(j+1)} = q_{i,y}^{(j)}.$$

The defining equations for the negative covering are

$$r_{1,t} = u_t(u_x - 1), \qquad r_{1,y} = u_x u_y,$$
$$r_{i,t} = u_t r_{i-1,x} - r_{i-1,t}, \qquad r_{i,y} = u_y r_{i-1,x},$$

$i > 1$. The auxiliary variables are $r_i^{(j)}$, defined by relations

$$r_i^{(0)} = r_i, \quad r_i^{(j+1)} = r_{i,y}^{(j)}.$$

4.2.4. *Pavlov's equation* (4.4). The Lax pair for the 3D Pavlov equation is

$$q_t = (\lambda^2 - \lambda u_x - u_y)q_x, \quad q_y = (\lambda - u_x)q_x.$$

Expanding q in integer powers of λ, we arrive to the covering

$$q_{i,t} = q_{i-2,x} - u_x q_{i-1,x} - u_y q_{i,x}, \quad q_{i,y} = q_{i-1,x} - u_x q_{i,x},$$

for all $i \in \mathbb{Z}$.

The positive covering corresponding to this system is

$$q_{0,t} + u_y q_{0,x} = 0, \qquad q_{0,y} + u_x q_{0,x} = 0;$$
$$q_{1,t} + u_y q_{1,x} = -u_x q_{0,x}, \qquad q_{1,y} + u_x q_{1,x} = q_{0,x},$$
$$q_{i,t} + u_y q_{i,x} = q_{i-2,x} - u_x q_{i-1,x}, \qquad q_{i,y} + u_x q_{i,x} = q_{i-1,x},$$

where $i \geq 2$, to which nonlocal variables $q_i^{(j)}$ defined by relations

$$q_i^{(0)} = q_i, \quad q_i^{(j+1)} = q_{i,x}^{(j)}$$

are added. This covering is not Abelian.

The negative covering is given by

$$r_{1,y} = u_t + u_x u_y, \qquad r_{1,x} = u_y + u_x^2;$$
$$r_{i,y} = r_{i-1,t} + u_y r_{i-1,x}, \qquad r_{i,x} = r_{i-1,y} + u_x r_{i-1,x},$$

$i \geq 2$, with additional nonlocal variables $r_i^{(j)}$ defined by relations

$$r_i^{(0)} = r_i, \quad r_i^{(j+1)} = r_{i,t}^{(j)}.$$

4.3. **Nonlocal symmetries, Lie algebra structure, recursion operators.** For each 4E equation, we obtained a full description of nonlocal symmetry algebras associated to above coverings. For all the coverings, the obtained Lie algebras of symmetries manifest similar (but not the same) structures. We also discuss actions of recursion operators on shadows of nonlocal symmetries. Let us briefly present the results on the rdDym equation obtained in [III]. The remaining equations are studied in [II] in a similar fashion.

All the local symmetries of the rdDym equation can be lifted both to the positive τ^+ and the negative τ^- covering and we denote the lifts by the corresponding capital letters: Ψ_0 for the lift of ψ_0, $\Theta_i(T)$ for $\theta_i(T)$, etc.

Three families of nonlocal symmetries are admitted in τ^+. The first one consists of invisible symmetries

$$\Phi^k_{\text{inv}}(Y) = (\underbrace{0, \dots, 0}_{k \text{ times}}, \varphi^1_{\text{inv}}, \dots, \varphi^i_{\text{inv}}, \dots)$$

where $\varphi^1_{\text{inv}} = Y(y)$, and another two are generated by the lifts Ψ_{-1} and Ψ_{-2} of the nonlocal shadows

$$\psi_{-1} = q_1 u_y + x, \quad \psi_{-2} = (2q_2 - q_1 q_1^{(1)})u_y$$

using the relations

$$\Psi_{-k} = [\Psi_{-k+1}, \Psi_{-1}], k \geq 3 \quad \text{and} \quad \Upsilon_{-k}(Y) = [\Psi_{-k-1}, \Phi^1_{\mathrm{inv}}(Y)].$$

Theorem 4.1. *There exist a basis in* $\mathrm{sym}_{\tau^+}(\mathscr{E})$ *consisting of the elements*

$$\{\mathbf{w}_i, \mathbf{v}_j(T), \mathbf{v}_k(Y)\}, \quad i \leq 0, j = 0, -1, -2, \quad k \in \mathbb{Z}$$

such that they commute as it is indicated in Table 5.

	$\mathbf{w}_j$	$\mathbf{v}_j(\bar{T})$	$\mathbf{v}_j(\bar{Y})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\bar{T})$, $-2 \leq i+j \leq 0$, 0, otherwise	$j\mathbf{v}_{i+j}(\bar{Y})$
$\mathbf{v}_i(T)$	...	$\mathbf{v}_{i+j}([T, \bar{T}])$, $-2 \leq i+j \leq 0$, 0, otherwise	0
$\mathbf{v}_i(Y)$	...	...	$\mathbf{v}_{i+j}([Y, \bar{Y}])$

Table 5: The rdDym equation: commutators in $\mathrm{sym}_{\tau^+}(\mathscr{E})$.

In a similar way, local symmetries are lifted to τ^- and three families of nonlocal symmetries arise in this covering. They are Ψ_k, $k \geq 1$, $\Theta_i(T)$, $i \geq -2$, Φ^l_{inv}.

The Lie algebra structure is then described by

Theorem 4.2. *There exist a basis in* $\mathrm{sym}_{\tau^-}(\mathscr{E})$ *consisting of the elements*

$$\{\mathbf{w}_i, \mathbf{v}_j(T), \mathbf{v}(Y)\}, \quad i \geq 0, \quad j \in \mathbb{Z},$$

that satisfy the commutator relations presented in Table 6.

	$\mathbf{w}_j$	$\mathbf{v}_j(\bar{T})$	$\mathbf{v}(\bar{Y})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\bar{T})$	0
$\mathbf{v}_i(T)$	...	$\mathbf{v}_{i+j}([T, \bar{T}])$	0
$\mathbf{v}(Y)$	...	...	$\mathbf{v}([Y, \bar{Y}])$

Table 6: The rdDym equation: commutators in $\mathrm{sym}_{\tau^-}(\mathscr{E})$.

Note that the components of the invisible symmetries are constructed using the operator

$$\mathscr{Y} = q_1 \frac{\partial}{\partial y} + \sum_{i=1}^{\infty} (i+1) q_{i+1} \frac{\partial}{\partial q_i}.$$

Similar operators will arise in the study of other equations.

The equation under study admits a recursion operator $\mathscr{R}_+$ defined by the system

$$\begin{aligned} D_t(\hat{\chi}) &= u_y^{-1} \left(u_y\, D_x(\chi) - u_x\, D_y(\chi) + (u_x u_{xy} - u_y u_{xx})\hat{\chi} \right), \\ D_x(\hat{\chi}) &= u_y^{-1} \left(u_{xy}\, \hat{\chi} - D_y(\chi) \right), \end{aligned} \tag{4.10}$$

see [33]. This means that $\hat{\chi}$ is a nonlocal shadow whenever χ is. Another recursion operator $\mathscr{R}_-$ is defined, in a fashion similar to $\mathscr{R}_+$, by the system

$$\begin{aligned} D_x(\chi) &= D_t(\hat{\chi}) - u_x\, D_x(\hat{\chi}) + u_{xx}\, \hat{\chi}, \\ D_y(\chi) &= -u_y\, D_x(\hat{\chi}) + u_{xy}\, \hat{\chi}. \end{aligned} \tag{4.11}$$

The operators $\mathscr{R}_+$ and $\mathscr{R}_-$ are mutually inverse.

The actions of $\mathscr{R}_+$ and $\mathscr{R}_-$ on $\mathrm{sym}(\mathscr{E})$ may be prolonged to the shadows of nonlocal symmetries from $\mathrm{sym}(\tilde{\mathscr{E}}^+)$ and $\mathrm{sym}(\tilde{\mathscr{E}}^-)$ if we replace the derivatives D_t, D_x and D_y in (4.10) and (4.11) by the total terivatives $\hat{D}_t$, $\hat{D}_x$ and $\hat{D}_y$ in the Whitney product of the coverings τ^+ and τ^- in the sense of [22]. The resulting operators will be still denoted by $\mathscr{R}_+$ and $\mathscr{R}_-$.

Note that the operators act nontrivially on 'vacuum': $\mathscr{R}_+(0) = \theta_{-2}(T)$, $\mathscr{R}_-(0) = \upsilon_0(Y)$, which immediately follows from Equations (4.10) and (4.11); thus it is reasonable to consider the actions of these operators modulo $\theta_{-2}(T)$ for $\mathscr{R}_+$ and $\upsilon_0(Y)$ for $\mathscr{R}_-$. Taking into account this remark, we have the following

Proposition 4.1. *Modulo the images of the trivial symmetry, the action of recursion operators is of the form*

$$\mathscr{R}_+(\theta_i(T)) = \begin{cases} \alpha_i^+\theta_{i-1}(T), & i > -2, \\ 0, & i = -2, \end{cases} \qquad \mathscr{R}_-(\theta_i(T)) = \alpha_i^-\theta_{i+1}(T), \quad i \geq -2,$$

$$\mathscr{R}_+(\upsilon_i(Y)) = \beta_i^+\upsilon_{i+1}(Y), \quad i \leq 0, \qquad \mathscr{R}_-(\upsilon_i(Y)) = \begin{cases} \beta_i^-\upsilon_{i+1}(Y), & i < 0, \\ 0, & i = 0, \end{cases}$$

$$\mathscr{R}_+(\psi_i) = \gamma_i^+\psi_{i-1}, \qquad \mathscr{R}_-(\psi_i) = \gamma_i^-\psi_{i+1}, \quad i \in \mathbb{Z},$$

where $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma_i^\pm$ are nonzero constants.

Note that the recursion operators $\mathscr{R}_+$ and $\mathscr{R}_-$ 'glue together' the shadows ψ_m of nonlocal symmetries in coverings $\tilde{\mathscr{E}}^+$ and $\tilde{\mathscr{E}}^-$ and 'tunnel' from the series of $\theta_k(T)$ to that of $\upsilon_j(Y)$, see Figure 2.

$$\dots \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \psi_{-1}^+ \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \psi_0^\pm \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \psi_1^- \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \dots$$

$$\dots \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \upsilon_{-1}^+ \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \upsilon_0^\pm \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} 0^\pm \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \theta_{-2}^\pm \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \theta_{-1}^\pm \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \theta_0^\pm \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \theta_1^- \xrightleftharpoons[\mathscr{R}_+]{\mathscr{R}_-} \dots$$

FIGURE 2. The rdDym equation: action of recursion operators (4.10) and (4.11). Straight arrows denote actions up to scalar multipliers and modulo the image of the trivial shadow. We write θ_i instead of $\theta_i(T)$, υ_k instead of $\upsilon_k(Y)$, etc. Notation $(\cdot)^+$ means that a shadow lives in τ^+, $(\cdot)^-$ is for those who live in τ^-; shadows marked by $(\cdot)^\pm$ live in both coverings.

5. 4E Symmetry reductions and its integrability properties

In the papers [IV, V], we study symmetry reductions of above mentioned 4E equations (4.1)–(4.4) and integrability properties of a 'nontrivial' subset of those reductions.

5.1. **The complete list of 2-dimensional reductions [V].** The paper [V] completely answered a natural question: What 2-dimensional equations are the reductions of 3-dimensional equations 4E? The result comprises 32 equations of which

- sixteen can be solved explicitly,

- one reduces to the Riccati equation,
- five can be linearized by the Legendre transformation,
- while the remaining ten are 'nontrivial'.

The latter are presented in Table 7 (in the third column, we exemplify the simplest relations).

Reduction	of Eq.	Relations with the initial eq.
$2\Phi = \Phi\Phi_{xz} - \Phi_x\Phi_z,$	(4.1)	$u = \dfrac{\Phi(x,z)}{y},$
$\Phi_{\xi\xi} = (\xi + \Phi_\xi)\Phi_{\eta\eta} - \Phi_\eta\Phi_{\xi\eta} - 2,$	(4.4)	$u = \Phi(\xi,\eta) + t^2\xi - 2t\eta,$ $\xi = y,\ \eta = x + ty,$
$\Phi_{\xi\xi} = \Phi_x\Phi_\xi - \Phi\Phi_{x\xi},$	(4.1)	$u = \Phi(x,\xi)e^{-z},\ \xi = ye^{-z},$
$(1 + \xi\Phi_z)\Phi_{\xi\xi} - \xi\Phi_\xi\Phi_{\xi z} + \Phi_\xi\Phi_z = 0,$	(4.1)	$u = \Phi(z,\xi)e^{-x},\ \xi = ye^{-x},$
$\Phi_\eta\Phi_{\xi\eta} - \Phi_\xi\Phi_{\eta\eta} = e^\eta\Phi_{\xi\xi},$	(4.1)	$u = \Phi(\xi,\eta)e^{-x},$ $\xi = ye^{-z},\ \eta = x - z,$
$(\xi + \Phi_\xi)\Phi_{\xi y} - \Phi_y(\Phi_{\xi\xi} + 2) = 0,$	(4.2)	$u = \Phi(\xi,y)e^{2t},\ \xi = xe^t,$
$\Phi_{\xi t} = 4\Phi\Phi_\xi - \xi\Phi_\xi^2 + 2\xi\Phi\Phi_{\xi\xi},$	(4.2)	$u = \Phi(\xi,t)x^2,\ \xi = xe^{-y},$
$\Phi_{\eta\eta} + (\xi + \Phi_\eta)\Phi_{\xi\eta} = \Phi_\eta(2 + \Phi_{\xi\xi}),$	(4.2)	$u = \Phi(\xi,\eta)e^{2t},$ $\xi = xe^{-t},\ \eta = y - t,$
$(4\xi^2 - 3\Phi)\Phi_{\xi\xi} - \Phi_{\xi t} - 6\xi\Phi_\xi + \Phi_\xi^2 + 6\Phi = 0,$	(4.4)	$u = \Phi(\xi,y)y^3,\ \xi = \dfrac{x}{y^2},$
$\Phi_{\xi\xi} = (\xi - \Phi_\eta)\Phi_{\xi\eta} + (2\eta + \Phi_\xi)\Phi_{\eta\eta} - \Phi_\eta = 0,$	(4.4)	$u = \Phi(\xi,\eta)e^{-3t},$ $\xi = ye^{\beta t}, \eta = xe^{2t}$

Table 7: 4E's 'nontrivial' reductions

The first two of these equations can be transformed to the Liouville equation [8] and the Gibbons-Tsarev equation [19], respectively. The other eight, studied in [IV], we will discuss later in Section 5.

A brief exposition of the results on all the reductions of 4E equations (4.1)–(4.4) obtained in [V] is given in Table 8.

Eqn	dim(sym $\mathscr{E}$)	Reductions	Comments
(4.1)	$1+\infty^{2\cdot x}+\infty^{2\cdot z}$	$X\Phi_{xz}-X'\Phi_z=0$	Solves explicitly
		$2\Phi=\Phi\Phi_{xz}-\Phi_x\Phi_z$	Transforms to the Liouville eq.
		$\Phi_{\xi\xi}=X'\Phi_\xi-X\Phi_{x\xi}$	Solves explicitly
		$\Phi_{\xi\xi}=\Phi_x\Phi_\xi-\Phi\Phi_{x\xi}$	
		$(1+Z\Phi_z)\Phi_{\xi\xi}=Z\Phi_\xi\Phi_{\xi z}+Z'\Phi_\xi^2$	Solves explicitly
		$(1+\xi\Phi_z)\Phi_{\xi\xi}-\xi\Phi_\xi\Phi_{\xi z}+\Phi_\xi\Phi_z=0$	
		$\Phi_{\xi\xi}=\Phi_\xi\Phi_{\eta\eta}-\Phi_\eta\Phi_{\xi\eta}$	Linearizes by the Legendre transform.
		$\Phi_\eta\Phi_{\xi\eta}-\Phi_\xi\Phi_{\eta\eta}=e^\eta\Phi_{\xi\xi}$	See $\mathscr{E}_1$ (5.1).
(4.2)	$1+\infty^{1\cdot y}+\infty^{3\cdot t}$	$\Phi_{yt}=T\Phi_y$	Solves explicitly
		$\Phi_{yt}=2\Phi\Phi_y$	Reduces to the Riccati eq.
		$(\alpha\xi+\Phi_\xi)\Phi_{\xi y}-\Phi_y(\Phi_{\xi\xi}+2\alpha)=0$	Solves explicitly for $\alpha=0$
		$T\Phi_{xx}=T'$	Solves explicitly
		$\left(\frac{T'}{T}\xi+\bar{T}\right)\Phi_{\xi\xi}+\Phi_{\xi t}=0$	Solves explicitly
		$\Phi_{\xi t}=4\Phi\Phi_\xi-\xi\Phi_\xi^2+2\xi\Phi\Phi_{\xi\xi}$	
		$\Phi_{\eta\eta}+(\alpha\xi+\Phi_\eta)\Phi_{\xi\eta}=\Phi_\eta(2\alpha+\Phi_{\xi\xi})$	Linearizes by the Legendre transform. for $\alpha=0$. See $\mathscr{E}_2$ (5.2).
(4.3)	$\infty^{2\cdot x}+\infty^{1\cdot y}+\infty^{1\cdot t}$	$\Phi_{yt}=0$	Solves explicitly
		$\bar{X}\Phi_{xt}-\bar{X}'\Phi_t=0$	Solves explicitly
		$\bar{X}\Phi_{xy}-\bar{X}'\Phi_y=0$	Solves explicitly
		$\Phi_{\xi\xi}=\bar{X}\Phi_{x\xi}-\bar{X}'\Phi_\xi$	Solves explicitly
		$(1+\Phi_\xi)\Phi_{y\xi}=\Phi_y\Phi_{\xi\xi}$	Solves explicitly
		$(1+\Phi_\xi)\Phi_{t\xi}=\Phi_t\Phi_{\xi\xi}$	Solves explicitly
		$\Phi_\eta\Phi_{\xi\xi}+(\Phi_\eta-\Phi_\xi-1)\Phi_{\xi\eta}-\Phi_\xi\Phi_{\eta\eta}=0$	Linearizes by the Legendre transform.
(4.4)	$2+\infty^{4\cdot t}$	$\Phi_{yy}=(T'-T^2)y+T\bar{T}-\bar{T}'$	Solves explicitly
		$\Phi_{yy}=\frac{2\Phi_y-T'}{y+T}+T''+\frac{\bar{T}^2}{(y+T)^3}$	Solves explicitly
		$\left(\left(\frac{T'}{T}+2\alpha T\right)\xi+\bar{T}^2+\bar{\bar{T}}\right)\Phi_{\xi\xi}$ $-\Phi_{\xi t}-\alpha T\Phi_\xi+\bar{T}'+2\alpha T\bar{T}=0$	Solves explicitly
		$(4\xi^2-3\Phi)\Phi_{\xi\xi}-\Phi_{\xi t}-6\xi\Phi_\xi+\Phi_\xi^2+6\Phi=0$	
		$\Phi_{\xi\xi}=(\beta\xi-\Phi_\eta)\Phi_{\xi\eta}+(2\beta\eta+\Phi_\xi)\Phi_{\eta\eta}-\beta\Phi_\eta$	Linearizes by the Legendre transform. for $\beta=0$. See $\mathscr{E}_3$ (5.3).
		$\Phi_{\xi\xi}=(\alpha\xi+\Phi_\xi)\Phi_{\eta\eta}-\Phi_\eta\Phi_{\xi\eta}-2\alpha$	Reduces to the Gibbons-Tsarev eq. for $\alpha\neq0$. Linearizes by the Legendre transform. for $\alpha=0$.

The notation $\infty^{k\cdot\tau}$ means the infinite-dimensional component corresponding to k arbitrary functions.

Table 8: Summary of 4E reductions

5.2. **Integrability properties of some reductions [IV].** The paper [IV] considers the above 8 'interesting' reductions (listed in the bottom part of Table 7). They can be divided in two groups by their symmetry properties: five equations admit infinite-dimensional Lie algebras of contact symmetries (with functional parameters) and three others possess finite-dimensional symmetry algebras. These are

- reduction $\mathscr{E}_1$ of the universal hierarchy equation (4.1)

$$u_y u_{xy} - u_x u_{yy} = e^y u_{xx}, \tag{5.1}$$

- reduction $\mathscr{E}_2$ of the 3D rdDym equation (4.2)

$$u_{yy} = (u_x + x)u_{xy} - u_y(u_{xx} + 2), \tag{5.2}$$

- reduction $\mathscr{E}_3$ of the Pavlov equation (4.4)

$$u_{xx} = (x - u_y)u_{xy} + (2y + u_x)u_{yy} - u_y. \tag{5.3}$$

We denote the variables in the reduced equations by u, x, y instead of Φ, ξ, η used in source equations listed in Table 8 above. All the reductions of the modified Veronese web equation (4.3) were either exactly solvable or linearizable. The above equations are pairwise inequivalent.

We deal with these three equations and study how the integrability properties of the initial 3D systems behave under reduction. More precisely, we construct the reductions of the zero-curvature representations for equations (5.1)–(5.3) and show that they result in differential coverings of the form

$$w_x = \frac{a_2 w^2 + a_1 w + a_0}{w^2 + c_1 w + c_0}, \qquad w_y = \frac{b_2 w^2 + b_1 w + b_0}{w^2 + c_1 w + c_0}, \tag{5.4}$$

where a_i, b_i, c_i are functions in x, y, u, u_x, and u_y. For every nonlinear covering we construct an infinite series of conservation laws and prove nontriviality of those.

We also study the behavior of the recursion operators for symmetries of three-dimensional systems and show that these operators do not survive under reduction. Local symmetries and cosymmetries of the reduction equations are described and the corresponding conservation laws are presented.

Using Lax representations of the 3D equations 4E, whose reductions are the equations at hand, we construct here nonlinear coverings of (5.1)–(5.3).

5.2.1. *Reductions of the Lax pairs, symmetries, cosymmetries.*

Equation $\mathscr{E}_1$: is obtained as the reduction of the universal hierarchy equation (4.1) with respect to the symmetry

$$\varphi = u_z + u_x + y u_y + u. \tag{5.5}$$

Equivalently, this reduction may be written in the form

$$u_{yy} = u_y u_{xx} - (u_x + u)u_{xy} + u_x u_y \tag{5.6}$$

and Equation (5.1) transforms to (5.6) by the change of variables $x \mapsto y$, $y \mapsto x$, $u \mapsto -e^y u$. In the further study of $\mathscr{E}_1$ we will use the form (5.6) rather than (5.1).

Equation (4.1) admits the Lax representation

$$\begin{aligned} w_z &= (w u_z - u_y) w^{-2} w_x, \\ w_y &= u_y w^{-1} w_x. \end{aligned} \tag{5.7}$$

The symmetry φ can be extended to a symmetry $\Phi = (\varphi, \chi)$ of (5.7), where

$$\chi = w_z + w_x + yw_y + w$$

and the corresponding reduction leads to the covering

$$(5.8)\qquad \begin{aligned} w_x &= -\frac{w^3}{w^2 - (u_x + u)w - u_y}, \\ w_y &= -\frac{u_y w^2}{w^2 - (u_x + u)w - u_y} \end{aligned}$$

of Equation (5.6).

The space $\mathrm{sym}(\mathscr{E}_1)$ spans the symmetries

$$\varphi_{-1} = u_y, \quad \varphi_0 = yu_y + u, \quad \varphi_0' = u_x, \quad \varphi_1 = e^{-x}.$$

The space $\mathrm{cosym}(\mathscr{E}_1)$ is 6-dimensional and is spanned by cosymmetries

$$\psi_{-3} = e^{4x}(3u_x^2 + 8u^2 + 10uu_x + 2u_y), \quad \psi_{-2} = e^{3x}(3u + 2u_x), \quad \psi_{-1} = e^{2x},$$

$$\psi_3 = \frac{1}{u_y^2}, \qquad \psi_4 = \frac{2u_x - yu_y + 2u}{u_y^3},$$

$$\psi_5 = \frac{-4u_x yu_y + 6uu_x + 3u_x^2 - 4yuu_y + 3u^2 + 2u_y + y^2u_y^2}{u_y^4}.$$

Equation $\mathscr{E}_2$: is obtained as the reduction of the 3D rdDym equation (4.2) with respect to the symmetry

$$(5.9)\qquad \varphi = u_t - xu_x - u_y + 2u.$$

The Lax representation of (4.2) is

$$(5.10)\qquad \begin{aligned} w_t &= (u_x + w)w_x, \\ w_y &= -u_y w^{-1} w_x. \end{aligned}$$

The symmetry φ extends to the one of (5.10): $\Phi = (\varphi, \chi)$, where

$$\chi = w_t - xw_x - w_y + u.$$

Reduction of the covering (5.10) with respect to Φ leads to the covering

$$(5.11)\qquad \begin{aligned} w_x &= -\frac{w^2}{w^2 + (u_x - x)w + u_y}, \\ w_y &= \frac{u_y w}{w^2 + (u_x - x)w + u_y}. \end{aligned}$$

over equation (5.2).

The space $\mathrm{sym}(\mathscr{E}_2)$ is generated by the symmetries

$$\varphi_{-2} = 1, \quad \varphi_{-1} = u_x + x, \quad \varphi_0 = u - \frac{1}{2}xu_x, \quad \varphi_0' = u_y.$$

The space $\mathrm{cosym}(\mathscr{E}_2)$ is 4-dimensional and is generated by the cosymmetries

$$\psi_{-3} = \frac{e^{-2y}(u_x + x)}{u_y^3}, \qquad \psi_2 = 1,$$

$$\psi_{-2} = \frac{e^{-y}}{u_y^2}, \qquad \psi_3 = u_x + 2x.$$

Equation $\mathscr{E}_3$: is the reduction of the Pavlov equation (4.4) with respect to the symmetry

$$\varphi = u_t - 2xu_x - yu_y + 3u. \tag{5.12}$$

The Pavlov equation (4.4) possesses the Lax pair

$$\begin{aligned} w_t &= (w^2 - wu_x - u_y)w_x, \\ w_y &= (w - u_x)w_x. \end{aligned} \tag{5.13}$$

The symmetry φ lifts to the symmetry $\Phi = (\varphi, \chi)$ of (5.13), where

$$\chi = w_t - 2xw_x - yw_y + w.$$

Reduction of the covering (5.13) with respect to this symmetry results in the nonlinear covering

$$\begin{aligned} w_x &= -\frac{w(w-u_y)}{w^2 - (u_y + x)w + xu_y - u_x - 2y}, \\ w_y &= -\frac{w}{w^2 - (u_y + x)w + xu_y - u_x - 2y} \end{aligned} \tag{5.14}$$

of Equation (5.3).

The space $\mathrm{sym}(\mathscr{E}_3)$ spans the symmetries

$$\varphi_0 = -\frac{1}{3}xu_x - \frac{2}{3}yu_y + u, \qquad \varphi_{-1} = u_x - xu_y + y - \frac{1}{2}x^2,$$
$$\varphi_{-2} = u_y + 2x, \qquad \varphi_{-3} = 1.$$

The space $\mathrm{cosym}(\mathscr{E}_3)$ is 6-dimensional and spans the elements

$$\begin{aligned} \psi_7 = {} & \frac{54}{5}xu_xu_y + \frac{164}{5}xu_yy + \frac{256}{5}x^2y + 2xu + \frac{4}{5}uu_y + \frac{12}{5}u_y^2u_x + 4yu_x + \frac{36}{5}u_y^2y \\ & + \frac{82}{5}x^2u_x + \frac{512}{15}x^3u_y + \frac{32}{5}xu_y^3 + \frac{96}{5}x^2u_y^2 + \frac{32}{5}y^2 + \frac{512}{15}x^4 + \frac{3}{5}u_x^2 + u_y^4, \\ \psi_6 = {} & \frac{49}{4}xy + 4xu_x + \frac{3}{2}u_yu_x + \frac{9}{2}u_yy + \frac{49}{4}x^2u_y + \frac{21}{4}xu_y^2 + \frac{343}{24}x^3 + \frac{1}{4}u + u_y^3, \\ \psi_5 = {} & 4xu_y + 6x^2 + 2y + \frac{2}{3}u_x + u_y^2, \\ \psi_4 = {} & \frac{5}{2}x + u_y, \quad \psi_3 = 1, \quad \psi_{-1} = \frac{1}{(-xu_y + u_x + 2y)^2}. \end{aligned}$$

5.2.2. *Conservation laws.* We listed local conservation laws of $\mathscr{E}_1$–$\mathscr{E}_3$ that corresponds to the cosymmetries described above in [IV, Sect. 6]. The dimension of the space of conservation laws for $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ is 6, 4 and 6, respectively.

5.2.3. *Hierarchies of nonlocal conservation laws.* Using above nonlinear coverings are in [IV, Sect. 3] constructed infinite hierarchies of nontrivial nonlocal conservation laws for $\mathscr{E}_1$–$\mathscr{E}_3$.

There is [IV, Sect. 3.1] a general construction of a hierarchy of nonlocal conservation laws over an equation $\mathscr{E}$ in two independent variables x and y and unknown function u, equipped by a differential covering

$$w_x = X(x, y, [u], w), \qquad w_y = Y(x, y, [u], w)$$

over $\mathscr{E}$, where $[u]$ denotes u itself and a collection of its derivatives up to some finite order. The initial step of the construction is the so-called *Pavlov reversing* [38].

Restricting the general covering to the Abelian case and assuming (5.4) we derived general recurrent formulae for the coefficients of the sought-for hierarchy.

Consequently, we apply this general construction on $\mathscr{E}_1$–$\mathscr{E}_3$. Moreover, we proved that the obtained conservation laws are nontrivial [IV, Proposition 3.1].

5.2.4. *On reductions of the recursion operators.* In [IV, Sect. 3] we proved that symmetry reductions of equations (4.1), (4.2), and (4.4) are incompatible with their recursion operators and thus the latter are not inherited by equations (5.1), (5.2), and (5.3), respectively.

Consider recursion operators for symmetries $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ of equations $\mathscr{E}_1$, $\mathscr{E}_1$, $\mathscr{E}_3$, i. e. (4.1), (4.2), and (4.4) found in [32, 33], see [IV, Sect. 4.2] for explicit formulae.

Proposition 5.1. *Recursion operators $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ are not invariant with respect to the natural lifts of the symmetries* (5.5), (5.9), *and* (5.12), *respectively.*

5.2.5. *Discussion on inequivalence.* In [IV, Section 5] we obtained the

Proposition 5.2. *Equations* (5.1), (5.2), *and* (5.3) *are pairwise inequivalent with respect to an arbitrary contact transformation.*

The proof is nothing but the comparison of Lie algebra structures of the spaces $\mathrm{sym}(\mathscr{E}_i)$ and dimensions of the Lie algebras $\mathrm{cosym}(\mathscr{E}_i)$.

Moreover, the equations under consideration are not equivalent to the Gibbons-Tsarev equation.

6. Integrable Weingarten surfaces

The classical geometry of immersed surfaces in the Euclidean space is well known to be closely connected with the modern theory of integrable systems [41]. The Gauss–Weingarten equations of a moving frame Ψ always take the form

$$\Psi_x = A\Psi, \quad \Psi_y = B\Psi. \tag{6.1}$$

where A, B are appropriate matrix-valued functions. Integrability conditions of (6.1) are called the Gauss–Mainardi–Codazzi equations and take the form of a *zero curvature representation*

$$A_y - B_x + [A, B] = 0. \tag{6.2}$$

The zero curvature representation (6.2) is the key ingredient in the soliton theory [13], where matrices A, B are additionally assumed to depend on what is called the *spectral parameter*. The essential requirement is that the spectral parameter cannot be removed by means of the gauge transformations. Consequently, if the matrices A, B can be modified so that they depend on a nonremovable parameter and still satisfy (6.2), then the corresponding Gauss–Mainardi–Codazzi equations are considered to be integrable in the sense of soliton theory, and their solutions are known as *integrable* or *soliton surfaces* [46].

Soliton-theoretic integrability can occur only when surfaces are subject to a constraint (such as being pseudospherical etc.). Here we employ a method due to Marvan [31]: we attempt to extend the given non-parametric zero curvature representation to a power series in terms of the spectral parameter.

To be 'geometric', the determining constraint on integrable surfaces must be invariant with respect to the changes of coordinates. The general non-differential invariant constraint is a functional relation $f(p,q) = 0$ between the principal curvatures p, q. Such a functional relation is characteristic of *Weingarten surfaces*. Well known to be integrable is the class of *linear Weingarten surfaces* [9, 41], characterized by a linear relation

$$ak + bh + c = 0, \qquad a, b, c = const \tag{6.3}$$

between the Gauss curvature $k = pq$ and the mean curvature $h = \frac{1}{2}(p + q)$. Other integrable classes of Weingarten surfaces sporadically occur in the literature.

So far, nothing contradicted the conjecture of Finkel [16, Conjecture 3.4] and Wu [48] that the only functional relation $f(p,q) = 0$ to determine an integrable class of Weingarten surfaces is the linear formula (6.3). Nevertheless, the main result of the paper [VII] asserts that the simple relation $\rho - \sigma = const$ between the principal radii of curvature, resp.

$$\frac{1}{p} - \frac{1}{q} = const \tag{6.4}$$

between the main curvatures $p = 1/\rho$, $q = 1/\sigma$, determines an integrable class of Weingarten surfaces. The associated nonlinear partial differential equation (6.11) has a zero curvature representation (6.12) (missed in [48]) with a nonremovable parameter, a third-order symmetry (6.14) (missed in [16]), and a recursion operator.

Paradoxically enough, surfaces satisfying relation (6.4) were not completely unknown to nineteenth century geometers. Ribaucour [40] established their most significant property, namely, that the corresponding focal surfaces (evolutes) are pseudospherical (i.e., have a constant Gaussian curvature $k < 0$). Consequently, surfaces satisfying equation (6.4) are involutes of pseudospherical surfaces. Moreover, the classical Bianchi transformation [6] is nothing but the induced correspondence between the two focal pseudospherical surfaces. Thus, our integrability result is not an entirely unexpected one.

The first examples of surfaces satisfying relation (6.4) also date to the nineteenth century. Lipschitz [26] derived a four-parametric family in terms of elliptic integrals. A particular subcase, the rotation surface of von Lilienthal [25], is the involute of the pseudosphere.

Ribaucour's theorems are covered in Darboux [9] and early twentieth-century monographs, such as [5, 12, 18, 47]. Later they became obsolete and forgotten as the induced Bianchi relation between pseudospherical surfaces became superseded by the classical Bäcklund transformation (the history is nicely reviewed by Prus and Sym in [39, Sect. 4]).

The left-hand side of Equation (6.4) is equal to the difference of the principal radii of curvature at a point. This geometric quantity has a definite physical meaning, being associated with the *interval of Sturm* [45], also known as the *astigmatic interval* or the *amplitude of astigmatism* or simply the *astigmatism* [20]. A mirror or a refracting surface satisfying relation (6.4) will feature a constant astigmatism in the normal directions.

In the sequel, surfaces satisfying condition (6.4) will be called *surfaces of constant astigmatism*. Accordingly, the equation (6.11) to determine the surfaces of constant astigmatism will be called the *constant astigmatism equation*.

6.1. **Weingarten surfaces.** We shall consider surfaces parametrized by curvature lines. As is well known, the fundamental forms can be written as

$$I = u^2\,dx^2 + v^2\,dy^2, \quad II = u^2 p\,dx^2 + v^2 q\,dy^2,$$

where p, q are the principal curvatures. Coordinates x, y are unique up to arbitrary changes $x = X(x)$, $y = Y(y)$. Let $\Psi = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ denote the orthonormal frame, given by $\mathbf{e}_1 = \mathbf{r}_x/u$, $\mathbf{e}_2 = \mathbf{r}_y/v$, $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$. The Gauss–Weingarten equations

$$\Psi_x = \begin{pmatrix} 0 & -\frac{u_y}{v} & up \\ \frac{u_y}{v} & 0 & 0 \\ -up & 0 & 0 \end{pmatrix} \Psi, \quad \Psi_y = \begin{pmatrix} 0 & \frac{v_x}{u} & 0 \\ -\frac{v_x}{u} & 0 & vq \\ 0 & -vq & 0 \end{pmatrix} \Psi, \tag{6.5}$$

or, more explicitly,

$$\begin{aligned} \mathbf{r}_{xx} &= \frac{u_x}{u}\mathbf{r}_x - \frac{u u_y}{v^2}\mathbf{r}_y + u^2 p\mathbf{n}, & \mathbf{n}_x &= -p\mathbf{r}_x, \\ \mathbf{r}_{xy} &= \frac{u_y}{u}\mathbf{r}_x + \frac{v_x}{v}\mathbf{r}_y, \\ \mathbf{r}_{yy} &= -\frac{v v_x}{u^2}\mathbf{r}_x + \frac{v_y}{v}\mathbf{r}_y + v^2 q\mathbf{n}, & \mathbf{n}_y &= -q\mathbf{r}_y \end{aligned}$$

are easily established. The Gauss equation is

$$u u_{yy} + v v_{xx} - \frac{v}{u} u_x v_x - \frac{u}{v} u_y v_y + u^2 v^2 pq = 0, \tag{6.6}$$

while the Mainardi–Codazzi equations are

$$(p - q)u_y + u p_y = 0, \quad (q - p)v_x + v q_x = 0 \tag{6.7}$$

and together they constitute the integrability conditions of the Gauss–Weingarten equations (6.5).

Let us impose a constraint $f(p, q) = 0$ determining the class of *Weingarten surfaces*. If nontrivial, it can be resolved with respect to one of the curvatures, say

$$q = F(p), \tag{6.8}$$

which we assume henceforth. Then the Gauss equation (6.6) becomes

$$p_{yy} = E^3 E'' p_{xx} + 2\frac{E'}{E} p_y^2 + E^2 (E E'')' p_x^2 + E E' p^2 - E^2 p, \tag{6.9}$$

where $E = E(p)$ is an arbitrary nonconstant function, $E' = dE/dp$ and the Gauss–Mainardi–Codazzi system of Weingarten surfaces reduces to the single equation (6.9).

The classification problem to be answered is: '*For which choices of the function $E(p)$ is the equation (6.9) integrable?*'

6.2. **Constant astigmatism equation [VII].** In the paper [VII], we found, besides the well-known linear Weingarten surfaces (6.3), another integrable class, consisting of surfaces with a constant difference between the principal radii of curvature (6.4), which we call *surfaces of constant astigmatism*. They emerge as a solution

$$E = \frac{p}{e^{1+c/p}}, \qquad c = const, \tag{6.10}$$

of the ordinary differential equation

$$\frac{E''}{E} - \left(\frac{E'}{E}\right)^2 + \frac{2}{p}\frac{E'}{E} - \frac{1}{p^2} = 0.$$

Using the solution (6.10) and assuming that the constant astigmatism condition (6.4) holds, the Gauss equation (6.9) simplifies to the *constant astigmatism equation*

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0. \tag{6.11}$$

The equation (6.11) has λ-dependent zero curvature representation

$$\begin{aligned} A &= \begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2+\lambda}\,z_y & (\lambda+1)z^{-\lambda} \\ \lambda z^{\lambda+1} & -\frac{1}{2}\sqrt{\lambda^2+\lambda}\,z_y \end{pmatrix}, \\ B &= \begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2+\lambda}\frac{z_x}{z^2} & \sqrt{\lambda^2+\lambda}\,z^{-\lambda-1} \\ \sqrt{\lambda^2+\lambda}\,z^{\lambda} & -\frac{1}{2}\sqrt{\lambda^2+\lambda}\frac{z_x}{z^2} \end{pmatrix}; \end{aligned} \tag{6.12}$$

it has obvious translational symmetries ∂_x, ∂_y, the scaling symmetry $2z\partial_z - x\partial_x + y\partial_y$, the discrete symmetry

$$x \to y, \quad y \to x, \quad z \to \frac{1}{z}. \tag{6.13}$$

and a recursion operator.

Computation reveals two third-order symmetries of equation (6.11), missed in [16]. One of them has the generator

$$\begin{aligned} &\frac{z^3}{K^3}(z_{xxx} - zz_{xxy}) - \frac{3}{K^5}z^3(z_x - zz_y)(z_{xx} - zz_{xy})^2 \\ &- \frac{2}{K^5}z^5(9z_x - zz_y)z_{xx} + \frac{1}{2K^5}z^2(9z_x^2 + 4zz_xz_y - z^2z_y^2)(z_x - zz_y)z_{xx} \\ &- \frac{2}{K^5}z^3z_x(z_x - zz_y)(4z_x - zz_y)z_{xy} + \frac{4}{K^5}z^6z_xz_{xy} \\ &+ \frac{3}{K^5}z^4(5z_x - zz_y)z_x^2 - \frac{3}{K^5}z(z_x - zz_y)z_x^4, \end{aligned} \tag{6.14}$$

where

$$K = \sqrt{(z_x - zz_y)^2 + 4z^3}.$$

The other is obtained by conjugation with the discrete symmetry (6.13).

6.3. **The classification [VI].** In the paper [VI] we completed the classification of integrable classes in the simplest possible case. The integrability criterion we adopt is the existence of an $\mathfrak{sl}(2)$-valued zero curvature representation depending on a nonremovable parameter. We apply method of formal spectral parameter, introduced in [31].

In [VI], we use the principal radii of curvature ρ, σ instead of the principal curvatures $p = 1/\rho$, $q = 1/\sigma$ used in [VII], since the radii transform in a very simple way under the offsetting symmetry of the integrability problem.

Employing the Maple package *Jets* [4], we completed the computer-aided cohomological classification outlined in [VII].

Proposition 6.1. *The third-order ordinary differential equation*

$$\rho''' = \frac{3}{2\rho'}\rho''^2 - \frac{\rho'-1}{\rho-\sigma}\rho'' + 2\frac{(\rho'-1)\rho'(\rho'+1)}{(\rho-\sigma)^2}. \tag{6.15}$$

determines a unique maximal class of Gauss–Mainardi–Codazzi equations of Weingarten surfaces whose initial $\mathfrak{sl}(2,\mathbb{C})$-valued zero curvature representation

$$A_0 = \begin{pmatrix} \frac{iu_y}{2v} & -\frac{u}{2\rho} \\ \frac{u}{2\rho} & -\frac{iu_y}{2v} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -\frac{iv_x}{2u} & -\frac{iv}{2\sigma} \\ -\frac{iv}{2\sigma} & \frac{iv_x}{2u} \end{pmatrix} \tag{6.16}$$

admits a second order formal spectral parameter under the condition that the normal form of the zero curvature representation can depend on derivatives of u, v, σ, ρ of no higher than the first order.

The above proposition provides a complete classification of integrable Weingarten surfaces under the following assumptions: the one-parameter zero curvature representation takes values in the Lie algebra $\mathfrak{sl}(2)$, includes the initial zero curvature representation (6.16) as a special case for some value of the parameter, depends analytically on the parameter, and its normal form involves derivatives of order no higher than one.

Proposition 6.2. *The nonremovable spectral parameter exists for all dependences $\rho(\sigma)$ allowed by the governing equation* (6.15)*.*

The governing equation (6.15) is explored in [VI, Sect. 4]. We identify two basic symmetries, scaling and translation (offsetting), and solve equation (6.15) in terms of elliptic integrals. The generic class of integrable Weingarten surfaces we obtained depends on one essential parameter (apart from the scaling and offsetting parameters).

In [VI, Sect. 5] we establish the integrable Gauss equation [VI, (39)] in the generic case as well as in a number of special cases when the elliptic integrals degenerate to elementary functions. All of these special cases could be located in the nineteenth century literature.

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Publications concerning the thesis

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Research paper

Infinitely many commuting nonlocal symmetries for modified Martínez Alonso–Shabat equation

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A B S T R A C T

We study the modified Martínez Alonso–Shabat equation

$$u_y u_{xz} + \alpha u_x u_{ty} - (u_z + \alpha u_t) u_{xy} = 0$$

and present its recursion operator and an infinite commuting hierarchy of full-fledged nonlocal symmetries. To date such hierarchies were found only for very few integrable systems in more than three independent variables.

1. Introduction

Integrable systems are well known to play an important role in modern mathematical physics, see e.g. [1–4]. An important feature of integrable partial differential systems is that any such system belongs to an infinite hierarchy of pairwise compatible systems that can be seen as symmetries of each other, cf. for example [1,3,5–7]. Such an infinite hierarchy of symmetries is an important sign of integrability. On the other hand, a useful structure attached to a given integrable system, as such a hierarchy provides, to an extent, the structure behind infinite families of explicit exact solutions like multisolitons, cf. e.g. the discussion in Fokas [1], Olver [2]; see also [2,3,6–9] for applications of symmetries in general.

For integrable partial differential systems in more than two independent variables the symmetries in question, as well as the conservation laws, are typically nonlocal, see e.g. [1,3–5,10,11], which makes the task of finding their commutation relations quite difficult, cf. e.g. [3,11]. There is a technique [12,13] allowing one to find an infinite hierarchy of nonlocal symmetries and establish its commutativity. This technique uses a Lax pair of the system under study for a fairly broad class of integrable multidimensional systems with isospectral Lax pairs involving an essential parameter. Given the importance of such hierarchies, as discussed above, it is natural to check whether indeed more examples of hierarchies of commuting nonlocal symmetries can be found using this technique. In the present paper we show that this can be done for the modified Martínez Alonso–Shabat equation in four independent variables (4D) and present an infinite commutative hierarchy of full-fledged nonlocal symmetries for this equation as well as a recursion operator.

Nonlocal symmetries may be used in the same way as local ones. For example, one can construct explicit solutions invariant w.r.t. nonlocal symmetries. Infinite-dimensional coverings of the presented type in many cases are infinite hydro-

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dynamical chains (cf. [14]) or systems very similar to the latter. The constructed nonlocal symmetries of the base equation are local for these chains and thus existence of commutative hierarchies proves S-integrability of the covering system (see [15]).

2. Modified Martínez Alonso–Shabat equation

Consider the modified Martínez Alonso–Shabat equation [13]

$$u_y u_{xz} + \alpha u_x u_{ty} - (u_z + \alpha u_t) u_{xy} = 0 \tag{1}$$

involving a nonzero real parameter α.

Eq. (1) is an integrable 4D PDE as it has [13] a Lax pair involving the spectral parameter $\lambda \neq \alpha$

$$r_y = \frac{\lambda}{\alpha}\frac{u_y}{u_x} r_x, \quad r_z = \frac{\lambda}{\alpha}\frac{u_z + \alpha u_t}{u_x} r_x - \lambda r_t, \tag{2}$$

cf. e.g. [3–5] and references therein for integrable 4D systems in general.

Identifying z and t in (1) yields [13] a 3D integrable reduction of the latter,

$$u_y u_{tx} - (\alpha + 1) u_t u_{xy} + \alpha u_x u_{ty} = 0. \tag{3}$$

In turn, (3) is, up to a possible relabeling of independent variables and multiplication by an overall constant, nothing but the Veronese web equation, also known as the ABC equation,

$$A u_x u_{ty} + B u_y u_{tx} + C u_t u_{xy} = 0, \quad A + B + C = 0, \tag{4}$$

which describes three-dimensional Veronese webs and is a subject of intense research, see e.g. [16,17] and references therein. Thus, (1) can be seen as a 4D generalization of (3) and hence of (4).

To simplify further computations, in what follows we shall work with Eq. (1) in the form

$$u_{ty} = \frac{\alpha u_t u_{xy} + u_z u_{xy} - u_y u_{xz}}{\alpha u_x} \tag{5}$$

resolved with respect to u_{ty}.

3. The recursion operator

Starting with (2) and using the deformation procedure described in Sergyeyev [18] (cf. also [5]) we readily find that (5) admits, in addition to (2), a Lax pair

$$\begin{aligned} q_y &= \frac{\lambda u_y q_x + (\alpha - \lambda) q u_{xy}}{\alpha u_x}, \\ q_z &= \frac{\lambda((\alpha u_t + u_z) q_x - \alpha u_x q_t - q u_{xz}) + \alpha q u_{xz}}{\alpha u_x}. \end{aligned} \tag{6}$$

In particular, for any given λ Eq. (6) define a covering, which we denote by $\mathcal{Q}_\lambda$, over (5); see e.g. [3] for general background on coverings.

Unlike r, if q satisfies (6) then it is a nonlocal symmetry shadow for (5) in the covering $\mathcal{Q}_\lambda$, i.e., roughly speaking, $\varphi = q$ satisfies the linearized version of (5),

$$\begin{aligned} &\frac{u_t u_{xy} D_x(\varphi) - u_x u_{xy} D_t(\varphi) + u_x^2 D_{ty}(\varphi) - u_t u_x D_{xy}(\varphi)}{u_x^2} + \frac{u_z u_{xy} D_x(\varphi) - u_y u_{xz} D_x(\varphi) + u_x u_{xz} D_y(\varphi) - u_x u_{xy} D_z(\varphi)}{\alpha u_x^2} \\ &+ \frac{u_x u_y D_{xz}(\varphi) - u_x u_z D_{xy}(\varphi)}{\alpha u_x^2} = 0 \end{aligned} \tag{7}$$

modulo (5) and (6) and differential consequences thereof.

Here D_x, D_y etc. denote total derivatives in the appropriate covering over (5), e.g. $\mathcal{Q}_\lambda$ for q, cf. e.g. [3] for relevant definitions.

Following [5,18], upon formally replacing λq by φ and q by ψ in (6), we readily arrive at the following

Proposition 1. *Eq. (5) admits a recursion operator $\mathcal{R}$ defined by the relations*

$$\begin{aligned} \psi_y &= \frac{u_y \varphi_x - u_{xy} \varphi + \alpha u_{xy} \psi}{\alpha u_x}, \\ \psi_z &= \frac{(\alpha u_t + u_z)\varphi_x - \alpha u_x \varphi_t - u_{xz} \varphi + \alpha u_{xz} \psi}{\alpha u_x}, \end{aligned} \tag{8}$$

meaning that for any nonlocal symmetry shadow $\mathcal{R}(\varphi) \stackrel{\text{def}}{=}$ for (5).

In other words, the above $\mathcal{R}$ defines a Bäcklund auto-transformation for the linearized version (7) of (5), see e.g. [3,5,19–21] and references therein for details on this approach to recursion operators.

While using $\mathcal{R}$ one readily can construct infinite hierarchies of nonlocal symmetry shadows for (5), this leaves one with the problem of finding a (minimal) covering in which all these shadows could be lifted to full-fledged nonlocal symmetries of (5), since only for those one can rigorously establish their commutation relations.

In what follows we shall take a slightly different route, using (6) rather than $\mathcal{R}$, to produce an infinite hierarchy of full-fledged nonlocal symmetries for (5) and establish their commutativity.

4. Nonlocal symmetries

While, as we have seen in the preceding section, q is a nonlocal symmetry shadow in the covering $\mathcal{Q}_\lambda$, this shadow cannot be lifted to a full-fledged nonlocal symmetry in the covering under study.

To circumvent this difficulty, consider a formal expansion $q = \sum_{i=0}^{\infty} q_i\lambda^i$. Substituting this expansion into (6) shows that $q_0 = Fu_x$, where $F(x,t)$ is an arbitrary function, while the remaining q_i are defined by the equations

$$(q_1)_y = \frac{\alpha u_{xy} q_1 + \left(u_{xx}u_y - u_{xy}u_x\right)F + u_x u_y F_x}{\alpha u_x},$$

$$(q_1)_z = \frac{\alpha u_{xz} q_1 + (\alpha(u_t u_x)_x + u_{xx}u_z - u_{xz}u_x)F + (\alpha u_t + u_z)u_x F_x - \alpha u_x^2 F_t}{\alpha u_x},$$

$$(q_i)_y = \frac{\alpha u_{xy} q_i - u_{xy}(q_{i-1}) + u_y(q_{i-1})_x}{\alpha u_x},$$

$$(q_i)_z = \frac{\alpha u_{xz} q_i - u_{xz}(q_{i-1}) - \alpha u_x(q_{i-1})_t + \alpha u_t(q_{i-1})_x + u_z(q_{i-1})_x}{\alpha u_x},$$

$i = 2, 3, \ldots$, that define an infinite-dimensional covering, which we denote by $\mathcal{Q}_\infty$, over (5).

Theorem 1. *Infinite prolongations of the vector fields*

$$Q_i = q_i\frac{\partial}{\partial u} + \sum_{j=1}^{\infty} B_i^j \frac{\partial}{\partial q_j}, \quad i = 1, 2, \ldots, \tag{9}$$

form an infinite hierarchy of commuting nonlocal symmetries for (5) in the covering $\mathcal{Q}_\infty$.

Here

$$B_i^j = \frac{\left(\left[u_x, q_{i+j-1}\right]_x - \alpha\left[u_x, q_{i+j}\right]_x\right)F - \left(\alpha\, q_{i+j} - q_{i+j-1}\right)u_x F_x}{\alpha\, u_x} + \frac{\left(q_{i+j-s(i,j)-1}\right)_x q_{s(i,j)+1}}{u_x} + \sum_{k=1}^{s(i,j)} \frac{\alpha\left[q_{i+j-k}, q_k\right]_x - \left[q_{i+j-k-1}, q_k\right]_x}{\alpha u_x}, \tag{10}$$

where $s(i,j) = \min(i-1, j-1)$ *and* $[A,B]_x = A_x B - AB_x$.

Before proceeding to the proof of the theorem note that by the very construction we have $q_{i+1} = \mathcal{R}(q_i)$, so the commutativity of infinite prolongations of Q_i suggests that the above recursion operator $\mathcal{R}$ could be hereditary (cf. e.g. [2,3] and references therein on the hereditary property in general), at least when restricted to the span of shadows q_i, $i = 1, 2, \ldots$, which could provide some additional insight into how the hereditary property works in the multidimensions.

Proof. First of all, it is immediate that q_i is a nonlocal symmetry shadow for (5) for each $i = 1, 2, \ldots$ since so is q.

Inspired by Sergyeyev [12], Morozov and Sergyeyev [13], we were able to find the lifts of q_i, $i = 1, 2, \ldots$, to the covering $\mathcal{Q}_\infty$. These lifts are nonlocal symmetries Q_i for (5) given by (9).

Now, commutativity of the infinite prolongations of Q_i is easily seen (cf. [12,13]) to be tantamount to that of the flows

$$\partial u/\partial\tau_i = q_i, \quad \partial q_i/\partial\tau_j = B_i^j, \quad i, j = 1, 2, \ldots \tag{11}$$

i.e., to the requirement that the relations

$$\partial^2 u/\partial\tau_i\partial\tau_j = \partial^2 u/\partial\tau_j\partial\tau_i, \quad \partial^2 q_k/\partial\tau_i\partial\tau_j = \partial^2 q_k/\partial\tau_j\partial\tau_i, \quad i, j, k = 1, 2, \ldots, \tag{12}$$

hold by virtue of (5) and (11) and their differential consequences, which in turn is readily verified by straightforward but tedious computation. □

Finding explicit form of the generators and providing rigorous proofs of commutation relations for infinite-dimensional algebras of nonlocal symmetries for multidimensional integrable PDEs, rather than merely finding shadows of nonlocal symmetries, appears to be quite rare, especially in the case of four (or more) independent variables. In particular, there are only a few earlier examples known to the present author where this was achieved in 4D, namely, the commutative hierarchies of nonlocal symmetries for the self-dual Yang–Mills equations [22] and for the Martínez Alonso–Shabat equation [13]. Interestingly, the situation appears to be quite different in 3D, where infinite-dimensional noncommutative algebras of nonlocal symmetries for a number of dispersionless integrable systems were found by direct computations, see e.g. [10,11,17,23].

Author statement

All the work done by the only author Hynek Baran.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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NONLOCAL SYMMETRIES OF INTEGRABLE LINEARLY DEGENERATE EQUATIONS: A COMPARATIVE STUDY

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We continue the study of Lax integrable equations. We consider four three-dimensional equations: (1) the rdDym equation $u_{ty} = u_x u_{xy} - u_y u_{xx}$, (2) the Pavlov equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$, (3) the universal hierarchy equation $u_{yy} = u_t u_{xy} - u_y u_{tx}$, and (4) the modified Veronese web equation $u_{ty} = u_t u_{xy} - u_y u_{tx}$. For each equation, expanding the known Lax pairs in formal series in the spectral parameter, we construct two differential coverings and completely describe the nonlocal symmetry algebras associated with these coverings. For all four pairs of coverings, the obtained Lie algebras of symmetries manifest similar (but not identical) structures; they are (semi)direct sums of the Witt algebra, the algebra of vector fields on the line, and loop algebras, all of which contain a component of finite grading. We also discuss actions of recursion operators on shadows of nonlocal symmetries.

Keywords: partial differential equation, integrable linearly degenerate equation, nonlocal symmetry, recursion operator

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1. Introduction and notation

In [1], we began a systematic study of the symmetry and integrability properties of Lax integrable three-dimensional equations, i.e., equations that admit a Lax pair with a nonremovable parameter. All the two-dimensional symmetry reductions of

- the rdDym equation $u_{ty} = u_x u_{xy} - u_y u_{xx}$,
- the three-dimensional Pavlov equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$,
- the universal hierarchy equation (UHE) $u_{yy} = u_t u_{xy} - u_y u_{tx}$, and
- the modified Veronese web equation (mVWE) $u_{ty} = u_t u_{xy} - u_y u_{tx}$

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were described. In [2], we studied the behavior of the Lax operators admitted by these equations under symmetry reductions and showed that some two-dimensional reductions (one of them is equivalent to the Gibbons–Tsarev equation [3]) inherit the Lax pairs. We also constructed infinite series of (nonlocal) conservation laws for these reductions. Finally, we recently used expansion of the Lax pair for the rdDym equation in formal series in the spectral parameter to construct two infinite-dimensional differential coverings of this equation and completely described the nonlocal symmetries in these coverings [4]. All these equations are linearly degenerate in the sense of [5], where such equations were classified.

Here, we use the same techniques to describe the Lie algebra structure of nonlocal symmetries for the other three equations. In Sec. 2, we briefly introduce the terminology used. In Sec. 3, to make the exposition self-contained, we briefly recall the results obtained in [4]. We consider the three-dimensional Pavlov equation in Sec. 4, discuss the results for the UHE in Sec. 5, and describe the symmetries of the mVWE in Sec. 6. In each case, we also discuss recursion operators and their action on the shadows of nonlocal symmetries. We also describe a Bäcklund autotransformation for the mVWE. Finally, in Sec. 7, we summarize the obtained results. We omit the proofs: a detailed exposition in the case of the rdDym equation can be found in [4], and all the other proofs are quite similar.

All the symmetry algebras in what follows have similar (but not identical) structures and are direct or semidirect sums of the following Lie algebras (see Table 13, where the main results are aggregated):

- the Witt algebra $\mathfrak{W}$ of vector fields $\mathbf{e}_i = z^{i+1}\partial/\partial z$, $i \in \mathbb{Z}$,
- its subalgebras $\mathfrak{W}_k^-$ spanned by $\mathbf{e}_i$, $i \le k \le 0$, and $\mathfrak{W}_k^+$ spanned by $\mathbf{e}_i$ with $i \ge k \ge 0$,
- the algebra $\mathfrak{V}[\rho]$ of vector fields $R(\rho)\,\partial/\partial\rho$ on $\mathbb{R}^1$ with a distinguished coordinate ρ (we use the notation $[R, \overline{R}] = R\overline{R}' - \overline{R}R'$ everywhere in what follows for functions R and $\overline{R}$ of ρ, where the prime denotes the derivative with respect to ρ),
- the loop algebra $\mathfrak{L}[\rho]$ spanned by the elements $z^i \otimes X$, $i \in \mathbb{Z}$, $X \in \mathfrak{V}[\rho]$, with the commutator $[z^i \otimes X, z^j \otimes Y] = z^{i+j} \otimes [X, Y]$, and
- the algebra $\mathfrak{L}_k^+[\rho]$ spanned by the elements $p(z) \otimes X$, where $X \in \mathfrak{V}[\rho]$, $p(z) \in \mathbb{R}[z]/(z^k)$ is a truncated polynomial. We similarly define $\mathfrak{L}_k^-[\rho]$ with $p(z) \in \mathbb{R}[z^{-1}]/(z^{-k})$.

Semidirect sums in the algebras of symmetries arise because of the natural actions of $\mathfrak{W}$ on $\mathfrak{L}[\rho]$, $\mathfrak{W}_k^-$ on $\mathfrak{L}_k^-[\rho]$, and $\mathfrak{W}_k^+$ on $\mathfrak{L}_k^+[\rho]$.

All the considered equations admit scaling symmetries that allow introducing natural weights in the space of polynomial functions on the equation. This structure is inherited by the symmetry algebras in all cases except the case of the mVWE. Perhaps, this is why the Lie algebra structure of symmetries for this equation differs a bit from the others.

2. Preliminaries

We everywhere consider second-order scalar differential equations in three independent variables x, y, and t (see [6] for a general coordinate-free exposition). For this, we consider the space $J^\infty(\mathbb{R}^3, \mathbb{R})$ of infinite jets of smooth functions $u = u(x, y, t)$ on $\mathbb{R}^3$. This space is endowed with the coordinates

$$x, y, t, u_{i,j,k} = \frac{\partial^{i+j+k} u}{\partial x^i\, \partial y^j\, \partial t^k}, \quad i, j, k \ge 0,$$

and its geometric structure is determined by the Cartan distribution spanned by the total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_{i,j,k\geq 0} u_{i+1,j,k}\frac{\partial}{\partial u_{i,j,k}}, \qquad D_y = \frac{\partial}{\partial y} + \sum_{i,j,k\geq 0} u_{i,j+1,k}\frac{\partial}{\partial u_{i,j,k}},$$

$$D_t = \frac{\partial}{\partial t} + \sum_{i,j,k\geq 0} u_{i,j,k+1}\frac{\partial}{\partial u_{i,j,k}}.$$

An equation $\mathcal{E} = \{F = 0\} \subset J^\infty(\mathbb{R}^3)$ is the subset defined by an infinite system of relations $D_\sigma(F) = 0$, where $F = F(x, y, t, u, u_x, u_y, u_t, u_{xx}, u_{xy}, \dots, u_{tt})$ is a smooth function and D_σ denotes all possible compositions of the total derivatives. Total derivatives and any differential operators in total derivatives can be restricted to $\mathcal{E}$, i.e., expressed in terms of internal coordinates on $\mathcal{E}$.

A symmetry of $\mathcal{E}$ is a vector field

$$S = \sum S_{i,j,k}\frac{\partial}{\partial u_{i,j,k}}$$

on $\mathcal{E}$ that commutes with the total derivatives (here and hereafter, summation is taken over all internal coordinates on $\mathcal{E}$). Any symmetry is an evolutionary vector field of the form

$$\mathbf{E}_\varphi = \sum D_x^i D_y^j D_t^k(\varphi)\frac{\partial}{\partial u_{i,j,k}},$$

where φ is an arbitrary smooth function on $\mathcal{E}$ that satisfies the equation $\ell_{\mathcal{E}}(\varphi) = 0$ and $\ell_{\mathcal{E}}$ is the restriction of the linearization operator

$$\ell_F = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x}D_x + \dots + \frac{\partial F}{\partial u_t}D_t + \frac{\partial F}{\partial u_{xx}}D_x^2 + \frac{\partial F}{\partial u_{xy}}D_xD_y + \dots + \frac{\partial F}{\partial u_{tt}}D_t^2$$

to $\mathcal{E}$. The function φ is the generating function (or the characteristic) of a symmetry. Symmetries form a Lie algebra $\mathrm{sym}(\mathcal{E})$ with respect to the commutator, and the commutator induces the Jacobi bracket on the space of generating functions: $\{\varphi_1, \varphi_2\} = \mathbf{E}_{\varphi_1}(\varphi_2) - \mathbf{E}_{\varphi_2}(\varphi_s)$. In what follows, we do not distinguish between symmetries and their generating functions.

A symmetry of the form $s = \delta u + \alpha x u_x + \beta y u_y + \gamma t u_t$, $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, is called a scaling symmetry of $\mathcal{E}$. If an equation admits such a symmetry, then we can assign weights to polynomial functions on $\mathcal{E}$ by $|x| = -\alpha$, $|y| = -\beta$, $|t| = -\gamma$, $|u_{i,j,k}| = \delta - i\alpha - j\beta - k\gamma$, with respect to which the space $\mathcal{P}(\mathcal{E})$ of such functions becomes graded: $\mathcal{P}(\mathcal{E}) = \bigoplus_{r\in\mathbb{Z}}\mathcal{P}_r(\mathcal{E})$. If $\mathbf{E}_\varphi$ is a symmetry and $\varphi \in \mathcal{P}(\mathcal{E})$, then we set $|\mathbf{E}_\varphi| = |\varphi| - |u|$. Then $\mathbf{E}_\varphi(\mathcal{P}_r(\mathcal{E})) \subset \mathcal{P}_{r+|\mathbf{E}_\varphi|}(\mathcal{E})$ and $|[\mathbf{E}_{\varphi_1}, \mathbf{E}_{\varphi_2}]| = |\mathbf{E}_{\varphi_1}| + |\mathbf{E}_{\varphi_2}|$. Hence, the space of polynomial symmetries becomes a $\mathbb{Z}$-graded Lie algebra.

Let $\mathcal{E}$ be an equation. A differential covering of $\mathcal{E}$ (see [7]) is an extension $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ by a system of first-order equations

$$\begin{aligned} w_x^\alpha &= X^\alpha(x, y, t, \dots, u_{i,j,k}, \dots, w^\beta, \dots), \\ w_y^\alpha &= Y^\alpha(x, y, t, \dots, u_{i,j,k}, \dots, w^\beta, \dots), \\ w_t^\alpha &= T^\alpha(x, y, t, \dots, u_{i,j,k}, \dots, w^\beta, \dots), \end{aligned} \tag{1}$$

$\alpha, \beta = 1, 2, \dots$, that are compatible modulo $\mathcal{E}$. The variables w^j are said to be nonlocal, and there exists a projection $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}$ such that the nonlocal variables are fiberwise coordinates of this projection. The number of independent nonlocal variables is the covering dimension. The total derivatives are lifted to $\widetilde{\mathcal{E}}$ by

$$\widetilde{D}_x = D_x + \sum X^j\frac{\partial}{\partial w^j}, \qquad \widetilde{D}_y = D_y + \sum Y^j\frac{\partial}{\partial w^j}, \qquad \widetilde{D}_t = D_t + \sum T^j\frac{\partial}{\partial w^j},$$

and any differential operator D in total derivatives can consequently be also lifted to $\widetilde{D}$. We say that a covering is Abelian if the right-hand sides of its defining equation are independent of nonlocal variables. In the case where system (1) can be written in the form of two equations, it is called a Lax pair.

Given a one-dimensional covering τ (i.e., a covering (1) with $w^\alpha = w$, $X^\alpha = X$, $Y^\alpha = Y$, and $T^\alpha = T$) that depends smoothly on $\lambda \in \mathbb{R}$, we can consider the expansion $w = \sum_{-\infty}^{\infty} \lambda^i w_i$ and also expand the defining equations of the covering in formal series in the parameter. An infinite-dimensional covering with the nonlocal variables w_i then arises. If $w_i = 0$ for $i < 0$, then we say that this is a positive covering associated with τ; if $w_i = 0$ for $i > 0$, then we have a negative covering.

A symmetry of $\widetilde{\mathcal{E}}$ is a nonlocal symmetry of $\mathcal{E}$. Nonlocal symmetries are vector fields

$$\mathbf{E}_\varphi + \sum_j \Phi^j \frac{\partial}{\partial w^j},$$

where φ and Φ^j are smooth functions on $\widetilde{\mathcal{E}}$ that satisfy $\widetilde{\ell}_{\mathcal{E}}(\varphi) = 0$ together with the system

$$\widetilde{D}_x(\varphi^\alpha) = \widetilde{\ell}_{X^\alpha}(\varphi) + \sum_\theta \frac{\partial X^\alpha}{\partial w^j}\Phi^j, \qquad \widetilde{D}_y(\varphi^\alpha) = \widetilde{\ell}_{Y^\alpha}(\varphi) + \sum_\theta \frac{\partial Y^\alpha}{\partial w^j}\Phi^j,$$

$$\widetilde{D}_t(\varphi^\alpha) = \widetilde{\ell}_{T^\alpha}(\varphi) + \sum_\theta \frac{\partial T^\alpha}{\partial w^j}\Phi^j.$$

A nonlocal symmetry is said to be invisible if $\varphi = 0$. Solutions of the equation $\widetilde{\ell}_{\mathcal{E}}(\varphi) = 0$ are called shadows. We say that a shadow φ is lifted (or reconstructed) if there exists a nonlocal symmetry $\Phi = (\varphi, \Phi^1, \dots, \Phi^j, \dots)$. Of course, lifts (if they exist) are defined up to an invisible symmetry.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be equations and $\tau_i \colon \widetilde{\mathcal{E}} \to \mathcal{E}_i$ be coverings. Then we have the diagram

$$\mathcal{E}_1 \xleftarrow{\tau_1} \widetilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2,$$

which is called a Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$. If $\mathcal{E}_1 = \mathcal{E}_2$, then it is a Bäcklund autotransformation. With any equation $\mathcal{E} = \mathcal{E}_F$, we associate the system $\mathcal{T}\mathcal{E}$

$$F(x, y, y, u, u_x, u_y, u_t, u_{xx}, u_{xy}, \dots, u_{tt}) = 0,$$

$$\frac{\partial F}{\partial u}v + \frac{\partial F}{\partial u_x}v_x + \frac{\partial F}{\partial u_y}v_y + \frac{\partial F}{\partial u_t}v_t + \frac{\partial F}{\partial u_{xx}}v_{xx} + \frac{\partial F}{\partial u_{xy}}v_{xy} + \dots + \frac{\partial F}{\partial u_{tt}}v_{tt} = 0,$$

which is called the tangent equation of $\mathcal{E}$. A Bäcklund autotransformation of this system is a recursion operator for shadows of symmetries of $\mathcal{E}$ (see [8]).

3. The rdDym equation: A synopsis

More information about the rdDym equation is available in [9]–[11], and a detailed discussion of coverings, nonlocal symmetries, and recursion operators for this equation can be found in [4]. Nevertheless, for completeness, we present a short overview of the previously obtained results. The equation is

$$u_{ty} = u_x u_{xy} - u_y u_{xx}. \tag{2}$$

We assign the weights $|x| = 1$, $|u| = 2$, and $|y| = |t| = 0$ to the variables x, y, t, and u. Consequently, the equation becomes homogeneous with respect to these weights. Local symmetries are solutions of the equation

$$\ell_{\mathcal{E}}(\varphi) \equiv D_t D_y(\varphi) - u_x D_x D_y(\varphi) + u_y D_x^2(\varphi) - u_{xy} D_x(\varphi) + u_{xx} D_y(\varphi) = 0.$$

The space of solutions is spanned by the functions

$$\psi_0 = -x\,u_x + 2\,u, \qquad \upsilon_0(Y) = Y u_y,$$

$$\theta_0(T) = T u_t + T'(x u_x - u) + \frac{1}{2} T'' x^2, \qquad \theta_{-1}(T) = T u_x + T' x, \qquad \theta_{-2}(T) = T,$$

where $T = T(t)$ and $Y = Y(y)$ are arbitrary functions of their arguments and the prime denotes the corresponding derivative. The corresponding evolutionary vector fields have the weights $|\mathbf{E}_{\psi_0}| = |\mathbf{E}_{\upsilon_0(Y)}| = 0$ and $|\mathbf{E}_{\theta_i(T)}| = i$, $i = 0, -1, -2$. Commutators of the symmetries are presented in Table 1.

Table 1

	ψ_0	$\upsilon_0(\overline{Y})$	$\theta_0(\overline{T})$	$\theta_{-1}(\overline{T})$	$\theta_{-2}(\overline{T})$
ψ_0	0	0	0	$\theta_{-1}(\overline{T})$	$2\theta_{-2}(\overline{T})$
$\upsilon_0(Y)$		$\upsilon_0([Y,\overline{Y}])$	0	0	0
$\theta_0(T)$			$\theta_0([T,\overline{T}])$	$\theta_{-1}([\overline{T},T])$	$\theta_{-2}([\overline{T},T])$
$\theta_{-1}(T)$				$\theta_{-2}([\overline{T},T])$	0
$\theta_{-2}(T)$					0

The rdDym equation: commutators of local symmetries.

The system

$$w_t = (u_x - \lambda) w_x, \qquad w_y = \lambda^{-1} u_y w_x \tag{3}$$

is a Lax pair for Eq. (2). Setting $w = \sum_{i=-\infty}^{+\infty} \lambda^i w_i$ and substituting this expansion in (3), we obtain $w_{i,t} = u_x w_{i,x} - w_{i-1,x}$ and $w_{i,y} = u_y w_{i+1,x}$. The corresponding positive covering is defined by the system

$$q_{1,t} = \frac{u_x}{u_y}, \qquad q_{1,x} = \frac{1}{u_y},$$

$$q_{i,t} = \frac{u_x}{u_y} q_{i-1,y} - q_{i-1,x}, \qquad q_{i,x} = \frac{q_{i-1,y}}{u_y}, \quad i \geq 2,$$

with the additional nonlocal variables $q_i^{(j)}$ defined by the equalities $q_i^{(0)} = q_i$ and $q_i^{(j+1)} = (q_i^{(j)})_y$. The weights assigned to the nonlocal variables are $|q_i^{(j)}| = -i$, $i \geq 1$, $j \geq 0$. The negative covering is defined by the system

$$r_{1,x} = u_x^2 - u_t, \qquad r_{1,y} = u_x u_y,$$

$$r_{i,x} = u_x r_{i-1,x} - r_{i-1,t}, \qquad r_{i,y} = u_y r_{i-1,x},$$

with the additional nonlocal variables $r_i^{(j)}$ obviously defined by $r_i^{(0)} = r_i$ and $r_i^{(j+1)} = (r_i^{(j)})_t$. We have $|r_i^{(j)}| = i + 2$, $i \geq 1$, $j \geq 0$.

All the local symmetries of the rdDym equation can be lifted to both τ^+ and τ^-, and we let the corresponding capital letters denote the lifts: Ψ_0 for the lift of ψ_0, $\Theta_i(T)$ for $\theta_i(T)$, and so on.

Three families of nonlocal symmetries are admitted in τ^+. The first consists of the invisible symmetries

$$\Phi^k_{\text{inv}}(Y) = (\underbrace{0, \dots, 0}_{k \text{ times}}, \varphi^1_{\text{inv}}, \dots, \varphi^i_{\text{inv}}, \dots),$$

where $\varphi^1_{\rm inv} = Y(y)$, and another two are generated by the lifts Ψ_{-1} and Ψ_{-2} of the nonlocal shadows $\psi_{-1} = q_1u_y + x$ and $\psi_{-2} = (2q_2 - q_1q_1^{(1)})u_y$ using the relations $\Psi_{-k} = [\Psi_{-k+1}, \Psi_{-1}]$, $k \geq 3$, and $\Upsilon_{-k}(Y) = [\Psi_{-k-1}, \Phi^1_{\rm inv}(Y)]$, $k \geq 0$. The constructed nonlocal symmetries have the weights $|\Psi_i| = |\Upsilon_i(Y)| = i$, $i \leq 0$, $|\Theta_j(T)| = j$, $j = 0, -1, -2$, and $|\Phi^k_{\rm inv}(Y)| = k$, $k \geq 1$.

We then have the following result.

Theorem 1. *There exists a basis in* $\mathrm{sym}_{\tau^+}(\mathcal{E})$ *consisting of the elements* $\mathbf{w}_i$, $i \leq 0$, $\mathbf{v}_j(T)$, $j = 0, -1, -2$, *and* $\mathbf{v}_k(Y)$, $k \in \mathbb{Z}$, *such that they commute as indicated in Table 2. Therefore, the algebra* $\mathrm{sym}_{\tau^+}(\mathcal{E})$ *is isomorphic to* $\mathfrak{W}_0^- \ltimes (\mathfrak{L}_3^-[t] \oplus \mathfrak{L}[y])$ *with the natural action of* $\mathfrak{W}_0^-$ *on* $\mathfrak{L}_3^-[t] \oplus \mathfrak{L}[y]$.

Table 2

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{T})$	$\mathbf{v}_j(\overline{Y})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\overline{T})$, $-2 \leq i+j \leq 0$, 0, otherwise	$j\mathbf{v}_{i+j}(\overline{Y})$
$\mathbf{v}_i(T)$		$\mathbf{v}_{i+j}([T,\overline{T}])$, $-2 \leq i+j \leq 0$, 0, otherwise	0
$\mathbf{v}_i(Y)$			$\mathbf{v}_{i+j}([Y,\overline{Y}])$

The rdDym equation: commutators in $\mathrm{sym}_{\tau^+}(\mathcal{E})$.

Similarly, local symmetries are lifted to τ^-, and three families of nonlocal symmetries arise in this covering. They are Ψ_k, $k \geq 1$, $\Theta_i(T)$, $i \geq -2$, and $\Phi^l_{\rm inv}$ and have the weights $|\Psi_k| = k$, $k \geq 0$, $|\Phi^l_{\rm inv}| = -l-2$, $l \geq 1$, $|\Theta_i(T)| = i$, $i \geq -3$, and $|\Upsilon_0(Y)| = 0$.

The following theorem then describes the Lie algebra structure.

Theorem 2. *There exists a basis in* $\mathrm{sym}_{\tau^-}(\mathcal{E})$ *consisting of the elements* $\mathbf{w}_i$, $i \geq 0$, $\mathbf{v}_j(T)$, $j \in \mathbb{Z}$, *and* $\mathbf{v}(Y)$ *satisfying the commutator relations presented in Table 3. Hence, the Lie algebra* $\mathrm{sym}_{\tau^-}(\mathcal{E})$ *is isomorphic to* $\mathfrak{W}_0^+ \ltimes \mathfrak{L}[t] \oplus \mathfrak{V}[y]$ *with the natural action of* $\mathfrak{W}_0^+$ *on* $\mathfrak{L}[t]$.

Table 3

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{T})$	$\mathbf{v}(\overline{Y})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\overline{T})$	0
$\mathbf{v}_i(T)$		$\mathbf{v}_{i+j}([T,\overline{T}])$	0
$\mathbf{v}(Y)$			$\mathbf{v}([Y,\overline{Y}])$

The rdDym equation: commutators in $\mathrm{sym}_{\tau^-}(\mathcal{E})$.

We note that the components of the invisible symmetries are constructed using the operator

$$\mathcal{Y} = q_1\frac{\partial}{\partial y} + \sum_{i=1}^{\infty}(i+1)q_{i+1}\frac{\partial}{\partial q_i}.$$

Similar operators arise in studying other equations in what follows.

The algebra $\operatorname{sym}(\mathcal{E})$ admits a recursion operator $\widehat{\chi} = \mathcal{R}_+(\chi)$ defined by the system

$$\begin{aligned} D_t(\widehat{\chi}) &= u_y^{-1}\big(u_y D_x(\chi) - u_x D_y(\chi) + (u_x u_{xy} - u_y u_{xx})\widehat{\chi}\big), \\ D_x(\widehat{\chi}) &= u_y^{-1}\big(u_{xy}\widehat{\chi} - D_y(\chi)\big) \end{aligned} \tag{4}$$

(see [12]). This means that $\widehat{\chi}$ is a nonlocal shadow if χ is. Another recursion operator $\chi = \mathcal{R}_-(\widehat{\chi})$ is given by the system

$$\begin{aligned} D_x(\chi) &= D_t(\widehat{\chi}) - u_x D_x(\widehat{\chi}) + u_{xx}\widehat{\chi}, \\ D_y(\chi) &= -u_y D_x(\widehat{\chi}) + u_{xy}\widehat{\chi}. \end{aligned} \tag{5}$$

The operators $\mathcal{R}_+$ and $\mathcal{R}_-$ are mutually inverse.

The actions of $\mathcal{R}_+$ and $\mathcal{R}_-$ on $\operatorname{sym}(\mathcal{E})$ can be prolonged to the shadows of nonlocal symmetries from $\operatorname{sym}(\widetilde{\mathcal{E}}^+)$ and $\operatorname{sym}(\widetilde{\mathcal{E}}^-)$ if we replace the derivatives D_t, D_x, and D_y in (4) and (5) with the total derivatives $\widehat{D}_t$, $\widehat{D}_x$, and $\widehat{D}_y$ in the Whitney product of the coverings τ^+ and τ^- in the sense of [7]. The resulting operators are also denoted by $\mathcal{R}_+$ and $\mathcal{R}_-$.

We note that the operators act nontrivially on the "vacuum," $\mathcal{R}_+(0) = \theta_{-2}(T)$ and $\mathcal{R}_-(0) = \upsilon_0(Y)$, which follows immediately from Eqs. (4) and (5). Therefore, the actions can be reasonably considered modulo $\theta_{-2}(T)$ for $\mathcal{R}_+$ and $\upsilon_0(Y)$ for $\mathcal{R}_-$. Taking this remark into account, we have the following proposition.

Proposition 1. *Modulo the images of the trivial symmetry, the action of recursion operators is of the form*

$$\mathcal{R}_+(\theta_i(T)) = \begin{cases} \alpha_i^+\theta_{i-1}(T), & i > -2, \\ 0, & i = -2, \end{cases} \qquad \mathcal{R}_-(\theta_i(T)) = \alpha_i^-\theta_{i+1}(T), \quad i \ge -2,$$

$$\mathcal{R}_+(\upsilon_i(Y)) = \beta_i^+\upsilon_{i+1}(Y), \quad i \le 0, \qquad \mathcal{R}_-(\upsilon_i(Y)) = \begin{cases} \beta_i^-\upsilon_{i+1}(Y), & i < 0, \\ 0, & i = 0, \end{cases}$$

$$\mathcal{R}_+(\psi_i) = \gamma_i^+\psi_{i-1}, \qquad \mathcal{R}_-(\psi_i) = \gamma_i^-\psi_{i+1}, \quad i \in \mathbb{Z},$$

where $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma_i^\pm$ are nonzero constants.

We note that the recursion operators $\mathcal{R}_+$ and $\mathcal{R}_-$ "glue" the shadows ψ_m of nonlocal symmetries in the coverings $\widetilde{\mathcal{E}}^+$ and $\widetilde{\mathcal{E}}^-$ and "tunnel" from the series of $\theta_k(T)$ to the series of $\upsilon_j(Y)$ (see Fig. 1). In all the figures here and hereafter, straight arrows denote actions up to scalar multipliers and modulo the image of a trivial shadow. We also "compress" the notation and write θ_i instead of $\theta_i(T)$, υ_k instead of $\upsilon_k(Y)$, and so on. The notation $(\,\cdot\,)^+$ is used for shadows in τ^+, $(\,\cdot\,)^-$, for shadows in τ^-, and $(\,\cdot\,)^\pm$, for shadows in both coverings.

4. The three-dimensional Pavlov equation

The three-dimensional Pavlov equation, which, for example, was discussed in [13], [14], has the form

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}. \tag{6}$$

We choose the internal coordinates on $\mathcal{E}$

$$u_{k,l}^0 = u_{\underbrace{x\ldots x}_{k\text{ times}}\underbrace{t\ldots t}_{l\text{ times}}}, \qquad u_{k,l}^1 = u_{\underbrace{x\ldots x}_{k\text{ times}}\underbrace{t\ldots t}_{l\text{ times}}}, \qquad k,l \ge 0.$$

$$\cdots \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi^+_{-1} \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi^\pm_0 \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi^-_1 \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \cdots$$

$$\cdots \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \upsilon^+_{-1} \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \upsilon^\pm_0 \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} 0^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \theta^\pm_{-2} \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \theta^\pm_{-1} \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \theta^\pm_0 \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \theta^-_1 \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \cdots$$

Fig. 1. The rdDym equation: action of recursion operators (4) and (5).

The total derivatives in these coordinates are

$$D_x = \frac{\partial}{\partial x} + \sum_{k,l}\left(u^0_{k+1,l}\frac{\partial}{\partial u^0_{k,l}} + u^1_{k+1,l}\frac{\partial}{\partial u^1_{k,l}}\right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_{k,l}\left(u^1_{k,l}\frac{\partial}{\partial u^0_{k,l}} + D^k_x D^l_t\left(u^0_{11} + u^1_{00}u^0_{20} - u^0_{10}u^1_{10}\right)\frac{\partial}{\partial u^1_{k,l}}\right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k,l}\left(u^0_{k,l+1}\frac{\partial}{\partial u^0_{k,l}} + u^1_{k,l+1}\frac{\partial}{\partial u^1_{k,l}}\right).$$

We assign the weights $|t| = 0$, $|y| = 1$, $|x| = 2$, and $|u| = 3$. Hence, $|u^0_{k,l}| = 3 - 2k$ and $|u^1_{k,l}| = 3 - 2k - 1$.

The symmetries of $\mathcal{E}$ are solutions of the equation

$$\ell_{\mathcal{E}}(\varphi) \equiv D^2_y(\varphi) - D_tD_x(\varphi) - u_yD^2_x(\varphi) + u_xD_xD_y(\varphi) - u_{xx}D_y(\varphi) + u_{xy}D_x(\varphi). \qquad (7)$$

The space $\mathrm{sym}(\mathcal{E})$ of solutions of Eq. (7) is spanned by the functions

$$\varphi_1 = 2x - yu_x, \qquad \varphi_2 = 3u - 2xu_x - yu_y,$$

$$\theta_0(T) = Tu_t + T'(xu_x + yu_y - u) + \frac{1}{2}T''(y^2u_x - 2xy) - \frac{1}{6}T'''y^3,$$

$$\theta_1(T) = Tu_y + T'(yu_x - x) - \frac{1}{2}T''y^2, \qquad \theta_2(T) = Tu_x - T'y, \qquad \theta_3(T) = T,$$

where T is a function of t and the prime denotes derivatives with respect to t. The commutators of these symmetries are presented in Table 4. The corresponding vector fields have the weights $|\mathbf{E}_{\varphi_1}| = -1$, $|\mathbf{E}_{\varphi_2}| = 0$, and $|\mathbf{E}_{\theta_i}| = -i$, $i = 0, -1, -2, -3$.

Table 4

	φ_1	φ_2	$\theta_0(\overline{T})$	$\theta_1(\overline{T})$	$\theta_2(\overline{T})$	$\theta_3(\overline{T})$
φ_1	0	φ_1	0	$-2\theta_2(\overline{T})$	$2\theta_3(\overline{T})$	0
φ_2		0	0	$-\theta_1(\overline{T})$	$-2\theta_2(\overline{T})$	$-3\theta_3(\overline{T})$
$\theta_0(T)$			$\theta_0([\overline{T},T])$	$\theta_1([\overline{T},T])$	$\theta_2([\overline{T},T])$	$\theta_3([\overline{T},T])$
$\theta_1(T)$				$\theta_2([\overline{T},T])$	$\theta_3([\overline{T},T])$	0
$\theta_2(T)$					0	0
$\theta_3(T)$						0

The Pavlov equation: commutators of local symmetries.

4.1. The Lax pair and hierarchies. The Lax pair for the three-dimensional Pavlov equation is $q_t = (\lambda^2 - \lambda u_x - u_y)q_x$, $q_y = (\lambda - u_x)q_x$. Expanding q in integer powers of λ, we obtain the covering $q_{i,t} = q_{i-2,x} - u_x q_{i-1,x} - u_y q_{i,x}$, $q_{i,y} = q_{i-1,x} - u_x q_{i,x}$, for all $i \in \mathbb{Z}$.

The positive covering corresponding to this system is

$$q_{0,t} + u_y q_{0,x} = 0, \qquad q_{0,y} + u_x q_{0,x} = 0,$$

$$q_{1,t} + u_y q_{1,x} = -u_x q_{0,x}, \qquad q_{1,y} + u_x q_{1,x} = q_{0,x},$$

$$q_{i,t} + u_y q_{i,x} = q_{i-2,x} - u_x q_{i-1,x}, \qquad q_{i,y} + u_x q_{i,x} = q_{i-1,x}, \quad i \geq 2,$$

with the additional nonlocal variables $q_i^{(j)}$ defined by $q_i^{(0)} = q_i$ and $q_i^{(j+1)} = q_{i,x}^{(j)}$. We have $|q_i^{(j)}| = -i - 2j$. This covering is not Abelian.

The negative covering is given by

$$r_{1,y} = u_t + u_x u_y, \qquad r_{1,x} = u_y + u_x^2,$$

$$r_{i,y} = r_{i-1,t} + u_y r_{i-1,x}, \qquad r_{i,x} = r_{i-1,y} + u_x r_{i-1,x}, \quad i \geq 2,$$

with the additional nonlocal variables $r_i^{(j)}$ defined by $r_i^{(0)} = r_i$ and $r_i^{(j+1)} = r_{i,t}^{(j)}$. We have $|r_i^{(j)}| = i + 3$.

4.2. Nonlocal symmetries in the positive covering.

4.2.1. Lifts of local symmetries. All the local symmetries can be lifted to τ^+. In more detail, we have the following results. The lift of $\varphi_1 = yu_x - 2x$ is $\Phi_1 = (\varphi_1, \varphi_1^0, \dots, \varphi_1^i, \dots)$, where $\varphi_1^i = yq_{i,x} + (i+1)q_{i+1}$. The symmetry $\varphi_2 = 2xu_x + yu_y - 3u$ is lifted by $\Phi_2 = (\varphi_2, \varphi_2^0, \dots, \varphi_2^i, \dots)$, where $\varphi_2^0 = -\varphi_1 q_{0,x}$ and $\varphi_2^i = -\varphi_1 q_{i,x} + yq_{i-1,x} + iq_i$, $i \geq 1$. The lift of $\theta_2(T) = Tu_x - T'y$ is $\Theta_2(T) = (\theta_2, Tq_{0,x}, \dots, Tq_{i,x}, \dots)$. The symmetry

$$\theta_1(T) = Tu_y + T'(yu_x - x) - \frac{1}{2}T''y^2$$

admits the lift $\Theta_1(T) = (\theta_1, \theta_1^0, \theta_1^1, \dots, \theta_1^i, \dots)$, where $\theta_1^0 = -\theta_1(T)q_{0,x}$ and $\theta_1^i = -\theta_2(T)q_{i,x} + Tq_{i-1,x}$, $i \geq 1$. The lift of

$$\theta_0(T) = Tu_t + T'(xu_x + yu_y - u) + T''\left(\frac{1}{2}y^2u_x - xy\right) - \frac{1}{6}T'''y^3$$

is $\Theta_0(T) = (\theta_0, \theta_0^0, \theta_0^1, \theta_0^2, \dots, \theta_0^i, \dots)$, where $\theta_0^0 = -\theta_1(T)q_{0,x}$, $\theta_0^1 = -\theta_1(T)q_{1,x} - \theta_2(T)q_{0,x}$, and $\theta_0^i = -\theta_1(T)q_{i,x} - \theta_2(T)q_{i-1,x} + Tq_{i-2,x}$, $i \geq 2$. Finally, for $\theta_3(T) = T$, we have $\Theta_3(T) = (\theta_3, 0 \dots, 0, \dots)$.

4.2.2. Nonlocal symmetries. Three families of nonlocal symmetries exist for the Pavlov equation in τ^+. The first consists of the invisible symmetries

$$\Phi_{\mathrm{inv}}^k(Y) = (\underbrace{0, \dots, 0}_{k \text{ times}}, \varphi_{\mathrm{inv}}^{k,k+1}, \varphi_{\mathrm{inv}}^{k,k+2}, \dots, \varphi_{\mathrm{inv}}^{k,k+i}, \dots), \quad k = 1, 2, \dots,$$

where $\varphi_{\mathrm{inv}}^{k,k+i} = R_{i-1}(Q)$ for every $i \geq 1$. Here, $R_0(Q) = Q(q_0)$ is an arbitrary function of q_0, and for $n \geq 1$, we set

$$R_n(Q) = \frac{1}{n}\mathcal{Y}(R_{n-1}(Q)),$$

where $\mathcal{Y}$ is the vector field

$$\mathcal{Y} = \sum_{i=0}^{\infty}(i+1)q_{i+1}\frac{\partial}{\partial q_i}.$$

We now explicitly define the nonlocal symmetry $\Psi_{-1} = (\psi_1, \psi_1^0, \dots, \psi_1^i, \dots)$ by setting

$$\psi_{-1} = \frac{q_1}{q_{0,x}} + y, \qquad \psi_{-1}^i = -(i+2)q_{i+2} + \frac{q_1 q_{i+1,x}}{q_{0,x}}.$$

The elements of the second nonlocal family are then $\Psi_{-k} = [\Phi_1, \Psi_{-1}]$, $k \geq 2$. We have $|\Psi_{-k}| = -k-1$. Finally, we define $\Xi_l(Q) = [\Psi_{-l}, \Phi_{\mathrm{inv}}^2(Q)]$, $l \geq 1$.

The distribution of symmetries over weights is $|\Psi_l| = -l-1$, $l \geq 1$, $|\Phi_1| = -1$, $|\Phi_2| = 0$, $|\Theta_k(T)| = k-2$, $k = 0,1,2,3$, $|\Xi_j(Q)| = -j+1$, $j \geq 1$, and $|\Phi_l^{\mathrm{inv}}(Q)| = l$, $l \geq 1$.

4.2.3. Lie algebra structure. We consider the spaces W spanned by Φ_1, Φ_2, and Ψ_i, $i \leq -1$, $V[t]$ spanned by $\Theta_i(T)$, $i = 0,1,2,3$, and $V[q_0]$ spanned by $\Phi_{\mathrm{inv}}^i(Q)$ and $\Xi_j(Q)$, $i, j \geq 1$. We then have the following result.

Theorem 3. *There exist bases $\mathbf{w}_i$ in W, $i \leq 0$, $\mathbf{v}_i(T)$ in $V[t]$, $i = 0,-1,-2,-3$, and $\mathbf{v}_i(Q)$ in $V[q_0]$, $i \in \mathbb{Z}$, such that their commutators satisfy the relations presented in Table 5. In other words, $\mathrm{sym}_{\tau^+}(\mathcal{E})$ is isomorphic to $\mathfrak{W}_0^- \ltimes (\mathfrak{L}[q_0] \oplus \mathfrak{L}_4^-[t])$ with the natural action of the Witt algebra $\mathfrak{W}_0^-$ on $\mathfrak{L}[q_0] \oplus \mathfrak{L}_4^-[t]$.*

Table 5

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{T})$	$\mathbf{v}_j(\overline{Q})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\overline{T})$, $-3 \leq i+j \leq 0$ 0, otherwise	$j\mathbf{v}_{i+j}(\overline{Q})$
$\mathbf{v}_i(T)$		$(j-i)\mathbf{v}_{i+j}([T,\overline{T}])$, $-3 \leq i+j \leq 0$ 0, otherwise	0
$\mathbf{v}_i(Q)$			$\mathbf{v}_{i+j}([Q,\overline{Q}])$

The Pavlov equation: commutators in $\mathrm{sym}_{\tau^+}(\mathcal{E})$.

4.3. Nonlocal symmetries in the negative covering.

4.3.1. Lifts of local symmetries. Similarly to the case of τ^+, all the local symmetries are lifted to the covering τ^-. Namely, the symmetry $\varphi_1 = yu_x - 2x$ has the lift $\Phi_1 = (\varphi_1, \varphi_1^1, \varphi_1^2, \dots, \varphi_1^i, \dots)$, where $\varphi_1^1 = yr_{1,x} - 3u$ and $\varphi_1^i = yr_{i,x} - (i+2)r_{i-1}$, $i \geq 2$. The symmetry $\varphi_2 = 2xu_x + yu_y - 3u$ has the lift $\Phi_2 = (\varphi_2, \varphi_2^1, \varphi_2^2, \dots, \varphi_2^i, \dots)$, where $\varphi_2^i = 2xr_{i,x} + yr_{i,y} - (i+3)r_i$, $i \geq 1$.

To describe the lift $\Theta_3(T) = (\theta_3, \theta_3^1, \theta_3^2, \dots, \theta_3^i, \dots)$ of $\theta_3(T) = T$, we consider the operator

$$\mathcal{Y} = y\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial x} + 4q_1\frac{\partial}{\partial u} + \sum_{i=1}^{\infty}(i+4)q_{i+1}\frac{\partial}{\partial q_i} \tag{8}$$

and set $\theta_3^1 = yT'$ and $\theta_3^i = i^{-1}\mathcal{Y}(\varphi_6^{i-1})$, $i \geq 2$.

To describe the lifts of

$$\theta_2(T) = Tu_x - T'y, \qquad \theta_1(T) = Tu_y + T'(yu_x - x) - \frac{1}{2}T''y^2,$$

$$\theta_0(T) = Tu_t + T'(xu_x + yu_y - u) + T''\left(\frac{1}{2}y^2u_x - xy\right) - \frac{1}{6}T'''y^33,$$

we need the nonlocal symmetry Ψ_0 (see Eq. (9) below). Namely, we set

$$\Theta_2(T) = \frac{1}{3}[\Psi_0, \Theta_3(T)], \qquad \Theta_1(T) = -\frac{1}{2}[\Psi_0, \Theta_2(T)], \qquad \Theta_0(T) = -[\Psi_0, \Theta_1(T)].$$

4.3.2. Nonlocal symmetries. The invisible symmetries in τ^- have the form

$$\Phi^k_{\text{inv}}(T) = (\underbrace{0,\dots,0}_{k \text{ times}}, \varphi^{k,k+1}_{\text{inv}}, \dots, \varphi^{k,k+i}_{\text{inv}}, \dots),$$

where $\varphi^{k,k+i}_{\text{inv}} = R_{i-1}(T)$ for every $i \ge 1$ and the sequence of functions R_n, $n \ge 0$ is defined as

$$R_0(T) = T, \qquad R_{n+1}(T) = \frac{1}{n+1}\mathcal{Y}(R_n(T)),$$

where the operator $\mathcal{Y}$ is defined by Eq. (8).

We now introduce the nonlocal symmetries

$$\Psi_0 = (\psi_0, \psi_0^1, \dots, \psi_0^i, \dots), \qquad \Psi_1 = (\psi_1, \psi_1^1, \dots, \psi_1^i, \dots), \tag{9}$$

by setting $\psi_0 = 4r_1 - 3uu_x - 2xu_y - yu_t$, $\psi_1 = 5r_2 - 4u_x r_1 - yr_{1,t} - 3uu_y - 2xu_t + yu_t u_x$, and $\psi_0^i = (i+4)r_{i+1} - 3ur_{i,x} - 2xr_{i,y} - yr_{i,t}$, $\psi_1^i = (i+5)r_{i+2} - yr_{i+1,t} - 3ur_{i,y} - 2xr_{i,t} - (4r_1 - yu_t)r_{i,x}$ for $i \ge 1$.

Using the symmetries Ψ_0 and Ψ_1 and induction, we define two new families of nonlocal symmetries by $\Psi_k = [\Psi_0, \Psi_{k-1}]$, $k \ge 2$, and $\Omega_l(T) = [\Psi_l, \Theta_1(T)]$.

The weights of the obtained symmetries are $|\Phi_1| = -1$, $|\Phi_2| = 0$, $|\Psi_k| = k+1$, $k \ge 0$, $|\Omega_l(T)| = l$, $l \ge 1$, $|\Phi_l^{\text{inv}}(T)| = -l-3$, $l \ge 1$, and $|\Theta_i(T)| = -i$, $i = 0, 1, 2, 3$.

4.3.3. Lie algebra structure. We consider the subspaces W spanned by Φ_1, Φ_2, and Ψ_i, $i \ge 0$, and $V[t]$ spanned by $\Phi^i_{\text{inv}}(T)$, $\Omega_j(T)$, $i, j \ge 1$, and $\Theta_k(T)$, $k \ge 0$, in $\operatorname{sym}_{\tau^-}(\mathcal{E})$.

Theorem 4. *There exist bases* $\mathbf{w}_i$ *in* W, $i \ge -1$, *and* $\mathbf{v}_j(T)$ *in* $V[t]$, $j \in \mathbb{Z}$, *that satisfy the commutator relations in Table* 6. *In other words, the Lie algebra* $\operatorname{sym}_{\tau^-}(\mathcal{E})$ *is isomorphic to* $\mathfrak{W}^+_{-1} \ltimes \mathfrak{L}[t]$ *with the natural action of* $\mathfrak{W}^+_{-1}$ *on* $\mathfrak{L}[t]$.

Table 6

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\overline{T})$
$\mathbf{v}_i(T)$		$\mathbf{v}_{i+j}([T, \overline{T}])$

The Pavlov equation: commutators in $\operatorname{sym}_{\tau^-}(\mathcal{E})$.

4.4. Recursion operators. We have the following result (see [12]).

Proposition 2. *Equation* (6) *admits the recursion operator for symmetries defined by the system*

$$D_t(\psi) = -u_y D_x(\psi) + u_{xy}\psi + D_y(\varphi), \qquad D_y(\psi) = -u_x D_x(\psi) + u_{xx}\psi + D_x(\varphi). \tag{10}$$

The inverse operator is defined by the system

$$D_x(\varphi) = u_x D_x(\psi) + D_y(\psi) - u_{xx}\psi, \qquad D_y(\varphi) = D_t(\psi) + u_y D_x(\psi) - u_{xy}\psi. \tag{11}$$

The action of the recursion operators on shadows is shown schematically in Fig. 2, where ξ_i^+ and ω_i^- are the respective shadows of $\Xi_i(T)$ and $\Omega_i(T)$.

$$\cdots \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi^+_{-2} \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi^+_{-1} \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \varphi^\pm_1 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \varphi^\pm_2 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi^-_0 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi^-_1 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi^-_2 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \cdots$$

$$\cdots \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \xi^+_2 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \xi^+_1 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} 0^\pm \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \theta^\pm_3 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \theta^\pm_2 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \theta^\pm_1 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \theta^\pm_0 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \omega^-_1 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \omega^-_2 \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \cdots$$

Fig. 2. The Pavlov equation: the action of recursion operators (10) and (11).

5. The universal hierarchy equation

The UHE was discussed in [15], [16] and is

$$u_{yy} = u_t u_{xy} - u_y u_{tx}. \tag{12}$$

We assign the weights $|x| = 0$, $|y| = 1$, $|t| = 0$, and $|u| = -1$ to the variables x, y, t, and u.

As in Sec. 4, we consider the internal coordinates

$$u^0_{k,l} = u_{\underbrace{x\dots x}_{k \text{ times}}\underbrace{t\dots t}_{l \text{ times}}}, \qquad u^1_{k,l} = u_{y\underbrace{x\dots x}_{k \text{ times}}\underbrace{t\dots t}_{l \text{ times}}}, \qquad k,l \geq 0,$$

on $\mathcal{E}$. Consequently, $|u^0_{k,l}| = -1$ and $|u^1_{k,l}| = -2$. The total derivatives in the chosen coordinates are

$$D_x = \frac{\partial}{\partial x} + \sum_{k,l}\left(u^0_{k+1,l}\frac{\partial}{\partial u^0_{k,l}} + u^1_{k+1,l}\frac{\partial}{\partial u^1_{k,l}}\right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_{k,l}\left(u^1_{k,l}\frac{\partial}{\partial u^0_{k,l}} + D^k_x D^l_t(u^0_{01}u^1_{10} - u^1_{00}u^0_{11})\frac{\partial}{\partial u^1_{k,l}}\right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k,l}\left(u^0_{k,l+1}\frac{\partial}{\partial u^0_{k,l}} + u^1_{k,l+1}\frac{\partial}{\partial u^1_{k,l}}\right).$$

Local symmetries of $\mathcal{E}$ are solutions of the equation $\ell_\mathcal{E}(\varphi) \equiv D^2_y(\varphi) - u_t D_x D_y(\varphi) + u_y D_x D_t(\varphi) - u_{xy}D_t(\varphi) + u_{xt}D_y(\varphi) = 0$. The space $\operatorname{sym}(\mathcal{E})$ is spanned by the functions $\theta_0(X) = Xu_x - X'u$, $\theta_1(X) = X$, $\varphi_0(T) = Tu_t + T'yu_y$, $\varphi_1(T) = Tu_y$, and $v = yu_y + u$, where X is a function of x, T is a function of t, and the prime denotes the corresponding derivatives. The commutators are presented in Table 7. Weights of the evolutionary vector fields are $|\mathbf{E}_v| = |\mathbf{E}_{\theta_0(X)}| = |\mathbf{E}_{\varphi_0(T)}| = 0$, $|\mathbf{E}_{\theta_1(X)}| = 1$, and $|\mathbf{E}_{\varphi_1(T)}| = -1$.

Table 7

	v	$\theta_0(\overline{X})$	$\theta_1(\overline{X})$	$\varphi_0(\overline{T})$	$\varphi_1(\overline{T})$
v	0	0	$-\theta_1(\overline{X})$	0	$\varphi_1(\overline{T})$
$\theta_0(X)$		$\theta_0([\overline{X},X])$	$\theta_1([\overline{X},X])$	0	0
$\theta_1(X)$			0	0	0
$\varphi_0(T)$				$\varphi_0([\overline{T},T])$	$\varphi_1([\overline{T},T])$
$\varphi_1(T)$					0

The UHE: commutators of local symmetries.

5.1. The Lax pair and hierarchies. The UHE admits the Lax representation $q_t = \lambda^{-2}(\lambda u_t - u_y)q_x$, $q_y = \lambda^{-1}u_y q_x$. Expanding in powers of λ leads to the system $q_{i,t} = u_t q_{i+1,x} - u_y q_{i+2,x}$, $q_{i,y} = u_y q_{i+1,x}$. The corresponding positive covering has the form

$$q_{1,y} = \frac{u_t}{u_y}, \qquad q_{1,x} = \frac{1}{u_y},$$

$$q_{i,y} = \frac{u_t}{u_y}q_{i-1,y} - q_{i-1,t}, \qquad q_{i,x} = \frac{q_{i-1,y}}{u_y}, \quad i > 1,$$

with the additional variables satisfying the relations $q_i^{(0)} = q_i$ and $q_i^{(j+1)} = q_{i,t}^{(j)}$. We have $|q_i^{(j)}| = i+1$.

The equations defining the negative covering are

$$r_{1,y} = u_x u_y, \qquad r_{1,t} = u_x u_t - u_y,$$

$$r_{i,y} = u_y r_{i-1,x}, \qquad r_{i,t} = u_t r_{i-1,x} - r_{i-1,y}, \quad i > 1,$$

with $r_i^{(j)}$ defined by $r_i^{(j+1)} = r_{i,x}^{(j)}$. The weights are $|r_i^{(j)}| = -i-1$.

5.2. Nonlocal symmetries in the positive covering.

5.2.1. Lifts of local symmetries. The local symmetries of the UHE are lifted as follows. The symmetry $\upsilon = yu_y + u$ is lifted to $\Upsilon = (\upsilon, \upsilon^1, \upsilon^2, \dots, \upsilon^i, \dots)$, where $\upsilon^i = -(i+1)q_i + yq_{i,y}$. The lift $\Theta_0(X) = (\theta_0, \theta_0^1, \theta_0^2, \dots, \theta_0^i, \dots)$ of $\theta_0(X) = Xu_x - X'u$ is defined by $\theta_0^i = (X/u_y)q_{i-1,y}$.

We now introduce the operator

$$\mathcal{Y} = -y\frac{\partial}{\partial t} + 2q_1\frac{\partial}{\partial y} + \sum_{k=1}^{\infty}(k+2)q_{k+1}\frac{\partial}{\partial q_k}$$

and define $R_i(T)$ by induction as

$$R_1(T) = -T'y, \qquad R_i(T) = \frac{1}{i}\mathcal{Y}(R_{i-1}(T)), \quad i \geq 2. \tag{13}$$

The lift of $\varphi_0(T) = Tu_t + T'yu_y$ is then $\Phi_0(T) = (\varphi_0, \varphi_0^1, \varphi_0^2, \dots, \varphi_0^i, \dots)$, where $\varphi_0^i = Tq_{i,t} + T'yq_{i,y} + R_{i+1}(T)$, and the symmetry $\varphi_1(T) = Tu_y$ is lifted by $\Phi_1(T) = (\varphi_1, \varphi_1^1, \varphi_1^2, \dots, \varphi_1^i, \dots)$ with $\varphi_1^i = Tq_{i,y} - R_i(T)$. Finally, the lift of $\theta_1(X)$ is $\Theta_1(X) = (\theta_1, 0, \dots, 0, \dots)$.

5.2.2. Nonlocal symmetries. There exists a family of invisible symmetries

$$\Phi_{\text{inv}}^i(T) = (\underbrace{0, \dots, 0}_{i \text{ times}}, \varphi_{\text{inv}}^1, \dots, \varphi_{\text{inv}}^k, \dots),$$

where $\varphi_{\text{inv}}^1 = T$ and $\varphi_{\text{inv}}^k = R_{i-1}(T)$, $i > 1$, $R_{i-1}(T)$ is given by Eq. (13).

The UHE also admits another two families of nonlocal symmetries in τ^+ defined as follows. We set $\Psi_0 = (\psi_0, \psi_0^1, \psi_0^2, \dots, \psi_0^i, \dots)$, where $\psi_0 = 2q_1u_y - yu_t$ and $\psi_0^i = -(i+2)q_{i+1} - yq_{i,t} + 2q_1q_{i,y}$. We also introduce $\Psi_1 = (\psi_1, \psi_1^1, \psi_1^2, \dots, \psi_1^i, \dots)$ with $\psi_1 = -3q_2u_y + 2q_1u_t - yu_yq_{1,t}$ and $\psi_1^i = (i+3)q_{i+2} + yq_{i+1,t} + 2q_1q_{i,t} - (3q_2 + yq_{1,t})q_{i,y}$. We then set $\Psi_k = [\Psi_0, \Psi_{k-1}]$, $k \geq 2$, and $\Xi_l(T) = [\Psi_l, \Phi_1(T)]$, $l \geq 1$. The distribution of the constructed symmetries over weights is given by $|\Upsilon| = |\Theta_0(X)| = |\Phi_0(T)| = 0$, $|\Theta_1(X)| = 1$, $|\Phi_1(T)| = -1$, $|\Psi_k| = k+1$, $k \geq 0$, $|\Phi_l^{\text{inv}}(T)| = -l-1$, and $|\Xi_l(T)| = l$, $l \geq 1$.

5.2.3. Lie algebra structure. We consider the following subspaces in $\mathrm{sym}_{\tau^+}(\mathcal{E})$: W spanned by Υ and Ψ_i, $i \geq 0$; $V[x]$ spanned by $\Theta_0(X)$ and $\Theta_1(X)$; $V[t]$ spanned by $\Phi_0(T)$, $\Phi_1(T)$, $\Phi^i_{\mathrm{inv}}(T)$, and $\Xi_j(T)$, $i, j \geq 1$. We then have the following theorem.

Theorem 5. *There exist bases* $\mathbf{w}_i$, $i \geq 0$, *in* W, $\mathbf{v}_0(X)$, $\mathbf{v}_1(X)$, *in* $V[x]$, *and* $\mathbf{v}_i(T)$, $i \in \mathbb{Z}$, *in* $V[t]$, *such that their commutators satisfy the relations in Table 8. Hence,* $\mathrm{sym}_{\tau^+}(\mathcal{E})$ *is isomorphic to* $\mathfrak{W}_0^+ \ltimes (\mathfrak{L}_2^+[x] \oplus \mathfrak{L}[t])$ *with the natural action of* $\mathfrak{W}_0^+$ *on* $\mathfrak{L}_2^+[x]$ *and* $\mathfrak{L}[t]$.

Table 8

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{X})$	$\mathbf{v}_j(\overline{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\overline{X})$, $0 \leq i+j \leq 1$, 0, otherwise	$j\mathbf{v}_{i+j}(\overline{T})$
$\mathbf{v}_i(X)$		$\mathbf{v}_{i+j}([X,\overline{X}])$, $0 \leq i+j \leq 1$, 0, otherwise	0
$\mathbf{v}_i(T)$			$\mathbf{v}_{i+j}([T,\overline{T}])$

The UHE: commutators in $\mathrm{sym}_{\tau^+}(\mathcal{E})$.

5.3. Nonlocal symmetries in the negative covering.

5.3.1. Lifts of local symmetries. The symmetry $\upsilon = yu_y + u$ is lifted to $\Upsilon = (\upsilon, \upsilon^1, \upsilon^2, \ldots, \upsilon^i, \ldots)$, where $\upsilon^i = (i+1)r_i + yu_y r_{i-1,x}$ and r_0 denotes u. The lift of $\theta_0(X) = Xu_x - X'u$ is $\Theta_0(X) = (\theta_0(X), \theta_0^1, \theta_0^2, \ldots, \theta_0^i, \ldots)$ with $\theta_0^i = Xr_{i,x} - R_{i+1}(X)$ and R_{i+1} is given by Eq. (13). For $\varphi_0(T) = Tu_t + T'yu_y$, we have $\Phi_0(T) = (\varphi_0(T), \varphi_0^1, \varphi_0^2, \ldots, \varphi_0^i, \ldots)$, where $\varphi_0^i = Tr_{i,t} + T'yu_y r_{i-1,x}$. The symmetry $\varphi_1(T) = Tu_y$ is lifted to $\Phi_1(T) = (\varphi_1(T), \varphi_1^1, \varphi_1^2, \ldots, \varphi_1^i, \ldots)$, where $\varphi_1^i = Tr_{i,y}$. Finally, for $\theta_1(X)$, we have $\Theta_1(X) = (\theta_1(X), R_1(X), \ldots, R_i(X), \ldots)$, where R_i is again given by (13).

5.3.2. Nonlocal symmetries. In the τ^- covering of the UHE, there exists a family of invisible symmetries of the form

$$\Phi^k_{\mathrm{inv}}(X) = (\underbrace{0, \ldots, 0}_{k \text{ times}}, \varphi^1_{\mathrm{inv}}, \ldots, \varphi^i_{\mathrm{inv}}, \ldots),$$

where $\varphi^1_{\mathrm{inv}} = X$ and $\varphi^i_{\mathrm{inv}} = R_{i-1}(X)$ (see Eq. (13) for the definition of R_i).

We now consider two nonlocal symmetries $\Psi_j = (\psi_j, \psi_j^1, \psi_j^2, \ldots, \psi_j^i, \ldots)$, $j = -1, -2$, defined by $\psi_{-1} = 2r_1 - uu_x$, $\psi^i_{-1} = (i+2)r_{i+1} - ur_{i,x}$ and $\psi_{-2} = 3r_2 - 2r_1u_x - ur_{1,x} + uu_x^2$, $\psi^i_{-2} = (i+3)r_{i+2} - ur_{i+1,x} + (uu_x - 2r_1)r_{i,x}$. We now introduce two families of nonlocal symmetries by setting $\Psi_{-k} = [\Psi_{-1}, \Psi_{-k+1}]$, $k \leq -3$, and $\Omega_l(X) = [\Psi_l, \Phi_5(X)]$, $l \leq -1$.

The τ^--nonlocal symmetries are distributed along weights as $|\Upsilon| = |\Theta_0(X)| = |\Phi_0(T)| = 0$, $|\Phi_1(T)| = -1$, $|\Theta_1(X)| = 1$, $|\Psi_k| = -k$, $k \leq -1$, $|\Phi_i^{\mathrm{inv}}(X)| = i+1$, $i \geq 1$, and $|\Omega_j(X)| = j$, $j \leq -1$.

5.3.3. Lie algebra structure. We consider the following subspaces in $\mathrm{sym}_{\tau^-}(\mathcal{E})$: W spanned by Υ and Ψ_k, $k \leq -1$; $V[x]$ spanned by $\Omega_l(X)$, $l \geq 1$, $\Theta_0(X)$, $\Theta_1(X)$, and $\Phi^k_{\mathrm{inv}}(X)$, $k \geq 1$; and $V[t]$ spanned by $\Phi_0(T)$, $\Phi_1(T)$. We then have the following result.

Theorem 6. *There exist bases* $\mathbf{w}_i$, $i \leq 0$, *in* W, $\mathbf{v}_i(X)$, $i \in \mathbb{Z}$, *in* $V[x]$, $\mathbf{v}_i(T)$, $i = 0, -1$, *in* $V[t]$ *such that their commutators satisfy the relations in Table 9. Hence,* $\mathrm{sym}_{\tau^-}(\mathcal{E})$ *is isomorphic to* $\mathfrak{W}_0^- \ltimes (\mathfrak{L}_2^-[t] \oplus \mathfrak{L}[x])$ *with the natural action of* $\mathfrak{W}_0^-$ *on* $\mathfrak{L}_2^-[t] \oplus \mathfrak{L}[x]$.

$$\cdots \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi_2^+ \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi_1^+ \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi_0^+ \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \upsilon^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi_{-1}^- \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi_{-2}^- \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \psi_{-3}^- \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \cdots$$

$$\cdots \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \xi_2^+ \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \xi_1^+ \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \varphi_0^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \varphi_1^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} 0^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \theta_1^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \theta_0^\pm \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \omega_1^- \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \omega_2^- \underset{\mathcal{R}_+}{\overset{\mathcal{R}_-}{\rightleftarrows}} \cdots$$

Fig. 3. The UHE: action of recursion operators (14) and (15).

Table 9

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{X})$	$\mathbf{v}_j(\overline{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\overline{X})$	$j\mathbf{v}_{i+j}(\overline{T}),\quad -1 \le i+j \le 0,$ $0,\quad$ otherwise
$\mathbf{v}_i(X)$		$\mathbf{v}_{i+j}([X,\overline{X}])$	0
$\mathbf{v}_i(T)$			$\mathbf{v}_{i+j}([T,\overline{T}])\quad -1 \le i+j \le 0,$ $0,\quad$ otherwise

The UHE: commutators in $\mathrm{sym}_{\tau^-}(\mathcal{E})$.

5.4. Recursion operators. The following proposition describes recursion operators for the symmetries of the UHE (see [17]).

Proposition 3. *Equation* (12) *admits the recursion operator* $\psi = \mathcal{R}_+(\varphi)$ *for symmetries defined by the system*

$$\begin{aligned} D_x(\psi) &= u_y^{-1}\big(-D_y(\varphi) + u_{xy}\psi\big), \\ D_y(\psi) &= D_t(\varphi) - u_y^{-1}\big(u_t\, D_y(\varphi) + (u_y\, u_{tx} - u_t\, u_{xy})\psi\big). \end{aligned} \tag{14}$$

The inverse operator $\varphi = \mathcal{R}_-(\psi)$ *is defined by the system*

$$D_t(\varphi) = D_y(\psi) - u_t D_x(\psi) + u_{tx}\psi, \qquad D_y(\varphi) = -u_y\, D_x(\psi) + u_{xy}\psi. \tag{15}$$

The action of the recursion operators on local symmetries and shadows is shown schematically in Fig. 3.

6. The modified Veronese web equation

The mVWE was studied in [18] and is related to the Veronese web equation, [19], [20] by a Bäcklund transformation (see below). The mVWE has the form

$$u_{ty} = u_t u_{xy} - u_y u_{tx}. \tag{16}$$

We assign zero weights to all the considered variables. Internal coordinates are chosen similarly to the preceding cases, i.e.,

$$u_k = u_{\underbrace{x\dots x}_{k \text{ times}}}, \qquad u_{k,l}^t = u_{\underbrace{x\dots x}_{k \text{ times}}\underbrace{t\dots t}_{l \text{ times}}}, \qquad u_{k,l}^y = u_{\underbrace{x\dots x}_{k \text{ times}}\underbrace{y\dots y}_{l \text{ times}}},$$

where $k \geq 0$ and $l > 0$. The total derivatives are then

$$D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k,l}\left(u^y_{k+1,l}\frac{\partial}{\partial u^y_{k,l}} + u^t_{k+1,l}\frac{\partial}{\partial u^t_{k,l}}\right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_k u^y_{k,1}\frac{\partial}{\partial u_k} + \sum_{k,l}\left(u^y_{k,l+1}\frac{\partial}{\partial u^y_{k,l}} + D^k_x D^{l-1}_t(u^t_{01}u^y_{11} - u^y_{01}u^t_{11})\frac{\partial}{\partial u^t_{k,l}}\right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_r u^t_{k,1}\frac{\partial}{\partial u_k} + \sum_{k,l}\left(D^k_x D^{l-1}_y(u^t_{01}u^y_{11} - u^y_{01}u^t_{11})\frac{\partial}{\partial u^y_{k,l}} + u^t_{k,l+1}\frac{\partial}{\partial u^t_{k,l}}\right).$$

Symmetries are defined by the equation

$$D_tD_y(\varphi) - u_tD_xD_y(\varphi) + u_yD_tD_x(\varphi) - u_{xy}D_t(\varphi) + u_{tx}D_y(\varphi) = 0. \tag{17}$$

The space of solutions is generated by the functions $\varphi(T) = Tu_t$, $\upsilon(Y) = Yu_y$, $\theta_0(X) = Xu_x - X'u$, and $\theta_1(X) = X$, where $X = X(x)$, $Y = Y(y)$, and $T = T(t)$ are arbitrary functions of their arguments. The commutators of the symmetries are presented in Table 10.

Table 10

	$\varphi(\overline{T})$	$\theta_0(\overline{X})$	$\theta_1(\overline{X})$	$\upsilon(\overline{Y})$
$\varphi(T)$	$\varphi([\overline{T},T])$	0	0	0
$\theta_0(X)$		$\theta_0([\overline{X},X])$	$\theta_1([\overline{X},X])$	0
$\theta_1(X)$			0	0
$\upsilon(Y)$				$\upsilon([\overline{Y},Y])$

The mVWE: commutators of local symmetries.

6.1. The Lax pair and hierarchies. The mVWE admits the Lax pair

$$q_t = (\lambda+1)^{-1}u_tq_x, \qquad q_y = \lambda^{-1}u_yq_x. \tag{18}$$

Expanding in powers of λ, we obtain $q_{i-1,t} + q_{i,t} = u_tq_{i,x}$ and $q_{i-1,y} = u_yq_{i,x}$. The positive covering then becomes

$$q_{1,t} = \frac{u_t}{u_y}, \qquad q_{1,x} = \frac{1}{u_y},$$

$$q_{i,x} = \frac{q_{i-1,y}}{u_y}, \qquad q_{i,t} = \frac{u_t}{u_y}q_{i-1,y} - q_{i-1,t}, \quad i > 1,$$

with the additional variables defined as usual: $q_i^{(0)} = q_i$ and $q_i^{(j+1)} = q_{i,y}^{(j)}$ with $|q_i^{(j)}| = 0$.

The defining equations for the negative covering are

$$r_{1,t} = u_t(u_x - 1), \qquad r_{1,y} = u_xu_y,$$

$$r_{i,t} = u_tr_{i-1,x} - r_{i-1,t}, \qquad r_{i,y} = u_yr_{i-1,x}, \quad i > 1.$$

The auxiliary variables are $r_i^{(j)}$, defined by $r_i^{(0)} = r_i$ and $r_i^{(j+1)} = r_{i,y}^{(j)}$. Similarly to the positive case, their weights are trivial.

6.2. Nonlocal symmetries in the positive covering.

6.2.1. Lifts of local symmetries. All the local symmetries can be lifted to the τ^+ covering. Namely, the lift of $\varphi_1(T) = Tu_t$ is $\Phi(T) = \big(\varphi(T), \varphi^1, \dots, \varphi^i, \dots\big)$, where $\varphi^i = Tq_{i,t}$. The lift of $\theta_0(X) = Xu_x - X'u$ is given by $\Theta_0(X) = \big(\theta_0(X), \theta_0^1, \dots, \theta_0^i, \dots\big)$, where $\theta_0^i = Xq_{i,x}$. To lift the symmetry $v(Y) = Yu_y$, we consider the operator

$$\mathcal{Y} = q_1\frac{\partial}{\partial y} + \sum_{k=1}^{\infty} q_{k+1}\frac{\partial}{\partial q_k}$$

and recursively set

$$R_1(Y) = Y'q_1, \qquad R_n(Y) = \frac{1}{n}\mathcal{Y}(R_{n-1}). \tag{19}$$

Then $\Upsilon(Y) = \big(v(Y), v^1, \dots, v^i, \dots\big)$, where $v^i = Yq_{i,y} - R_i(Y)$. Finally, we have $\Theta_1 = (\theta_1(X), 0, \dots, 0, \dots)$ for the lift of $\theta_1(X) = X$.

6.2.2. Nonlocal symmetries. There exist three families of "purely nonlocal" symmetries in τ^+. The first consists of the invisible symmetries of the form

$$\Phi^k_{\text{inv}}(Y) = \big(\underbrace{0, \dots, 0}_{k \text{ times}}, \varphi^1_{\text{inv}}, \dots, \varphi^i_{\text{inv}}, \dots\big),$$

where $\varphi^1_{\text{inv}} = Y$ and $\varphi^i_{\text{inv}} = R_{i-1}(Y)$, $i > 1$, and $R_l(Y)$ is given by (19).

The second family is constructed as follows. The symmetries Ψ_0 and Ψ_1 are defined by $\Psi_0 = (\psi_0^0, \psi_0^1, \dots, \psi_0^i, \dots)$, where $\psi_0^0 = q_1u_y + u$ and $\psi_0^i = -(i+1)q_{i+1} - iq_i + q_1q_{1,y}$, $i > 0$, and $\Psi_1 = (\psi_1^0, \psi_1^1, \dots, \psi_1^i, \dots)$, where $\psi_1^0 = (-2q_2 - q_1 + q_1q_{1,y})u_y$ and $\psi_1^i = (i+2)q_{i+2} + (i+1)q_{i+1} - q_1q_{i+1,y} + (-2q_2 - q_1 + q_1q_{1,y})q_{i,y}$, $i > 0$. We also set $\Psi_k = [\Psi_0, \Psi_{k-1}] + k\Psi_{k-1}$, $k > 1$, by induction.

The third family consists of the symmetries $\Xi_k(Y) = [\Psi_k, \Phi^1_{\text{inv}}(Y)] - (k-1)!\Upsilon(Y)$, $k = 0, 1, \dots$.

6.2.3. Lie algebra structure. We consider the following subspaces in $\text{sym}_{\tau^+}(\mathcal{E})$: $V[x]$ spanned by $\Theta_0(X)$ and $\Theta_1(X)$; $V[t]$ spanned by $\Phi(T)$; $V[y]$ spanned by $\Xi_k(Y)$, $\Upsilon(Y)$, and $\Phi^l_{\text{inv}}(Y)$; and W spanned by Ψ_k. We have the following result.

Theorem 7. *There exist bases* $\mathbf{w}_i$, $i \geq 1$, *in* W, $\mathbf{v}_i(X)$, $i = 0, 1$, *in* $V[x]$, $\mathbf{v}_i(Y)$, $i \in \mathbb{Z}$, *in* $V[y]$, *and* $\mathbf{v}(T)$ *in* $V[t]$ *such that their commutators satisfy the relations in Table* 11. *In other words,* $\text{sym}_{\tau^+}(\mathcal{E})$ *is isomorphic to* $\widetilde{\mathfrak{W}}_0^+ \ltimes (\mathfrak{L}[y] \oplus \mathfrak{L}_2^+[x]) \oplus \mathfrak{V}[t]$ *with the natural action of the Witt algebra* $\mathfrak{W}_0^+$ *on* $\mathfrak{L}[y] \oplus \mathfrak{L}_2^+[x]$. *Here,* $\widetilde{\mathfrak{W}}_0^+$ *denotes the subalgebra in* $\mathfrak{W}_0^+$ *generated by* $\mathbf{e}_i - \mathbf{e}_0$, $i \geq 1$.

Table 11

	$\mathbf{w}_j$	$\mathbf{v}_0(\overline{X})$	$\mathbf{v}_1(\overline{X})$	$\mathbf{v}_j(\overline{Y})$	$\mathbf{v}(\overline{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}+$ $i\mathbf{w}_i - j\mathbf{w}_j$	0	$-\mathbf{v}_1(\overline{X})$	$j(\mathbf{v}_{i+j}(\overline{Y}) - \mathbf{v}_j(\overline{Y}))$	0
$\mathbf{v}_0(X)$		$\mathbf{v}_0([X,\overline{X}])$	$\mathbf{v}_1([X,\overline{X}])$	0	0
$\mathbf{v}_1(X)$			0	0	0
$\mathbf{v}_i(Y)$				$\mathbf{v}_{i+j}([Y,\overline{Y}])$	
$\mathbf{v}(T)$					$\mathbf{v}([T,\overline{T}])$

The mVWE: commutators in $\text{sym}_{\tau^+}(\mathcal{E})$.

Remark 1. The Lie algebra $\widetilde{\mathfrak{W}}_0^+$ is an example of a two-point Krichever–Novikov type algebra [21]. In the basis $\widetilde{\mathbf{e}}_i = \mathbf{e}_i - \mathbf{e}_{i-1} = z^i(z-1)\,\partial/\partial z$, $i \geq 1$, it has an almost-graded structure $[\widetilde{\mathbf{e}}_i, \widetilde{\mathbf{e}}_j] = (j-i)(\widetilde{\mathbf{e}}_{i+j} - \widetilde{\mathbf{e}}_{i+j-1})$.

6.3. Nonlocal symmetries in the negative covering.

6.3.1. Lifts of local symmetries. The symmetry $\varphi(T) = Tu_t$ is lifted to $\Phi(T) = (\varphi(T), \varphi^1, \dots, \varphi^i, \dots)$, where $\varphi^i = Tr_{i,t}$. To define the lift of $\theta_0(X) = Xu_x - X'u$, we consider the operator

$$\mathcal{Y} = u\frac{\partial}{\partial x} + 2r_1\frac{\partial}{\partial u} + \sum_{k=1}^{\infty}(k+2)r_{k+1}\frac{\partial}{\partial r_k}$$

and define the quantities $R_n(X)$ by induction, setting

$$R_1(X) = X'u, \qquad R_n(X) = \frac{1}{n}\mathcal{Y}(R_{n-1}). \tag{20}$$

Then $\Theta_0(X) = (\theta_0(X), \theta_0^1, \dots, \theta_0^i, \dots)$, where $\theta_0^i = Xr_{i,x} - R_{i+1}(X)$ and R_n is given by (20). For $v(Y) = Yu_y$, we have $\Upsilon(Y) = (v, v^1, \dots, v^i, \dots)$ with $v^i = Yr_{i,y}$. The symmetry $\theta_1(X) = X$ is lifted to $\Theta_1(X) = (\theta_1, \theta_1^1, \dots, \theta_1^i, \dots)$, where $\theta_1^i = R_i(X)$.

6.3.2. Nonlocal symmetries. Similarly to the positive case, three families of nonlocal symmetries arise in $\tau^-(\mathcal{E})$. The first consists of the invisible symmetries

$$\Phi_{\text{inv}}^k(X) = (\underbrace{0, \dots, 0}_{k \text{ times}}, \varphi_{\text{inv}}^1, \dots, \varphi_{\text{inv}}^i, \dots),$$

where $\varphi_{\text{inv}}^1 = X$ and $\varphi_{\text{inv}}^i = R_{i-1}(X)$, $i \geq 2$. In addition, two nonlocal symmetries,

$$\Psi_{-1} = (\psi_{-1}, \psi_{-1}^1, \dots, \psi_{-1}^i, \dots), \qquad \Psi_{-2} = (\psi_{-2}, \psi_{-2}^1, \dots, \psi_{-2}^i, \dots),$$

are constructed explicitly. Namely, we set $\psi_{-1} = 2r_1 - uu_x + u$, $\psi_{-1}^i = (i+2)r_{i+1} + (i+1)r_i - ur_{i,x}$ and $\psi_{-2} = 3r_2 - 2r_1u_x - ur_{1,x} + uu_x^2 - u$, $\psi_{-2}^i = (i+3)r_{i+2} - (i+1)r_i - ur_{i+1,x} + (uu_x - 2r_1)r_{i,x}$. The second family is then defined by $\Psi_{-k-1} = [\Psi_{-1}, \Psi_{-k}] - k\Psi_{-k} + (-1)^{k+1}(k-3)!\Psi_{-1}$, $k > 1$, and the third family is $\Omega_{-l}(X) = [\Psi_{-l}, \Phi_4(X)] + (-1)^{l+1}(l-2)!\Theta_1(X)$, $l \geq 0$.

6.3.3. Lie algebra structure. Let W spanned by Ψ_k, $V[x]$ spanned by $\Omega_l(X)$, $\Theta_0(X)$, $\Theta_1(X)$, and $\Phi_{\text{inv}}^i(X)$, $V[t]$ spanned by $\Phi(T)$, and $V[y]$ spanned by $\Upsilon(Y)$ be subspaces in $\operatorname{sym}_{\tau^-}(\mathcal{E})$.

Theorem 8. *There exist bases $\mathbf{w}_i$, $i \leq -1$, in W, $\mathbf{v}_i(X)$, $i \in \mathbb{Z}$, in $V[x]$, $\mathbf{v}(T)$ in $V[t]$, and $\mathbf{v}(Y)$ in $V[y]$ such that their commutators satisfy the relations in Table 12. Hence, $\operatorname{sym}_{\tau^-}(\mathcal{E})$ is isomorphic to $\widetilde{\mathfrak{W}}_0^- \ltimes \mathfrak{L}[x] \oplus \mathfrak{V}[y] \oplus \mathfrak{V}[t]$ with the natural action of the Witt algebra $\mathfrak{W}$ on $\mathfrak{L}[x]$. Here, $\widetilde{\mathfrak{W}}_0^-$ denotes the subalgebra in $\mathfrak{W}_0^-$ generated by the elements $\mathbf{e}_i - \mathbf{e}_0$, $i \leq -1$.*

Table 12

	$\mathbf{w}_j$	$\mathbf{v}_j(\overline{X})$	$\mathbf{v}(\overline{Y})$	$\mathbf{v}(\overline{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j} + i\mathbf{w}_i - j\mathbf{w}_j$	$j(\mathbf{v}_{i+j}(\overline{X}) - \mathbf{v}_j(\overline{X}))$	0	0
$\mathbf{v}_i(X)$		$\mathbf{v}_{i+j}([X, \overline{X}])$	0	0
$\mathbf{v}(Y)$			$\mathbf{v}([Y, \overline{Y}])$	0
$\mathbf{v}(T)$				$\mathbf{v}([T, \overline{T}])$

The mVWE: commutators in $\operatorname{sym}_{\tau^-}(\mathcal{E})$.

Remark 2. The Lie algebra $\widetilde{\mathfrak{W}}_0^-$ is obviously isomorphic to $\widetilde{\mathfrak{W}}_0^+$. The isomorphism $\mathbf{e}_{-k} - \mathbf{e}_0 \mapsto -(\mathbf{e}_k - \mathbf{e}_0)$, $k \geq 1$, is given by the change of variable $z \mapsto z^{-1}$.

6.4. Recursion operators. To construct a recursion operator for Eq. (16), we use the techniques in [22] (also cf. [23]–[26]). We find a shadow for Eq. (16) in covering (18). It has the form $s = H(q)q_x^{-1}$, where H is an arbitrary function in q. Because system (18) is invariant under the transformation $q \mapsto H(q)$, we set $s = q_x^{-1}$ without loss of generality. Differentiating (18) with respect to x and substituting $q_x = s^{-1}$, we obtain another covering

$$s_t = (\lambda + 1)^{-1}(u_t s_x - u_{tx} s), \qquad s_y = \lambda^{-1}(u_y s_x - u_{xy} s) \tag{21}$$

for Eq. (16). We note that s is a solution of linearization (17) of Eq. (16). We now set

$$s = \sum_{n=-\infty}^{\infty} s_n \lambda^n. \tag{22}$$

Because (17) is independent of λ, each s_n is a solution of (17). Substituting (22) in (21) yields $s_{n-1,t} + s_{n,t} = u_t s_{n,x} - u_{tx} s_n$, $s_{n-1,y} = u_y$, and $s_{n,x} - u_{xy} s_n$. Setting $s_{n-1} = \varphi$ and $s_n = \psi$, we obtain the following proposition.

Proposition 4. *The system*

$$\begin{aligned} D_t(\psi) &= -D_t(\varphi) + u_y^{-1}\big(u_t\, D_y(\varphi) + (u_t u_{xy} - u_y u_{tx})\psi\big), \\ D_x(\psi) &= u_y^{-1}\big(D_y(\varphi) + u_y \psi\big) \end{aligned} \tag{23}$$

defines a recursion operator $\psi = \mathcal{R}_+(\varphi)$ for symmetries of Eq. (16). The inverse operator $\varphi = \mathcal{R}_-(\psi)$ is given by the system

$$D_t(\varphi) = -D_t(\psi) + u_t D_x(\psi) - u_{tx}\psi, \qquad D_y(\varphi) = u_y D_x(\psi) - u_{xy}\psi. \tag{24}$$

The action of the recursion operators $\mathcal{R}_+$ and $\mathcal{R}_-$ on the shadows of nonlocal symmetries is more complicated than in Secs. 3–5. It is described by the following proposition.

Proposition 5. *The action of (23) and (24) on the shadows $\psi_i^\pm$, ξ_i^+, and ω_i^- has the forms*

$$\mathcal{R}_+(\psi_i^+) = \sum_{j=1}^{i+1} \alpha_{ij}\psi_j^+, \qquad \alpha_{i\,i+1} \neq 0, \quad i \geq 0, \tag{25}$$

$$\mathcal{R}_+(\xi_i^+) = \sum_{j=1}^{i+1} \beta_{ij}\xi_j^+, \qquad \beta_{ii+1} \neq 0, \quad i \geq 0, \tag{26}$$

$$\mathcal{R}_-(\psi_{-k}^-) = \sum_{j=1}^{k+1} \gamma_{kj}\psi_{-j}^-, \qquad \gamma_{kk+1} \neq 0, \quad k \geq 1, \tag{27}$$

$$\mathcal{R}_-(\omega_i^-) = \sum_{j=0}^{i+1} \delta_{ij}\omega_j^- + \varepsilon_i\theta_0^-, \qquad \delta_{ii+1} \neq 0, \quad i \geq 0, \tag{28}$$

where α_{ij}, β_{ij}, γ_{kj}, and ε_i are constants. To find the action of $\mathcal{R}_-$ on ψ_i^+ and ξ_i^+, we must apply $\mathcal{R}_-$ to both sides of (25) and (26) and then solve the obtained triangular systems. The action of $\mathcal{R}_+$ on ψ_{-i}^- and ω_i^- can be found in the same way.

These results are shown schematically in Fig. 4, where the wavy arrows indicate actions (25)–(28).

$$\cdots \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi_2^+ \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi_1^+ \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi_0^+ \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi_{-1}^- \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi_{-2}^- \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \psi_{-3}^- \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \cdots$$

$$\cdots \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \xi_1^+ \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \xi_0^+ \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} v^\pm \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} 0^\pm \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \theta_1^\pm \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \theta_0^\pm \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \omega_0^- \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \omega_1^- \overset{\mathcal{R}_-}{\underset{\mathcal{R}_+}{\rightleftarrows}} \cdots$$

Fig. 4. The mVWE: action of recursion operators (23) and (24).

6.5. Bäcklund autotransformation. We again consider the first and the second equations in the positive covering of Eq. (16) (see Sec. 6.1) and replace q_1 with v in them:

$$v_t = \frac{u_t}{u_y}, \qquad v_x = \frac{1}{u_y}. \tag{29}$$

This gives the expressions for u_t and u_y

$$u_t = \frac{v_t}{v_x}, \qquad u_y = \frac{1}{v_x}. \tag{30}$$

Cross-differentiation of this system with respect to y and t gives $v_{tx} = v_t v_{xy} - v_x v_{ty}$. This equation differs from Eq. (16) just by the change of variables

$$x \mapsto y, \qquad y \mapsto x. \tag{31}$$

We thus obtain the following proposition.

Proposition 6. *The superposition of* (29) *and* (31) *gives a Bäcklund autotransformation for Eq.* (16). *The inverse transformation is given by the superposition of* (31) *and* (30).

7. Conclusions

The equations discussed above have many common features:

1. They all admit differential coverings with a nonremovable parameter.
2. They are all linearly degenerate.
3. Each of these equations can be obtained as a symmetry reduction of the five-dimensional equation $u_{zs} + u_{yz} - u_{ts} + u_z u_{xs} - u_s u_{xz} = 0$ (see [27]).
4. As shown in [28], they are pairwise related by Bäcklund transformations.

This similarity is manifested in a striking resemblance of their symmetry algebra structures (see Table 13). Perhaps, the mVWE equation is somewhat unique: its symmetries are not graded in the same sense as the symmetries of the other three equations.

Table 13

	τ^+	τ^-
rdDym equation	$\mathfrak{W}_0^- \ltimes (\mathfrak{L}_3^-[t] \oplus \mathfrak{L}[y])$	$\mathfrak{W}_0^+ \ltimes \mathfrak{L}[t] \oplus \mathfrak{V}[y]$
3D Pavlov equation	$\mathfrak{W}_0^- \ltimes (\mathfrak{L}[q_0] \oplus \mathfrak{L}_4^-[t])$	$\mathfrak{W}_{-1}^+ \ltimes \mathfrak{L}[t]$
UHE	$\mathfrak{W}_0^+ \ltimes (\mathfrak{L}_2^+[x] \oplus \mathfrak{L}[t])$	$\mathfrak{W}_0^- \ltimes (\mathfrak{L}_2^-[t] \oplus \mathfrak{L}[x])$
mVWE	$\widetilde{\mathfrak{W}}_0^+ \ltimes (\mathfrak{L}[y] \oplus \mathfrak{L}_2^+[x]) \oplus \mathfrak{V}[t]$	$\widetilde{\mathfrak{W}}_0^- \ltimes \mathfrak{L}[x] \oplus \mathfrak{V}[y] \oplus \mathfrak{V}[t]$

Lie algebras of nonlocal symmetries.

We think that it would be extremely interesting to learn which properties of these equations, in addition to their linear degeneracy, are responsible for such symmetry structures, and we plan to shed light on this problem in future research. We also intend to clarify the invariant meaning of the operators $\mathcal{Y}$ that play such an important role in the constructions discussed above.

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COVERINGS OVER LAX INTEGRABLE EQUATIONS AND THEIR NONLOCAL SYMMETRIES

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We consider the three-dimensional rdDym equation $u_{ty} = u_x u_{xy} - u_y u_{xx}$. Using the known Lax representation with a nonremovable parameter and two hierarchies of nonlocal conservation laws associated with it, we describe the algebras of nonlocal symmetries in the corresponding coverings.

Keywords: partial differential equation, three-dimensional rdDym equation, nonlocal symmetry, recursion operator

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1. Introduction

The three-dimensional rdDym (the rth dispersionless Dym) equation $\mathscr{E}$ (see [1]–[3]) is an example of nonlinear integrable equations in three independent variables. Integrability here means the existence of a Lax pair with a nonremovable parameter. Such equations have been studied quite intensively (see, e.g., [4], [5]). In particular, in [6], [7], we recently fully described two-dimensional reductions of such equations and studied the integrability of the equations obtained as a result of the reductions.

Using the known Lax pair with a nonremovable parameter for the three-dimensional rdDym equation, we construct two infinite hierarchies of two-component nonlocal conservation laws corresponding to nonnegative and nonpositive powers of the spectral parameter. Two coverings correspond to these hierarchies; these coverings are Abelian and infinite-dimensional in the sense in [8], [9]. We call them *positive* and *negative* coverings and let $\tilde{\mathscr{E}}^+$ and $\tilde{\mathscr{E}}^-$ denote them. Our main result is a complete description of the algebras of nonlocal symmetries in these coverings.

The equation $\mathscr{E}$ itself has an infinite-dimensional Lie algebra of local symmetries parameterized by three arbitrary functions of t and one depending on y and also has an "isolated" scaling symmetry (which allows assigning a weight to all considered polynomial objects; see Table 1 below). We show that all these symmetries can be lifted to both the positive and the negative coverings. Nevertheless, in addition to the lifts of the local symmetries, new purely nonlocal symmetries arise in both cases.

In the covering $\tilde{\mathscr{E}}^+$, a new series isomorphic to the nonpositive part $\mathfrak{W}^-$ of the Witt algebra arises from the scaling symmetry, while the y-dependent symmetries become a part of the loop algebra $\mathfrak{L}[y]$, whose

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coefficients also depend on y; moreover, $\mathfrak{W}^-$ acts naturally on $\mathfrak{L}[y]$. No new t-dependent symmetry arises in $\tilde{\mathcal{E}}^+$, and the local symmetries form a graded ideal in $\mathrm{sym}(\tilde{\mathcal{E}}^+)$. The exact formulation of these results is contained in Theorem 1.

In the case of $\tilde{\mathcal{E}}^-$ (see Theorem 2), the scaling symmetry yields the beginning of the nonnegative part $\mathfrak{W}^+$ of the Witt algebra, and the t-dependent symmetries are included in the loop algebra $\mathfrak{L}[t]$. The algebra $\mathfrak{W}^+$ acts on $\mathfrak{L}[t]$, and the local y-dependent symmetries form a direct summand in the Lie algebra $\mathrm{sym}(\tilde{\mathcal{E}}^-)$.

Finally, we show that the mutually inverse recursion operators found in [10] act in $\mathfrak{L}[y]$ and $\mathfrak{L}[t]$ and also connect the two parts $\mathfrak{W}^-$ and $\mathfrak{W}^+$ of the Witt algebra with each other.

This paper is organized as follows. In Sec. 2, we present basic definitions and facts needed for the further exposition. In Sec. 3, we discuss the three-dimensional rdDym equation: local symmetries, the Lax pair, and the coverings. We formulate and prove the main results in Sec. 4. In Sec. 5, we discuss the recursion operators.

2. Preliminaries

In this section in a simplified coordinate form, we expound the basics of the geometric approach to differential equations and differential coverings. We follow [9], [8].

2.1. Jets and equations. We consider $\mathbb{R}^n$ with the coordinates $x^1,\dots,x^n$ and $\mathbb{R}^m$ with the coordinates $u^1,\dots,u^m$. The space of *k-jets* $J^k(n,m)$, $k=0,1,\dots,\infty$, has the coordinates $x^1,\dots,x^n$ and u^j_σ, where $j=1,\dots,m$ and σ is a symmetric multi-index of length $|\sigma|\le k$; we set $u^j_\varnothing=u^j$. If $u^j=f(x^1,\dots,x^n)$ is a vector function, then the collection of partial derivatives

$$u^j_\sigma=\frac{\partial^{|\sigma|}u^j}{\partial x^\sigma},\quad j=1,\dots,m,\quad |\sigma|\le k,$$

is called its *k-jet*.

We fix a point $\theta\in J^k(n,m)$. The linear span $\mathcal{C}_\theta$ of tangent spaces to the graphs of all k-jets passing through this point is called the *Cartan plane*, and the correspondence $\mathcal{C}\colon\theta\mapsto\mathcal{C}_\theta$ is called the *Cartan distribution*. If $k=\infty$, then a basis of $\mathcal{C}$ consists of the vector fields

$$D_{x^i}=\frac{\partial}{\partial x^i}+\sum_{j,\sigma}u^j_{\sigma i}\frac{\partial}{\partial u^j_\sigma},\quad i=1,\dots,n,$$

called *total derivatives*. Total derivatives pairwise commute, and this means that the Cartan distribution on $J^\infty(n,m)$ is formally integrable.

We consider a submanifold in $J^k(n,m)$ defined by the relations

$$F^1(x^i,u^j_\sigma)=\dots=F^r(x^i,u^j_\sigma)=0. \tag{1}$$

This is a differential equation of order k. Its infinite prolongation $\mathcal{E}\subset J^\infty(n,m)$ is given by

$$D_\sigma(F^j)=0,\quad j=1,\dots,r,\quad |\sigma|\ge 0,$$

where $D_\sigma=D_{x^{i_1}}\circ\dots\circ D_{x^{i_k}}$ for $\sigma=i_1\dots i_k$. Everywhere below, we deal with only infinite prolongations and identify them with differential equations themselves.

The total derivatives can be restricted to infinite prolongations, and these restrictions span the Cartan distribution on $\mathcal{E}$. Maximal integral manifolds of the obtained distribution are solutions of the considered equation.

2.2. Symmetries. We consider an equation $\mathscr{E} \subset J^\infty(n,m)$. Below, we always assume that the natural projection $\mathscr{E} \to J^0(n,m) = \mathbb{R}^n \times \mathbb{R}^m$ is a surjective map onto its image.[1] Consequently, the algebra $C^\infty(J^0(n,m))$ of functions is embedded in the algebra $C^\infty(\mathscr{E})$.

A vector field $X\colon C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ is said to be *vertical* if $X\big|_{C^\infty(J^0(n,m))} = 0$, i.e., X does not contain components of the form $\partial/\partial x^i$. A vertical field X is a (*higher* or *generalized*) *symmetry* of $\mathscr{E}$ if it preserves the Cartan distribution, i.e., $[X, \mathscr{C}] \subset \mathscr{C}$. Symmetries of $\mathscr{E}$ form a Lie $\mathbb{R}$-algebra, denoted by $\operatorname{sym}(\mathscr{E})$.

A vector field is a symmetry if and only if it is *evolutionary*, i.e., has the form

$$\mathbf{E}_\varphi = \sum D_\sigma(\varphi^j)\,\frac{\partial}{\partial u^j_\sigma}, \tag{2}$$

where the summation is over the internal coordinates on $\mathscr{E}$. Here, $\varphi = (\varphi^1, \ldots, \varphi^m)$ is a vector function on $\mathscr{E}$ called the *generating section* (or *characteristic*) of the symmetry. It must satisfy the equation

$$\ell_{\mathscr{E}}(\varphi) = 0,$$

where $\ell_{\mathscr{E}}$ is the *linearization* of $\mathscr{E}$ defined as the restriction of the operator

$$\ell_F = \left\| \sum_\sigma \frac{\partial F^j}{\partial u^l_\sigma} D_\sigma \right\| \tag{3}$$

to $\mathscr{E}$. Generating functions of symmetries form a Lie algebra with respect to the *Jacobi bracket*

$$\{\varphi, \psi\}^j = \sum \left(D_\sigma(\varphi^l)\frac{\partial \psi^j}{\partial u^l_\sigma} - D_\sigma(\psi^l)\frac{\partial \varphi^j}{\partial u^l_\sigma} \right).$$

The Jacobi bracket can be defined without using local coordinates by setting $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$.

2.3. Differential coverings. We consider the space $\tilde{\mathscr{E}} = \mathbb{R}^s \times \mathscr{E}$, $s \le \infty$, and the natural projection $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$. We say that τ is an s-dimensional (*differential*) *covering* over $\mathscr{E}$ if $\tilde{\mathscr{E}}$ is endowed with vector fields $\widetilde{D}_{x^1}, \ldots, \widetilde{D}_{x^n}$ such that

$$[\widetilde{D}_{x^i}, \widetilde{D}_{x^j}] = 0, \qquad \tau_*(\widetilde{D}_{x^i}) = D_{x^i}, \quad i,j = 1, \ldots, n.$$

Let $\{w^\alpha\}$ be coordinates in $\mathbb{R}^s$ (they are called *nonlocal variables*). Then the covering structure is given by $\widetilde{D}_{x^i} = D_{x^i} + X_i$ such that

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0,$$

where

$$X_i = \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}$$

are τ-vertical vector fields.

There exists a special class of coverings that are associated with two-component conservation laws of $\mathscr{E}$. We fix two integers i and j, $1 \le i < j \le n$, and consider a differential form

$$\omega = X_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n + X_j\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n,$$

[1] This means that the differential consequences of Eq. (1) do not contain functional relations.

where $\widehat{dx^i}$ means that the corresponding term is omitted. Let the form ω be closed with respect to the horizontal de Rham differential, i.e.,

$$D_{x^i}(X_i) = (-1)^{i+j-1} D_{x^j}(X_j).$$

We consider the Euclidean space V with the coordinates w^σ, where σ is a symmetric multi-index whose entries are any integers from 1 to n except i and j. Therefore, $\dim V = 1$ if $n = 2$ and $\dim V = \infty$ otherwise. The system of vector fields

$$\widetilde{D}_{x^k} = D_{x^k} + \sum_\sigma w^{\sigma k} \frac{\partial}{\partial w^\sigma}, \quad k \neq i, j,$$

$$\widetilde{D}_{x^i} = D_{x^i} + \sum_\sigma \widetilde{D}_\sigma(X_j) \frac{\partial}{\partial w^\sigma},$$

$$\widetilde{D}_{x^j} = D_{x^j} + (-1)^{i+j-1} \sum_\sigma \widetilde{D}_\sigma(X_i) \frac{\partial}{\partial w^\sigma}$$

then defines a covering structure on $\tilde{\mathscr{E}}_\omega = V \times \mathscr{E}$. Such coverings are said to be *Abelian*.

2.4. Nonlocal symmetries. We let $\widetilde{\mathscr{C}}$ denote the distribution on $\tilde{\mathscr{E}}$ spanned by the fields $\widetilde{D}_{x^1}, \dots, \widetilde{D}_{x^n}$ and let X be a field vertical with respect to the composition $\tilde{\mathscr{E}} \to \mathscr{E} \to \mathbb{R}^n$. Such a field is called a *nonlocal symmetry* if it preserves $\widetilde{\mathscr{C}}$. These symmetries form a Lie algebra on $\mathbb{R}$ denoted by $\mathrm{sym}_\tau(\mathscr{E})$. The restriction $X\big|_{C^\infty(\mathscr{E})}\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ is called a nonlocal *τ-shadow*. A nonlocal symmetry is said to be *invisible* if its shadow vanishes.

In local coordinates, any $X \in \mathrm{sym}_\tau(\mathscr{E})$ has the form

$$X = \widetilde{\mathbf{E}}_\varphi + \sum_\alpha \psi^\alpha \frac{\partial}{\partial w^\alpha},$$

where $\varphi = (\varphi^1, \dots, \varphi^m)$ and ψ^α are functions on $\tilde{\mathscr{E}}$ satisfying the equations

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0,$$

$$\widetilde{D}_{x^i}(\psi^\alpha) = \sum_{j,\sigma} \frac{\partial X_i^\alpha}{\partial u_\sigma^j} \widetilde{D}_\sigma(\varphi^j) + \sum_\beta \frac{\partial X_i^\alpha}{\partial w^\beta} \psi^\beta,$$

where $\widetilde{\mathbf{E}}_\varphi$ and $\tilde{\ell}_{\mathscr{E}}$ are obtained from the respective expressions (2) and (3) by changing D_{x^i} to $\widetilde{D}_{x^i}$. Nonlocal shadows are the operators $\widetilde{\mathbf{E}}_\varphi$, and invisible symmetries are obtained from general symmetries by setting $\varphi = 0$.

In particular, for coverings of the form $\tilde{\mathscr{E}}_\omega$, where ω is a two-component conservation law, the symmetries become

$$X = \widetilde{\mathbf{E}}_\varphi + \sum_\sigma D_\sigma(\psi) \frac{\partial}{\partial w^\sigma},$$

where φ and ψ satisfy

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0,$$

$$\widetilde{D}_{x^i}(\psi) = \sum_{\sigma,k} \frac{\partial X_j}{\partial u_\sigma^k} \widetilde{D}_\sigma(\varphi^k) + \sum_\sigma \frac{\partial X_j}{\partial w^\sigma} \widetilde{D}_\sigma(\psi),$$

$$\widetilde{D}_{x^j}(\psi) = (-1)^{i+j-1} \Bigl(\sum_{\sigma,k} \frac{\partial X_i}{\partial u_\sigma^k} \widetilde{D}_\sigma(\varphi^k) + \sum_\sigma \frac{\partial X_i}{\partial w^\sigma} \widetilde{D}_\sigma(\psi) \Bigr).$$

2.5. Bäcklund transformations and recursion operators. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be equations. A *Bäcklund transformation* between $\mathscr{E}_1$ and $\mathscr{E}_2$ is the diagram

$$\begin{array}{ccc} & \tilde{\mathscr{E}} & \\ {}^{\tau_1}\swarrow & & \searrow^{\tau_2} \\ \mathscr{E}_1 & & \mathscr{E}_2 \end{array},$$

where τ_1 and τ_2 are coverings. If $\mathscr{E}_1 = \mathscr{E}_2$, then it is called a *Bäcklund autotransformation.* If τ_1 is finite-dimensional and $\gamma \subset \mathscr{E}_1$ is the graph of a solution, then $\tau_2(\tau_1^{-1}(\gamma))$ in the general case is a finite-dimensional manifold endowed with an integrable n-dimensional distribution whose integral manifolds are solutions of $\mathscr{E}_2$.

We now consider an equation $\mathscr{E}$ given by (1) and the system

$$F(x^i, u^j_\sigma) = 0, \qquad \ell_F(x^i, u^j_\sigma, q^j_\sigma) = 0,$$

where $F = (F^1, \dots, F^r)$. This system is called the *tangent* equation to $\mathscr{E}$ and is denoted by $\mathscr{T}\mathscr{E}$, and the projection $\mathrm{t}\colon \mathscr{T}\mathscr{E} \to \mathscr{E}$ is called the *tangent covering.* Sections of this covering that preserve the Cartan distribution are identified with generating functions of symmetries of $\mathscr{E}$.

Let $\mathscr{R}$ be a Bäcklund transformation between $\mathscr{T}\mathscr{E}_1$ and $\mathscr{T}\mathscr{E}_2$. It then follows from the above that it establishes a correspondence between symmetries of the two equations $\mathscr{E}_1$ and $\mathscr{E}_2$. If $\mathscr{E}_1 = \mathscr{E}_2$, then this correspondence is called a *recursion operator* [11].

3. The equation

The three-dimensional rdDym equation $\mathscr{E}$ has the form

$$u_{ty} = u_x u_{xy} - u_y u_{xx}. \tag{4}$$

As the internal coordinates in $\mathscr{E}$, we can choose the functions

$$u_k = u_{\underbrace{x\dots x}_{k \text{ times}}}, \qquad u^t_{k,l} = u_{\underbrace{x\dots x}_{k \text{ times}}\underbrace{t\dots t}_{l \text{ times}}}, \qquad u^y_{k,l} = u_{\underbrace{x\dots x}_{k \text{ times}}\underbrace{y\dots y}_{l \text{ times}}}, \qquad k \geq 0, \quad l \geq 0.$$

Hence, $u_0 = u$, $u_1 = u_x$, $u^y_{0,1} = u_y$, $u^t_{0,1} = u_t$, etc. The total derivatives become

$$D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k,l}\left(u^y_{k+1,l}\frac{\partial}{\partial u^y_{k,l}} + u^t_{k+1,l}\frac{\partial}{\partial u^t_{k,l}}\right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_k u^y_{k,1}\frac{\partial}{\partial u_k} + \sum_{k,l}\left(u^y_{k,l+1}\frac{\partial}{\partial u^y_{k,l}} + D^k_x D^{l-1}_t(u_x u_{xy} - u_y u_{xx})\frac{\partial}{\partial u^t_{k,l}}\right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_r u^t_{k,1}\frac{\partial}{\partial u_k} + \sum_{k,l}\left(D^k_x D^{l-1}_y(u_x u_{xy} - u_y u_{xx})\frac{\partial}{\partial u^y_{k,l}} + u^t_{k,l+1}\frac{\partial}{\partial u^t_{k,l}}\right)$$

in these coordinates.

3.1. Local symmetries. Local symmetries of Eq. (4) are solutions of the linearized equation

$$\ell_{\mathscr{E}}(\varphi) \equiv D_t D_y(\varphi) - u_x D_x D_y(\varphi) + u_y D_x^2(\varphi) - u_{xy} D_x(\varphi) + u_{xx} D_y(\varphi) = 0. \tag{5}$$

The space of solutions is spanned by the functions

$$\psi_0 = xu_x - 2u, \qquad \upsilon_0(B) = Bu_y,$$

$$\theta_0(A) = Au_t + A'(xu_x - u) + \frac{1}{2}A''x^2, \qquad \theta_{-1}(A) = Au_x + A'x, \qquad \theta_{-2}(A) = A,$$

where $A = A(t)$, $B = B(y)$, and the prime denotes the derivative with respect to t. To any solution φ, there corresponds the evolutionary vector field

$$\mathbf{E}_\varphi = \sum_k D_x^k(\varphi)\frac{\partial}{\partial u_k} + \sum_{k,l}\left(D_x^k D_y^l(\varphi)\frac{\partial}{\partial u_{k,l}^y} + D_x^k D_t^l(\varphi)\frac{\partial}{\partial u_{k,l}^t}\right) \tag{6}$$

on $\mathscr{E}$.

The Lie algebra structure in the space $\mathrm{sym}(\mathscr{E})$ is presented in Table 1.

Table 1

	ψ_0	$\upsilon_0(\bar{B})$	$\theta_0(\bar{A})$	$\theta_{-1}(\bar{A})$	$\theta_{-2}(\bar{A})$
ψ_0	0	0	0	$-\theta_{-1}(\bar{A})$	$-2\theta_{-2}(\bar{A})$
$\upsilon_0(B)$	$\cdots$	$\upsilon_0(B\bar{B}' - \bar{B}B')$	0	0	0
$\theta_0(A)$	$\cdots$	$\cdots$	$\theta_0(\bar{A}A' - A\bar{A}')$	$\theta_{-1}(\bar{A}A' - A\bar{A}')$	$\theta_{-2}(\bar{A}A' - A\bar{A}')$
$\theta_{-1}(A)$	$\cdots$	$\cdots$	$\cdots$	$\theta_{-2}(\bar{A}A' - A\bar{A}')$	0
$\theta_{-2}(A)$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0

The Lie algebra structure of $\mathrm{sym}(\mathscr{E})$.

3.2. Coverings. Three-dimensional rdDym equation (4) has the linear Lax representation

$$\begin{aligned} w_t &= (u_x - \lambda)w_x, \\ w_y &= \lambda^{-1}u_y w_x, \end{aligned} \tag{7}$$

where $\lambda \neq 0$ is a nonremovable parameter. Expanding w in a formal series in λ,

$$w = \sum_{i=-\infty}^{\infty} w_i \lambda^i,$$

yields (cf. [2], [12])

$$\begin{aligned} w_{i,t} &= u_x w_{i,x} - w_{i-1,x}, \\ w_{i,y} &= u_y w_{i+1,x}. \end{aligned} \tag{8}$$

This system is infinite in both directions, and the nonlocal quantities w_i are therefore not defined properly. To define them appropriately, we consider two reductions of (8): (*a*) $w_i = 0$ for $i < 0$ and (*b*) $w_i = 0$ for $i > 0$. Two hierarchies of nonlocal two-component conservation laws thus arise [2], respectively called the *positive* and *negative* hierarchies. Our aim is to describe nonlocal symmetries of the corresponding Abelian coverings.

We note that the positive hierarchy corresponds to the Taylor expansion of w and the negative hierarchy corresponds to the Laurent expansion.

3.2.1. The positive hierarchy. We assume that $w_i = 0$ for $i < 0$ and rewrite (8) in the form

$$w_{i,t} = \frac{u_x}{u_y} w_{i-1,y} - w_{i-1,x},$$
$$w_{i,x} = \frac{w_{i-1,y}}{u_y}.$$

From this assumption, we then have $w_{0,t} = w_{0,x} = 0$ or $w_0 = G(y)$, and the defining equations of the covering are

$$w_{1,t} = \frac{u_x}{u_y} G', \qquad w_{i,t} = \frac{u_x}{u_y} w_{i-1,y} - w_{i-1,x},$$
$$w_{1,x} = \frac{G'}{u_y}, \qquad w_{i,x} = \frac{w_{i-1,y}}{u_y},$$

where $i > 0$ and the prime denotes the derivative with respect to y.

Without loss of generality, we can assume that $G' \neq 0$ and change the variable $y \mapsto G(y)$. This transformation preserves our equation (because of the symmetry $v_0(B)$). Letting q_i, $i > 0$, denote the resulting nonlocal variables, we obtain the covering defined by

$$q_{1,t} = \frac{u_x}{u_y}, \qquad q_{1,x} = \frac{1}{u_y} \tag{9}$$

and

$$q_{i,t} = \frac{u_x}{u_y} q_{i-1,y} - q_{i-1,x}, \qquad q_{i,x} = \frac{q_{i-1,y}}{u_y}. \tag{10}$$

We note that the quantities q_i do not form a complete set of nonlocal variables in the covering under consideration. To have a complete collection, we introduce functions $q_i^{(j)}$ such that

$$q_i^{(0)} = q_i, \qquad q_i^{(j+1)} = \big(q_i^{(j)}\big)_y.$$

The total derivatives on the space $\tilde{\mathscr{E}}^+$ of the covering are then given by

$$\widetilde{D}_x = D_x + \sum_{j=0}^{\infty} \widetilde{D}_y^j \left(\frac{1}{u_y}\right) \frac{\partial}{\partial q_1^{(j)}} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \widetilde{D}_y^j \left(\frac{q_{i-1}^{(1)}}{u_y}\right) \frac{\partial}{\partial q_i^{(j)}},$$
$$\widetilde{D}_y = D_y + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} q_i^{(j+1)} \frac{\partial}{\partial q_i^{(j)}},$$
$$\widetilde{D}_t = D_t + \sum_{j=0}^{\infty} \widetilde{D}_y^j \left(\frac{u_x}{u_y}\right) \frac{\partial}{\partial q_1^{(j)}} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \widetilde{D}_y^j \left(\frac{u_x}{u_y} q_{i-1}^{(1)} - \widetilde{D}_x\left(q_{i-1}^{(0)}\right)\right) \frac{\partial}{\partial q_i^{(j)}},$$

where D_x, D_y, and D_t are the total derivatives on $\mathscr{E}$ given above.

3.2.2. The negative hierarchy. In the case of the negative hierarchy, we set $w_i = 0$ for $i > 0$. It then follows from (8) that

$$w_{0,x} = 0, \qquad w_{-1,x} = u_x w_{0,x} - w_{0,t}, \qquad w_{-2,x} = u_x w_{-1,x} - w_{-1,t},$$
$$w_{0,y} = 0, \qquad w_{-1,y} = u_y w_{0,x}, \qquad w_{-2,y} = u_y w_{-1,x},$$

and consequently

$$w_0 = \widetilde{F}(t), \qquad w_{-1} = -x\widetilde{F}' + G(t), \qquad w_{-2} = -\widetilde{F}'u + \frac{1}{2}x^2\widetilde{F}'' - G'x + H(t).$$

Without loss of generality, we can assume that $G = H = 0$. Introducing the notation $r_i = w_{-i-2}$, $i = 1, 2, \dots$, we then obtain the defining equations for the negative hierarchy:

$$\begin{aligned} r_{1,x} &= F(u_t - u_x^2) + F'(u + xu_x) - \frac{1}{2}x^2F'', &\qquad r_{i,x} &= u_x r_{i-1,x} - r_{i-1,t}, \\ r_{1,y} &= u_y(xF' - Fu_x), & r_{i,y} &= u_y r_{i-1,x}, \end{aligned} \tag{11}$$

for $i > 1$, where $F = \widetilde{F}'$. The defining equations can be simplified.

Proposition 1. *There exists a gauge transformation of the space* $\tilde{\mathscr{E}}^-$ *that "suppresses" the function* F, *i.e., transforms* (11) *into*

$$\begin{aligned} r_{1,x} &= u_x^2 - u_t, &\qquad r_{i,x} &= u_x r_{i-1,x} - r_{i-1,t}, \\ r_{1,y} &= u_x u_y, & r_{i,y} &= u_y r_{i-1,x}. \end{aligned} \tag{12}$$

Proof. We define the new nonlocal variable $\bar{r}_1$ by

$$r_1 = -F\bar{r}_1 - F'xu + \frac{1}{6}F''x^3. \tag{13}$$

Substituting (13) in the first and third equations in system (11), we see that

$$\bar{r}_{1,x} = u_x^2 - u_t, \qquad \bar{r}_{1,y} = u_x u_y.$$

We now introduce the operator

$$\mathscr{Y}_- = -x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial x} - 3\bar{r}_1\frac{\partial}{\partial u} + \sum_{i\ge 1}(i+3)\bar{r}_{i+1}\frac{\partial}{\partial \bar{r}_i}$$

and by induction set

$$r_k = \frac{1}{k+2}\mathscr{Y}_-(r_{k-1}) \tag{14}$$

for $k \ge 2$. Obviously,

$$r_k = F\bar{r}_k + o(k-1),$$

where $o(k-1)$ denotes the terms that depend only on $\bar{r}_1, \dots, \bar{r}_{k-1}$.

We now assume that $k > 1$ and the statement holds for the defining equations on $\bar{r}_1, \dots, \bar{r}_{k-1}$. Substituting (14) in the equations for r_k, we see that it transforms into

$$F\bar{r}_{k,x} = F(u_x\bar{r}_{k-1,x} - \bar{r}_{k-1,t}), \qquad F\bar{r}_{k,y} = Fu_y\bar{r}_{k-1,x}$$

by the induction assumption. ■

We forget about the "old" variables r_k and change the notation from $\bar{r}_k$ to r_k. A complete set of nonlocal variables consists of the quantities $r_i^{(j)}$ defined by

$$r_i^{(0)} = r_i, \qquad r_i^{(j+1)} = \left(r_i^{(j)}\right)_t.$$

The total derivatives on the covering space $\tilde{\mathscr{E}}^-$ have the forms

$$\widetilde{D}_x = D_x + \sum_{j=0}^{\infty} \widetilde{D}_t^j(u_x^2 - u_t)\frac{\partial}{\partial r_1^{(j)}} + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \widetilde{D}_t^j(u_x r_{i-1,x} - r_{i-1,t})\frac{\partial}{\partial r_i^{(j)}},$$

$$\widetilde{D}_y = D_y + \sum_{j=0}^{\infty} \widetilde{D}_t^j(u_x u_y)\frac{\partial}{\partial r_1^{(j)}} + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \widetilde{D}_t^j(u_y r_{i-1,x})\frac{\partial}{\partial r_i^{(j)}},$$

$$\widetilde{D}_t = D_t + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} r_i^{(j+1)}\frac{\partial}{\partial r_i^{(j)}}$$

in these coordinates.

3.3. Weights. We assign the weights

$$|x| = 1, \qquad |u| = 2, \qquad |y| = |t| = 0$$

to the dependent and independent variables. Then

$$|u_k| = |u_{k,l}^y| = |u_{k,l}^t| = 2 - k,$$

and any monomial obtains the weight equal to the sum of its factors.

We say that a vector field X is *homogeneous* if

$$|X(f)| = |X| + |f|$$

for any homogeneous function f, where the integer $|X|$ depends only on X and is the weight of X. All local symmetries from Sec. 3.1 are homogeneous in this sense, and their weights are presented in Table 2. Obviously,

$$|[X,Y]| = |X| + |Y|$$

for any homogeneous X and Y.

Table 2

Weights	-2	-1	0
	$\theta_{-2}(A)$	$\theta_{-1}(A)$	ψ_0 $\theta_0(A)$ $v_0(B)$

Distribution of local symmetries by weight.

From Eqs. (9)–(11), we immediately calculate the weights

$$|q_i| = -i, \qquad |r_i| = i + 2, \quad i = 1, 2, \ldots,$$

of the nonlocal variables in $\tilde{\mathscr{E}}^+$ and $\tilde{\mathscr{E}}^-$.

4. Symmetries

In this section, we describe the nonlocal symmetry Lie algebras $\operatorname{sym}\tilde{\mathscr{E}}^+$ and $\operatorname{sym}\tilde{\mathscr{E}}^-$.

4.1. Symmetries in the positive hierarchy. Any symmetry of $\tilde{\mathscr{E}}^+$ is a vector field

$$\mathbf{X}_\Phi = \widetilde{\mathbf{E}}_\varphi + \sum_{i=1}^{\infty}\left(\varphi_i \frac{\partial}{\partial q_i} + \sum_{j=1}^{\infty} \widetilde{D}_y^j(\varphi_i)\frac{\partial}{\partial q_i^{(j)}}\right), \tag{15}$$

where $\widetilde{\mathbf{E}}_\varphi$ is given by (6) with the total derivatives $\widetilde{D}_\bullet$ instead of $D_\bullet$ and the collection of functions

$$\Phi = \langle \varphi_0 = \varphi, \varphi_1, \dots, \varphi_i, \dots\rangle, \quad \varphi_i \in C^\infty(\tilde{\mathscr{E}}^+),$$

satisfies the equations

$$\tilde{\ell}_{\mathscr{E}}(\varphi) \equiv \widetilde{D}_t\widetilde{D}_y(\varphi) - u_x\widetilde{D}_x\widetilde{D}_y(\varphi) + u_y\widetilde{D}_x^2(\varphi) - u_{xy}\widetilde{D}_x(\varphi) + u_{xx}\widetilde{D}_y(\varphi) = 0, \tag{16}$$

$$\widetilde{D}_y(\varphi)q_{1,t} + u_y\widetilde{D}_t(\varphi_1) = \widetilde{D}_x(\varphi), \tag{17}$$

$$\widetilde{D}_y(\varphi)q_{1,x} + u_y\widetilde{D}_x(\varphi_1) = 0, \tag{18}$$

$$\widetilde{D}_y(\varphi)(q_{i,t} + q_{i-1,x}) + u_y(\widetilde{D}_t(\varphi_i) + \widetilde{D}_x(\varphi_{i-1})) = \widetilde{D}_x(\varphi)q_{i-1,y} + u_x\widetilde{D}_x(\varphi_{i-1}), \tag{19}$$

$$\widetilde{D}_y(\varphi)q_{i,x} + u_y\widetilde{D}_x(\varphi_i) = \widetilde{D}_y(\varphi_{i-1}), \tag{20}$$

$i > 1$. For any two symmetries Φ and Ψ, their *Jacobi bracket* $\{\Phi, \Psi\}$ is defined by

$$\mathbf{X}_{\{\Phi,\Psi\}} = [\mathbf{X}_\Phi, \mathbf{X}_\Psi].$$

4.1.1. Lifts of local symmetries and hierarchies of nonlocal ones. We begin with the following statement.

Proposition 2. *The local symmetries ψ_0, $\theta_{-2}(A)$, $\theta_{-1}(A)$, and $\theta_0(A)$ can be lifted to $\tilde{\mathscr{E}}^+$.*

Proof. We let

$$\Psi_0 = \langle \psi_0, \psi_0^1, \dots, \psi_0^i, \dots\rangle,$$
$$\Theta_{-2}(A) = \langle \theta_{-2}(A), \theta_{-2}^1(A), \dots, \theta_{-2}^i(A), \dots\rangle,$$
$$\Theta_{-1}(A) = \langle \theta_{-1}(A), \theta_{-1}^1(A), \dots, \theta_{-1}^i(A), \dots\rangle,$$
$$\Theta_0(A) = \langle \theta_0(A), \theta_0^1(A), \dots, \theta_0^i(A), \dots\rangle$$

denote the desired lifts and set

$$\psi_0^i = iq_i + xq_{i,x}, \quad i \geq 1,$$
$$\theta_{-2}^i(A) = 0, \qquad \theta_{-1}^i(A) = Aq_{i,x}, \quad i \geq 1,$$
$$\theta_0^1(A) = \theta_{-1}(A)q_{1,x}, \qquad \theta_0^i(A) = \theta_{-1}(A)q_{i,x} - Aq_{i-1,x}, \quad i > 1.$$

To establish that the functions introduced above are symmetries, we directly verify that they satisfy Eqs. (17)–(20). For example, we prove that Ψ_0 is a symmetry.

For Eq. (17), we have[2]

$$D_y(xu_x - 2u)q_{1,x} + u_y D_x(q_1 + xq_{1,x}) = \Big(xu_{xy}\boxed{-2u_y}\Big)q_{1,x} + u_y\Big(\boxed{2q_{1,x}} + xq_{1,xx}\Big) =$$

$$= xu_{xy}q_{1,x} + u_y x\left(\frac{1}{u_y}\right)_x = xu_{xy}\frac{1}{u_y} - xu_y\frac{u_{xy}}{u_y^2} = 0.$$

Now Eq. (18) is

$$D_y(xu_x - 2u)q_{1,t} + u_y D_t(q_1 + xq_{1,x}) - D_x(xu_x - 2u) =$$

$$= \Big(xu_{xy}\boxed{-2u_y}\Big)q_{1,t} + u_y\Big(\boxed{q_{1,t}} + xq_{1,xt}\Big) + u_x - xu_{xx} =$$

$$= (xu_{xy} - u_y)\frac{u_x}{u_y} + xu_y\left(\frac{u_x}{u_y}\right)_x +$$

$$+ u_x - xu_{xx} = (xu_{xy} - u_y)\frac{u_x}{u_y} + xu_y\frac{u_{xx}u_y - u_x u_{xy}}{u_y^2} + u_x = 0.$$

Equation (19) becomes

$$D_y(xu_x - 2u)q_{i,x} + u_y D_x(iq_i + xq_{i,x}) - D_y((i-1)q_{i-1} + xq_{i-1,x}) =$$

$$= \Big(xu_{xy}\boxed{-2u_y}\Big)q_{i,x} + u_y\Big(\boxed{(i+1)q_{i,x}} + xq_{i,xx}\Big) - (i-1)q_{i-1,y} - xq_{i-1,xy} =$$

$$= \Big(xu_{xy} + \boxed{(i-1)u_y}\Big)\frac{q_{i-1,y}}{u_y} + u_y x\left(\frac{q_{i-1,y}}{u_y}\right)_x \boxed{-(i-1)q_{i-1,y}} - xq_{i-1,xy} =$$

$$= xu_{xy}\frac{q_{i-1,y}}{u_y} + u_y x\frac{q_{i-1,xy}u_y - q_{i-1,y}u_{xy}}{u_y^2} - xq_{i-1,xy} = 0.$$

Finally, for Eq. (20), we have

$$D_y(xu_x - 2u)\frac{u_x}{u_y}q_{i-1,y} + u_y(D_t(iq_i + xq_{i,x}) + D_x((i-1)q_{i-1} + xq_{i-1,x})) -$$

$$- D_x(xu_x - 2u)q_{i-1,y} - u_x D_y((i-1)q_{i-1} + xq_{i-1,x}) =$$

$$= (xu_{xy} - 2u_y)\frac{u_x}{u_y}q_{i-1,y} + \Big(\boxed{iq_{i,t}} + xq_{i,xt} + \boxed{iq_{i-1,x}} + xq_{i-1xx}\Big) +$$

$$+ (u_x - xu_{xx})q_{i-1,y} - u_x((i-1)q_{i-1,y} + xq_{i-1,xy}) =$$

$$= \Big(xu_{xy}\boxed{-2u_y}\Big)\frac{u_x}{u_y}q_{i-1,y} + u_y\left(\boxed{i\frac{u_x}{u_y}q_{i-1,y}} + xq_{i,xt} + xq_{i-1,xx}\right) +$$

$$+ \Big(\boxed{u_x} - xu_{xx}\Big)q_{i-1,y} - u_x\Big(\boxed{(i-1)q_{i-1,y}} + xq_{i-1,xy}\Big) =$$

$$= xu_{xy}\frac{u_x}{u_y}q_{i-1,y} + xu_y\Big(\boxed{q_{i,t} + q_{i-1,x}}\Big)_x - xu_{xx}q_{i-1,y} - u_x xq_{i-1,xy} =$$

[2] Here and hereafter, boxed terms cancel each other.

$$= xu_{xy}\frac{u_x}{u_y}q_{i-1,y} + xu_y\left(\frac{u_x}{u_y}q_{i-1,y}\right)_x - xu_{xx}q_{i-1,y} - u_xxq_{i-1,xy} =$$

$$= xu_{xy}\frac{u_x}{u_y}q_{i-1,y} + xu_y\left(\left(\frac{u_x}{u_y}\right)_x q_{i-1,y} + \boxed{\frac{u_x}{u_y}q_{i-1,xy}}\right) -$$

$$- xu_{xx}q_{i-1,y}\boxed{-u_xxq_{i,xy}} =$$

$$= xu_{xy}\frac{u_x}{u_y}q_{i-1,y} + xu_y\frac{u_{xx}u_y - u_xu_{xy}}{u_y^2}q_{i-1,y} - xu_{xx}q_{i-1,y}.$$

and this finishes the proof.

The proofs for the other symmetries are similar. ■

We now need a description of invisible symmetries in $\tilde{\mathscr{E}}^+$. We say that Φ is an invisible symmetry of *depth k* if its first k components vanish, i.e.,

$$\Phi = \langle\underbrace{0,\dots,0}_{k\text{ times}},\varphi_1^{\text{inv}},\dots,\varphi_i^{\text{inv}},\dots\rangle.$$

The defining equations for invisible symmetries are

$$\widetilde{D}_x(\varphi_1^{\text{inv}}) = 0, \qquad \widetilde{D}_t(\varphi_1^{\text{inv}}) = 0,$$

$$u_y\widetilde{D}_x(\varphi_i^{\text{inv}}) = \widetilde{D}_y(\varphi_{i-1}^{\text{inv}}), \qquad u_y(\widetilde{D}_t(\varphi_i^{\text{inv}}) + \widetilde{D}_x(\varphi_{i-1}^{\text{inv}})) = u_x\widetilde{D}_x(\varphi_{i-1}^{\text{inv}}), \quad i > 1.$$

Then $\varphi_1^{\text{inv}} = B(y)$ and any homogeneous symmetry of depth k is completely determined by the function B. We let $\Upsilon_k(B)$ denote such a symmetry. We have

$$|\Upsilon_k(B)| = k.$$

Proposition 3. *For any integer $k \geq 1$ and a function $B = B(y)$, the symmetry $\Upsilon_k(B)$ exists.*

Proof. We consider the operator

$$\mathscr{X} = q_1\frac{\partial}{\partial y} + \sum_{i=1}^{\infty}(i+1)q_{i+1}\frac{\partial}{\partial q_i}$$

and define

$$\varphi_1^{\text{inv}} = B(y), \qquad \varphi_i^{\text{inv}} = \frac{1}{i-1}\mathscr{X}(\varphi_{i-1}^{\text{inv}}), \quad i > 1. \tag{21}$$

We note that the defining equations for invisible symmetries can be rewritten in the form

$$\frac{\partial\varphi_2^{\text{inv}}}{\partial q_1} = \frac{\partial B}{\partial y},$$

$$\vdots$$

$$\frac{\partial\varphi_i^{\text{inv}}}{\partial q_{i-1}} = \frac{\partial\varphi_{i-1}^{\text{inv}}}{\partial q_{i-2}}, \qquad \dots, \qquad \frac{\partial\varphi_i^{\text{inv}}}{\partial q_1} = \frac{\partial\varphi_{i-1}^{\text{inv}}}{\partial y},$$

$$\vdots$$

We prove the equalities

$$\frac{\partial \varphi_i^{\mathrm{inv}}}{\partial q_j} = \frac{\partial \varphi_{i-1}^{\mathrm{inv}}}{\partial q_{j-1}}$$

by induction (we formally set $q_0 = y$). The case $i = 2$ is verified by direct computation. We now suppose that the statement holds for some $i > 2$ and note that

$$\left[\frac{\partial}{\partial q_j}, \mathscr{X}\right] = j\frac{\partial}{\partial q_{j-1}}.$$

Then

$$\begin{aligned}\frac{\partial \varphi_{i+1}^{\mathrm{inv}}}{\partial q_j} &= \frac{1}{i}\left(j\frac{\partial \varphi_i^{\mathrm{inv}}}{\partial q_{j-1}} + \mathscr{X}\left(\frac{\partial \varphi_i^{\mathrm{inv}}}{\partial q_j}\right)\right) = \frac{1}{i}\left(j\frac{\partial \varphi_i^{\mathrm{inv}}}{\partial q_{j-1}} + \mathscr{X}\left(\frac{\partial \varphi_{i-1}^{\mathrm{inv}}}{\partial q_{j-1}}\right)\right) = \\ &= \frac{1}{i}\left(j\frac{\partial \varphi_i^{\mathrm{inv}}}{\partial q_{j-1}} - (j-1)\frac{\partial \varphi_{i-1}^{\mathrm{inv}}}{\partial q_{j-2}} + \frac{\partial \mathscr{X}(\varphi_{i-1}^{\mathrm{inv}})}{\partial q_{j-1}}\right) = \\ &= \frac{1}{i}\frac{\partial}{\partial q_{j-1}}\left(\varphi_i^{\mathrm{inv}} + \mathscr{X}(\varphi_{i-1}^{\mathrm{inv}})\right) = \frac{1}{i}\frac{\partial}{\partial q_{j-1}}(\varphi_i^{\mathrm{inv}} + (i-1)\varphi_i^{\mathrm{inv}}) = \frac{\partial \varphi_i^{\mathrm{inv}}}{\partial q_{j-1}},\end{aligned}$$

which finishes the proof. ■

Direct computations now show that the functions

$$\psi_{-1} = q_1 u_y + x, \qquad \psi_{-2} = (2q_2 - q_1 q_1^{(1)})u_y \tag{22}$$

are shadows in the positive covering, i.e., they satisfy Eq. (16).

Proposition 4. *Shadows* (22) *can be extended to symmetries of* $\tilde{\mathscr{E}}^{+}$.

Proof. We set

$$\Psi_{-1} = \langle \psi_{-1}, \psi_{-1}^1, \dots, \psi_{-1}^i, \dots \rangle, \qquad \Psi_{-2} = \langle \psi_{-2}, \psi_{-2}^1, \dots, \psi_{-2}^i, \dots \rangle,$$

where

$$\psi_{-1}^i = -(i+1)q_{i+1} + q_i^{(1)}q_1, \qquad \psi_{-2}^i = -(i+2)q_{i+2} + q_1 q_{i+1}^{(1)} + (2q_2 - q_1 q_1^{(1)})q_i^{(1)}.$$

The rest of the proof is similar to the proof of Proposition 2. ■

Obviously,

$$|\Psi_{-1}| = -1, \qquad |\Psi_{-2}| = -2.$$

We now define two hierarchies of nonlocal symmetries by

$$\begin{aligned}\Psi_{-k} &= \mathrm{ad}_{-1}^{k-2}(\Psi_{-2}), & k \geq 3,\\ \Upsilon_{-k}(B) &= \{\Psi_{-k-1}, \Upsilon_1(B)\}, & k \geq 0,\end{aligned}$$

where

$$\mathrm{ad}_{-1}(\Phi) = \{\Phi, \Psi_{-1}\}.$$

Obviously,

$$|\Psi_{-k}| = |\Upsilon_{-k}(B)| = -k,$$

and $\Upsilon_0(B)$ is an extension of the local symmetry $\upsilon_0(B)$ to $\tilde{\mathscr{E}}^+$. Elements of the algebra $\mathrm{sym}(\tilde{\mathscr{E}}^+)$ are distributed by weight as shown in Table 3.

Table 3

Weights	...	$-l$	...	-2	-1	0	1	...	k	...
	...	Ψ_{-l}	...	Ψ_{-2}	Ψ_{-1}	Ψ_0				
				$\Theta_{-2}(A)$	$\Theta_{-1}(A)$	$\Theta_0(A)$				
	...	$\Upsilon_{-l}(B)$	...	$\Upsilon_{-2}(B)$	$\Upsilon_{-1}(B)$	$\Upsilon_0(B)$	$\Upsilon_1(B)$	...	$\Upsilon_k(B)$	...

Distribution of nonlocal symmetries in $\tilde{\mathscr{E}}^+$ by weight.

4.1.2. The Lie algebra structure. To compute the commutators, we need asymptotic estimates for the coefficients of symmetries that constitute a basis of $\mathrm{sym}(\tilde{\mathscr{E}}^+)$.

We begin with the symmetries Ψ_{-k}, $k \geq 1$, and we are interested in the higher-order terms (with respect to q_j) of the coefficients of $\partial/\partial q_i$. Using notation (15), by definition, we have

$$\mathbf{X}_{\Psi_{-1}} = \cdots + \left(-(i+1)q_{i+1} + q_1 q_i^{(1)} + o(i-1)\right)\frac{\partial}{\partial q_i} + \dots,$$

$$\mathbf{X}_{\Psi_{-2}} = \cdots + \left(-(i+2)q_{i+2} + q_1 q_{i+1}^{(1)} + o(i)\right)\frac{\partial}{\partial q_i} + \dots,$$

where $o(k)$ denotes terms containing q_j with $j \leq k$. We now assume that

$$\mathbf{X}_{\Psi_{-k}} = \cdots + \left(a_k^i q_{i+k} + b_k^i q_1 q_{i+k-1}^{(1)} + o(i+k-2)\right)\frac{\partial}{\partial q_i} + \dots.$$

Then

$$\mathbf{X}_{\Psi_{-k-1}} = [\mathbf{X}_{\Psi_{-k}}, \mathbf{X}_{\Psi_{-1}}] = \cdots + \big((i+k+1)a_k^i - (i+1)a_k^{i+1}q_{i+k+1} +$$
$$+ \left((i+k)b_k^i - (i+1)b_k^{i+1}\right)q_1 q_{i+k}^{(1)} + o(i+k-1)\big)\frac{\partial}{\partial q_i} + \dots.$$

Hence,

$$a_{k+1}^i = (i+k+1)a_k^i - (i+1)a_k^{i+1}, \qquad b_{k+1}^i = (i+k)b_k^i - (i+1)b_k^{i+1},$$

and by elementary induction with the base $a_2^i = -(i+2)$, $b_2^i = 1$, we immediately obtain

$$a_k^i = -(k-2)!(k+i), \qquad b_k^i = (k-2)! \tag{23}$$

for all $i \geq 1$ (we formally set $(-1)! = 1$). To comply with this result, we change the basic element Ψ_0 by $\Psi_0 \mapsto -\Psi_0$.

We now estimate the elements $\Upsilon_k(B)$. For $k > 0$, we use Definition (21) and by simple computations obtain

$$\varphi_i^{\mathrm{inv}} = B' q_{i-1} + B'' q_1 q_{i-2} + o(i-3)$$

and consequently

$$\mathbf{X}_{\Upsilon_k(B)} = \varphi_1^{\mathrm{inv}}\frac{\partial}{\partial q_k} + \cdots + \varphi_{i-k+1}^{\mathrm{inv}}\frac{\partial}{\partial q_i} + \cdots =$$
$$= B\frac{\partial}{\partial q_k} + \cdots + (B' q_{i-k} + B'' q_1 q_{i-k-1} + o(i-k-2))\frac{\partial}{\partial q_i} + \dots.$$

Further,

$$\mathbf{X}_{\Upsilon_{-k}(B)} = [\mathbf{X}_{\Psi_{-k-1}}, \mathbf{X}_{\Upsilon_1(B)}] =$$

$$= \Big[\dots + \big(a^i_{k+1} q_{i+k+1} + b^i_{k+1} q_1 q^{(1)}_{i+k} + o(i+k-1)\big)\frac{\partial}{\partial q_i} + \dots,$$

$$B\frac{\partial}{\partial q_1} + \dots + \big(B' q_{i-1} + B'' q_1 q_{i-2} + o(i-3)\big)\frac{\partial}{\partial q_i} + \dots\Big] =$$

$$= \dots + \big((a^{i-1}_{k+1} - a^i_{k+1})B' q_{i+k} - b^i_{k+1} B q^{(1)}_{i+k} +$$

$$+ (a^{i-2}_{k+1} - a^i_{k+1} - b^i_{k+1})B'' q_1 q_{i+k-1} +$$

$$+ (b^{i-1}_{k+1} - b^i_{k+1})B' q_1 q^{(1)}_{i+k-1} + o(i+k-2)\big)\frac{\partial}{\partial q_i} + \dots =$$

$$= (k-1)!\Big(\dots + \big(B' q_{i+k} - B q^{(1)}_{i+k} + B'' q_1 q_{i+k-1}\big)\frac{\partial}{\partial q_i} + \dots\Big).$$

We are now ready to compute the commutators using the obtained estimates.[3]

Proposition 5. *We have the commutation relations*

$$\{\Psi_{-k}, \Psi_{-l}\} = \frac{(k-2)!(l-2)!}{(k+l-2)!}(k-l)\Psi_{-k-l}, \quad k, l \geq 0,$$

$$\{\Psi_{-k}, \Upsilon_l(B)\} = \frac{l(-l-1)!(k-2)!}{(l-k-1)!}\Upsilon_{l-k}(B), \quad k \geq 0,\ l \in \mathbb{Z},$$

$$\{\Upsilon_k(B), \Upsilon_l(\tilde{B})\} = \frac{(-k-1)!(-l-1)!}{(-k-l-1)!}\Upsilon_{k+l}(B\tilde{B}' - B'\tilde{B}), \quad k, l \in \mathbb{Z}.$$

Proof. The proof is a neat use of the above deduced estimates. ■

We change the initial basis of the algebra $\mathrm{sym}(\tilde{\mathscr{E}}^+)$ by

$$\Psi_{-k} \mapsto \frac{1}{(k-2)!}\Psi_{-k}, \qquad \Upsilon_l(B) \mapsto \frac{1}{(-l-1)!}\Upsilon_l(B)$$

and recall a standard construction. Let $\mathfrak{g}$ be a Lie $\mathbb{R}$-algebra and $\mathbb{R}_n[z] = \mathbb{R}[z]/(z^n)$ be the ring of truncated polynomials. Then the Lie algebra $\mathfrak{g}_{[n]} = \mathbb{R}_n[z] \otimes_{\mathbb{R}} \mathfrak{g}$ with the bracket

$$[a \otimes g, b \otimes h] = ab \otimes [g, h], \quad g, h \in \mathfrak{g}, \quad a, b \in \mathbb{R}_n[z],$$

is a graded Lie algebra, where $\mathfrak{g}_0 = \dots = \mathfrak{g}_{n-1} = \mathfrak{g}$ and all other components $\mathfrak{g}_i$ are trivial. A similar construction for polynomials in z^{-1} is denoted by $\mathfrak{g}_{[-n]}$. We also let $\mathfrak{V}[t]$ denote the Lie algebra of vector fields $A(t)\partial/\partial t$ on $\mathbb{R}$. We then have the following result.

Theorem 1. *The Lie algebra* $\mathrm{sym}(\tilde{\mathscr{E}}^+)$ *is isomorphic to the semidirect product of the nonpositive part*

$$\mathfrak{W}^- = \left\{ Z_k = z^{-k+1}\frac{\partial}{\partial z} \;\middle|\; k \in \mathbb{N} \cup \{0\} \right\}$$

[3] Everywhere below, we assume that $s! = 1$ for $s < 0$.

of the Witt algebra times the direct sum $\mathfrak{L}[y] \oplus \mathfrak{V}_{[-3]}[t]$ of

$$\mathfrak{L}[y] = \left\{ Y_m(B) = z^m\, B(y)\, \frac{\partial}{\partial y} \;\middle|\; m \in \mathbb{Z},\ B \in C^\infty(\mathbb{R}) \right\}$$

and

$$\mathfrak{V}[t]_{[-3]} = \left\{ X_s(A) = z^s\, A(t)\, \frac{\partial}{\partial t} \;\middle|\; s \in \{0,1,2\},\ A \in C^\infty(\mathbb{R}) \right\}$$

with the natural action of $z^{-k+1}\partial/\partial z$ on $\mathfrak{L}[y]$ and $\mathfrak{V}[t]_{[-3]}$.

In Theorem 1, the isomorphism maps Ψ_{-k} to Z_k, $\Upsilon_m(B)$ to $Y_m(B)$, and $\Theta_{-s}(A)$ to $X_s(A)$.

4.2. Symmetries in the negative hierarchy. Using Proposition 1, we set $F = 1$ in the defining equations of the negative hierarchy. After such a simplification, the study of the negative case becomes quite similar to that of the positive case. Any symmetry in $\tilde{\mathscr{E}}^-$ is a vector field

$$\mathbf{X}_\varphi = \widetilde{\mathbf{E}}_\varphi + \sum_{i=1}^{\infty}\left(\varphi_i \frac{\partial}{\partial r_i} + \sum_{j=1}^{\infty} \widetilde{D}_t^j(\varphi_i)\frac{\partial}{\partial r_i^{(j)}}\right), \tag{24}$$

where $\widetilde{\mathbf{E}}_\varphi$ with the total derivatives on $\tilde{\mathscr{E}}^-$ and

$$\Phi = \langle \varphi_0 = \varphi, \varphi_1, \ldots, \varphi_i, \ldots\rangle, \qquad \varphi_i \in C^\infty(\tilde{\mathscr{E}}^-),$$

satisfies the equations

$$\tilde{\ell}_{\mathscr{E}}(\varphi) \equiv \widetilde{D}_t\widetilde{D}_y(\varphi) - u_x\widetilde{D}_x\widetilde{D}_y(\varphi) + u_y\widetilde{D}_x^2(\varphi) - u_{xy}\widetilde{D}_x(\varphi) + u_{xx}\widetilde{D}_y(\varphi) = 0, \tag{25}$$

$$\widetilde{D}_x(\varphi_1) = \widetilde{D}_t(\varphi) - 2u_x\widetilde{D}_x(\varphi), \tag{26}$$

$$\widetilde{D}_y(\varphi_1) = -u_y\widetilde{D}_x(\varphi) - u_x\widetilde{D}_y(\varphi), \tag{27}$$

$$\widetilde{D}_x(\varphi_i) = r_{i-1,x}\widetilde{D}_x(\varphi) + u_x\widetilde{D}_x(\varphi_{i-1}) - \widetilde{D}_t(\varphi_{i-1}), \tag{28}$$

$$\widetilde{D}_y(\varphi_i) = r_{i-1,x}\widetilde{D}_y(\varphi) + u_y\widetilde{D}_x(\varphi_{i-1}), \tag{29}$$

$i > 1$. As in Sec. 4.1, the Jacobi bracket $\{\Phi, \Psi\}$ for any two symmetries Φ and Ψ is defined by

$$\mathbf{X}_{\{\Phi,\Psi\}} = [\mathbf{X}_\Phi, \mathbf{X}_\Psi].$$

4.2.1. Lifts of local symmetries and hierarchies of nonlocal ones. In what follows, we need the operator

$$\mathscr{Y}_+ = -x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial x} + 3r_1\frac{\partial}{\partial u} + \sum_{i\ge 1}(i+3)r_{i+1}\frac{\partial}{\partial r_i}. \tag{30}$$

Proposition 6. *The symmetries ψ_0, $\upsilon(B)$, and $\theta_{-2}(A)$ can be lifted to $\tilde{\mathscr{E}}^-$.*

Proof. We let

$$\Psi_0 = \langle \psi_0, \psi_0^1, \ldots, \psi_0^i, \ldots\rangle,$$

$$\Upsilon_0(B) = \langle \upsilon_0(B), \upsilon_0^1(B), \ldots, \upsilon_0^i(B), \ldots\rangle,$$

$$\Theta_{-2}(A) = \langle \theta_{-2}(A), \theta_{-2}^1(A), \ldots, \theta_{-2}^i(A), \ldots\rangle$$

denote the lifts and set

$$\psi_0^i = -(i+2)r_i + xr_{i,x}, \quad i \geq 1,$$

$$\upsilon_0^i(B) = Br_{i,y}, \quad i \geq 1,$$

$$\theta_{-2}^1(A) = -xA', \qquad \theta_{-2}^i(A) = \frac{1}{i}\mathscr{Y}_+(\theta_{-2}^{i-1}(A)), \quad i > 1.$$

The rest of the proof is similar to the proof of Proposition 2. ∎

The next step is to describe invisible symmetries. These symmetries must satisfy

$$\widetilde{D}_x(\varphi_1) = 0, \qquad \widetilde{D}_y(\varphi_1) = 0,$$

$$\widetilde{D}_x(\varphi_i) = u_x\widetilde{D}_x(\varphi_{i-1}) - \widetilde{D}_t(\varphi_{i-1}), \qquad \widetilde{D}_y(\varphi_i) = u_y\widetilde{D}_x(\varphi_{i-1}),$$

where $i > 1$.

Proposition 7. *For every $A = A(t)$ and $k \geq 3$, there exists a unique invisible symmetry $\Theta_{-k}(A)$ of weight $|\Theta_{-k}(A)| = -k$.*

Proof. We use notation (24) and set

$$\mathbf{X}_{\Theta_{-k}(A)} = \varphi_1^{\text{inv}}\frac{\partial}{\partial r_{k-2}} + \dots + \varphi_i^{\text{inv}}\frac{\partial}{\partial r_{k+i-3}} + \dots,$$

where $\varphi_1^{\text{inv}} = A$ and

$$\varphi_i^{\text{inv}} = \frac{1}{i-1}\mathscr{Y}_+(\varphi_{i-1}^{\text{inv}}), \quad i > 1.$$

The proof is by induction on i. ∎

We now consider the two functions

$$\psi_1 = 3r_1 + xu_t - 2uu_x, \qquad \psi_2 = 4r_2 + xr_1^{(1)} + 2uu_t - (xu_t + 3r_1)u_x.$$

It can be directly verified that they are shadows in $\tilde{\mathscr{E}}^-$, i.e., satisfy Eq. (25).

Proposition 8. *The shadows ψ_1 and ψ_2 can be extended to nonlocal symmetries of $\tilde{\mathscr{E}}^-$.*

Proof. It suffices to set

$$\Psi_1 = \langle \psi_1, \psi_1^1, \dots, \psi_1^i, \dots \rangle,$$

where

$$\psi_1^i = (i+3)r_{i+1} + xr_i^{(1)} - 2ur_{i,x},$$

and

$$\Psi_2 = \langle \psi_2, \psi_2^2, \dots, \psi_2^i, \dots \rangle,$$

where

$$\psi_2^i = (i+4)r_{i+2} + xr_{i+1}^{(1)} + 2ur_i^{(1)} - (xu_t + 3r_1)r_{i,x}.$$

The rest of the proof is a direct verification of Eqs. (26)–(29). ∎

Obviously,

$$|\Psi_1| = 1, \qquad |\Psi_2| = 2.$$

Similarly to the positive case, we now define the first hierarchy of nonlocal symmetries by setting

$$\Psi_k = \mathrm{ad}_{+1}^{k-2}(\Psi_2), \quad k \geq 3,$$

where $\mathrm{ad}_{+1}(\Phi) = \{\Psi_1, \Phi\}$. We have

$$|\Psi_k| = k.$$

We define the second hierarchy in Sec. 4.2.2 (see equality (31)).

4.2.2. The Lie algebra structure. As above, we need asymptotic estimates to compute the commutators. Similarly to the positive case, we establish the estimates for the symmetries Ψ_k by induction:

$$\mathbf{X}_{\Psi_k} = \cdots + \left(a_k^i r_{i+k} + b_k^i x r_{i+k-1}^{(1)} + o(i+k-2)\right)\frac{\partial}{\partial r_i} + \cdots,$$

where

$$a_k^i = (k-2)!(i+k+2), \qquad b_k^i = (k-2)!.$$

To unify the signs, we change $\Psi_0 \mapsto -\Psi_0$. Using this estimate, we easily prove the following statement.

Proposition 9. *We have the commutation relations*

$$\{\Psi_k, \Psi_l\} = \frac{(l-2)!(k-2)!(l-k)}{(k+l-2)!}\,\Psi_{k+l}$$

for all k and $l \geq 0$.

After the natural change $\Psi_k \mapsto \Psi_k/(k-2)!$, we obtain the commutators

$$\{\Psi_k, \Psi_l\} = (l-k)\Psi_{k+l}.$$

Moreover, for the new Ψ_k, the estimate becomes

$$\mathbf{X}_{\Psi_k} = \cdots + \left((i+k+2)r_{i+k} + x r_{i+k-1}^{(1)} + o(i+k-2)\right)\frac{\partial}{\partial r_i} + \cdots.$$

We now complete the hierarchy of symmetries $\{\Theta_{-k}(A)\}$ by setting

$$\Theta_k(A) = -\frac{1}{3}\{\Psi_{k+3}, \Theta_{-3}(A)\}, \quad k \geq -2. \tag{31}$$

We have $|\Theta_k(A)| = k$, and the elements of $\mathrm{sym}(\tilde{\mathscr{E}}^-)$ are distributed by weight as shown in Table 4.

Table 4

Weights	$\cdots$	$-l$	$\cdots$	-2	-1	0	1	$\cdots$	k	$\cdots$
	$\cdots$	$\Theta_{-l}(A)$	$\cdots$	$\Theta_{-2}(A)$	$\Theta_{-1}(A)$	Ψ_0 $\Theta_0(A)$ $\Upsilon_0(B)$	Ψ_1 $\Theta_1(A)$	$\cdots$	Ψ_k $\Theta_k(A)$	$\cdots$

Distribution of $\mathrm{sym}(\tilde{\mathscr{E}}^-)$ by weight.

The coefficients of the invisible symmetries are

$$\varphi_1^{\text{inv}} = A,$$
$$\varphi_2^{\text{inv}} = -xA',$$
$$\varphi_3^{\text{inv}} = -uA + \frac{1}{2}x^2A'',$$
$$\varphi_4^{\text{inv}} = -r_1A' + uxA'' - \frac{1}{6}x^3A''',$$

and we have the estimates

$$\varphi_i^{\text{inv}} = -A'r_{i-3} + xA''r_{i-4} + o(i-5)$$

for $i \geq 5$. Therefore,

$$\mathbf{X}_{\Theta_{-k}(A)} = A\frac{\partial}{\partial r_{k-2}} + \cdots + \varphi_{i-k+3}^{\text{inv}}\frac{\partial}{\partial r_i} + \cdots =$$
$$= A\frac{\partial}{\partial r_{k-2}} + \cdots + \left(-A'r_{i-k} + xA''r_{i-k-1} + o(i-k-2)\right)\frac{\partial}{\partial r_i} + \cdots$$

for $k \geq 3$.

Using the obtained estimates for Ψ_k and $\Theta_{-3}(A)$, we obtain

$$\mathbf{X}_{\Theta_k(A)} = -\frac{1}{3}[\mathbf{X}_{\Psi_{k+3}}, \mathbf{X}_{\Theta_{-3}(A)}] =$$
$$= \left[\cdots + \left((i+k+2)r_{i+k} + xr_{i+k-1}^{(1)} + o(i+k-2)\right)\frac{\partial}{\partial r_i} + \ldots,\right.$$
$$\left.\cdots + \left(-A'r_{i-3} + xA''r_{i-4} + o(i-5)\right)\frac{\partial}{\partial r_i} + \cdots\right] =$$
$$= \cdots + (-A'r_{i+k} + xA''r_{i+k-1} + o(i+k-2))\frac{\partial}{\partial r_i} + \cdots$$

for all $k \geq -2$. The following statement follows from these estimates.

Proposition 10. *We have*

$$\{\Psi_k, \Theta_l(A)\} = l\,\Theta_{k+l}(A)$$

for all $k \geq 0$ and $l \in \mathbb{Z}$.

Finally, we have the following statement.

Proposition 11. *We have*

$$\{\Theta_k(A), \Theta_l(\tilde{A})\} = \Theta_{k+l}(A\tilde{A}' - A'\tilde{A})$$

for all k, $l \in \mathbb{Z}$, and smooth functions $A = A(t)$ and $\tilde{A} = \tilde{A}(t)$.

Proof. The result easily follows from the above estimates for $k \le -3$ or $l \le -3$, but the method does not work when both $k > -3$ and $l > -3$. Nevertheless, in this case, we have

$$\begin{aligned}
\{\Theta_k(A), \Theta_l(\tilde{A})\} &= -\frac{1}{3}\{\{\Psi_{k+3}, \Theta_{-3}(A)\}, \Theta_l(\tilde{A})\} = \\
&= -\frac{1}{3}(\{\{\Psi_{k+3}, \Theta_l(\tilde{A})\}, \Theta_{-3}(A)\} + \{\Psi_{k+3}, \{\Theta_{-3}(A), \Theta_l(\tilde{A})\}\}) = \\
&= -\frac{1}{3}(l\,\{\Theta_{k+l+3}(\tilde{A}), \Theta_{-3}(A)\} + \{\Psi_{k+3}, \Theta_{l-3}(A\tilde{A}' - A'\tilde{A})\}) = \\
&= -\frac{1}{3}(-l\,\Theta_{k+l}(A\tilde{A}' - A'\tilde{A}) + (l-3)\,\Theta_{k+l}(A\tilde{A}' - A'\tilde{A})) = \\
&= \Theta_{k+l}(A\tilde{A}' - A'\tilde{A}),
\end{aligned}$$

and this finishes the proof. ■

We hence have a result similar to Theorem 1.

Theorem 2. *The Lie algebra* $\mathrm{sym}(\tilde{\mathscr{E}}^-)$ *is isomorphic to the direct sum* $(\mathfrak{W}^+ \ltimes \mathfrak{L}[t]) \oplus \mathfrak{V}[y]$ *of the semidirect product of the positive part*

$$\mathfrak{W}^+ = \left\{ z^{k+1} \frac{\partial}{\partial z} \,\middle|\, k \in \mathbb{N} \cup \{0\} \right\}$$

of the Witt algebra times

$$\mathfrak{L}[t] = \left\{ z^m A(t) \frac{\partial}{\partial t} \,\middle|\, m \in \mathbb{Z},\ A \in C^\infty(\mathbb{R}) \right\},$$

where the vector fields $z^{k+1}\partial/\partial z$ *act naturally on* $\mathfrak{L}[t]$ *and*

$$\mathfrak{V}[y] = \left\{ B(y) \frac{\partial}{\partial y} \,\middle|\, B \in C^\infty(\mathbb{R}) \right\}$$

is the Lie algebra of vector fields on the line.

5. Action of the recursion operators

We discuss the action of recursion operators in the hierarchies of nonlocal symmetries described above.

5.1. Action of the recursion operator to local symmetries and shadows. The algebra $\mathrm{sym}(\mathscr{E})$ admits a recursion operator $\hat{\chi} = \mathscr{R}_+(\chi)$ defined by the system (see [10])

$$\begin{aligned}
D_t(\hat{\chi}) &= u_y^{-1}\left(u_y\, D_x(\chi) - u_x\, D_y(\chi) + (u_x u_{xy} - u_y u_{xx})\hat{\chi}\right), \\
D_x(\hat{\chi}) &= u_y^{-1}\left(u_{xy}\,\hat{\chi} - D_y(\chi)\right).
\end{aligned} \tag{32}$$

This means that $\hat{\chi}$ is a solution of (5) whenever χ is. Another recursion operator $\chi = \mathscr{R}_-(\hat{\chi})$ is given by the system

$$\begin{aligned}
D_x(\chi) &= D_t(\hat{\chi}) - u_x\, D_x(\hat{\chi}) + u_{xx}\,\hat{\chi}, \\
D_y(\chi) &= -u_y\, D_x(\hat{\chi}) + u_{xy}\,\hat{\chi}.
\end{aligned} \tag{33}$$

The operators $\mathscr{R}_+$ and $\mathscr{R}_-$ are mutually inverse.

The actions of $\mathscr{R}_+$ and $\mathscr{R}_-$ on $\mathrm{sym}(\mathscr{E})$ can be prolonged to the shadows of nonlocal symmetries in $\mathrm{sym}(\tilde{\mathscr{E}}^+)$ and $\mathrm{sym}(\tilde{\mathscr{E}}^-)$ if we replace the derivatives D_t, D_x, and D_y in (32) and (33) with $\widehat{D}_t$, $\widehat{D}_x$, and $\widehat{D}_y$ defined as

$$\begin{aligned}\widehat{D}_x = D_x &+ \sum_{j=0}^{\infty} \widehat{D}_y^j\left(\frac{1}{u_y}\right)\frac{\partial}{\partial q_1^{(j)}} + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \widehat{D}_y^j\left(\frac{q_{i-1}^{(1)}}{u_y}\right)\frac{\partial}{\partial q_i^{(j)}} + \\ &+ \sum_{j=0}^{\infty} \widehat{D}_t^j(u_x^2 - u_t)\frac{\partial}{\partial r_1^{(j)}} + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \widehat{D}_t^j(u_x r_{i-1,x} - r_{i-1,t})\frac{\partial}{\partial r_i^{(j)}},\end{aligned}$$

$$\widehat{D}_y = D_y + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} q_i^{(j+1)}\frac{\partial}{\partial q_i^{(j)}} + \sum_{j=0}^{\infty} \widehat{D}_t^j(u_x\, u_y)\frac{\partial}{\partial r_1^{(j)}} + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \widehat{D}_t^j(u_y r_{i-1,x})\frac{\partial}{\partial r_i^{(j)}},$$

$$\widehat{D}_t = D_t + \sum_{j=0}^{\infty} \widehat{D}_y^j\left(\frac{u_x}{u_y}\right)\frac{\partial}{\partial q_1^{(j)}} + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \widehat{D}_y^j\left(\frac{u_x}{u_y} q_{i-1}^{(1)} - \widehat{D}_x\left(q_{i-1}^{(0)}\right)\right)\frac{\partial}{\partial q_i^{(j)}} + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} r_i^{(j+1)}\frac{\partial}{\partial r_i^{(j)}},$$

i.e., if we consider the Whitney product of the coverings $\tilde{\mathscr{E}}^+$ and $\tilde{\mathscr{E}}^-$. The results of the replacement are also denoted by $\mathscr{R}_+$ and $\mathscr{R}_-$.

We note that the operators act nontrivially on the "vacuum":

$$\mathscr{R}_+(0) = \theta_{-2}(A), \qquad \mathscr{R}_-(0) = \upsilon_0(B),$$

which immediately follows from Eqs. (32) and (33). Hence, the actions are reasonable to consider modulo $\theta_{-2}(A)$ for $\mathscr{R}_+$ and $\upsilon_0(B)$ for $\mathscr{R}_-$. Taking this remark into account, we have the following statement.

Proposition 12. *Modulo images of a trivial symmetry, the action of the recursion operators has the form*

$$\mathscr{R}_+(\theta_i(A)) = \begin{cases}\alpha_i^+\theta_{i-1}(A), & i > -2,\\ 0, & i = -2,\end{cases} \qquad \mathscr{R}_-(\theta_i(A)) = \alpha_i^-\theta_{i+1}(A), \quad i \ge -2,$$

$$\mathscr{R}_+(\upsilon_i(B)) = \beta_i^+\upsilon_{i+1}(B), \quad i \le 0, \qquad \mathscr{R}_-(\upsilon_i(B)) = \begin{cases}\beta_i^-\upsilon_{i+1}(B), & i < 0,\\ 0, & i = 0,\end{cases}$$

$$\mathscr{R}_+(\psi_i) = \gamma_i^+\psi_{i-1}, \qquad \mathscr{R}_-(\psi_i) = \gamma_i^-\psi_{i+1}, \quad i \in \mathbb{Z},$$

where $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma_i^\pm$ are nonzero constants.

Proof. It suffices to note that the weights of $\mathscr{R}_+$ and $\mathscr{R}_-$ are respectively -1 and $+1$, that their action (modulo images of 0) does not change the dependence of shadows on y and t, and that the only shadows that can be taken to 0 are $\theta_{-2}(A)$ and $\upsilon_0(B)$. ■

We note that the recursion operators $\mathscr{R}_+$ and $\mathscr{R}_-$ "glue" the shadows ψ_m of nonlocal symmetries in the coverings $\tilde{\mathscr{E}}^+$ and $\tilde{\mathscr{E}}^-$ to each other and connect the series of $\theta_k(A)$ to the series of $\upsilon_j(B)$:

$$\cdots \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \psi_{-1} \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \psi_0 \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \psi_1 \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \cdots$$

$$\cdots \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \upsilon_{-1}(B) \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \upsilon_0(B) \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} 0 \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \theta_{-2}(A) \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \theta_{-1}(A) \underset{\mathscr{R}_+}{\overset{\mathscr{R}_-}{\rightleftarrows}} \cdots.$$

5.2. Recursion relations for symmetries of the positive covering. We describe an operator providing an alternative way to construct elements of $\mathrm{sym}(\tilde{\mathscr{E}}^+)$. For this, we express the functions u_x and u_y in (9):

$$u_x = \frac{q_{1,t}}{q_{1,x}},\qquad u_y = \frac{1}{q_{1,x}}. \tag{34}$$

The compatibility condition for this system is the equation

$$q_{1,xx} = q_{1,t}q_{1,xy} - q_{1,x}q_{1,ty}, \tag{35}$$

known as the universal hierarchy equation (see [13], [14]). This means that systems (9) and (34) define a Bäcklund transformation between (4) and (35) (see [15]). Substituting (34) in (10), we obtain

$$q_{k,t} = q_{1,t}q_{k-1,y} - q_{k-1,x},\qquad q_{k,x} = q_{1,x}q_{k-1,y}, \tag{36}$$

where $k \geq$. The compatibility conditions for this system after the change $k-1 \mapsto k$ become

$$q_{k,xx} = q_{1,t}q_{k,xy} - q_{1,x}q_{k,ty},\quad k \geq 2. \tag{37}$$

Proposition 13. *Systems* (34) *and* (36) *define a Bäcklund autotransformation for infinite system of partial differential equations* (35) *and* (37).

Proof. The compatibility conditions of (34) and (36) are definitions of (35) and (37). From (36), we obtain the inverse transformation

$$q_{k-1,x} = -q_{k,t} + \frac{q_{1,t}}{q_{1,x}}q_{k,x},\qquad q_{k-1,y} = \frac{q_{k,x}}{q_{1,x}},$$

whose compatibility conditions also coincide with (37). ■

Corollary 1. *The linearizations of* (9) *and* (36)

$$D_t(\hat{\chi}_1) = u_y^{-2}\big(u_y\, D_x(\chi_0) - u_x\, D_y(\chi_0)\big), \tag{38}$$

$$D_x(\hat{\chi}_1) = -u_y^{-2}\, D_y(\chi_0), \tag{39}$$

$$D_t(\hat{\chi}_k) = q_{1,t}D_y(\chi_{k-1}) + q_{k-1,y}D_t(\chi_1) - D_x(\chi_{k-1}), \tag{40}$$

$$D_x(\hat{\chi}_k) = q_{1,x}D_y(\chi_{k-1}) + q_{k-1,y}D_x(\chi_1) \tag{41}$$

define a recursion operator

$$\mathscr{Q}((\chi_0, \chi_1, \chi_2, \dots, \chi_k, \dots)) = (\chi_0, \hat{\chi}_1, \hat{\chi}_2, \dots, \hat{\chi}_k, \dots)$$

for $\mathrm{sym}(\tilde{\mathscr{E}}^+)$.

We note that the symmetries ξ and $\mathscr{Q}(\xi)$ have the same shadows and consequently differ by an invisible symmetry. Therefore, the recursion operator $\mathscr{Q}$ seems useless at first glance, but this is not the case: it provides an alternative way to lift shadows to nonlocal symmetries in $\tilde{\mathscr{E}}^+$. More precisely, we take a local symmetry or shadow χ_0, then (38) and (39) allow calculating χ_1; applying (40) and (41) with $k = 2$ to χ_1, we obtain χ_2; applying (40) and (41) with $k = m$ to χ_{m-1}, we obtain χ_m, and so on.

Proposition 14. *For all $k \geq 1$ and $j = 0, 1, 2,$ we have the relations*

$$\mathscr{Q}(\psi_0) = \psi_0^1, \qquad \mathscr{Q}(\psi_0^k) = \psi_0^{k+1},$$

$$\mathscr{Q}(\theta_{-j}(A)) = \theta_{-j}^1(A), \qquad \mathscr{Q}(\theta_{-j}^k(A)) = \theta_{-j}^{k+1}(A).$$

Moreover, we have the equalities

$$\mathscr{Q}(v_0(B)) = v_0^1(B), \qquad \mathscr{Q}(v_0^k(B)) = v_0^{k+1}(B).$$

This proposition is proved in the same way as Proposition 2.

Unfortunately, we could not construct a similar recursion operator for symmetries in the negative covering.

6. Conclusion

We have completely described nonlocal symmetries associated with the Lax representation of the three-dimensional rdDym equation. The revealed Lie algebra structure of these symmetries seems quite interesting, and we intend to further study nonlocal symmetries of other Lax integrable equations in [5].

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Integrability properties of some equations obtained by symmetry reductions

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In our recent paper [1], we gave a complete description of symmetry reduction of four Lax-integrable (i.e., possessing a zero-curvature representation with a non-removable parameter) 3-dimensional equations. Here we study the behavior of the integrability features of the initial equations under the reduction procedure. We show that the ZCRs are transformed to nonlinear differential coverings of the resulting 2D-systems similar to the one found for the Gibbons-Tsarev equation in [17]. Using these coverings we construct infinite series of (nonlocal) conservation laws and prove their nontriviality. We also show that the recursion operators are not preserved under reductions.

Keywords: Partial differential equations, symmetry reductions, solutions, the Gibbons-Tsarev equation, Lax-integrable equations

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Introduction

In [1] we gave a complete description of symmetry reductions for four three dimensional systems: the universal hierarchy equation, the 3D rdDym equation, the modified Veronese web equation, and Pavlov's equation. The result comprised more than 30 equations, but the majority of them were either exactly solvable or linearized by the generalized Legendre transformations. Nevertheless, there were 10 'interesting' reductions, among which two well-known equations, i.e., the Liouville

*Permanent address: Independent University of Moscow, B. Vlasevsky 11, 119002 Moscow, Russia and Gibbons-Tsarev equations, [3,5]. The rest eight can be divided in two groups by their symmetry properties: five equations admit infinite-dimensional Lie algebras of contact symmetries (with functional parameters) and three others possess finite-dimensional symmetry algebras. These are

$$u_y u_{xy} - u_x u_{yy} = e^y u_{xx} \tag{0.1}$$

(reduction of the universal hierarchy equation),

$$u_{yy} = (u_x + x)u_{xy} - u_y(u_{xx} + 2) \tag{0.2}$$

(reduction of the 3D rdDym equation), and

$$u_{xx} = (x - u_y)u_{xy} + (2y + u_x)u_{yy} - u_y \tag{0.3}$$

(reduction of the Pavlov equation)[a]. These equations are pair-wise inequaivalent (see Section 5).

We deal with these three equations below and study how the integrability properties of the initial 3D systems behave under reduction. More precisely, we construct (Section 1) the reductions of the zero-curvature representations for Equations (0.1)–(0.2) and show that they result in differential coverings of the form

$$w_x = \frac{a_2 w^2 + a_1 w + a_0}{w^2 + c_1 w + c_0}, \qquad w_y = \frac{b_2 w^2 + b_1 w + b_0}{w^2 + c_1 w + c_0},$$

where a_i, b_i, c_i are functions in x, y, u, u_x, and u_y. These coverings are similar to the one found in [17] for the Gibbons-Tsarev equation and this resemblance, by all means, reflects the relations between generalized Gibbons-Tsarev equations and integrable 3D-systems [18]. In Section 3, for every nonlinear covering we construct an infinite series of conservation laws and prove their non-triviality.

We also study the behavior of the recursion operators for symmetries of three-dimensional systems and show that these operators do not survive under reduction (Section 4).

In Section 2 local symmetries and cosymmetries of the reduction equations are described. The corresponding conservation laws are presented in the Appendix.

Throughout the text the notion of (differential) covering is understood in the sense of [9].

1. Reduction of the Lax pairs

Using Lax representations of the 3D equations, whose reductions are the equations at hand, we construct here nonlinear coverings of Equations (0.1)–(0.3).

1.1. *Equation* (0.1)

This equation is obtained as the reduction of the universal hierarchy equation[b]

$$u_{yy} = u_z u_{xy} - u_y u_{xz} \tag{1.1}$$

with respect to the symmetry

$$\varphi = u_z + u_x + y u_y + u. \tag{1.2}$$

[a] All the reductions of the modified Veronese web equation were either exactly solvable or linearizable.

[b] To save the notation here and below, we denote by u the dependent and by x, y the independent variables. These are *not* the same as in the initial equation; see the details in [1].

Equivalently, this reduction may be written in the form

$$u_{yy} = u_y u_{xx} - (u_x + u)u_{xy} + u_x u_y \tag{1.3}$$

and Equation (0.1) transforms to (1.3) by the change of variables $x \mapsto y$, $y \mapsto x$, $u \mapsto -e^y u$.

Equation (1.1) admits the following Lax representation

$$\begin{aligned} w_z &= (wu_z - u_y)w^{-2}w_x, \\ w_y &= u_y w^{-1} w_x. \end{aligned} \tag{1.4}$$

The symmetry φ can be extended to a symmetry $\Phi = (\varphi, \chi)$ of (1.4), where

$$\chi = w_z + w_x + yw_y + w$$

and the corresponding reduction leads to the covering

$$\begin{aligned} w_x &= -\frac{w^3}{w^2 - (u_x + u)w - u_y}, \\ w_y &= -\frac{u_y w^2}{w^2 - (u_x + u)w - u_y} \end{aligned} \tag{1.5}$$

of Equation (1.3). Note that the first equation above is cubic in w, but by an appropriate gauge transformation it can be converted to a quadratic one, see Subsection 3.2 below.

Remark 1.1. Equation (0.1) can be written in the potential form

$$\left(\frac{u_y}{u_x}\right)_y = \left(\frac{e^y}{u_x}\right)_x,$$

the corresponding Abelian covering being

$$v_x = \frac{u_y}{u_x}, \qquad v_y = \frac{e^y}{u_x}. \tag{1.6}$$

Then v enjoys the equation

$$v_y - v_{yy} = v_y v_{xx} - v_x v_{xy}, \tag{1.7}$$

which also admits the rational covering

$$\begin{aligned} w_x &= \frac{wv_x - xv_x + v_y}{w^2 + (-2x + v_x)w + x^2 - xv_x + v_y}, \\ w_y &= \frac{wv_y - xv_y}{w^2 + (-2x + v_x)w + x^2 - xv_x + v_y}. \end{aligned}$$

of the same type. □

1.2. *Equation* (0.2)

This equation was obtained as the reduction of the 3D rdDym equation

$$u_{ty} = u_x u_{xy} - u_y u_{xx} \tag{1.8}$$

with respect to the symmetry

$$\varphi = u_t - xu_x - u_y + 2u. \tag{1.9}$$

The Lax representation for Equation (1.8) is

$$\begin{aligned} w_t &= (u_x + w)w_x, \\ w_y &= -u_y w^{-1} w_x. \end{aligned} \tag{1.10}$$

The symmetry φ extends to the one of (1.10): $\Phi = (\varphi, \chi)$, where

$$\chi = w_t - xw_x - w_y + u.$$

Reduction of the covering (1.10) with respect to Φ leads to the covering

$$\begin{aligned} w_x &= -\frac{w^2}{w^2 + (u_x - x)w + u_y}, \\ w_y &= \frac{u_y w}{w^2 + (u_x - x)w + u_y}. \end{aligned} \tag{1.11}$$

over Equation (0.2).

1.3. *Equation* (0.3)

Finally, Equation (0.3) is the reduction of the Pavlov equation

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \tag{1.12}$$

with respect to the symmetry

$$\varphi = u_t - 2xu_x - yu_y + 3u. \tag{1.13}$$

The Pavlov equation possesses the Lax pair

$$\begin{aligned} w_t &= (w^2 - wu_x - u_y)w_x, \\ w_y &= (w - u_x)w_x. \end{aligned} \tag{1.14}$$

The symmetry φ lifts to the symmetry $\Phi = (\varphi, \chi)$ of (1.14), where

$$\chi = w_t - 2xw_x - yw_y + w.$$

Reduction of the covering (1.14) with respect to this symmetry results in the nonlinear covering

$$\begin{aligned} w_x &= -\frac{w(w - u_y)}{w^2 - (u_y + x)w + xu_y - u_x - 2y}, \\ w_y &= -\frac{w}{w^2 - (u_y + x)w + xu_y - u_x - 2y} \end{aligned} \tag{1.15}$$

of Equation (0.3).

Remark 1.2. Equation (0.3) has a close relative. Namely, if we accomplish reduction of the Pavlov equation using another symmetry

$$\varphi' = u_t - yu_x + 2x$$

the resulting equation will be

$$u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2. \tag{1.16}$$

The symmetry φ' can also be lifted to (1.14) by $\Phi' = (\varphi', \chi')$, where

$$\chi' = w_t - yw_x + 1,$$

and the reduction of (1.14) will be

$$\begin{aligned} w_x &= -\frac{1}{w^2 - u_x w - u_y - y}, \\ w_y &= -\frac{w - u_x}{w^2 - u_x w - u_y - y}. \end{aligned} \tag{1.17}$$

By the change of variables $u \mapsto u - y^2/2$, Equation (1.16) transforms to the Gibbons-Tsarev equation, see [5],

$$u_{yy} = u_y u_{xx} - u_x u_{xy} - 1,$$

while (1.15) becomes

$$\begin{aligned} w_x &= -\frac{1}{w^2 - u_x w - u_y}, \\ w_y &= -\frac{w - u_x}{w^2 - u_x w - u_y}, \end{aligned}$$

cf. [17]. □

Remark 1.3. Equations (0.1), (0.2) and (0.3) are known to admit linear Lax representations with non-removable parameter (see [10,11,19] for the universal hierarchy equation, [13,20] for the 3Drd-Dym equation, and [4, 19] for the Pavlov equation). Nonlinear Lax pairs (1.4), (1.10), and (1.14) can be obtained from their linear counterparts by the standard procedure proposed in [21] or by the methods used in [13, 14].

2. Local symmetries and cosymmetries of the reduced equations

We present here computational results on classical symmetries and cosymmetries of Equations (0.1)–(0.3), i.e., solutions of the equations

$$\ell_{\mathscr{E}}(\varphi) = 0$$

and

$$\ell^*_{\mathscr{E}}(\psi) = 0,$$

where $\ell_{\mathscr{E}}$ is the linearization of the equation at hand and $\ell^*_{\mathscr{E}}$ is its formally adjoint and φ and ψ depend on x, y, u, u_x, u_y (see, e.g., [7]). The conservation laws corresponding to classical cosymmetries are presented in the Appendix below. The spaces of solutions are denoted by $\mathrm{sym}_c(\mathscr{E})$ and $\mathrm{cosym}_c(\mathscr{E})$, respectively.

All the equations under consideration happen to possess a scaling symmetry and thus admit weights (which we denote by $|\cdot|$) with respect to which they become homogeneous.

2.1. *Equation* (0.1)

We consider this equation in the form (1.3), i.e.,

$$u_{yy} = u_y u_{xx} - (u_x + u)u_{xy} + u_x u_y.$$

The weights are

$$|x| = 0, \quad |y| = 1, \quad |u| = -1, \quad |u_x| = -1, \quad |u_y| = -2.$$

Symmetries

The defining equation for symmetries is[c]

$$D_y^2(\varphi) = u_y D_x^2(\varphi) - (u_x + u)D_x D_y(\varphi) + (u_y - u_{xy})D_x(\varphi) + (u_{xx} + u_x)D_y(\varphi) - u_{xy}\varphi.$$

The space $\mathrm{sym}_c(\mathscr{E})$ is generated by the symmetries

$$\varphi_{-1} = u_y, \quad \varphi_0 = yu_y + u, \quad \varphi_0' = u_x, \quad \varphi_1 = e^{-x},$$

where the subscripts coincide with the weights[d].

Cosymmetries

The defining equation for cosymmetries of Equation (0.1) is

$$D_y^2(\psi) = u_y D_x^2(\psi) - (u_x + u)D_x D_y(\psi) + 2(u_{xy} + u_y)D_x(\psi) - 2(u_{xx} + u_x)D_y(\psi) - 3u_{xy}\psi.$$

The space $\mathrm{cosym}_c(\mathscr{E})$ is 6-dimensional and is spanned by the following cosymmetries:

$$\psi_{-3} = e^{4x}(3u_x^2 + 8u^2 + 10uu_x + 2u_y), \quad \psi_{-2} = e^{3x}(3u + 2u_x), \quad \psi_{-1} = e^{2x}$$

and

$$\psi_3 = \frac{1}{u_y^2}, \qquad \psi_4 = \frac{2u_x - yu_y + 2u}{u_y^3},$$

$$\psi_5 = \frac{-4u_x y u_y + 6uu_x + 3u_x^2 - 4yuu_y + 3u^2 + 2u_y + y^2 u_y^2}{u_y^4},$$

where superscript coincides with the weight[e].

2.2. *Equation* (0.2)

The weights are

$$|x| = 1, \quad |y| = 0, \quad |u| = 2, \quad |u_x| = 1, \quad |u_y| = 2.$$

[c] Here and below D_x and D_y denote the total derivatives with respect to x and y.
[d] To a symmetry φ we assign the weight of the corresponding evolutionary vector field $\mathbf{E}_\varphi$.
[e] To every cosymmetry we assign the weight of the corresponding variational form, see [8]

Symmetries

The linearized equation is

$$D_y^2(\varphi) = (u_x + x)D_xD_y(\varphi) - u_yD_x^2(\varphi) + u_{xy}D_x(\varphi) - (u_{xx} + 2)D_y(\varphi).$$

The space $\mathrm{sym}_c(\mathscr{E})$ is generated by the symmetries

$$\varphi_{-2} = 1, \quad \varphi_{-1} = u_x + x, \quad \varphi_0 = u - \frac{1}{2}xu_x, \quad \varphi_0' = u_y.$$

Cosymmetries

The defining equation for cosymmetries reads

$$D_y^2(\psi) = (u_x + x)D_xD_y(\psi) - u_yD_x^2(\psi) - 2u_{xy}D_x(\psi) + (2u_{xx} + 3)D_y(\psi).$$

The space $\mathrm{cosym}_c(\mathscr{E})$ is generated by the cosymmetries

$$\psi_{-3} = \frac{e^{-2y}(u_x + x)}{u_y^3}, \qquad \psi_2 = 1,$$

$$\psi_{-2} = \frac{e^{-y}}{u_y^2}, \qquad \psi_3 = u_x + 2x.$$

2.3. *Equation* (0.3)

The weights of variables are

$$|x| = 1, \quad |y| = 2, \quad |u| = 3, \quad |u_x| = 2, \quad |u_y| = 1.$$

in this case.

Symmetries

The symmetries are defined by the equation

$$D_x^2(\varphi) = (x - u_y)D_xD_y(\varphi) + (2y + u_x)D_y^2(\varphi) - D_y(\varphi)$$

and the space $\mathrm{sym}_c(\mathscr{E})$ is generated the symmetries

$$\varphi_0 = -\frac{1}{3}xu_x - \frac{2}{3}yu_y + u, \qquad \varphi_{-1} = u_x - xu_y + y - \frac{1}{2}x^2,$$

$$\varphi_{-2} = u_y + 2x, \qquad \varphi_{-3} = 1.$$

Cosymmetries

The defining equation for cosymmetries is of the form

$$D_x^2(\psi) = (x - u_y)D_xD_y(\psi) + (2y + u_x)D_y^2 - u_{yy}D_x + 3(2 - u_{xy})D_y.$$

The space $\mathrm{cosym}_c(\mathscr{E})$ is 6-dimensional and and is spanned by the elements

$$\psi_7 = \frac{54}{5}xu_xu_y + \frac{164}{5}xu_yy + \frac{256}{5}x^2y + 2xu + \frac{4}{5}uu_y + \frac{12}{5}u_y^2u_x + 4yu_x + \frac{36}{5}u_y^2y$$

$$
\begin{aligned}
&+\frac{82}{5}x^2u_x+\frac{512}{15}x^3u_y+\frac{32}{5}xu_y^3+\frac{96}{5}x^2u_y^2+\frac{32}{5}y^2+\frac{512}{15}x^4+\frac{3}{5}u_x^2+u_y^4,\\
\psi_6 &= \frac{49}{4}xy+4xu_x+\frac{3}{2}u_yu_x+\frac{9}{2}u_yy+\frac{49}{4}x^2u_y+\frac{21}{4}xu_y^2+\frac{343}{24}x^3+\frac{1}{4}u+u_y^3,\\
\psi_5 &= 4xu_y+6x^2+2y+\frac{2}{3}u_x+u_y^2,\\
\psi_4 &= \frac{5}{2}x+u_y,\\
\psi_3 &= 1,\\
\psi_{-1} &= \frac{1}{(-xu_y+u_x+2y)^2}.
\end{aligned}
$$

3. Hierarchies of nonlocal conservation laws

Using the nonlinear coverings presented in Section 1 we construct here infinite hierarchies of nonlocal conservation laws for Equations (0.1)–(0.1).

3.1. *A general construction*

The initial step of the construction is the so-called *Pavlov reversing*, [21] (see [6] for the invariant geometrical interpretation). Let $\mathcal{E}$ be an equation in two independent variables x and y and unknown function u and

$$w_x = X(x,y,[u],w), \qquad w_y = Y(x,y,[u],w)$$

be a differential covering over $\mathcal{E}$, where $[u]$ denotes u itself and a collection of its derivatives up to some finite order. Then the system

$$\psi_x = -X(x,y,[u],\lambda)\psi_\lambda, \qquad \psi_y = -Y(x,y,[u],\lambda)\psi_\lambda \tag{3.1}$$

is also compatible modulo $\mathcal{E}$ (thus, the nonlocal variable w turns into a formal parameter in the new setting).

Assume now that

$$
\begin{aligned}
X &= X_{-1}\lambda + X_0 + \frac{X_1}{\lambda} + \dots + \frac{X_i}{\lambda^i} + \dots,\\
Y &= Y_{-1}\lambda + Y_0 + \frac{Y_1}{\lambda} + \dots + \frac{Y_i}{\lambda^i} + \dots,
\end{aligned}
$$

where X_i, Y_i, $i \geq -1$, are functions in x, y and $[u]$, and also expand ψ in formal Laurent series

$$\psi = \psi_{-1}\lambda + \psi_0 + \frac{\psi_1}{\lambda} + \dots + \frac{\psi_i}{\lambda^i} + \dots$$

Then (3.1) implies

$$\psi_{i,x} = -\sum_{j+k=i+1} kX_j\psi_k, \qquad \psi_{i,y} = -\sum_{j+k=i+1} kY_j\psi_k,$$

or

$$\psi_{-1,x} = -X_{-1}\psi_{-1}, \qquad \psi_{-1,y} = -Y_{-1}\psi_{-1};$$

$$\begin{aligned}
&\psi_{0,x} = -X_0\psi_{-1}, && \psi_{0,y} = -Y_0\psi_{-1};\\
&\psi_{1,x} = X_{-1} - X_1\psi_{-1}, && \psi_{1,y} = Y_{-1} - Y_1\psi_{-1};\\
&\psi_{2,x} = 2X_{-1}\psi_2 + X_0\psi_1 - X_2\psi_{-1}, && \psi_{2,y} = 2Y_{-1}\psi_2 + Y_0\psi_1 - Y_2\psi_{-1};\\
&\dots && \dots
\end{aligned}$$

and

$$\begin{aligned}
\psi_{k,x} &= kX_{-1}\psi_k + (k-1)X_0\psi_{i-1} + \dots + X_{k-2}\psi_1 - X_k\psi_{-1},\\
\psi_{k,y} &= kY_{-1}\psi_k + (k-1)Y_0\psi_{i-1} + \dots + Y_{k-2}\psi_1 - Y_k\psi_{-1}
\end{aligned}$$

for all $k > 2$.

In general, this system defines an infinite-dimensional non-Abelian covering (which may be trivial generally) over the base equation $\mathscr{E}$, but in the particular case $X_{-1} = Y_{-1} = 0$ the covering becomes Abelian, i.e., transforms to an infinite series of (nonlocal) conservation laws. Indeed, the first pair of equations reads

$$\psi_{-1,x} = 0, \qquad \psi_{-1,y} = 0$$

in this case and without loss of generality we may set $\psi_{-1} = 1$. The rest equations read

$$\begin{aligned}
&\psi_{0,x} = -X_0, && \psi_{0,y} = -Y_0;\\
&\psi_{1,x} = -X_1, && \psi_{1,y} = -Y_1;\\
&\psi_{2,x} = X_0\psi_1 - X_2, && \psi_{2,y} = Y_0\psi_1 - Y_2;\\
&\psi_{3,x} = 2X_0\psi_2 + X_1\psi_1 - X_3, && \psi_{3,x} = 2Y_0\psi_2 + Y_1\psi_1 - Y_3;\\
&\dots && \dots
\end{aligned}$$

and

$$\begin{aligned}
\psi_{k,x} &= (k-1)X_0\psi_{k-1} + (k-2)X_1\psi_{k-2} + \dots + X_{k-2}\psi_1 - X_k,\\
\psi_{k,y} &= (k-1)Y_0\psi_{k-1} + (k-2)Y_1\psi_{k-2} + \dots + Y_{k-2}\psi_1 - Y_k
\end{aligned} \tag{3.2}$$

for all $k > 3$.

Remark 3.1. The first two pairs of equations define local conservation laws (probably, trivial) and the potential ψ_0 does not enter the other equations. This means that the obtained covering is the Whitney product of the one-dimensional Abelian covering τ_0 associated to ψ_0 and the infinite-dimensional τ_* related to $\psi_1, \psi_2, \dots$ We shall deal with τ_* below. □

We now confine ourselves to the case

$$X = \frac{a_2w^2 + a_1w + a_0}{w^2 + c_1w + c_0}, \qquad Y = \frac{b_2w^2 + b_1w + b_0}{w^2 + c_1w + c_0}, \tag{3.3}$$

where a_i, b_i, and c_i are functions in x, y, and $[u]$, and deduce the needed Laurent expansions. One has

$$\frac{a_2\lambda^2 + a_1\lambda + a_0}{\lambda^2 + c_1\lambda + c_0} = \left(a_2 + \frac{a_1}{\lambda} + \frac{a_0}{\lambda^2}\right)\cdot\left(\frac{1}{1 + \frac{c_1\lambda + c_0}{\lambda^2}}\right)$$

$$= \left(a_2 + \frac{a_1}{\lambda} + \frac{a_0}{\lambda^2}\right) \cdot \sum_{i \geq 0} \left(-\frac{c_1\lambda + c_0}{\lambda^2}\right)^i.$$

Let us present temporally the second factor in the form

$$\sum_{i \geq 0} \left(-\frac{c_1\lambda + c_0}{\lambda^2}\right)^i = \sum_{i \geq 0} \frac{d_i}{\lambda^i}.$$

Then

$$\frac{a_2\lambda^2 + a_1\lambda + a_0}{\lambda^2 + c_1\lambda + c_0} = \left(a_2 + \frac{a_1}{\lambda} + \frac{a_0}{\lambda^2}\right) \cdot \sum_{i \geq 0} \frac{d_i}{\lambda^i}$$
$$= a_2 d_0 + \frac{a_2 d_1 + a_1 d_0}{\lambda} + \frac{a_2 d_2 + a_1 d_1 + a_0 d_0}{\lambda^2} + \dots + \frac{a_2 d_i + a_1 d_{i-1} + a_0 d_{i-2}}{\lambda^i} + \dots$$

Compute the coefficients d_i now. One has

$$\left(-\frac{c_1\lambda + c_0}{\lambda^2}\right)^i = (-1)^i \sum_{j=0}^{i} \binom{i}{j} \frac{c_1^j c_0^{i-j}}{\lambda^{2i-j}},$$

from where it follows that

$$d_0 = 1, \qquad d_1 = -c_1$$

and

$$d_i = \begin{cases} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+j}{2j} c_0^{k-j} c_1^{2j} & \text{if } i = 2k, \\ \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k+j+1}{2j+1} c_0^{k-j} c_1^{2j+1} & \text{if } i = 2k+1 \end{cases} \tag{3.4}$$

for $i > 1$, Or, in shorter notation

$$d_i = \sum_{j=0}^{[i/2]} (-1)^{[i/2]-j+p(i)} \binom{[i/2]+j+p(i)}{2j+p(i)} c_0^{[i/2]-j} c_1^{2j+p(i)}, \tag{3.5}$$

where $p(i) = i \bmod 2$ is the parity of i and $[k/2]$ is the integer part.

Gathering together the results of the above computations, one obtains that in the case of coverings (3.3) we have $X_{-1} = Y_{-1} = 0$, while other coefficients are

$$\begin{aligned} X_0 &= a_2, & Y_0 &= b_2; \\ X_1 &= a_1 - a_2 c_1, & Y_1 &= b_1 - b_2 c_1; \\ X_2 &= a_0 - a_1 c_1 + a_2(c_1^2 - c_0), & Y_2 &= b_0 - b_1 c_1 + b_2(c_1^2 - c_0); \\ &\dots & &\dots \\ X_i &= a_0 d_{i-2} + a_1 d_{i-1} + a_2 d_i, & Y_i &= b_0 d_{i-2} + b_1 d_{i-1} + b_2 d_i; \\ &\dots & &\dots, \end{aligned}$$

where the functions d_i are given by (3.4).

Let us now show how these general constructions look like in the particular cases of the equations under consideration.

3.2. *Equation* (0.1)

Note first that the covering (1.5) is not of the form (3.3). Nevertheless, it can be transformed to the needed form by the gauge transformation $w \mapsto we^{-x}$. Then (1.5) acquires the form

$$w_x = \frac{(u_x+u)e^x w^2 - u_y e^{2x} w}{w^2 - (u_x+u)e^x w - u_y e^{2x}}, \qquad w_y = -\frac{u_y e^x w^2}{w^2 - (u_x+u)e^x w - u_y e^{2x}}.$$

We have $|w| = -1$.

Thus,

$$\begin{aligned} a_0 &= 0, & a_1 &= -u_y e^{2x} & a_2 &= (u_x+u)e^x, \\ b_0 &= 0, & b_1 &= 0, & b_2 &= -u_y e^x, \\ c_0 &= -u_y e^{2x}, & c_1 &= -(u_x+u)e^x. \end{aligned}$$

Let us compute the coefficients d_i. By (3.4), we have

$$d_{2k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k+j}{2j} \left(-u_y e^{2x}\right)^{k-j} (-(u_x+u)e^x)^{2j}$$
$$= e^{2kx} \sum_{j=0}^{k} \binom{k+j}{2j} u_y^{k-j} (u_x+u)^{2j}$$

and

$$d_{2k+1} = \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k+j+1}{2j+1} \left(-u_y e^{2x}\right)^{k-j} (-(u_x+u)e^x)^{2j+1}$$
$$= e^{(2k+1)x} \sum_{j=0}^{k} \binom{k+j+1}{2j+1} u_y^{k-j} (u_x+u)^{2j+1},$$

or

$$d_i = e^{ix} \sum_{j=0}^{[i/2]} \binom{[i/2]+j+p(i)}{2j+p(i)} u_y^{[i/2]-j} (u_x+u)^{2j+p(i)}. \tag{3.6}$$

Hence,

$$\begin{aligned} X_0 &= (u_x+u)e^x, & Y_0 &= -u_y e^x; \\ X_1 &= \left((u_x+u)^2 - u_y\right) e^{2x}, & Y_1 &= (u_x+u)u_y e^{2x} \end{aligned}$$

and

$$X_i = e^{(i+1)x} \left((u_x+u)^{i+1} + \sum_{j=1}^{[(i+1)/2]} \left(\binom{i-j}{i-2j} - \binom{i-j}{i-2j+1} \right) u_y^j (u_x+u)^{i-2j+1} \right),$$
$$Y_i = -e^{(i+1)x} \sum_{j=0}^{[i/2]} \binom{[i/2]+j+p(i)}{2j+p(i)} u_y^{[i/2]-j+1} (u_x+u)^{2j+p(i)}$$

for $i > 1$ (we assume $\binom{\alpha}{\beta} = 0$ for $\beta < 0$). Obviously,

$$|X_i| = -i-1, \qquad |Y_i| = -i-2.$$

The functions X_i, Y_i define, by Equations (3.2), the infinite number of nonlocal variables ψ_i for Equation (0.1) with

$$|\psi_i| = -i-1.$$

The corresponding conservation laws have the same weights and the first three of them coincide (up to equivalence) with the local conservation laws ω_{-2}, ω_{-3}, ω_{-4} described in Section 2.1. The first essentially nonlocal one is associated to ψ_3.

3.3. *Equation* (0.2)

Due to Equations (1.11), one has

$$\begin{aligned} &a_0 = 0, && a_1 = 0 && a_2 = -1, \\ &b_0 = 0, && b_1 = u_y, && b_2 = 0, \\ &c_0 = u_y, && c_1 = u_x - x. \end{aligned}$$

Hence,

$$\begin{aligned} &X_0 = -1, && Y_0 = 0; \\ &X_1 = u_x - x, && Y_1 = u_y; \\ &X_2 = -(u_x-x)^2 + u_y, && Y_2 = -u_y(u_x-x) \end{aligned}$$

and

$$X_i = -d_i = \sum_{j=0}^{[i/2]} (-1)^{[i/2]-j+p(i)+1} \binom{[i/2]+j+p(i)}{2j+p(i)} u_y^{[i/2]-j} (u_x-x)^{2j+p(i)},$$

$$Y_i = u_y d_{i-1} = \sum_{j=0}^{[(i-1)/2]} (-1)^{[(i-1)/2]-j+p(i-1)} \times$$

$$\times \binom{[(i-1)/2]+j+p(i-1)}{2j+p(i-1)} u_y^{[(i-1)/2]-j+1} (u_x-x)^{2j+p(i-1)}$$

for $i > 2$. Consequently,

$$\begin{aligned} &\psi_{0,x} = -X_0 = 1, && \psi_{0,y} = -Y_0 = 0; \\ &\psi_{1,x} = -X_1 = -u_x + x, && \psi_{1,y} = -Y_1 = -u_y \end{aligned}$$

and one may set

$$\psi_0 = x, \qquad \psi_1 = -u + \frac{x^2}{2},$$

while

$$\psi_{2,x} = (u_x - x)^2 + u_y + u - \frac{x^2}{2}, \quad \psi_{2,y} = (u_x - x)u_y$$

and for $i > 2$

$$\psi_{i,x} = -(i-1)\psi_{i-1} + (i-2)X_1\psi_{i-2} + \cdots + X_{i-3}\psi_2 + \left(\frac{x^2}{2} - u\right)X_{i-2} - X_i,$$

$$\psi_{i,y} = (i-2)Y_1\psi_{i-2} + \cdots + Y_{i-3}\psi_2 + \left(\frac{x^2}{2} - u\right)Y_{i-2} - Y_i,$$

where X_k, Y_k are given by the above formulas.

One has

$$|X_i| = i, \quad |Y_i| = i+1, \quad |\psi_i| = i+1.$$

The conservation law corresponding to ψ_i is of the weight $i+1$ and the first two ones, up to equivalence coincide with those described in Section 2.2, while all the others are essentially nonlocal.

3.4. *Equation* (0.3)

By Equation (1.15), we have

$$\begin{aligned} &a_0 = 0, && a_1 = u_y && a_2 = -1, \\ &b_0 = 0, && b_1 = -1, && b_2 = 0, \\ &c_0 = xu_y - u_x - 2y, && c_1 = -(u_y + x). && \end{aligned}$$

Consequently,

$$\begin{aligned} &X_0 = -1, && Y_0 = 0; \\ &X_1 = -x, && Y_1 = -1; \\ &X_2 = -u_x - x^2 - 2y, && Y_2 = -u_y - x \end{aligned}$$

and

$$X_i = u_y d_{i-1} - d_i, \qquad Y_i = -d_{i-1}$$

for $i > 2$, where

$$d_i = \sum_{j=0}^{[i/2]} (-1)^{[i/2]-j} \binom{[i/2] + j + p(i)}{2j + p(i)} (xu_y - u_x - 2y)^{[i/2]-j} (u_y + x)^{2j+p(i)}.$$

One has

$$|X_i| = i, \qquad |Y_i| = i-1.$$

Thus we have

$$\psi_{1,x} = x, \qquad \psi_{1,y} = 1;$$

$$\psi_{2,x} = u_x + \frac{x^2}{2} + y, \qquad \psi_{2,y} = u_y + x$$

and we may set

$$\psi_1 = \frac{x^2}{2} + y, \qquad \psi_2 = u + xy + \frac{x^3}{6}.$$

Then the other potentials are defined by

$$\begin{aligned}
\psi_{i,x} &= -(i-1)\psi_{i-1} - (i-2)\psi_{i-2}(i-3)X_2\psi_{i-3} + \dots \\
&\dots + 3X_{i-4}\psi_3 + \left(2u + 2xy + \frac{x^3}{3}\right)X_{i-3} + \left(\frac{x^2}{2} + y\right)X_{i-2} - X_i, \\
\psi_{i,y} &= -(i-2)\psi_{i-2}(i-3)Y_2\psi_{i-3} + \dots \\
&\dots + 3Y_{i-4}\psi_3 + \left(2u + 2xy + \frac{x^3}{3}\right)Y_{i-3} + \left(\frac{x^2}{2} + y\right)Y_{i-2} - Y_i,
\end{aligned}$$

$i > 2$. We have

$$|\psi_i| = i + 1.$$

The conservation laws associated with $\psi_3, \dots, \psi_7$ are equivalent to $\omega_4, \dots, \omega_8$ introduced in Section 2.3. The first essentially nonlocal conservation law corresponds to ψ_8.

3.5. *Proof of nontriviality*

We shall now prove that the above constructed conservation laws are nontrivial. To this end, introduce the notation $\mathscr{E}_\alpha$, $\alpha = 1, 2, 3$, for Equations (0.1), (0.2) and (0.3), respectively, and

$$\tau_{i,\alpha} \colon \mathscr{E}_{i,\alpha} \to \mathscr{E}_\alpha$$

for the coverings defined by the nonlocal variables $\psi_\alpha, \dots, \psi_i$. Let

$$D_x^{i,\alpha}, \qquad D_y^{i,\alpha}$$

be the total derivatives on $\mathscr{E}_{i,\alpha}$.

Proposition 3.1. *For all $i \geq \alpha$, the only solutions of the system*

$$D_x^{i,\alpha}(f) = 0, \qquad D_y^{i,\alpha}(f) = 0 \tag{3.7}$$

are constants.

Proof. Let us present the total derivatives in the form

$$D_x^{i,\alpha} = D_x^\alpha + X^{i,\alpha}, \quad D_y^{i,\alpha} = D_y^\alpha + Y^{i,\alpha},$$

where D_x^α, D_y^α are the total derivatives on $\mathscr{E}_\alpha$ and $X^{i,\alpha}$, $Y^{i,\alpha}$ are the 'nonlocal tails':

$$X^{i,\alpha} = \sum_{j=\alpha}^{i} X_j^{i,\alpha} \frac{\partial}{\partial \psi_j}, \qquad Y^{i,\alpha} = \sum_{j=\alpha}^{i} Y_j^{i,\alpha} \frac{\partial}{\partial \psi_j},$$

$X_j^{i,\alpha}$, $Y_j^{i,\alpha}$ being the right-hand sides of the defining equations (3.2) for the potentials ψ.

From the constructions of Sections 3.2–3.4 one readily sees that the quantities $X_j^{i,\alpha}$ and $Y_j^{i,\alpha}$ are polynomials in u_x and u_y and, moreover,

$$X^{i,1} = \pm e^{(i+1)x} u_x^{i+1} \frac{\partial}{\partial \psi_i} + o, \qquad Y^{i,1} = \pm e^{(i+1)x} u_x^i u_y \frac{\partial}{\partial \psi_i} + o;$$

$$X^{i,2} = \pm u_x^i \frac{\partial}{\partial \psi_i} + o; \qquad Y^{i,2} = \pm u_x^{i-1} \frac{\partial}{\partial \psi_i} + o;$$

$$X^{i,3} = \pm u_y^{i-2} u_x \frac{\partial}{\partial \psi_i} + o, \qquad Y^{i,3} = \pm u_y^{i-1} \frac{\partial}{\partial \psi_i} + o,$$

where o denotes terms of lower degree.

Now, the proof goes by induction. For small i's the result follows from the fact that the cosymmetries corresponding to the local conservation laws do not vanish and these conservation laws are of different weights. Assume now that the statement is valid for all $k < i$ and consider Equation (3.7). Then from the above estimates it follows that $\partial f / \partial \psi_i = 0$. □

Evidently, nontriviality of the constructed conservation laws is a direct consequence of the Proposition 3.1.

4. On reductions of the recursion operators

We show here that symmetry reductions of Equations (1.1), (1.8), and (1.12) are incompatible with their recursion operators and thus the latter are not inherited by Equations (0.1), (0.2), and (0.3), respectively.

4.1. *A general construction*

We treat here recursion operators for symmetries as Bäcklund transformations of the tangent coverings, cf. [12]. More precisely, let $\mathscr{E}$ be a differential equation given by the system

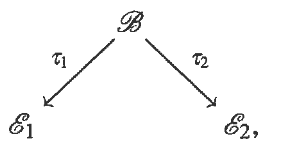
$$\mathscr{E} = \{F = 0\}, \quad F = (F^1(x,y,[u]), \dots, F^s(x,y,[u])),$$

F^j being functions on some jet space, [7]. Here, as above, $[u]$ denotes the collection of u and its derivatives. The tangent covering $\mathrm{t} = \mathrm{t}_{\mathscr{E}} \colon \mathscr{T}\mathscr{E} \to \mathscr{E}$ is the projection $(x,y,[u],[q]) \mapsto (x,y,[u])$ of the system

$$\mathscr{T}\mathscr{E} = \{F(x,y,[u]) = 0,\ \ell_F(x,y,[u],[q]) = 0\}$$

to $\mathscr{E}$. The characteristic property of t is that its sections that preserve the Cartan (higher contact) distribution are identified with symmetries of $\mathscr{E}$.

A Bäcklund transformation between equations $\mathscr{E}_1$ and $\mathscr{E}_2$ is a diagram



where τ_1 and τ_2 are coverings. It relates solutions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to each other. A recursion operator between symmetries of $\mathcal{E}_1$ and $\mathcal{E}_2$ is a Bäcklund transformation of the form

$$\begin{array}{ccc} & & \mathcal{T}\mathcal{E}_1 \xrightarrow{\mathrm{t}_{\mathcal{E}_1}} \mathcal{E}_1 \\ & \nearrow^{\tau_1} & \\ \mathcal{R} & & \\ & \searrow_{\tau_2} & \\ & & \mathcal{T}\mathcal{E}_2 \xrightarrow{\mathrm{t}_{\mathcal{E}_2}} \mathcal{E}_2. \end{array}$$

In particular, if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ it relates symmetries of $\mathcal{E}$ to each other. Then $\mathcal{R}$ may be considered as an equation

$$\mathcal{R} \subset \mathcal{T}\mathcal{E} \otimes_{\mathcal{E}} \mathcal{T}\mathcal{E}$$

in the Whitney product of $\mathrm{t}_{\mathcal{E}}$ with itself.

Any symmetry $\varphi = \varphi(x, y, [u])$ of $\mathcal{E}$ admits a natural lift $\Phi = (\varphi, \varphi')$ to $\mathcal{T}\mathcal{E}$. To this end, it suffices to set

$$\varphi' = \frac{\partial \varphi}{\partial u} q + \dots + \frac{\partial \varphi}{\partial u_\sigma} q_\sigma + \dots$$

Choose a symmetry φ of $\mathcal{E}$ and denote by $\mathrm{r}_\varphi \colon \mathcal{E} \to \mathcal{E}_\varphi$ the corresponding reduction map. Then the diagram

$$\begin{array}{ccc} \mathcal{T}\mathcal{E} & \xrightarrow{\mathrm{t}_{\mathcal{E}}} & \mathcal{E} \\ \downarrow \mathrm{r}_\Phi & & \downarrow \mathrm{r}_\varphi \\ (\mathcal{T}\mathcal{E})_\Phi = \mathcal{T}(\mathcal{E}_\varphi) & \xrightarrow{\mathrm{t}_{\mathcal{E}_\varphi}} & \mathcal{E}_\varphi \end{array}$$

is commutative. An immediate consequence of this fact is

Proposition 4.1. *Let $\mathcal{R} \subset \mathcal{T}\mathcal{E} \otimes_{\mathcal{E}} \mathcal{T}\mathcal{E}$ be a recursion operator for symmetries of equation $\mathcal{E}$ and φ be a symmetry of $\mathcal{E}$. If $\mathcal{R}$ is invariant with respect to φ then $\mathcal{R}_\Phi$ is a recursion operator for symmetries of $\mathcal{E}_\varphi$.*

4.2. *Recursion operators for symmetries of 3D systems*

We briefly recall here the results on recursion operators for symmetries of Equation (1.1), (1.8), and (1.12) obtained in [15, 16]

The universal hierarchy equation

Equation (1.1) admits the following recursion operator

$$\begin{aligned} D_y(\tilde{\varphi}) &= u_y D_x(\varphi) - u_{xy}\,\varphi, \\ D_z(\tilde{\varphi}) &= u_z D_x(\varphi) - D_y(\varphi) - u_{xz}\,\varphi \end{aligned} \tag{4.1}$$

that acts on its symmetries.

The 3DrdDym equation

The Bäcklund transformation

$$\begin{aligned} D_x(\tilde{\varphi}) &= u_x D_x(\varphi) - D_t(\varphi) - u_{xx}\varphi, \\ D_y(\tilde{\varphi}) &= u_y D_x(\varphi) - u_{xy}\varphi \end{aligned} \tag{4.2}$$

is a recursion operator for symmetries of Equation (1.8).

The Pavlov equation

The relations

$$\begin{aligned} D_x(\tilde{\varphi}) &= u_x D_x(\varphi) + D_y(\varphi) - u_{xx}\varphi, \\ D_y(\tilde{\varphi}) &= D_t(\varphi) + u_y D_x(\varphi) - u_{xy}\varphi. \end{aligned} \tag{4.3}$$

are a recursion operator for symmetries of Equation (1.12).

4.3. *The negative result*

Here we show that the general construction of Section 4.1 produces no recursion operator for the reduced equations under consideration.

Proposition 4.2. *Recursion operators* (4.1), (4.2) *and* (4.3) *are not invariant with respect to the natural lifts of the symmetries* (1.2), (1.9), *and* (1.13), *respectively.*

Proof. By direct check. □

Remark 4.1. The same fact holds for the reduction of the Pavlov equation that leads to the Gibbons-Tsarev equation.

Remark 4.2. We also tried to construct recursion operators for all the equations at hand directly, but this did not lead us to positive results either.

5. Discussion

Let us first establish the following fact:

Proposition 5.1. *Equations* (0.1), (0.2), *and* (0.3) *are pair-wise inequivalent with respect to an arbitrary contact transformation.*

Proof. Let us first compare dimensions (see Table 1). Consequently, only Equations (0.1) and (0.3)

	$\dim \mathrm{sym}_c(\mathscr{E})$	$\dim \mathrm{cosym}_c(\mathscr{E})$
Equation (0.1)	4	6
Equation (0.2)	4	4
Equation (0.3)	4	6

Table 1. Dimensions of symmetry and cosymmetry spaces

may be equivalent. Now, the Lie algebra structure of $\mathrm{sym}_c(\mathscr{E})$ for Equations (0.1) and (0.3) is

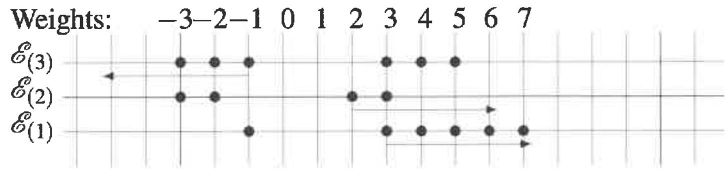


Fig. 1. Distribution of cosymmetries

presented in Table 2. One can see that dimension of the commutant in the first case is 2, while in the second case it equals 3. Thus, the algebras are not isomorphic. □

Remark 5.1. The equations under consideration are not equivalent to the Gibbons-Tsarev equation, because the symmetry algebra of the latter is five-dimensional.

Nevertheless, as we saw, all these equations have several common features. In particular, we would like to indicate how local cosymmetries of our equations are distributed with respect to weights (see Figure 1). In all three cases, they fit into two disjoint groups with certain gaps between them: the first one consist of cosymmetries whose corresponding conservation laws are members of infinite series (these are underlined by arrows, and the arrow itself indicates the direction to which the sequence of conservation laws goes). The second group includes 'standing-alone' cosymmetries.

Remark 5.2. A similar picture is observed in the case of the Gibbons-Tsarev equation. It also possesses a 'standing-alone' cosymmetry of order three.

A natural question arises: does there exist a construction, similar to the one of Section 3, that allows to embed the conservation laws corresponding to the 'standing-alone' cosymmetries into other infinite hierarchies?

Another question relates to the algebras of nonlocal symmetries in the infinite-dimensional coverings constructed above. It seems that such an algebra for Equation (0.3) should be similar (or isomorphic to that of the Gibbons-Tsarev equation), while the algebras for Equations (0.1) and (0.2) are different: all these Lie algebras are graded, but in the first two cases all homogeneous components are one-dimensional and for other equations this is not the case.

Finally, it is interesting to study the structure of symmetries and cosymmetries of the reductions that admit symmetry algebras with functional parameters (see the Introduction) and compare them with the results described here.

All these problems are subject to future research.

6. Appendix: Conservation laws

We present here the conservation laws that correspond to the cosymmetries described above. Everywhere below $|\omega_i| = i$. We also use the notation $\psi_\omega \in \mathrm{cosym}_c(\mathscr{E})$ for the generating function of a conservation law ω.

Eq. (0.1)	φ_0	φ_0'	φ_1
φ_{-1}	φ_{-1}	0	0
φ_0	$*$	0	φ_1
φ_0'	$*$	$*$	$-\varphi_1$

Eq. (0.3)	φ_{-2}	φ_{-1}	φ_0
φ_{-3}	0	0	$-\varphi_{-3}$
φ_{-2}	$*$	$-\varphi_{-3}$	$\frac{2}{3}\varphi_{-2}$
φ_{-1}	$*$	$*$	$-\frac{1}{3}\varphi_{-1}$

Table 2. Commutators in $\mathrm{sym}_c\,\mathscr{E}_{(0.1)}$ and $\mathrm{sym}_c\,\mathscr{E}_{(0.3)}$

Equation (0.1)

The space of corresponding conservation laws is 6-dimensional and is spanned by the following elements $\omega_i = P_i\,dx + Q_i\,dy$:

$$
\begin{aligned}
P_{-4} &= e^{4x}(u_x^2u_y + 8u^3u_x + 13u^2u_x^2 + 2uu_x^3 + 8u^2u_y + u_y^2 - 3uu_x^2u_{xx} + 2uu_xu_y \\
&\quad - 2uu_xu_{xy} - 2uu_{xx}u_y),\\
Q_{-4} &= ue^{4x}(-2u_xu_yu_{xx} + 3u_x^2u_y - u_x^2u_{xy} + 8uu_xu_y + 2uu_xu_{xy} + 4u_y^2 \\
&\quad - 2u_yu_{xy});\\
P_{-3} &= e^{3x}(-uu_{xy} + u_xu_y + 3u^2u_x + uu_x^2 - 2uu_xu_{xx}),\\
Q_{-3} &= ue^{3x}(-u_yu_{xx} - u_xu_{xy} + uu_{xy} + 2u_xu_y);\\
P_{-2} &= -e^{2x}(-u_y + uu_x + uu_{xx}),\\
Q_{-2} &= -ue^{2x}u_{xy};\\
P_2 &= -\frac{1}{u_y},\\
Q_2 &= \frac{1}{u_y}(u_x + u);\\
P_3 &= \frac{1}{u_y^3}(u_y^2y + 2uu_{xy} - uu_y - 2u_xu_y),\\
Q_3 &= -\frac{1}{u_y^3}(uu_y^2y + u_xu_y^2y + 2u^2u_{xy} - u^2u_y + 2uu_xu_{xy} - 4uu_xu_y - 2uu_{xx}u_y \\
&\quad - u_x^2u_y);\\
P_4 &= \frac{1}{u_y^4}(-u_y^3y^2 - 4uu_{xy}u_yy + 2uu_y^2y + 4u_xu_y^2y - u^2u_y + 6uu_xu_{xy} \\
&\quad - 2uu_xu_y - 2uu_{xx}u_y - 3u_x^2u_y - u_y^2),\\
Q_4 &= \frac{1}{u_y^4}(uu_y^3y^2 + u_xu_y^3y^2 + 4u^2u_{xy}u_yy - 2u^2u_y^2y + 4uu_xu_{xy}u_yy \\
&\quad - 8uu_xu_y^2y - 4uu_{xx}u_y^2y - 2u_x^2u_y^2y + u^3u_y - 6u^2u_xu_{xy} + 3u^2u_xu_y \\
&\quad - 6uu_x^2u_{xy} + 9uu_x^2u_y + 6uu_xu_{xx}u_y + u_x^3u_y - 2uu_{xy}u_y + 4uu_y^2).
\end{aligned}
$$

Here $|\psi_\omega| = |\omega| + 1$.

Equation (0.2)

The space of conservation laws is 4-dimensional and is generated by $\omega_i = P_i\,dx + Q_i\,dy$ of the form

$$
\begin{aligned}
P_{-2} &= \frac{1}{2}(2uu_{xy} - 2u_xu_y - u_yx)\frac{e^{-2y}}{u_y^3},\\
Q_{-2} &= \frac{1}{2}(2uu_xu_{xy} - 2uu_{xx}u_y + 2uu_{xy}x - u_x^2u_y - 2u_xu_yx - u_yx^2 - 2uu_y)\frac{e^{-2y}}{u_y^3};
\end{aligned}
$$

$$
\begin{aligned}
P_{-1} &= -\frac{e^{-y}}{u_y},\\
Q_{-1} &= -(u_x+x)\frac{e^{-y}}{u_y};\\
P_3 &= uu_{xx}+3u+u_y,\\
Q_3 &= uu_{xy}+u_yx;\\
P_4 &= -\frac{1}{2}uu_{xy}+2u_yx+\frac{1}{2}u_xu_y+\frac{5}{2}uxu_{xx}+uu_xu_{xx}+8ux+\frac{1}{2}uu_x,\\
Q_4 &= 2u_yx^2+\frac{1}{2}u_xu_yx+2uu_{xy}x+\frac{1}{2}uu_xu_{xy}+\frac{1}{2}uu_{xx}u_y+uu_y.
\end{aligned}
$$

Again, $|\psi_\omega| = |\omega|-1$.

Equation (0.3)

The space of conservation laws is 6-dimensional; elements $\omega_i = P_i\,dx+Q_i\,dy$ of a basis are

$$
\begin{aligned}
P_8 = {}& u_y^3u_xu_{yy}u+\frac{1}{5}uxu_y^3u_{xy}+\frac{116}{5}ux^2u_xu_{xy}+\frac{162}{5}uxu_xu_y+\frac{229}{15}ux^3u_yu_{xy}\\
&+\frac{8}{5}ux^2u_y^2u_{xy}+\frac{3}{5}u_y^2u_xu_{xy}u+\frac{379}{15}u_xu_{yy}ux^3+\frac{758}{15}u_{yy}ux^3y+2u_y^3u_{yy}uy\\
&+\frac{184}{5}u_{yy}uxy^2+\frac{348}{5}ux^2yu_{xy}-\frac{48}{5}xyu_xu_y^2+\frac{6}{5}uyu_y^2u_{xy}+\frac{72}{5}u_yu_{yy}uy^2\\
&+\frac{12}{5}u_yu_x^2u_{yy}u+80uxyu_y+\frac{36}{5}uyu_xu_{xy}-\frac{164}{5}x^2yu_xu_y-\frac{6}{5}yu_xu_y^3-\frac{8}{5}x^2u_xu_y^3\\
&-\frac{1024}{15}x^4yu_y+43ux^3u_y+\frac{48}{5}uyu_y^2+\frac{18}{5}uu_xu_y^2-\frac{164}{5}xy^2u_y^2\\
&-\frac{1}{5}xu_xu_y^4+\frac{52}{5}uy^2u_{xy}+\frac{14}{5}x^2u_x^2u_y-\frac{64}{5}x^2yu_y^3+\frac{2048}{5}ux^2y+2yu_x^2u_y+\frac{16}{5}uxu_y^3\\
&+\frac{82}{5}uyu_x+\frac{32}{5}u_xu_y^2u_{yy}ux+\frac{64}{5}u_y^2u_{yy}uxy+24uxyu_yu_{xy}+\frac{132}{5}u_xu_{yy}uxy\\
&+12u_xu_yu_{yy}uy+\frac{96}{5}u_xu_yu_{yy}ux^2+\frac{192}{5}u_yu_{yy}ux^2y+\frac{56}{5}uxu_xu_yu_{xy}+\frac{1}{5}u_x^2u_y^3\\
&+\frac{3}{5}u_x^3u_y+\frac{256}{5}uy^2+\frac{4096}{15}ux^4-\frac{241}{5}u^2x+\frac{2}{5}uu_y^4-\frac{24}{5}y^2u_y^3-\frac{64}{5}y^3u_y-\frac{2}{5}yu_y^5\\
&+\frac{8}{5}uu_x^2-\frac{512}{5}x^2y^2u_y+\frac{64}{5}ux^2u_y^2+\frac{113}{5}ux^2u_x-\frac{512}{15}x^3yu_y^2-\frac{16}{5}xyu_y^4-4y^2u_xu_y\\
&-\frac{32}{5}x^3u_xu_y^2+\frac{6}{5}u_x^2u_{xy}u-\frac{256}{15}x^4u_xu_y+\frac{127}{3}ux^4u_{xy}+xu_x^2u_y^2,\\
Q_8 = {}& \frac{36}{5}uyu_xu_{yy}-\frac{72}{5}xyu_xu_y+\frac{42}{5}uyu_y^2u_{yy}+\frac{92}{5}uxyu_{xy}+\frac{32}{5}uxu_y^2u_{xy}+4uxu_xu_{xy}\\
&+\frac{256}{5}ux^2yu_{yy}+\frac{12}{5}u_xu_{xy}u_yu+\frac{64}{3}ux^3u_yu_{yy}+\frac{72}{5}ux^2u_y^2u_{yy}+\frac{36}{5}uyu_yu_{xy}\\
&+\frac{28}{5}uxu_y^3u_{yy}+\frac{96}{5}ux^2u_xu_{yy}+\frac{96}{5}ux^2u_yu_{xy}+3u_y^2u_xu_{yy}u+\frac{52}{5}uy^2u_{yy}\\
&+\frac{6}{5}u_x^2u_{yy}u+u_y^4u_{yy}u+\frac{256}{15}ux^4u_{yy}+\frac{32}{5}xy^2u_y+\frac{94}{5}uyu_y+\frac{379}{15}ux^3u_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{256}{5}x^2yu_x+\frac{82}{5}uxu_x+16uxu_y^2-\frac{17}{5}xu_x^2u_y+u_y^3u_{xy}u+\frac{32}{5}uu_xu_y\\
&-\frac{133}{15}x^3u_xu_y+\frac{256}{5}x^3yu_y+\frac{512}{5}uxy+\frac{176}{5}uxyu_yu_{yy}+\frac{64}{5}uxu_xu_yu_{yy}+\frac{12}{5}uu_y^3\\
&+\frac{2048}{15}ux^3+\frac{512}{15}x^5u_y-2yu_x^2-\frac{32}{5}y^2u_x-\frac{41}{5}x^2u_x^2-\frac{512}{15}x^4u_x-\frac{1}{5}u_x^3;
\end{aligned}
$$

$$
\begin{aligned}
P_7=&\ \frac{13}{4}uyu_xu_{yy}-\frac{25}{4}xyu_xu_y+2uyu_y^2u_{yy}+\frac{65}{4}uxyu_{xy}+\frac{1}{4}uxu_y^2u_{xy}+\frac{23}{4}uxu_xu_{xy}\\
&+\frac{65}{4}ux^2yu_{yy}+\frac{3}{2}u_xu_{xy}u_yu+\frac{13}{4}uyu_yu_{xy}+\frac{65}{8}ux^2u_xu_{yy}+\frac{47}{8}ux^2u_yu_{xy}\\
&+u_y^2u_xu_{yy}u+\frac{9}{2}uy^2u_{yy}+\frac{1}{2}u_x^2u_{yy}u-\frac{49}{2}xy^2u_y+\frac{45}{4}uyu_y+\frac{391}{24}ux^3u_{xy}+2uxu_x\\
&+\frac{7}{2}uxu_y^2+\frac{5}{4}xu_x^2u_y+\frac{9}{2}uu_xu_y-\frac{49}{8}x^3u_xu_y-\frac{343}{12}x^3yu_y+\frac{343}{4}uxy-\frac{1}{4}xu_xu_y^3\\
&-yu_xu_y^2+\frac{21}{2}uxyu_yu_{yy}+\frac{21}{4}uxu_xu_yu_{yy}+\frac{1}{2}uu_y^3+\frac{2401}{24}ux^3\\
&-\frac{9}{2}y^2u_y^2+\frac{1}{4}u_x^2u_y^2-\frac{1}{2}yu_y^4-\frac{53}{8}u^2-\frac{7}{2}xyu_y^3-\frac{49}{4}x^2yu_y^2-\frac{7}{4}x^2u_xu_y^2+\frac{131}{8}ux^2u_y,
\end{aligned}
$$

$$
\begin{aligned}
Q_7=&\ \frac{21}{4}uxu_xu_{yy}+\frac{21}{4}uxu_yu_{xy}+\frac{11}{2}uyu_yu_{yy}+\frac{35}{4}ux^2u_yu_{yy}+\frac{9}{2}uxu_y^2u_{yy}\\
&+2u_yu_xu_{yy}u+14uxyu_{yy}-\frac{49}{4}xyu_x+\frac{343}{8}ux^2+\frac{49}{4}uy-2xu_x^2-\frac{1}{2}u_x^2u_y-\frac{343}{24}x^3u_x\\
&+2uu_x+\frac{343}{24}x^4u_y+\frac{5}{2}uu_y^2+\frac{1}{2}u_xu_{xy}u+\frac{49}{4}x^2yu_y+\frac{49}{6}ux^3u_{yy}+\frac{65}{8}ux^2u_{xy}\\
&-\frac{9}{4}yu_xu_y+\frac{9}{4}uyu_{xy}-\frac{33}{8}x^2u_xu_y+u_y^3u_{yy}u+u_y^2u_{xy}u;
\end{aligned}
$$

$$
\begin{aligned}
P_6=&\ 12uy+\frac{2}{3}uu_y^2+36ux^2+\frac{1}{3}u_x^2u_y-\frac{2}{3}yu_y^3+\frac{7}{3}uxu_xu_{yy}+\frac{14}{3}uxyu_{yy}+u_yu_xu_{yy}u\\
&+2uyu_yu_{yy}+\frac{17}{3}uxu_y+2uxu_yu_{xy}-\frac{2}{3}yu_xu_y+\frac{8}{3}uyu_{xy}+u_xu_{xy}u+\frac{19}{3}ux^2u_{xy}\\
&-12x^2yu_y-2x^2u_xu_y-4xyu_y^2-\frac{1}{3}xu_xu_y^2-4y^2u_y-\frac{1}{3}uu_x,
\end{aligned}
$$

$$
\begin{aligned}
Q_6=&\ 12xu-6x^2u_x-2yu_x+6x^3u_y-\frac{1}{3}u_x^2+\frac{10}{3}uxu_yu_{yy}-\frac{5}{3}xu_xu_y+4ux^2u_{yy}\\
&+u_y^2u_{yy}u+\frac{7}{3}uxu_{xy}+\frac{8}{3}uyu_{yy}+u_xu_{yy}u+u_{xy}u_yu+2xu_yy;
\end{aligned}
$$

$$
\begin{aligned}
P_5=&-5xu_yy-\frac{5}{2}xu_xu_y-u_y^2y-\frac{1}{2}u_y^2u_x+\frac{25}{2}xu+\frac{1}{2}u_xu_{yy}u+uyu_{yy}-\frac{1}{2}u_{xy}u_yu\\
&+\frac{1}{2}uxu_{xy}-\frac{1}{2}uu_y,
\end{aligned}
$$

$$
Q_5=\frac{1}{2}uu_{xy}+\frac{5}{2}u-\frac{5}{2}xu_x-\frac{1}{2}u_yu_x+\frac{5}{2}x^2u_y-2xu_y^2-\frac{1}{2}u_y^3;
$$

$$
P_4=-u_yu_x-2u_yy+4u,
$$

$$
Q_4=-u_y^2+xu_y-u_x;
$$

$$
P_0=\frac{u_y}{xu_y-u_x-2y},
$$

$$Q_0 = -\frac{1}{xu_y - u_x - 2y}.$$

Here $|\psi_\omega| = |\omega| - 1$.

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Symmetry reductions and exact solutions of Lax integrable 3-dimensional systems

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We present a complete description of 2-dimensional equations that arise as symmetry reductions of four 3-dimensional Lax-integrable equations: (1) the universal hierarchy equation $u_{yy} = u_z u_{xy} - u_y u_{xz}$; (2) the 3D rdDym equation $u_{ty} = u_x u_{xy} - u_y u_{xx}$; (3) the equation $u_{ty} = u_t u_{xy} - u_y u_{tx}$, which we call modified Veronese web equation; (4) Pavlov's equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$.

Keywords: Partial differential equations; symmetry reductions, solutions.

2010 Mathematics Subject Classification: 35B06

Introduction

We consider four 3-dimensional Lax-integrable[a] equations:

- the *universal hierarchy equation* (Sec. 2)

$$u_{yy} = u_z u_{xy} - u_y u_{xz}, \tag{0.1}$$

  see [11].

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[a]We say that an equation is *Lax-integrable* if it admits a zero-curvature representation with a non-removable parameter.

- the *3D rdDym equation* (Sec. 3)

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \tag{0.2}$$

  see [3, 13, 15].
- the equation (Sec. 4)

$$u_{ty} = u_t u_{xy} - u_y u_{tx}, \tag{0.3}$$

  see [1, 6, 8, 16]. In [8] it was shown that the equation at hand is related to a particular case of the 3D Veronese web equation by a Bäcklund transformation. Below we call Eq. (0.3) the *modified Veronese web equation*.
- *Pavlov's equation* (Sec. 5)

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \tag{0.4}$$

  see [5, 14].

Some of these equations arise also in [7] as integrable hydrodynamic reductions of multi-dimensional dispersionless PDEs.

All the above listed equations may be obtained as the symmetry reductions of the following Lax-integrable 4-dimensional systems:

$$u_{yz} = u_{tx} + u_x u_{xy} - u_y u_{xx}$$

and

$$u_{ty} = u_z u_{xy} - u_y u_{xz}$$

introduced in [7] and [11], respectively, while the latter two, in turn, are the reductions of

$$u_{yz} = u_{ts} + u_s u_{xz} - u_z u_{xs}.$$

Here we give a complete answer to a natural question: what 2-dimensional equations are the reductions of the 3-dimensional ones? The result comprises 32 equations of which

- sixteen can be solved explicitly,
- one reduces to the Riccati equation,
- five can be linearized by the Legendre transformation,
- while the rest ten are 'nontrivial'.

The latter are presented in Table 1 (in the third column, we exemplify the simplest relations). The first two of these equations can be transformed to the Liouville equation and the Gibbons-Tsarev equation, respectively. The other eight, to our strong opinion, may possess interesting integrability properties and we plan to study them in the nearest future. More detailed, but also concise, information on the reductions may be also found in Table 6.

In Sec. 1, we briefly expose necessary preliminaries (see, e.g., [10]). In Sec. 6, we present the obtained results in a concise form.

Reduction	of Eq.	Relations with the initial equation
$2\Phi=\Phi\Phi_{xz}-\Phi_x\Phi_z$	(0.1)	$u=\dfrac{\Phi(x,z)}{y}$,
$\Phi_{\xi\xi}=(\xi+\Phi_\xi)\Phi_{\eta\eta}-\Phi_\eta\Phi_{\xi\eta}-2$	(0.4)	$u=\Phi(\xi,\eta)+t^2\xi-2t\eta$, $\xi=y$, $\eta=x+ty$,
$\Phi_{\xi\xi}=\Phi_x\Phi_\xi-\Phi\Phi_{x\xi}$	(0.1)	$u=\Phi(x,\xi)e^{-z}$, $\xi=ye^{-z}$,
$(1+\xi\Phi_z)\Phi_{\xi\xi}-\xi\Phi_\xi\Phi_{\xi z}+\Phi_\xi\Phi_z=0$	(0.1)	$u=\Phi(z,\xi)e^{-x}$, $\xi=ye^{-x}$,
$\Phi_\eta\Phi_{\xi\eta}-\Phi_\xi\Phi_{\eta\eta}=e^\eta\Phi_{\xi\xi}$	(0.1)	$u=\Phi(\xi,\eta)e^{-x}$, $\xi=ye^{-z}$, $\eta=x-z$,
$(\xi+\Phi_\xi)\Phi_{\xi y}-\Phi_y(\Phi_{\xi\xi}+2)=0$	(0.2)	$u=\Phi(\xi,y)e^{2t}$, $\xi=xe^t$,
$\Phi_{\xi t}=4\Phi\Phi_\xi-\xi\Phi_\xi^2+2\xi\Phi\Phi_{\xi\xi}$	(0.2)	$u=\Phi(\xi,t)x^2$, $\xi=xe^{-y}$,
$\Phi_{\eta\eta}+(\xi+\Phi_\eta)\Phi_{\xi\eta}=\Phi_\eta(2+\Phi_{\xi\xi})$	(0.2)	$u=\Phi(\xi,\eta)e^{2t}$, $\xi=xe^{-t}$, $\eta=y-t$,
$(4\xi^2-3\Phi)\Phi_{\xi\xi}-\Phi_{\xi t}-6\xi\Phi_\xi+\Phi_\xi^2+6\Phi=0$	(0.4)	$u=\Phi(\xi,y)y^3$, $\xi=\dfrac{x}{y^2}$,
$\Phi_{\xi\xi}=(\xi-\Phi_\eta)\Phi_{\xi\eta}+(2\eta+\Phi_\xi)\Phi_{\eta\eta}-\Phi_\eta$	(0.4)	$u=\Phi(\xi,\eta)e^{-3t}$, $\xi=ye^{\beta t}$, $\eta=xe^{2t}$

Table 1. 'Nontrivial' reductions

1. Preliminaries

Let $\mathscr{E}$ be a differential equation given by

$$F\Big(x,\dots,\frac{\partial^{|\sigma|}u}{\partial x^\sigma},\dots\Big)=0, \tag{1.1}$$

where $u(x)$ is the unknown function in the variables $x=(x^1,\dots,x^n)$. A *symmetry* of $\mathscr{E}$ is a function $\varphi=\varphi(x,\dots,u_\sigma,\dots)$ in the *jet variables* u_σ, σ being a multi-index, $u_\varnothing=u$, that satisfies the *linearized equation*

$$\ell_{\mathscr{E}}(\varphi)\equiv\sum_\sigma\frac{\partial F}{\partial u_\sigma}D_\sigma(\varphi)=0, \tag{1.2}$$

where $D_\sigma=D_{i_1}\circ\cdots\circ D_{i_k}$ for $\sigma=i_1\dots i_k$, while

$$D_i=\frac{\partial}{\partial x^i}+\sum_\sigma u_{\sigma i}\frac{\partial}{\partial u_\sigma} \tag{1.3}$$

are the *total derivatives* restricted to $\mathscr{E}$. Symmetries of $\mathscr{E}$ form a Lie algebra $\operatorname{sym}\mathscr{E}$ over $\mathbb{R}$ with respect to the *Jacobi bracket*

$$\{\varphi,\bar\varphi\}=\sum_\sigma\Big(\frac{\partial\varphi}{\partial u_\sigma}D_\sigma(\bar\varphi)-\frac{\partial\bar\varphi}{\partial u_\sigma}D_\sigma(\varphi)\Big). \tag{1.4}$$

A solution u to Eq. (1.1) is said to be *invariant* with respect to a symmetry $\varphi\in\operatorname{sym}\mathscr{E}$ if it enjoys the equation

$$\varphi\Big(x,\dots,\frac{\partial^{|\sigma|}u}{\partial x^\sigma},\dots\Big)=0. \tag{1.5}$$

The *reduction* of $\mathscr{E}$ with respect to φ is Eq. (1.1) rewritten in terms of first integrals of Eq. (1.5).

2. The universal hierarchy equation

Recall that the equation is

$$\mathscr{E}_{(0.1)}: \quad u_{yy} = u_z u_{xy} - u_y u_{xz}.$$

2.1. *Symmetries*

The defining equation for symmetries for this equation is

$$D_y^2(\varphi) = u_z D_x D_y(\varphi) - u_y D_x D_z(\varphi) + u_{xy} D_z(\varphi) - u_{xz} D_y(\varphi). \tag{2.1}$$

Its solutions are

$$\begin{aligned}
\varphi_1 &= y u_y + u,\\
\varphi_2(X_2) &= X_2 u_x - X_2' u,\\
\varphi_3(Z_3) &= Z_3 u_z + Z_3' y u_y,\\
\varphi_4(Z_4) &= Z_4 u_y,\\
\varphi_5(X_5) &= X_5,
\end{aligned}$$

where X_i are functions of x, Z_i are functions of z and 'prime' denotes the derivative with respect to the corresponding variable. The commutator relations are given in Table 2.

	φ_1	$\varphi_2(\bar{X}_2)$	$\varphi_3(\bar{Z}_3)$	$\varphi_4(\bar{Z}_4)$	$\varphi_5(\bar{X}_5)$
φ_1	0	0	0	$\varphi_4(\bar{Z}_4)$	$-\varphi_5(\bar{X}_5)$
$\varphi_2(X_2)$	...	$\varphi_2(\bar{X}_2 X_2' - X_2 \bar{X}_2')$	0	0	$\varphi_5(\bar{X}_5 X_2' - X_2 \bar{X}_5')$
$\varphi_3(Z_3)$	...	...	$\varphi_3(\bar{Y}_3 Y_3' - Y_3 \bar{Y}_3')$	$\varphi_4(\bar{Y}_4 Y_3' - Y_3 \bar{Y}_4')$	0
$\varphi_4(Z_4)$	...	...	...	0	0
$\varphi_5(X_5)$	...	...	...	...	0

Table 2. Lie algebra structure of $\operatorname{sym} \mathscr{E}_{(0.1)}$

2.2. *Reductions*

Thus, the general symmetry of Eq. (0.1) is

$$\varphi = X_2 u_x + (\alpha y + Z_3' y + Z_4) u_y + Z_3 u_z + (\alpha - X_2') u + X_5,$$

where $\alpha \in \mathbb{R}$ is a constant. Thus, invariant with respect to φ solutions are given by the system

$$\frac{dx}{X_2} = \frac{dy}{(\alpha + Z_3')y + Z_4} = \frac{dz}{Z_3} = -\frac{du}{(\alpha - X_2')u + X_5}. \tag{2.2}$$

We consider the following basic cases below:

Case 00 $X_2 = 0$, $Z_3 = 0$;

Case 01 $X_2 = 0, Z_3 \neq 0$;
Case 10 $X_2 \neq 0, Z_3 = 0$;
Case 11 $X_2 \neq 0, Z_3 \neq 0$.

Let us study them in detail.

2.2.1. *Case* 00

System (2.2) takes the form

$$\frac{dx}{0} = \frac{dy}{\alpha y + Z_4} = \frac{dz}{0} = -\frac{du}{\alpha u + X_5}.$$

Its integrals are

$$(\alpha y + Z_4)u + X_5 y = \text{const}, \quad x = \text{const}, \quad z = \text{const}$$

and the general solution is given by

$$\Psi((\alpha y + Z)u + Xy, x, z) = 0,$$

where $Z = Z_4$, $X = X_5$. Hence,

$$u = \frac{\Phi(x,z) - Xy}{\alpha y + Z}. \tag{2.3}$$

To simplify the subsequent exposition, we consider two subcases:

Subcase 00.0 $\alpha = 0$;
Subcase 00.1 $\alpha \neq 0$.

Then we have:

Subcase 00.0 After redenoting $\Phi \mapsto \Phi/Z$, $Z \neq 0$, we have

$$u = \Phi(x,z) - \frac{Xy}{Z}. \tag{2.4}$$

Substituting to Eq. (0.1), one obtains

$$\frac{1}{Z} \cdot (X\Phi_{xz} - X'\Phi_z) = 0,$$

which leads to the following class of solutions

$$\boxed{u = \begin{cases} \Phi(x,z), & \text{if } X = 0, \\ XP(z) + Q(x) - \dfrac{Xy}{Z}, & \text{if } X \neq 0. \end{cases}}$$

Subcase 00.1 Making the change $\Phi \mapsto \Phi - XZ$, one gets

$$u = \frac{\Phi}{y+Z} - X.$$

Substituting to (0.1), one arrives to the equation

$$\boxed{2\Phi = \Phi\Phi_{xz} - \Phi_x\Phi_z.}$$

After the change $\Phi = e^{\Psi}$ we obtain the Liouville equation

$$\boxed{\Psi_{xz} = 2e^{-\Psi},}$$

see, e.g. [4].

2.2.2. *Case* 01

Now we have

$$\frac{dx}{0} = \frac{dy}{(\alpha + Z_3')y + Z_4} = \frac{dz}{Z_3} = -\frac{du}{\alpha u + X_5}.$$

The integrals of the system are

$$u\exp\left(\int \frac{\alpha dz}{Z_3}\right) + \int \frac{X_5}{Z_3}\exp\left(\int \frac{\alpha dz}{Z_3}\right) dz = \text{const}, \quad x = \text{const},$$
$$y\exp\left(-\int \frac{\alpha + Z_3'}{Z_3} dz\right) - \int \frac{Z_4}{Z_3}\exp\left(-\int \frac{\alpha + Z_3'}{Z_3} dz\right) dz = \text{const}.$$

We introduce new functions

$$Z = \int \frac{dz}{Z_3}, \quad \bar{Z} = \int \frac{Z_4}{Z_3}\exp\left(-\int \frac{\alpha + Z_3'}{Z_3} dz\right) dz, \quad X = X_5.$$

Note that $Z' \neq 0$, We again distinguish two subcases:

Subcase 01.0 $\alpha = 0$;
Subcase 01.1 $\alpha \neq 0$.

Let us study them.

Subcase 01.0 In this case, the system of integrals transforms to

$$u + XZ = \text{const}, \quad x = \text{const}, \quad yZ' - \bar{Z} = \text{const},$$

and thus

$$\Psi(u + XZ, x, yZ' - \bar{Z}) = 0 \tag{2.5}$$

is the general solution. Consequently,

$$u = \Phi(x, \xi) - XZ,$$

where $\xi = yZ' - \bar{Z}$. Substituting the last expression to Eq. (0.1), we obtain the equation

$$\boxed{\Phi_{\xi\xi} = X'\Phi_{\xi} - X\Phi_{x\xi}.} \tag{2.6}$$

When $X = 0$, we obtain the solutions

$$\boxed{u = a_1(yZ' - \bar{Z}) + a_0, \qquad a_i = a_i(x).}$$

If $X \neq 0$ Eq. (2.6) can also be solved and the general solution is

$$\boxed{u = \Phi\left(yZ' - \bar{Z} + \int \frac{dx}{X}\right) + a(x),}$$

where Φ is an arbitrary function in one argument.

Subcase 01.1 We can set $\alpha = 1$ and the general solution is

$$\Psi((u+X)e^{Z}, x, Z'e^{-Z}y - \bar{Z}) = 0.$$

Hence, after the change $Z \mapsto \ln Z$, we have

$$u = \frac{1}{Z}\Phi(x,\xi) - X, \tag{2.7}$$

where $\xi = Z'y/Z^2 - \bar{Z}$. Substituting (2.7) to Eq. (0.1), one obtains

$$\boxed{\Phi_{\xi\xi} = \Phi_x\Phi_{\xi} - \Phi\Phi_{x\xi}.}$$

2.2.3. *Case* 10

System (2.2) is now of the form

$$\frac{dx}{X_2} = \frac{dy}{\alpha y + Z_4} = \frac{dz}{0} = -\frac{du}{(\alpha - X_2')u + X_5}.$$

Then the integrals are

$$u\exp\left(\int \frac{\alpha - X_2'}{X_2}dx\right) + \int \frac{X_5}{X_2}\exp\left(\int \frac{\alpha - X_2'}{X_2}dx\right)dx = \text{const},$$
$$y\exp\left(-\int \frac{\alpha\, dx}{X_2}\right) - \int \frac{Z_4}{X_2}\exp\left(-\int \frac{\alpha\, dx}{X_2}\right)dx = \text{const}, \; z = \text{const}.$$

Let us introduce the notation

$$\int \frac{dx}{X_2} = X, \quad \frac{X_5}{X_2}\exp\left(\int \frac{\alpha - X_2'}{X_2}dx\right)dx = \bar{X}, \quad Z_4 = Z$$

and consider the subcases

Subcase 10.0 $\alpha = 0$;
Subcase 10.1 $\alpha \neq 0$.

Consider them in detail.

Subcase 10.0 In this case the general solution is given by

$$\Psi(X'u+\bar{X},y-XZ,z)=0, \qquad X'\neq 0,$$

and thus

$$u=\frac{1}{X'}\Phi(\xi,z)-\bar{X},$$

where $\xi=y-XZ$. Substituting this expression to Eq. (0.1), one obtains

$$\boxed{(1+Z\Phi_z)\Phi_{\xi\xi}=Z\Phi_\xi\Phi_{\xi z}+Z'\Phi_\xi^2.}$$

The equation can be solved explicitly. Indeed, dividing by Φ_ξ^2 one obtains

$$\frac{\Phi_{\xi\xi}}{\Phi_\xi^2}-Z'=Z\frac{\Phi_\xi\Phi_{\xi z}-\Phi_z\Phi_{\xi\xi}}{\Phi_\xi^2},$$

or

$$-\left(\frac{1}{\Phi_\xi}\right)_\xi-Z'=Z\left(\frac{\Phi_z}{\Phi_\xi}\right)_\xi.$$

Hence,

$$-\frac{1}{\Phi_\xi}-Z'\xi=Z\frac{\Phi_z}{\Phi_\xi}+\varphi,$$

where $\varphi=\varphi(z)$ is an arbitrary function. Thus,

$$Z\Phi_z+(Z'\xi+\varphi)\Phi_\xi=-1$$

and

$$\boxed{\Phi=\Upsilon(\xi-\bar{\varphi})-\int\frac{dz}{Z}}$$

is the general solution, where $\bar{\varphi}=Z\int\frac{\varphi dz}{Z^2}$.

Subcase 10.1 We may set $\alpha=1$ and then obtain the general solution in the form

$$\Psi\left(X'ue^X+\bar{X},e^{-X}(y+Z),z\right)=0,$$

or, after the change $X\mapsto\ln X$,

$$u=\frac{\Phi(\xi,z)-\bar{X}}{X'}, \tag{2.8}$$

where $\xi=(y+Z)/X$. Substituting to (0.1), one has

$$\boxed{(1+\xi\Phi_z)\Phi_{\xi\xi}-\xi\Phi_\xi\Phi_{\xi z}+\Phi_\xi\Phi_z=0.}$$

2.2.4. *Case* 11

We have here

$$\frac{dx}{X_2} = \frac{dy}{(\alpha+Z_3')y+Z_4} = \frac{dz}{Z_3} = -\frac{du}{(\alpha-X_2')u+X_5},$$

where $X_2 \neq 0$, $Z_3 \neq 0$. The integrals are

$$u\exp\left(\int\frac{\alpha-X_2'}{X_2}dx\right)+\int\frac{X_5}{X_2}\exp\left(\int\frac{\alpha-X_2'}{X_2}dx\right)dx = \text{const}, \int\frac{dx}{X_2}-\int\frac{dx}{Z_3} = \text{const},$$
$$y\exp\left(-\int\frac{\alpha+Z_3'}{Z_3}dx\right)-\int\frac{Z_4}{Z_3}\exp\left(-\int\frac{\alpha+Z_3'}{Z_3}dx\right)dx = \text{const}.$$

As before, we introduce the notation

$$\int\frac{dx}{X_2} = X,\ \int\frac{X_5}{X_2}\exp\left(\int\frac{\alpha-X_2'}{X_2}dx\right)dx = \bar{X},\quad \int\frac{dx}{Z_3} = Z,\ \int\frac{Z_4}{Z_3}\exp\left(-\int\frac{\alpha+Z_3'}{Z_3}dx\right)dx = \bar{Z}$$

and consider two subcases

Subcase 11.0 $\alpha = 0$;
Subcase 11.1 $\alpha \neq 0$.

Then we have:

Subcase 11.0 The general solution is given by

$$\Psi(X'u+\bar{X}, Z'y-\bar{Z}, X-Z) = 0$$

in this case and thus

$$u = \frac{\Phi(\xi,\eta)-\bar{X}}{X'}, \qquad \xi = Z'y-\bar{Z}, \quad \eta = X-Z.$$

After substituting to Eq. (0.1), one has

$$\boxed{\Phi_{\xi\xi} = \Phi_\xi\Phi_{\eta\eta}-\Phi_\eta\Phi_{\xi\eta}.}$$

The equation can be linearized by the Legendre transformation, [12].

Subcase 11.1 Setting $\alpha = 1$, we obtain the general solution

$$\Psi(X'e^X u+\bar{X}, Z'e^{-Z}y-\bar{Z}, X-Z) = 0$$

and, after the change $X \mapsto \ln X$, $Z \mapsto -\ln Z$, one has

$$u = \frac{\Phi(\xi,\eta)-\bar{X}}{X'}, \qquad \xi = Z'y-\bar{Z}, \quad \eta = \ln(XZ). \tag{2.9}$$

Substituting (2.9) to (0.1), we obtain the equation

$$\boxed{\Phi_\eta\Phi_{\xi\eta}-\Phi_\xi\Phi_{\eta\eta} = e^\eta\Phi_{\xi\xi}.}$$

3. The 3D rdDym equation

As it was said above, the equation is

$$\mathscr{E}_{(0.2)}: \quad u_{ty} = u_x u_{xy} - u_y u_{xx}.$$

3.1. *Symmetries*

Symmetries of Eq. (0.2) are defined by

$$D_t D_y(\varphi) = u_x D_x D_y(\varphi) - u_y D_x^2(\varphi) + u_{xy} D_x(\varphi) - u_{xx} D_y(\varphi). \tag{3.1}$$

Solutions of (3.1) are

$$\begin{aligned}
\varphi_1 &= x u_x - 2u,\\
\varphi_2(T_2) &= T_2 u_t + T_2'(x u_x - u) + \frac{1}{2} T_2'' x^2,\\
\varphi_3(Y_3) &= Y_3 u_y,\\
\varphi_4(T_4) &= T_4 u_x + T_4' x,\\
\varphi_5(T_5) &= T_5,
\end{aligned}$$

where $Y_i = Y_i(y)$, $T_i = T_i(t)$ and 'primes' denote the derivatives. The commutator relations are given in Table 3.

	φ_1	$\varphi_2(\bar{T}_2)$	$\varphi_3(\bar{Y}_3)$	$\varphi_4(\bar{T}_4)$	$\varphi_5(\bar{T}_5)$
φ_1	0	0	0	$\varphi_4(\bar{T}_4)$	$2\varphi_5(\bar{T}_5)$
$\varphi_2(T_2)$	...	$\varphi_2(\bar{T}_2 X_2' - T_2 \bar{T}_2')$	0	$\varphi_4(\bar{T}_4 T_2' - T_2 \bar{T}_4')$	$\varphi_5(\bar{T}_5 T_2' - T_2 \bar{T}_5')$
$\varphi_3(Y_3)$	...	...	0	0	0
$\varphi_4(T_4)$	...	...	...	$\varphi_5(\bar{T}_4 T_4' - T_4 \bar{T}_4')$	0
$\varphi_5(T_5)$	...	...	...	...	0

Table 3. Lie algebra structure of sym $\mathscr{E}_{(0.2)}$

3.2. *Reductions*

The general symmetry of Eq. (0.2) is

$$\varphi = (\alpha x + T_2' x + T_4) u_x + Y_3 u_y + T_2 u_t - (2\alpha + T_2') u + T_5 + T_4' x + \frac{1}{2} T_2'' x^2,$$

where $\alpha \in \mathbb{R}$ is a constant. Consequently, φ-invariant solutions are defined by the system

$$\frac{dx}{(\alpha + T_2')x + T_4} = \frac{dy}{Y_3} = \frac{dt}{T_2} = \frac{du}{(2\alpha + T_2')u - T_5 - T_4' x - \frac{1}{2} T_2'' x^2}. \tag{3.2}$$

In what follows, we consider the following cases

Case 00 $Y_3 = 0$, $T_2 = 0$;
Case 01 $Y_3 = 0$, $T_2 \neq 0$;
Case 10 $Y_3 \neq 0$, $T_2 = 0$;
Case 11 $Y_3 \neq 0$, $T_2 \neq 0$.

3.2.1. *Case* 00

Eq. (4.2) takes the form

$$\frac{dx}{\alpha x + T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2\alpha u - T_5 - T_4' x}.$$

As before, two subcases must be considered:

Subcase 00.0 $\alpha = 0$;
Subcase 00.1 $\alpha \neq 0$.

One has the following:

Subcase 00.0 Here we have

$$\frac{dx}{T_4} = \frac{dy}{0} = \frac{dt}{0} = -\frac{du}{T_5 + T_4' x}$$

and the general solution is given by

$$\Psi\left(u + \frac{1}{2}Tx^2 + \bar{T}x, y, t\right) = 0,$$

or

$$u = \Phi(y,t) - \frac{1}{2}Tx^2 - \bar{T}x, \tag{3.3}$$

where $T = T_4'/T_4$, $\bar{T} = T_5/T_4$. Substituting (3.3) in Eq. (2.9), we obtain

$$\boxed{\Phi_{yt} = T\Phi_y.}$$

The general solution is

$$\Phi = \varphi(y)e^{\int T\,dt} + \psi(t),$$

which leads to the following family of solutions to Eq. (2.9):

$$\boxed{u = \varphi(y)e^{\int T\,dt} + \psi(t) - \frac{1}{2}Tx^2 - \bar{T}x.}$$

Subcase 00.1 Setting $\alpha = 1$, we obtain

$$\frac{dx}{x + T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2u - T_5 - T_4' x}.$$

The general solution of this system is

$$u = (x+T)^2\Phi(y,t) + T'(x+T) + \bar{T}, \tag{3.4}$$

where $T = T_4$, $\bar{T} = (T_5 - T_4T_4')/2$. Substituting to (0.2), we obtain the equation

$$\boxed{\Phi_{yt} = 2\Phi\Phi_y.}$$

Integrating over y, we come to the Riccati equation

$$\Phi_t = \Phi^2 + \varphi(t).$$

Thus, to any choice of φ there corresponds a family of solutions to Eq. (0.2).

Examples. Let us consider some particular cases.

(1) If $\varphi = 0$ then

$$\Phi = \frac{1}{\psi - t}.$$

Here and in all the examples below ψ is an arbitrary function of y.

(2) For $\varphi = a^2$, $a = \mathrm{const}$, one has

$$\Phi = a\tan\big(a(t+\psi)\big).$$

(3) If $\varphi = -a^2$ then

$$\Phi = a\frac{1+e^{2a(t+\psi)}}{1-e^{2a(t+2\psi)}}$$

(4) For $\varphi = t^\kappa$, $\kappa \in \mathbb{R}$, one has

$$\Phi = \frac{-\psi J(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}) + \psi J(\frac{3+\kappa}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2})t^{\frac{1}{2}\kappa+1} - Y(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}) + Y(\frac{3+\kappa}{\kappa+2}, 2\frac{t^{\frac{1}{2}\kappa+1}}{\kappa+2})t^{\frac{1}{2}\kappa+1}}{t\big(\psi J(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}) + Y(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1)}}{\kappa+2})\big)},$$

where

$$J(a,\theta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+a+1)}\left(\frac{\theta}{2}\right)^{2m+a},$$

$$Y(a,\theta) = \frac{J(a,\theta)\cos(a\pi) - J(-a,\theta)}{\sin(a\pi)}$$

are Bessel functions of the first and second kinds, respectively.

(5) If $\varphi = e^t$, then

$$\Phi = \left(\frac{\psi Y(1,2e^{\frac{t}{2}})}{\psi Y(0,2e^{\frac{t}{2}}) + J(0,2e^{\frac{t}{2}})} + \frac{J(1,2e^{\frac{t}{2}})}{\psi Y(0,2e^{\frac{t}{2}}) + J(0,2e^{\frac{t}{2}})}\right)e^{\frac{t}{2}}$$

(6) For $\varphi = (1-t)/(1+t)$ the solution is

$$\Phi = \frac{2\psi \operatorname{Ei}(1,-2-2t)t + \psi e^{2+2t} + 2t}{2\psi(1+t)\operatorname{Ei}(1,-2-2t) + 2t + \psi e^{2+2t} + 2},$$

where

$$\operatorname{Ei}(a,t) = \int_1^\infty \frac{e^{-\theta t}\,d\theta}{\theta^a}$$

is the exponential integral function.

3.2.2. *Case* 01

We have

$$\frac{dx}{(\alpha+T_2')x+T_4}=\frac{dy}{0}=\frac{dt}{T_2}=\frac{du}{(2\alpha+T_2')u-T_5-T_4'x-\frac{1}{2}T_2''x^2},$$

where $T_2\neq 0$. Its integrals are

$$T'xe^{-\alpha T}-\bar{T}=\text{const},\ y=\text{const},$$

$$T'ue^{-2\alpha T}+\bar{\bar{T}}+\left(\alpha\bar{T}+\left(\frac{\bar{T}}{T'}\right)'\right)(T'xe^{-\alpha T}-\bar{T})-\frac{1}{2}\frac{T''}{(T')^2}(T'xe^{-\alpha T}-\bar{T})^2=\text{const},$$

where

$$T=\int\frac{dt}{T_2},\quad \bar{T}=\int\left(T_4\cdot(T')^2\cdot e^{-\alpha T}\right)dt,$$

$$\bar{\bar{T}}=\int\left(T_5\cdot(T')^2\cdot e^{-2\alpha T}+T_4'\cdot T\cdot T'\cdot e^{-\alpha T}+\frac{1}{2}T_2''\cdot\bar{T}^2\right)dt.$$

Then the general solution is

$$\Psi\left(T'xe^{-\alpha T}-\bar{T},y,T'ue^{-2\alpha}T+\bar{\bar{T}}+\left(\alpha\bar{T}+\left(\frac{\bar{T}}{T'}\right)'\right)(T'xe^{-\alpha T}-\bar{T})\right.$$
$$\left.-\frac{1}{2}\frac{T''}{(T')^2}(T'xe^{-\alpha T}-\bar{T})^2\right)=0,$$

or

$$u=\left(\Phi(\xi,y)-\bar{\bar{T}}-\left(\alpha\bar{T}+\left(\frac{\bar{T}}{T'}\right)'\right)\xi+\frac{1}{2}\frac{T''}{(T')^2}\xi^2\right)\frac{e^{2\alpha T}}{T'},$$

where

$$\xi=T'xe^{-\alpha T}-\bar{T}.$$

Substituting to Eq. (0.2), one obtains

$$\boxed{(\alpha\xi+\Phi_\xi)\Phi_{\xi y}-\Phi_y(\Phi_{\xi\xi}+2\alpha)=0.}\qquad(3.5)$$

3.2.3. *Case* 10

The defining equations are

$$\frac{dx}{\alpha x+T_4}=\frac{dy}{Y_3}=\frac{dt}{0}=\frac{du}{2\alpha u-T_5-T_4'x},\qquad(3.6)$$

where $Y_3\neq 0$. Below we consider the following subcases:

Subcase 10.00 $\alpha=0$, $T_4=0$;
Subcase 10.01 $\alpha=0$, $T_4\neq 0$;
Subcase 10.1 $\alpha\neq 0$.

Subcase 10.00 In this case, $T_5 \neq 0$ and System (3.6) takes the form

$$\frac{dx}{0} = Y'\,dy = \frac{dt}{0} = -\frac{du}{T_5}.$$

Denote $T_5 = T$. Then the integrals are

$$x = \text{const}, \quad t = \text{const}, \quad u + YT = \text{const}.$$

Then the general solution is given by

$$\Psi(u + YT, x, t) = 0,$$

or

$$u = \Phi(x,t) - YT.$$

Substituting to Eq. (0.2), one obtains

$$-Y'T' = Y'T\Phi_{xx},$$

or, since $Y' = 1/Y_3 \neq 0$,

$$\Phi_{xx} = -\frac{T'}{T}.$$

This delivers us the following family of solutions:

$$\boxed{u = -\frac{T'}{2T}x^2 + \varphi(t)x + \psi(t) - YT.}$$

Subcase 10.01 The defining equations are now

$$\frac{dx}{T_4} = Y'\,dy = \frac{dt}{0} = -\frac{du}{T_5 + T_4' x}.$$

Let us introduce the notation $T_4 = T$, $T_5/T_4 = \bar{T}$. Then the integrals are

$$x - YT = \text{const}, \quad t = \text{const}, \quad u + \frac{T'}{2T}x^2 + \bar{T}x = \text{const}$$

and the general solution is

$$\Psi(u + \frac{T'}{2T}x^2 + \bar{T}x, x - YT, t) = 0,$$

or

$$u = \Phi(\xi, t) - \frac{T'}{2T}x^2 - \bar{T}x,$$

where $\xi = x - YT$. Substituting to (0.2), we obtain the linear equation

$$\boxed{\left(\frac{T'}{T}\xi + \bar{T}\right)\Phi_{\xi\xi} + \Phi_{\xi t} = 0.}$$

The general solution of this equation is

$$\Phi = \varphi(\eta)T + \psi(t), \qquad \eta = \frac{\xi}{T} - \int \frac{\bar{T}}{T}\,dt,$$

which gives the family of solutions to (0.2):

$$\boxed{u = \varphi(\eta)T + \psi(t) - \frac{T'}{2T}x^2 - \bar{T}x.}$$

Subcase 10.1 We can assume $\alpha = 1$ and the defining equations become

$$\frac{dx}{x+T_4} = Y'\,dy = \frac{dt}{0} = \frac{du}{2u - T_5 - T_4'x}.$$

The integrals of this system are

$$(x+T)e^{-Y} = \text{const}, \quad t = \text{const}, \quad \frac{u-\bar{T}}{(x+T)^2} - \frac{T'}{x+T} = \text{const},$$

where $T = T_4$, $\bar{T} = (T_5 - T'T)/2$, and thus the general solution is

$$\Psi\left(\frac{u-\bar{T}}{(x+T)^2} - \frac{T'}{x+T}(x+T), e^{-Y}, t\right) = 0,$$

or

$$u = (x+T)^2\Phi(\xi,t) + T'(x+T) + \bar{T}, \qquad \xi = (x+T)e^{-Y}.$$

Substituting to (0.2), one obtains the equation

$$\boxed{\Phi_{\xi t} = 4\Phi\Phi_\xi - \xi\Phi_\xi^2 + 2\xi\Phi\Phi_{\xi\xi}.}$$

3.2.4. *Case* 11

Let us set $Y' = 1/Y_3 \neq 0$ and $T' = 1/T_2 \neq$. Then System (3.2) becomes

$$\frac{dx}{(\alpha + T_2')x + T_4} = Y'\,dy = T'\,dt = \frac{du}{(2\alpha + T_2')u - T_5 - T_4'x - \frac{1}{2}T_2''x^2}.$$

The integrals are

$$T'xe^{-\alpha T} - \bar{T} = \text{const}, \quad Y - T = \text{const},$$

$$T'ue^{-2\alpha} + \bar{\bar{T}} + \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'}\right)'\right)(T'xe^{-\alpha T} - \bar{T}) - \frac{1}{2}\frac{T''}{(T')^2}(T'xe^{-\alpha T} - \bar{T})^2 = \text{const},$$

where, as before,

$$T = \int \frac{dt}{T_2}, \quad \bar{T} = \int \left(T_4 \cdot (T')^2 \cdot e^{-\alpha T}\right) dt,$$

$$\bar{\bar{T}} = \int \left(T_5 \cdot (T')^2 \cdot e^{-2\alpha T} + T_4' \cdot T \cdot T' \cdot e^{-\alpha T} + \frac{1}{2}T_2'' \cdot \bar{T}^2\right) dt.$$

Thus, the general solution is given by

$$\Psi\left(T'xe^{-\alpha T}-\bar{T},Y-T,T'ue^{-2\alpha T}+\bar{\bar{T}}+\left(\alpha\bar{T}+\left(\frac{\bar{T}}{T'}\right)'\right)(T'xe^{-\alpha T}-\bar{T})\right.$$
$$\left.-\frac{1}{2}\frac{T''}{(T')^2}(T'xe^{-\alpha T}-\bar{T})^2\right)=0,$$

or

$$u=\left(\Phi(\xi,\eta)-\bar{\bar{T}}-\left(\alpha\bar{T}+\left(\frac{\bar{T}}{T'}\right)'\right)\xi+\frac{1}{2}\frac{T''}{(T')^2}\xi^2\right)\frac{e^{2\alpha T}}{T'},$$

where

$$\xi=T'xe^{-\alpha T}-\bar{T},\qquad \eta=Y-T.$$

Substituting the last expression to Eq. (0.2), one obtains

$$\boxed{\Phi_{\eta\eta}+(\alpha\xi+\Phi_\eta)\Phi_{\xi\eta}=\Phi_\eta(2\alpha+\Phi_{\xi\xi}).}$$

4. The modified Veronese web equation

The equation is

$$\mathscr{E}_{(0.3)}:\quad u_{ty}=u_tu_{xy}-u_yu_{tx}.$$

4.1. *Symmetries*

Symmetries of (0.3) are defined by

$$D_tD_y(\varphi)=u_tD_xD_y(\varphi)-u_yD_tD_x(\varphi)+u_{xy}D_t(\varphi)-u_{tx}D_y(\varphi),\tag{4.1}$$

whose solutions are

$$\begin{aligned}\varphi_1(T_1)&=T_1u_t,\\ \varphi_2(X_2)&=X_2u_x-X_2'u,\\ \varphi_3(Y_3)&=Y_3u_y,\\ \varphi_4(X_4)&=X_4,\end{aligned}$$

where $X_i=X_i(x)$, $Y_i=Y_i(y)$, and $T_i=T_i(t)$. The commutator relations in $\operatorname{sym}\mathscr{E}_{(0.3)}$ are given in Table 4.

4.2. *Reductions*

The general symmetry of Eq. (0.3) is

$$\varphi=X_2u_x+Y_3u_y+T_1u_t-X_2'u+X_4$$

and the corresponding invariant solutions must satisfy the system

$$\frac{dx}{X_2}=\frac{dy}{Y_3}=\frac{dt}{T_1}=\frac{du}{X_2'u-X_4}.\tag{4.2}$$

We consider below the following cases:

	$\varphi_1(\bar{T}_1)$	$\varphi_2(\bar{X}_2)$	$\varphi_3(\bar{Y}_3)$	$\varphi_4(\bar{X}_4)$
$\varphi_1(T_1)$	$\varphi_1(\bar{T}_1T_1'-T_1\bar{T}_1')$	0	0	0
$\varphi_2(X_2)$	...	$\varphi_2(\bar{X}_2X_2'-X_2\bar{X}_2')$	0	$\varphi_4(\bar{X}_4X_2'-X_2\bar{X}_4')$
$\varphi_3(Y_3)$	...	...	$\varphi_3(\bar{Y}_3Y_3'-Y_3\bar{Y}_3')$	0
$\varphi_4(X_4)$	...	...	...	0

Table 4. Lie algebra structure of $\operatorname{sym}\mathscr{E}_{(0.3)}$

Case 100 $X_2\neq 0, Y_3=0, Z_4=0$;
Case 010 $X_2=0, Y_3\neq 0, Z_4=0$;
Case 001 $X_2=0, Y_3=0, Z_4\neq 0$;
Case 011 $X_2=0, Y_3\neq 0, Z_4\neq 0$;
Case 101 $X_2\neq 0, Y_3=0, Z_4\neq 0$;
Case 110 $X_2\neq 0, Y_3\neq 0, Z_4=0$;
Case 111 $X_2\neq 0, Y_3\neq 0, Z_4\neq 0$;

and use the notation $1/X_2=X'$, $1/Y_3=Y'$, $1/Z_4=Z'$ when it is well defined.

4.2.1. *Case* 100

The defining equation is

$$\frac{dx}{X_2}=\frac{dy}{0}=\frac{dt}{0}=\frac{du}{X_2'u-X_4}.$$

The integrals are

$$Xu-\bar{X}=\text{const},\qquad y=\text{const},\qquad t=\text{const},$$

and thus

$$\Psi(Xu-\bar{X},y,t)=0$$

is the general solution, where $\bar{X}=\int X_4X'\,dx$. Consequently,

$$u=\frac{\Phi(y,t)+\bar{X}}{X}.$$

Substituting to (0.3), one obtains

$$\boxed{\Phi_{yt}=0.}$$

Hence, $\Phi=\varphi(y)+\psi(t)$ and

$$\boxed{u=\frac{\varphi(y)+\psi(t)+\bar{X}}{X}}$$

is a family of solutions to (0.3).

4.2.2. *Case* 010

The defining equation is

$$\frac{dx}{0} = \frac{dy}{Y_3} = \frac{dt}{0} = -\frac{du}{X_4}.$$

The integrals are

$$u + \bar{X}Y = \text{const}, \quad x = \text{const}, \quad t = \text{const},$$

where $\bar{X} = X_4$. Then

$$\Psi(u + \bar{X}Y, x, t) = 0$$

is the general solution and

$$u = \Phi(x,t) - \bar{X}Y.$$

Substituting to (0.3), one obtains $Y'(\bar{X}\Phi_{xt} - \bar{X}'\Phi_t) = 0$ and since $Y' \neq 0$,

$$\boxed{\bar{X}\Phi_{xt} - \bar{X}'\Phi_t = 0.}$$

Thus, if $\bar{X} = 0$ we obtain the obvious family of solutions

$$\boxed{u = \Phi(x,t).}$$

If $\bar{X} \neq 0$ the corresponding family of solutions is

$$\boxed{u = \bar{X}\varphi(t) + \psi(x) - \bar{X}Y.}$$

4.2.3. *Case* 001

The defining equation is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{T_1} = -\frac{du}{X_4}.$$

Then, again denoting $\bar{X} = X_4$, we get the integrals

$$u + \bar{X}T = \text{const}, \quad x = \text{const}, \quad y = \text{const}$$

and the general solution in the form

$$\Psi(u + \bar{X}T, x, y) = 0,$$

or

$$u = \Phi(x,y) - \bar{X}T.$$

Substituting to (0.3), one obtains

$$\boxed{\bar{X}\Phi_{xy} - \bar{X}'\Phi_y = 0,}$$

since $T' \neq 0$. Then in the case $\bar{X} = 0$ we get the family of solutions

$$\boxed{u = \Phi(x,y)}$$

and when $\bar{X} \neq 0$ the family

$$\boxed{u = \bar{X}\varphi(y) + \psi(x) - \bar{X}T.}$$

4.2.4. *Case* 011

The defining equation is

$$\frac{dx}{0} = \frac{dy}{Y_3} = \frac{dt}{T_1} = -\frac{du}{X_4}.$$

Its integrals are

$$x = \text{const}, \quad Y - T = \text{const}, \quad u + \bar{X}Y = \text{const}$$

and the general solution is

$$\Psi(u + \bar{X}Y, x, Y - T) = 0,$$

or

$$u = \Phi(x,\xi) - \bar{X}Y, \qquad \xi = Y - T.$$

Substituting to Eq. (0.3), one obtains

$$\boxed{\Phi_{\xi\xi} = \bar{X}\Phi_{x\xi} - \bar{X}'\Phi_{\xi}.}$$

If $\bar{X} = 0$ then

$$\boxed{u = \varphi(x) + \psi(Y - T) - \bar{X}Y.}$$

In the case $\bar{X} \neq 0$ the corresponding family is

$$\boxed{u = \bar{X}\varphi\left(Y - T + \int \frac{dx}{\bar{X}}\right) + \psi(x) - \bar{X}Y.}$$

4.2.5. *Case* 101

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{0} = \frac{dt}{T_1} = \frac{du}{X_2'u - X_4}.$$

The integrals of this system are

$$X'u + \bar{X} = \text{const}, \quad X - T = \text{const}, \quad y = \text{const},$$

where $\bar{X} = \int (X')^2 X_4\, dx$ and the general solution is given by

$$\Psi(X'u - \bar{X}, X - T, y) = 0,$$

or

$$u = \frac{\Phi(y,\xi)+\bar{X}}{X'}, \qquad \xi = X - T.$$

After substitution to Eq. (0.3) one obtains

$$\boxed{(1+\Phi_\xi)\Phi_{y\xi} = \Phi_y\Phi_{\xi\xi}.}$$

The general solution to this equation is

$$\Psi = \Upsilon(\xi + \psi(y)) - \xi,$$

where Υ is an arbitrary function in one argument, and thus we get the family of solutions

$$\boxed{u = \frac{\Upsilon\big(X - T + \psi(y)\big) - X + T + \bar{X}}{X'}}$$

to Eq. (0.3).

4.2.6. *Case* 110

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{Y_3} = \frac{dt}{0} = \frac{du}{X_2'u - X_4}.$$

The integrals of this system are

$$X'u - \bar{X} = \text{const}, \quad X - Y = \text{const}, \quad y = \text{const},$$

where $\bar{X} = \int (X')^2 X_4\,dx$ and the general solution is given by

$$\Psi(X'u + \bar{X}, X - Y, y) = 0,$$

or

$$u = \frac{\Phi(y,\xi)+\bar{X}}{X'}, \qquad \xi = X - Y.$$

Substitution to Eq. (0.3) leads to

$$\boxed{(1+\Phi_\xi)\Phi_{t\xi} = \Phi_t\Phi_{\xi\xi}.}$$

Similar to the previous case, we solve this equation and obtain the following family of solutions to Eq. (0.3):

$$\boxed{u = \frac{\Upsilon\big(X - Y + \psi(t)\big) - X + Y + \bar{X}}{X'}.}$$

4.2.7. *Case* 111

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{Y_3} = \frac{dt}{T_1} = \frac{du}{X_2' u - X_4}.$$

The integrals are

$$X'u - \bar{X} = \text{const}, \quad X - Y = \text{const}, \quad X - T = \text{const},$$

where, as before, $\bar{X} = \int (X')^2 X_4 \, dx$. This delivers the general solution

$$\Psi(X'u - \bar{X}, X - Y, X - T) = 0,$$

i.e.,

$$u = \frac{\Phi(\xi, \eta) + \bar{X}}{X'}, \qquad \xi = X - Y, \quad \eta = X - T.$$

Substituting to (0.3), we obtain the equation

$$\boxed{\Phi_\eta \Phi_{\xi\xi} + \left(\Phi_\eta - \Phi_\xi - 1\right) \Phi_{\xi\eta} - \Phi_\xi \Phi_{\eta\eta} = 0.} \tag{4.3}$$

The equation linearizes by the Legendre transformation, [12].

5. Pavlov's equation

The Pavlov equation reads

$$\mathscr{E}_{(0.4)}: \qquad u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}.$$

5.1. *Symmetries*

The defining equation for symmetries of (0.4) is

$$D_y^2(\varphi) = D_t D_x(\varphi) + u_y D_x^2(\varphi) - u_x D_x D_y(\varphi) + u_{xx} D_y(\varphi) - u_{xy} D_x(\varphi). \tag{5.1}$$

Its solutions are

$$\begin{aligned}
\varphi_1 &= 2x - y u_x, \\
\varphi_2 &= 3u - 2x u_x - y u_y, \\
\varphi_3(T_3) &= T_3 u_t + T_3'(x u_x + y u_y - u) + \frac{1}{2} T_3''(y^2 u_x - 2xy) - \frac{1}{6} T_3''' y^3, \\
\varphi_4(T_4) &= T_4 u_x - T_4' y, \\
\varphi_5(T_5) &= T_5 u_y + T_5'(y u_x - x) - \frac{1}{2} T_5'' y^2, \\
\varphi_6(T_6) &= T_6,
\end{aligned}$$

where T_i are functions of t. The Lie algebra structure in $\operatorname{sym} \mathscr{E}_{(0.4)}$ is given in Table 5.

	φ_1	φ_2	$\varphi_3(\bar{T}_3)$	$\varphi_4(\bar{T}_4)$	$\varphi_5(\bar{T}_5)$	$\varphi_6(\bar{T}_6)$
φ_1	0	φ_1	0	$2\varphi_6(\bar{T}_4)$	$-2\varphi_4(\bar{T}_5)$	0
φ_2	...	0	0	$-2\varphi_4(\bar{T}_4)$	$-\varphi_5(\bar{T}_5)$	$-3\varphi_6(\bar{T}_6)$
$\varphi_3(T_3)$	...	...	$\varphi_3(\bar{T}_3T_3' - T_3\bar{T}_3')$	$\varphi_4(\bar{T}_4T_3' - T_3\bar{T}_4')$	$\varphi_5(\bar{T}_5T_3' - T_3\bar{T}_5')$	$\varphi_6(\bar{T}_6T_3 - T_3\bar{T}_6')$
$\varphi_4(T_4)$	...	...	...	0	$\varphi_6(\bar{T}_5T_4' - T_4\bar{T}_5')$	0
$\varphi_5(T_5)$	...	...	...	...	$\varphi_4(\bar{T}_5T_5' - T_5\bar{T}_5')$	0
$\varphi_6(T_6)$	...	...	...	...	...	0

Table 5. Lie algebra structure of $\operatorname{sym}\mathscr{E}_{(0.4)}$

5.2. *Reductions*

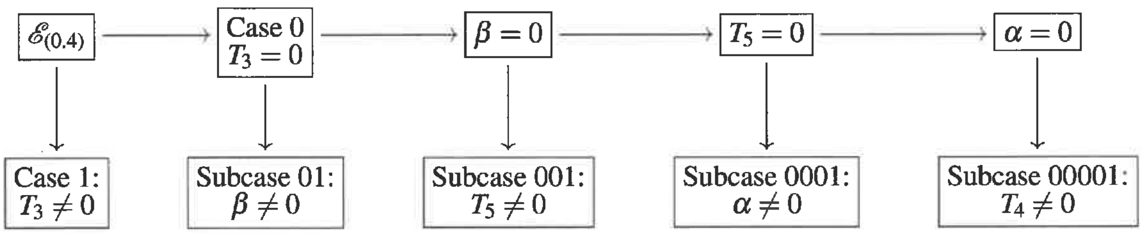
The general symmetry of Eq. (0.4) is

$$\varphi = \left(-\alpha y - 2\beta x + T_3'x + \frac{1}{2}T_3''y^2 + T_4 + T_5'y\right)u_x + (-\beta y + T_3'y + T_5)u_y + T_3u_t$$
$$+ (3\beta - T_3')u + 2\alpha x - T_3''xy - \frac{1}{6}T_3'''y^3 - T_5'x - \frac{1}{2}T_5''y^2 + T_6 - T_4'y,$$

where α, $\beta \in \mathbb{R}$ are constants. Then the φ-invariant solutions are determined by the system

$$\frac{dx}{(T_3' - 2\beta)x + (T_5' - \alpha)y + \frac{1}{2}T_3''y^2 + T_4} = \frac{dy}{(T_3' - \beta)y + T_5} = \frac{dt}{T_3}$$
$$= \frac{du}{(T_3' - 3\beta)u + (T_5' - 2\alpha)x + T_4'y + T_3''xy + \frac{1}{2}T_5''y^2 + \frac{1}{6}T_3'''y^3 - T_6}. \quad (5.2)$$

We consider the following tree of options:



5.2.1. *Case* 0

The defining equations are

$$\frac{dx}{-2\beta x + (T_5' - \alpha)y + T_4} = \frac{dy}{-\beta y + T_5} = \frac{dt}{0} = \frac{du}{-3\beta u + (T_5' - 2\alpha)x + T_4'y + \frac{1}{2}T_5''y^2 - T_6}.$$

Due to the above picture, consider the following subcases:

Subcase 00001 $\beta = 0$, $T_5 = 0$, $\alpha = 0$, $T_4 \neq 0$;
Subcase 0001 $\beta = 0$, $T_5 = 0$, $\alpha \neq 0$;
Subcase 001 $\beta = 0$, $T_5 \neq 0$;
Subcase 01 $\beta \neq 0$.

Subcase 00001 The defining equations are

$$\frac{dx}{T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{T_4'y - T_6}.$$

Then the integrals are

$$u - (Ty - \bar{T})x = \text{const}, \qquad y = \text{const}, \qquad t = \text{const},$$

where $T = T_4'/T_4$, $\bar{T} = T_6/T_4$. Thus,

$$\Psi(u - (Ty - \bar{T})x, y, t) = 0$$

is the general solution and

$$u = \Phi(y,t) + (Ty - \bar{T})x.$$

Substituting to Eq. (0.4), one obtains

$$\Phi_{yy} = (T' - T^2)y + T\bar{T} - \bar{T}',$$

which gives the family

$$\boxed{u = \frac{1}{6}(T' - T^2)y^3 + \frac{1}{2}(T\bar{T} - \bar{T}')y^2 + \varphi(t)y + \psi(t) + (Ty - \bar{T})x.}$$

of exact solutions to (0.4).

Subcase 0001 We may assume $\alpha = -1$ and the defining equations become

$$\frac{dx}{y + T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2x + T_4'y - T_6}.$$

Then the integrals are

$$(y + T_4)u - x^2 - (T_4'y - T_6)x = \text{const}, \quad y = \text{const}, \quad t = \text{const}.$$

Consequently, the general solution is given by

$$\Psi\left((y + T_4)u - x^2 - (T_4'y - T_6)x, y, t\right) = 0$$

and thus

$$u = \frac{\Phi(y,t) + x^2 + (T_4'y - T_6)x}{y + T_4}, \tag{5.3}$$

or

$$u = \Phi(y,t) + T'x + \frac{x^2 - \bar{T}x}{y + T},$$

where $T = T_4$, $\bar{T} = T_4T_4' + T_6$. After substituting to (0.4), we obtain the equation

$$\boxed{\Phi_{yy} = \frac{2\Phi_y}{y + T} + T'' - \frac{\bar{T}'}{y + T} + \frac{\bar{T}^2}{(y + T)^3}.}$$

Solving this equation, we obtain the following family of solutions to Eq. (0.4):

$$\boxed{u=\varphi(t)(y+T)^3-\frac{1}{2}T''(y+T)^2+\frac{1}{2}\bar{T}'(y+T)+\frac{2x^2-2\bar{T}x+\bar{T}^2}{y+T}+T'x+\psi(t).}$$

Subcase 001 The defining equations are

$$\frac{dx}{(T_5'-\alpha)y+T_4}=\frac{dy}{T_5}=\frac{dt}{0}=\frac{du}{(T_5'-2\alpha)x+T_4'y+\frac{1}{2}T_5''y^2-T_6}.$$

Let us introduce the notation $T=1/T_5$, $\bar{T}=T_4/T_5$, $\bar{\bar{T}}=T_6/T_5$. Then the integrals acquire the form

$$t=\text{const},\quad x+\frac{1}{2}\left(\frac{T'}{T}+\alpha T\right)y^2-\bar{T}y=\text{const},$$
$$u+\left(\frac{1}{6}\frac{T''}{T}+\alpha T'+\frac{2}{3}\alpha^2T^2\right)y^3-\frac{1}{2}\left(\bar{T}'+2\alpha T\bar{T}\right)y^2+\left(\left(\frac{T'}{T}+2\alpha T\right)x+\bar{\bar{T}}\right)y=\text{const}.$$

Hence, the general solution is

$$\Psi\left(u+\left(\frac{1}{6}\frac{T''}{T}+\alpha T'+\frac{2}{3}\alpha^2T^2\right)y^3-\frac{1}{2}\left(\bar{T}'+2\alpha T\bar{T}\right)y^2+\left(\left(\frac{T'}{T}+2\alpha T\right)x+\bar{\bar{T}}\right)y,\right.$$
$$\left.x+\frac{1}{2}\left(\frac{T'}{T}+\alpha T\right)y^2-\bar{T}y,t\right)=0,$$

or

$$u=\Phi(\xi,t)-\left(\frac{1}{6}\frac{T''}{T}+\alpha T'+\frac{2}{3}\alpha^2T^2\right)y^3+\frac{1}{2}\left(\bar{T}'+2\alpha T\bar{T}\right)y^2-\left(\left(\frac{T'}{T}+2\alpha T\right)x+\bar{\bar{T}}\right)y,$$

where

$$\xi=x+\frac{1}{2}\left(\frac{T'}{T}+\alpha T\right)y^2-\bar{T}y. \tag{5.4}$$

Substituting to Eq. (0.4), one obtains

$$\boxed{\left(\left(\frac{T'}{T}+2\alpha T\right)\xi+\bar{T}^2+\bar{\bar{T}}\right)\Phi_{\xi\xi}-\Phi_{\xi t}-\alpha T\Phi_\xi+\bar{T}'+2\alpha T\bar{T}=0.}$$

Of course, the equation can be solved explicitly, though the final result is too cumbersome: it is easily shown that

$$\Phi_\xi=\left(\bar{Z}\left(\xi e^{\int a\,dt}-\int be^{\int a\,dt}\,dt\right)+\int ce^{\alpha\int T\,dt}\,dt\right)e^{-\alpha\int T\,dt},$$

where $\bar{Z}$ is an arbitrary function in one variable and

$$a=\frac{T'}{T}+2\alpha T,\quad b=\bar{T}^2+\bar{\bar{T}},\quad c=\bar{T}'+2\alpha T\bar{T}.$$

Thus,

$$\Phi=Z\left(\xi e^{\int a\,dt}-\int be^{\int a\,dt}\,dt\right)e^{-2\alpha\int T\,dt}+\left(\int ce^{\alpha\int T\,dt}\,dt\right)\xi e^{-\alpha\int T\,dt}+\varphi(t),$$

and the corresponding family of solutions is

$$u = Z\left(\xi e^{\int a\,dt} - \int b e^{\int a\,dt}\,dt\right) e^{-2\alpha\int T\,dt} + \left(\int c e^{\alpha\int T\,dt}\,dt\right)\xi e^{-\alpha\int T\,dt} + \varphi(t)$$
$$-\left(\frac{1}{6}\frac{T''}{T} + \alpha T' + \frac{2}{3}\alpha^2 T^2\right)y^3 + \frac{1}{2}(\bar{T}' + 2\alpha T\bar{T})y^2 - \left(\left(\frac{T'}{T} + 2\alpha T\right)x + \bar{\bar{T}}\right)y$$

with ξ given by (5.4).

<u>**Subcase** 01</u> Since $\beta \neq 0$, we can set $\beta = -1$ and the defining equations become

$$\frac{dx}{2x + (T_5' - \alpha)y + T_4} = \frac{dy}{y + T_5} = \frac{dt}{0} = \frac{du}{3u + (T_5' - 2\alpha)x + T_4' y + \frac{1}{2}T_5'' y^2 - T_6}. \tag{5.5}$$

Let us introduce the notation $T = T_5$, $\bar{T} = T_4 - T_5(T_5' - \alpha)$. Then the integrals of (5.5) are

$$t = \text{const}, \qquad \frac{x + \frac{1}{2}\bar{T}}{(y+T)^2} + \frac{T' - \alpha}{y + T} = \text{const}$$
$$\frac{u + (x + \frac{1}{2}\bar{T})(T' - 2\alpha) + \frac{1}{3}\bar{\bar{T}}}{(y+T)^3} + \frac{(T' - \alpha)^2 + \frac{1}{2}\bar{T}'}{(y+T)^2} + \frac{1}{2}\frac{T''}{y+T} = \text{const},$$

where

$$\bar{\bar{T}} = -\frac{1}{2}(T' - 2\alpha)\bar{T} - (\bar{T}' + T'(T' - \alpha) + TT'')T + \frac{1}{2}T^2 T'' - T_6.$$

Consequently, the general solution is

$$\Psi\left(\frac{u + (x + \frac{1}{2}\bar{T})(T' - 2\alpha) + \frac{1}{3}\bar{\bar{T}}}{(y+T)^3} + \frac{(T' - \alpha)^2 + \frac{1}{2}\bar{T}'}{(y+T)^2} + \frac{1}{2}\frac{T''}{y+T},\ \frac{x + \frac{1}{2}\bar{T}}{(y+T)^2} + \frac{T' - \alpha}{y+T},\ t\right) = 0,$$

or

$$u = (y+T)^3\Phi(\xi, t) - \frac{1}{2}T''(y+T)^2 - \left((T' - \alpha)^2 + \frac{1}{2}\bar{T}'\right)(y+T) - (T' - 2\alpha)\left(x + \frac{1}{2}\bar{T}\right) - \frac{1}{3}\bar{\bar{T}},$$

where

$$\xi = \frac{x + \frac{1}{2}\bar{T}}{(y+T)^2} + \frac{T' - \alpha}{y+T}.$$

Substituting to Eq. (0.4), one obtains

$$\boxed{(4\xi^2 - 3\Phi)\Phi_{\xi\xi} - \Phi_{\xi t} - 6\xi\Phi_\xi + \Phi_\xi^2 + 6\Phi = 0.}$$

5.2.2. *Case* 1

The defining equation is now

$$\frac{dx}{(T_3'-2\beta)x+(T_5'-\alpha)y+\frac{1}{2}T_3''y^2+T_4}=\frac{dy}{(T_3'-\beta)y+T_5}=\frac{dt}{T_3}$$
$$=\frac{du}{(T_3'-3\beta)u+(T_5'-2\alpha)x+T_4'y+T_3''xy+\frac{1}{2}T_5''y^2+\frac{1}{6}T_3'''y^3-T_6}$$

and since $T_3 \neq 0$ we may set $T' = 1/T_3$. The integrals are

$$\begin{aligned}
&T'ye^{\beta T}-\bar{T}=\text{const},\\
&T'xe^{2\beta T}+\frac{1}{2}T_3'\xi^2-\left(\left(\frac{\bar{T}}{T'}\right)'-\beta\bar{T}'-\alpha k(\beta)\right)\xi-\bar{\bar{T}}=\text{const},\\
&T'ue^{3\beta T}-T_{00}-T_{10}\xi-T_{01}\eta-T_{20}\xi^2-T_{11}\xi\eta-T_{30}\xi^3=\text{const},
\end{aligned}$$

where

$$\begin{aligned}
\xi&=T'ye^{\beta T}-\bar{T},\\
\eta&=T'xe^{2\beta T}+\frac{1}{2}T_3'\xi^2-\left(\left(\frac{\bar{T}}{T'}\right)'-\beta\bar{T}'-\alpha k(\beta)\right)\xi-\bar{\bar{T}},\\
\bar{T}&=\int T_5(T')^2e^{\beta T}\,dt,\\
\bar{\bar{T}}&=\int\left(T_4(T')^2e^{2\beta T}+(T_5'-\alpha)\bar{T}T'e^{\beta T}+\frac{1}{2}T_3''(\bar{T})^2\right)dt,\\
T_{00}&=\int\left(T_3''\bar{\bar{T}}\bar{T}+\frac{T_3'''\bar{T}^3}{6T'}+\left(T_4'\bar{T}T'+(T_5'-2\alpha)T'+\frac{1}{2T_5''(\bar{T})^2}\right)e^{\beta T}-T_6(T')^2e^{3\beta T}\right)dt,\\
T_{10}&=\int\left(\frac{T_3'''(\bar{T})^2}{2T'}+T_3''\left(\bar{\bar{T}}+\bar{T}\left(\left(\frac{\bar{T}}{T'}\right)'-\alpha k(\beta)-\beta\bar{T}\right)\right)\right.\\
&\quad\left.+\left(T_5''\bar{T}+(T_5'-2\alpha)\left(\left(\frac{\bar{T}}{T'}\right)'-\alpha k(\beta)-\beta\bar{T}\right)\right)e^{\beta T}+T_4'T'e^{2\beta T}\right)dt,\\
T_{01}&=\int\left(T_3'''\bar{T}+(T_5'-2\alpha)T'e^{\beta T}\right)dt,\\
T_{20}&=\int\left(T_3''\left(\frac{\bar{T}}{2T'}+\left(\frac{\bar{T}}{T'}\right)'-\alpha k(\beta)-\beta\bar{T}-\frac{1}{2}T_3'\bar{T}\right)+\frac{1}{2}\left(T_5''-(T_5'-2\alpha)T'\right)e^{\beta T}\right)dt,\\
T_{11}&=\int T_3''\,dt=T_3',\\
T_{30}&=\int\left(\frac{T_3'''}{6T'}-\frac{1}{2}T_3''T_3'\right)dt=\frac{1}{6}T_3''T_3-\frac{1}{3}(T_3')^2,
\end{aligned}$$

and

$$k(\beta) = \int T' e^{\beta T}\, dt = \begin{cases} \dfrac{e^{\beta T}}{\beta}, & \beta \neq 0, \\ T, & \beta = 0. \end{cases}$$

Thus, the general solution is

$$u = \left(\frac{\Phi(\xi,\eta) + T_{00} + T_{10}\xi + T_{01}\eta + T_{20}\xi^2 + T_{11}\xi\eta + T_{30}\xi^3}{T'} \right) e^{3\beta T}. \tag{5.6}$$

Substituting (5.6) to Eq. (0.4), one obtains

$$\Phi_{\xi\xi} = (\beta\xi - \Phi_\eta)\Phi_{\xi\eta} + \left(2\beta\eta + \alpha\kappa + \Phi_\xi\right)\Phi_{\eta\eta} - \beta\Phi_\kappa - 2\alpha\kappa,$$

where

$$\kappa = e^{\beta T} - \beta k(\beta),$$

i.e.,

$$\kappa = \begin{cases} 0, & \beta \neq 0, \\ \xi, & \beta = 0. \end{cases}$$

Thus, the reductions are

$$\boxed{\Phi_{\xi\xi} = (\beta\xi - \Phi_\eta)\Phi_{\xi\eta} + (2\beta\eta + \Phi_\xi)\Phi_{\eta\eta} - \beta\Phi_\eta}$$

for $\beta \neq 0$ and

$$\boxed{\Phi_{\xi\xi} = (\alpha\xi + \Phi_\xi)\Phi_{\eta\eta} - \Phi_\eta\Phi_{\xi\eta} - 2\alpha} \tag{5.7}$$

for $\beta = 0$. Note that in the case $\alpha \neq 0$ Eq. (5.7) transforms to the Gibbons-Tsarev equation (see [9])

$$\boxed{\Phi_{\xi\xi} = \Phi_\xi\Phi_{\eta\eta} - \Phi_\eta\Phi_{\xi\eta} - \alpha}$$

by $\Phi \mapsto \Phi - \alpha\xi^2/2$.

6. Summary of results

Below, a concise exposition of the obtained results is given[b] in Table 6 on p. 28.

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[b]We use the notation $\infty^{k\cdot\tau}$ to indicate the infinite-dimensional component corresponding to k arbitrary functions in τ and the abbreviation 'LLT' means 'Linearizes by the Legendre transformation'.

Eqn	dim(sym $\mathscr{E}$)	Reductions	Comments
(0.1)	$1+\infty^{2\cdot x}+\infty^{2\cdot z}$	$X\Phi_{xz}-X'\Phi_z=0$	Solves explicitly
		$2\Phi=\Phi\Phi_{xz}-\Phi_x\Phi_z$	Transforms to the Liouville eq.
		$\Phi_{\xi\xi}=X'\Phi_\xi-X\Phi_{x\xi}$	Solves explicitly
		$\Phi_{\xi\xi}=\Phi_x\Phi_\xi-\Phi\Phi_{x\xi}$	
		$(1+Z\Phi_z)\Phi_{\xi\xi}=Z\Phi_\xi\Phi_{\xi z}+Z'\Phi_\xi^2$	Solves explicitly
		$(1+\xi\Phi_z)\Phi_{\xi\xi}-\xi\Phi_\xi\Phi_{\xi z}+\Phi_\xi\Phi_z=0$	
		$\Phi_{\xi\xi}=\Phi_\xi\Phi_{\eta\eta}-\Phi_\eta\Phi_{\xi\eta}$	LLT
		$\Phi_\eta\Phi_{\xi\eta}-\Phi_\xi\Phi_{\eta\eta}=e^\eta\Phi_{\xi\xi}$	
(0.2)	$1+\infty^{1\cdot y}+\infty^{3\cdot t}$	$\Phi_{yt}=T\Phi_y$	Solves explicitly
		$\Phi_{yt}=2\Phi\Phi_y$	Reduces to the Riccati eq.
		$(\alpha\xi+\Phi_\xi)\Phi_{\xi y}-\Phi_y(\Phi_{\xi\xi}+2\alpha)=0$	Solves explicitly
		$T\Phi_{xx}=T'$	Solves explicitly for $\alpha=0$
		$T\Phi_{xx}=T'$	Solves explicitly
		$\left(\frac{T'}{T}\xi+\bar{T}\right)\Phi_{\xi\xi}+\Phi_{\xi t}=0$	Solves explicitly
		$\Phi_{\xi t}=4\Phi\Phi_\xi-\xi\Phi_\xi^2+2\xi\Phi\Phi_{\xi\xi}$	
		$\Phi_{\eta\eta}+(\alpha\xi+\Phi_\eta)\Phi_{\xi\eta}=\Phi_\eta(2\alpha+\Phi_{\xi\xi})$	LLT for $\alpha=0$
(0.3)	$\infty^{2\cdot x}+\infty^{1\cdot y}+\infty^{1\cdot t}$	$\Phi_{yt}=0$	Solves explicitly
		$\bar{X}\Phi_{xt}-\bar{X}'\Phi_t=0$	Solves explicitly
		$\bar{X}\Phi_{xy}-\bar{X}'\Phi_y=0$	Solves explicitly
		$\Phi_{\xi\xi}=\bar{X}\Phi_{x\xi}-\bar{X}'\Phi_\xi$	Solves explicitly
		$(1+\Phi_\xi)\Phi_{y\xi}=\Phi_y\Phi_{\xi\xi}$	Solves explicitly
		$(1+\Phi_\xi)\Phi_{t\xi}=\Phi_t\Phi_{\xi\xi}$	Solves explicitly
		$\Phi_\eta\Phi_{\xi\xi}+(\Phi_\eta-\Phi_\xi-1)\Phi_{\xi\eta}-\Phi_\xi\Phi_{\eta\eta}=0$	LLT
(0.4)	$2+\infty^{4\cdot t}$	$\Phi_{yy}=(T'-T^2)y+T\bar{T}-\bar{T}'$	Solves explicitly
		$\Phi_{yy}=\frac{2\Phi_y-T'}{y+T}+T''+\frac{\bar{T}^2}{(y+T)^3}$	Solves explicitly
		$\left(\left(\frac{T'}{T}+2\alpha T\right)\xi+\bar{T}^2+\bar{\bar{T}}\right)\Phi_{\xi\xi}$ $-\Phi_{\xi t}-\alpha T\Phi_\xi+\bar{T}'+2\alpha T\bar{T}=0$	Solves explicitly
		$(4\xi^2-3\Phi)\Phi_{\xi\xi}-\Phi_{\xi t}-6\xi\Phi_\xi+\Phi_\xi^2+6\Phi=0$	
		$\Phi_{\xi\xi}=(\beta\xi-\Phi_\eta)\Phi_{\xi\eta}+(2\beta\eta+\Phi_\xi)\Phi_{\eta\eta}-\beta\Phi_\eta$	LLT for $\beta=0$
		$\Phi_{\xi\xi}=(\alpha\xi+\Phi_\xi)\Phi_{\eta\eta}-\Phi_\eta\Phi_{\xi\eta}-2\alpha$	Reduces to the Gibbons-Tsarev eq. for $\alpha\neq 0$ LLT for $\alpha=0$

Table 6. Summary of reductions

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Classification of integrable Weingarten surfaces possessing an $\mathfrak{sl}(2)$-valued zero curvature representation

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Abstract
In this paper we classify Weingarten surfaces integrable in the sense of soliton theory. The criterion is that the associated Gauss equation possesses an $\mathfrak{sl}(2)$-valued zero curvature representation with a nonremovable parameter. Under certain restrictions on the jet order, the answer is given by a third order ordinary differential equation to govern the functional dependence of the principal curvatures. Employing the scaling and translation (offsetting) symmetry, we give a general solution of the governing equation in terms of elliptic integrals. We show that the instances when the elliptic integrals degenerate to elementary functions were known to nineteenth-century geometers. Finally, we characterize the associated normal congruences.

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1. Introduction

Already the classical works of nineteenth-century geometers have established a major connection between differential geometry and the theory of partial differential equations. Powerful solution-generating techniques, such as the Bäcklund and Darboux transformations [36], have origins in the prototypical relationship between pseudospherical surfaces and solutions of the sine–Gordon equation.

Methods available for solving nonlinear partial differential equations were substantially extended in the 1970s to include the inverse scattering transform and its numerous developments; see, e.g., [8, 15, 29, 42]. An important open problem is to describe the class

of partial differential equations solvable by these powerful methods. Indirect detectors such as the symmetry analysis have been involved in obtaining extensive complete classifications of integrable evolution equations and systems; see [31] and references therein. The known theoretical answer given in terms of the existence of the associated one-parametric zero curvature representation

$$A_y - B_x + [A, B] = 0$$

has been considered as a classification tool in conjunction with the gauge cohomology by one of us [28]. These methods are not limited to evolution equations, although the necessary computations are rather complex, resource consuming and unthinkable without substantial use of computer algebra. However, certain partial differential equations of geometric origin are particularly well suited for this classification method, namely the Gauss–Mainardi–Codazzi equations of immersed surfaces. These equations always possess an associated linear zero curvature representation, albeit without the spectral parameter. If a nonremovable parameter can be incorporated, then the corresponding class of surfaces is said to be integrable, see [7, 36, 37] and references therein. There exists a remarkable way to associate surfaces with solutions of integrable equations—the generalized Sym–Tafel formula [10, 18, 19, 37].

Since their introduction by Weingarten [39], immersed surfaces in $\mathbb{R}^3$ that satisfy a functional relation between the principal curvatures have been of continuing interest in differential geometry, see, e.g., [21, 23, 25, 38]. It is therefore not surprising that attempts have been made to identify classes of Weingarten surfaces such that the corresponding Gauss equation is integrable in the sense of soliton theory. The work of Wu [41] and Finkel [17] indicated that all integrable cases are classical, characterized by a linear relation between the Gauss and the mean curvatures (linear Weingarten surfaces [13, section 812]; see also [20, 40] and references therein). In other words, the integrable Weingarten surfaces were conjectured to be either minimal or parallel to surfaces of constant Gaussian curvature. This conjecture was, however, disproved by the present authors in [1], henceforth referred to as part I. In part I we found another integrable class, consisting of surfaces with a constant difference between the principal radii of curvature, which we called surfaces of constant astigmatism. Surprisingly enough, this extra class turned out to be classical as well, apparently first mentioned by Beltrami [3, chapter 9, section 20], covered by Bianchi [4] and Darboux [13], see also [34], yet forgotten today.

In this paper we continue the work begun in part I and complete the classification of integrable classes in the simplest possible case. The integrability criterion we adopt is the existence of an $\mathfrak{sl}(2)$-valued zero curvature representation depending on a nonremovable parameter. We apply the same method of formal spectral parameter, introduced in [28] and briefly reproduced in part I. The underlying symbolic computations, done with the help of *Maple* and our own package *Jets* [2], are omitted. To stay within the limits given by available computing resources we had to restrict the jet order (order of derivatives).

The answer is given by a third-order nonlinear ordinary differential equation (10) to govern the functional dependence of the principal curvatures. Incorporation of the actual spectral parameter is achieved in section 3. This can be considered a proof of integrability, opening up the possibility to obtain explicit solutions by the methods of soliton theory [8, 15, 42]. However, we had to resign ourselves to following this road. Neither were we able to establish a Bäcklund or Darboux transformation [26, 29, 36], which would allow us to construct families of exact solutions depending on an arbitrary number of parameters. We only remark that seed solutions could be conveniently found among the rotational surfaces, see [25, equation (1)].

The governing equation (10) is explored in section 4. We identify two basic symmetries, scaling and translation (offsetting), and solve equation (10) in terms of elliptic integrals. The generic class of integrable Weingarten surfaces we obtained depends on one essential parameter (apart from the scaling and offsetting parameters) and is believed to be new. In section 5 we establish the integrable Gauss equation (39 in the generic case as well as in a number of special cases when the elliptic integrals degenerate to elementary functions. All of these special cases could be located in the nineteenth-century literature.

Geometrically, surfaces are related by an offsetting symmetry if they are parallel to each other, i.e. if they share the same normal line congruence. Therefore, the offsetting symmetry indicates that the concept of integrability naturally extends from surfaces to their normal line congruences. Section 7 grew out of our attempt to characterize the normal congruences of the integrable Weingarten surfaces. We obtain certain relations satisfied by suitably chosen metric invariants of the pair of the focal surfaces. Naturally, we expect the corresponding focal surfaces to be integrable as well, but a detailed investigation had to be postponed to the next paper.

2. Preliminaries

We consider surfaces $\boldsymbol{r}(x, y)$, parametrized by the lines of curvature. This is a regular parametrization except at umbilic points. The umbilic points are isolated by the Hartman–Wintner theorem [21] except for spheres and planes, which are, therefore, the only surfaces excluded from consideration.

The fundamental forms can be written as

$$\begin{aligned} \mathrm{I} &= u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2, \\ \mathrm{II} &= \frac{u^2}{\rho}\,\mathrm{d}x^2 + \frac{v^2}{\sigma}\,\mathrm{d}y^2, \end{aligned} \tag{1}$$

where ρ, σ are the principal radii of curvature. The radii transform in a very simple way under the offsetting symmetry (21) of the integrability problem (unlike the principal curvatures $p = 1/\rho$, $q = 1/\sigma$ we used in part I).

Choosing the orthonormal frame $\Psi = (\boldsymbol{r}_x/u, \boldsymbol{r}_y/v, \boldsymbol{n})$, we consider the Gauss–Weingarten equations

$$\Psi_x = \begin{pmatrix} 0 & -\dfrac{u_y}{v} & \dfrac{u}{\rho} \\ \dfrac{u_y}{v} & 0 & 0 \\ -\dfrac{u}{\rho} & 0 & 0 \end{pmatrix} \Psi, \qquad \Psi_y = \begin{pmatrix} 0 & \dfrac{v_x}{u} & 0 \\ -\dfrac{v_x}{u} & 0 & \dfrac{v}{\sigma} \\ 0 & -\dfrac{v}{\sigma} & 0 \end{pmatrix} \Psi \tag{2}$$

or, more explicitly,

$$\begin{aligned} \boldsymbol{r}_{xx} &= \frac{u_x}{u}\boldsymbol{r}_x - \frac{uu_y}{v^2}\boldsymbol{r}_y + \frac{u^2}{\rho}\boldsymbol{n}, & \boldsymbol{n}_x &= -\frac{1}{\rho}\boldsymbol{r}_x, \\ \boldsymbol{r}_{xy} &= \frac{u_y}{u}\boldsymbol{r}_x + \frac{v_x}{v}\boldsymbol{r}_y, \\ \boldsymbol{r}_{yy} &= -\frac{vv_x}{u^2}\boldsymbol{r}_x + \frac{v_y}{v}\boldsymbol{r}_y + \frac{v^2}{\sigma}\boldsymbol{n}, & \boldsymbol{n}_y &= -\frac{1}{\sigma}\boldsymbol{r}_y. \end{aligned} \tag{3}$$

Consequently, the Gauss–Mainardi–Codazzi equations, which are the compatibility conditions for (3), read as

$$uu_{yy} + vv_{xx} - \frac{v}{u}u_x v_x - \frac{u}{v}u_y v_y + \frac{u^2v^2}{\rho\sigma} = 0, \tag{4}$$

and

$$\frac{u_y}{u} + \frac{\sigma\rho_y}{\rho(\rho-\sigma)} = 0, \qquad \frac{v_x}{v} + \frac{\rho\sigma_x}{\sigma(\sigma-\rho)} = 0. \tag{5}$$

As with part I, we concentrate on Weingarten surfaces, which are characterized by the existence of a functional dependence between ρ and σ. We often resort to a parametric representation $\rho(w), \sigma(w)$ of the dependence.

Recall that parameters x, y label the lines of curvature; otherwise they are arbitrary. In line with Finkel's approach [17], we use this reparametrization freedom to solve the Mainardi–Codazzi subsystem (5). The following proposition is a mixture of classical and new results.

Proposition 1. *Away from umbilic points, a Weingarten surface can be parametrized by the lines of curvature in such a way that*

$$u = \exp \int \frac{\rho'\sigma}{(\sigma - \rho)\rho} \, dw, \qquad v = \exp \int \frac{\rho\sigma'}{(\rho - \sigma)\sigma} \, dw. \tag{6}$$

The Mainardi–Codazzi subsystem (5) is then identically satisfied, while the remaining Gauss equation can be written in the compact form

$$R_{yy} + S_{xx} + T = 0, \tag{7}$$

where R, S, T are appropriate functions of the unknown w. Moreover, the constraint

$$\left(\frac{1}{\rho} - \frac{1}{\sigma}\right) uv = 1 \tag{8}$$

can be imposed as an additional condition, and then $T = 1/(\sigma - \rho)$.

Proof. Writing $\rho(w), \sigma(w)$ for some function $w(x, y)$, the general solution of the Mainardi–Codazzi subsystem (5) is

$$u = u_0(x) \exp \int \frac{\rho'\sigma}{(\sigma - \rho)\rho} \, dw, \qquad v = v_0(y) \exp \int \frac{\rho\sigma'}{(\rho - \sigma)\sigma} \, dw.$$

Obviously from formulae (1), the multipliers $u_0(x)$, $v_0(y)$ can be removed by an appropriate relabelling $\tilde{x} = \tilde{x}(x)$, $\tilde{y} = \tilde{y}(y)$ of the surface's curvature lines. With $u_0 = v_0 = 1$, we have

$$uv = \exp \int \left(\frac{\rho'\sigma}{(\rho - \sigma)\rho} + \frac{\rho\sigma'}{(\sigma - \rho)\sigma} \right) dw = c \frac{\rho\sigma}{\sigma - \rho},$$

where c is an arbitrary constant multiplier. Setting $c = 1$ by the same relabelling argument proves the last relation.

Having solved the Mainardi–Codazzi subsystem, we are left with the Gauss equation (4) alone. Multiplied by $1/\rho - 1/\sigma$, equation (4) can be written in the compact form (7), where

$$R = \int \frac{\rho'}{\rho^2} u^2 \, dw, \qquad S = -\int \frac{\sigma'}{\sigma^2} v^2 \, dw, \qquad T = u^2 v^2 \frac{\sigma - \rho}{\rho^2 \sigma^2}. \tag{9}$$

Substituting $1/(1/\rho - 1/\sigma)$ for uv completes the proof. □

3. The classification result

Employing the Maple package *Jets* [2], we completed the computer-aided cohomological classification outlined in part I. We have no computer-independent proof of the following result.

Proposition 2. *The third-order ordinary differential equation*

$$\rho''' = \frac{3}{2\rho'} \rho''^2 - \frac{\rho' - 1}{\rho - \sigma} \rho'' + 2 \frac{(\rho' - 1)\rho'(\rho' + 1)}{(\rho - \sigma)^2} \tag{10}$$

determines a unique maximal class of Gauss–Mainardi–Codazzi equations of Weingarten surfaces whose initial $\mathfrak{sl}(2,\mathbb{C})$*-valued zero curvature representation*

$$A_0 = \begin{pmatrix} \dfrac{\mathrm{i}u_y}{2v} & -\dfrac{u}{2\rho} \\ \dfrac{u}{2\rho} & -\dfrac{\mathrm{i}u_y}{2v} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -\dfrac{\mathrm{i}v_x}{2u} & -\dfrac{\mathrm{i}v}{2\sigma} \\ -\dfrac{\mathrm{i}v}{2\sigma} & \dfrac{\mathrm{i}v_x}{2u} \end{pmatrix} \tag{11}$$

admits a second-order formal spectral parameter under the condition that the normal form of the zero curvature representation can depend on derivatives of u, v, σ, ρ *of no higher than the first order.*

Here and in what follows we assume that ρ is a function of σ and the prime refers to derivatives with respect to σ. A kth order formal parameter λ means a power series in terms of λ up to order k. Part I should be consulted for the other unexplained notions.

Remark 1.

(1) The last proposition provides a complete classification of integrable Weingarten surfaces under the following assumptions: the one-parametric zero curvature representation takes values in the Lie algebra $\mathfrak{sl}(2)$, includes the initial zero curvature representation (11) as a member, depends analytically on the parameter and its normal form involves derivatives of no higher than the first order. All these limitations can be overcome, in principle [27], at the cost of requiring significantly more computational resources.
(2) We would like to stress that the only part relying on machine computations is the completeness of the classification. All the other proofs in this paper are traditional.

In the rest of this section we establish integrability of the class determined by equation (10). The equation itself will be solved in the next section.

Proposition 3. *The nonremovable spectral parameter exists for all dependences* $\rho(\sigma)$ *allowed by the governing equation (10).*

Proof. Inspired by the results of the computer-aided classification, we depart from the following ansatz for the parameter-dependent zero curvature representation:

$$A = \begin{pmatrix} a_{111}\dfrac{u_y}{v} + a_{110}\sigma_x & a_{12}u \\ a_{21}u & -a_{111}\dfrac{u_y}{v} - a_{110}\sigma_x \end{pmatrix},$$

$$B = \begin{pmatrix} b_{111}\dfrac{v_x}{u} + b_{110}\sigma_y & b_{12}v \\ b_{12}v & -b_{111}\dfrac{v_x}{u} - b_{110}\sigma_y \end{pmatrix},$$

with $a_{111}, b_{111}, a_{110}, b_{110}, a_{12}, a_{21}, b_{12}$ being the unknown functions of σ. The problem is to solve the zero curvature condition $D_yA - D_xB + [A, B] = 0$ for matrix functions A, B of u, v, σ, ρ and their derivatives. However, the derivatives are not independent quantities, being subject to the Gauss–Mainardi–Codazzi equations. The proper way to deal with this situation is to introduce the manifold determined by the equation and its derivatives (a diffiety [9]). This is fairly easy if the order of derivatives is restricted as it is. Initially the derivatives are considered to be independent (jet space coordinates). Considering ρ as a function of σ and solving the Mainardi–Codazzi equations (5) for u_y, v_x, we can express u_y, v_x as functions of $u, v, \sigma, \sigma_x, \sigma_y$. Similarly, the derivatives of the Mainardi–Codazzi equations (5) can be solved for $u_{xy}, u_{yy}, v_{xx}, v_{xy}$, giving $u_{xy}, u_{yy}, v_{xx}, v_{xy}$ as functions of $u, u_x, v, v_y, \sigma, \sigma_x, \sigma_y$.

Consequently, the Gauss equation (4) can be written in terms of $u, u_x, v, v_y, \sigma, \sigma_x, \sigma_y, \sigma_{xx}, \sigma_{yy}$, and then solved for σ_{yy}. The explicit formulae are somewhat cumbersome, hence omitted.

With A, B chosen as above, the left-hand side $S := D_y A - D_x B + [A, B]$ of the zero curvature condition $S = 0$ is a matrix function of $u, u_x, v, v_y, \sigma, \sigma_x, \sigma_y, \sigma_{xx}, \sigma_{xy}$. From $\partial S/\partial\sigma_{xx} = 0$ and $\partial S/\partial\sigma_{xy} = 0$ we obtain

$$b_{111} = -a_{111}, \qquad b_{110} = a_{110}.$$

From either $\partial^2 S/\partial\sigma_x^2 = 0$ or $\partial^2 S/\partial\sigma_y^2 = 0$ we get $a'_{111} = 0$. Hence, a_{111} is a constant, which we rename λ in anticipation of its role as the spectral parameter.

Now, $\partial S/\partial\sigma_x = 0$ if and only if

$$a_{110} = \frac{\lambda\rho}{2\sigma(\sigma-\rho)}\frac{a_{12}+a_{21}}{b_{12}}, \qquad b'_{12} = \frac{\rho}{\sigma(\sigma-\rho)}[b_{12} + \lambda(a_{21} - a_{12})], \tag{12}$$

while $\partial S/\partial\sigma_y = 0$ can be rewritten as

$$\begin{aligned} a'_{12} &= 2a_{110}a_{12} + \frac{\sigma\rho'}{\rho(\rho-\sigma)}(a_{12} + 2\lambda b_{12}), \\ a'_{21} &= -2a_{110}a_{21} + \frac{\sigma\rho'}{\rho(\rho-\sigma)}(a_{21} - 2\lambda b_{12}). \end{aligned} \tag{13}$$

Modulo these relations, vanishing of S is equivalent to

$$b_{12} = \frac{\lambda}{\rho\sigma(a_{12} - a_{21})}. \tag{14}$$

We claim that the governing equation (10) arises as the condition that system (12)–(14) be compatible for arbitrary $\lambda \neq 0$. To prove this, we denote $P = a_{12} + a_{21}$, $Q = a_{12} - a_{21}$. With a_{110} and b_{12} taken from formulae (12) and (14), respectively, equations (13) turn into

$$P' = P\frac{\sigma\rho' - Q^2\rho^3}{\rho(\rho-\sigma)}, \qquad Q' = Q\frac{\sigma\rho' - P^2\rho^3}{\rho(\rho-\sigma)} + \frac{4\lambda^2\rho'}{\rho^2(\rho-\sigma)}\frac{1}{Q}, \tag{15}$$

and the second equation in (12) into

$$\rho^4(Q^2 - P^2)Q^2 + \rho^2(\rho' - 1)P^2 + 4\lambda^2\rho' = 0. \tag{16}$$

Now the question is whether equations (15) and (16) are compatible. Modulo equation (15), the derivative of (16) with respect to σ is

$$\begin{aligned} &2\rho^6(P^2 - Q^2)P^2Q^2 + 2(1 - 3\rho')\rho^4P^2Q^2 - 4\rho'\lambda^2P^2 \\ &\quad + (4\lambda^2 + \rho^2Q^2)[4\rho'\rho^2Q^2 + (\rho - \sigma)\rho'' + 2\rho'^2 - 2\rho'] = 0. \end{aligned} \tag{17}$$

This is equivalent to

$$[(\rho-\sigma)\rho'' - 2\rho'^2 + 2(1 + 8\lambda^2)\rho']\rho^2Q^2 + 4\lambda^2[(\rho-\sigma)\rho'' - 2\rho'^2 - 2\rho'] = 0 \tag{18}$$

modulo (16), since (18) is the remainder after division of (17) by (16) as polynomials in P. Similarly, dividing (16) by (18) as polynomials in Q, we get

$$\begin{aligned} &[(\rho-\sigma)\rho'' - 2\rho'^2 - 2\rho'][(\rho-\sigma)\rho'' - 2\rho'^2 + 2(1 + 8\lambda^2)\rho']\rho^2P^2 \\ &\quad - 4(1 + 4\lambda^2)[(\sigma-\rho)^2\rho''^2 - 4\rho'^4 + 8(1 + 8\lambda^2)\rho'^3 - 4\rho'^2] = 0. \end{aligned} \tag{19}$$

Differentiating (17) once more and taking the result modulo (15), (19) and (18), we get the governing equation (10) immediately.

Summing up, we obtain a zero curvature representation

$$A = \begin{pmatrix} -\dfrac{\lambda\sigma\rho'}{\rho(\rho-\sigma)}\dfrac{u}{v}\sigma_y - \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_x & \dfrac{1}{2}(P+Q)u \\ \dfrac{1}{2}(P-Q)u & \dfrac{\lambda\sigma\rho'}{\rho(\rho-\sigma)}\dfrac{u}{v}\sigma_y + \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_x \end{pmatrix},$$

$$B = \begin{pmatrix} -\dfrac{\lambda\rho}{\sigma(\rho-\sigma)}\dfrac{v}{u}\sigma_x - \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_y & \dfrac{\lambda}{\sigma\rho Q}v \\ \dfrac{\lambda}{\sigma\rho Q}v & \dfrac{\lambda\rho}{\sigma(\rho-\sigma)}\dfrac{v}{u}\sigma_x + \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_y \end{pmatrix},$$

where P and Q are the square roots to be determined from equations (19) and (18), respectively. Away from umbilic points (where $\rho = \sigma$), matrices A, B actually exist unless $(\rho-\sigma)\rho'' - 2\rho'^2 - 2\rho' = 0$ when P is undefined. This excludes exactly spheres and the linear Weingarten surfaces. The latter surfaces are, however, well known to be integrable, being parallel to surfaces of constant curvature (either Gaussian or mean), see [41] or [36, section 1.5.2].

If $\lambda = \mathrm{i}/2$, then we have $P = 0$ and $Q = 1/r^2$, which reproduces the parameterless zero curvature representation (11) we started with. □

Nonremovability of the parameter is ensured by the method [28] (follows from nontriviality of the first gauge cohomology group).

4. Solution of the governing equation

Apart from the discrete symmetry $\rho \leftrightarrow \sigma$, the governing equation (10) has two obvious continuous symmetries, which should be expected in every integrable class of surfaces: the *scaling symmetry*

$$\rho \mapsto \mathrm{e}^T\rho, \qquad \sigma \mapsto \mathrm{e}^T\sigma \tag{20}$$

and the *translational symmetry*

$$\rho \mapsto \rho + T, \qquad \sigma \mapsto \sigma + T. \tag{21}$$

The geometric meaning of the latter symmetry is *offsetting*, also known as taking the *parallel surface*. In terms of position vectors, $\boldsymbol{r}$ is transformed to $\boldsymbol{r} + T\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal vector and T is the distance.

With the help of these symmetries we can reduce the order of equation (10) by two. This can be done by rewriting the equation in terms of the symmetry invariants. Since rescaling applies also to the offset, the translational reduction should precede the scaling reduction. For the two lowest order translational invariants we choose

$$\xi = \rho - \sigma, \qquad \eta = \rho' \tag{22}$$

(recall that the prime denotes the derivative with respect to σ).

1. If $\xi' = 0$ (equivalently, $\rho' = 1$), then $\rho - \sigma = \text{const}$, which are the surfaces of constant astigmatism we dealt with in part I.

2. Otherwise, more translational invariants can be computed as derivatives of η with respect to ξ:

$$\eta_\xi = \frac{\eta'}{\xi'} = \frac{\rho''}{\rho'-1}, \qquad \eta_{\xi\xi} = \frac{\rho'''}{(\rho'-1)^2} - \frac{\rho''^2}{(\rho'-1)^3}, \tag{23}$$

etc. In terms of these invariants, the governing equation (10) reduces to the second-order equation

$$2\xi^2(\eta-1)\eta\eta_{\xi\xi} - \xi^2(\eta-3)\eta_\xi^2 + 2\xi(\eta-1)\eta\eta_\xi - 4(\eta+1)\eta^2 = 0. \tag{24}$$

As expected, this equation is scaling invariant. To reduce it with respect to scaling, we proceed as follows. In addition to η, one more scaling invariant is

$$\zeta = \xi(\eta-1)\eta_\xi. \tag{25}$$

Although dispensable, the factor $\eta - 1$ simplifies the computations to follow.

2.1. If $\eta' = 0$, i.e. $\rho'' = 0$, then (10) reduces to $\rho' = c$, where c is either of $-1, 0, 1$. The corresponding surfaces are, respectively, the constant mean curvature surfaces (a subclass of linear Weingarten surfaces), the tubular surfaces (surfaces swept by spheres of constant radius moving along a space curve) and once more the constant astigmatism surfaces.

2.2. Otherwise $\rho'' \neq 0$ and we have

$$\zeta_\eta = \frac{\rho'''}{\rho''}(\rho-\sigma) + \rho' - 1.$$

In terms of η, ζ, the reduced governing equation (24) becomes the Bernoulli equation

$$\zeta_\eta = \frac{3}{2}\frac{\zeta}{\eta} + 2\frac{\eta^3-\eta}{\zeta}$$

with the general solution $\zeta^2 = 4(\eta^2 + 2c_0\eta + 1)\eta^2$, where c_0 is the integration constant. Substituting from equation (25) yields the separable first-order equation

$$\xi\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \pm 2\frac{\eta}{\eta-1}\sqrt{\eta^2+2c_0\eta+1} \tag{26}$$

containing the parameter c_0. Being written in terms of the scaling and translation invariants, this equation determines the integrable Weingarten surfaces up to rescaling and offsetting. Depending on the value of the parameter c_0 and on the choice of the '$\pm$' sign, we obtain the following cases.

2.2.1 Let $c_0 = 1$. Equation (26) becomes

$$\xi\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \pm 2\frac{(\eta+1)\eta}{\eta-1}. \tag{27}$$

2.2.1.1. With the choice of the plus sign in (27), the general solution is $(\eta+1)^2 = c_1\eta\xi^2$. Substituting from equation (22), we obtain

$$(\rho'+1)^2 = c_1(\rho-\sigma)^2\rho'.$$

If $c_1 = 0$, the general solution is $\rho + \sigma = \text{const}$. Otherwise, we apply the transformation

$$\kappa = \rho + \sigma, \qquad \xi = \rho - \sigma \tag{28}$$

to get

$$(c_1\xi^2 - 4)\left(\frac{\mathrm{d}\kappa}{\mathrm{d}\xi}\right)^2 = c_1\xi^2.$$

The equation is separable with a general solution $(\kappa - c_2)^2 - \xi^2 + 4/c_1 = 0$, i.e.

$$4\rho\sigma - 2c_2(\rho+\sigma) + \frac{4+c_1c_2^2}{c_1} = 0.$$

In both cases, $c_1 = 0$ and $c_1 \neq 0$, solutions correspond to the linear Weingarten surfaces.

2.2.1.2. With the choice of the minus sign in (27), the general solution is $(\eta + 1)^2\xi^2 = c_1\eta$. Substituting from equation (22), we obtain $(\rho' + 1)^2(\rho - \sigma)^2 = c_1\rho'$. For $c_1 = 0$ we have the special linear Weingarten surfaces $\rho + \sigma = \text{const}$ again. Otherwise, we apply transformation (28) to get

$$(4\xi^2 - c_1)\left(\frac{d\kappa}{d\xi}\right)^2 + c_1 = 0.$$

The solutions are

$$\kappa = \pm\frac{1}{2}\sqrt{-c_1}\ln\left(2\sqrt{-c_1}\xi + \sqrt{c_1^2 - 4c_1\xi^2}\right) + c_2,$$

where c_2 is the integration constant.

2.2.1.2.1. For $c_1 < 0$ we can write

$$\xi = \frac{\sqrt{-c_1}}{2}\sinh\left(\pm\frac{2}{\sqrt{-c_1}}(\kappa - c_2) - \ln(-c_1)\right)$$

or

$$\frac{\rho - \sigma}{C_1} = \pm\sinh\left(\frac{\rho + \sigma}{C_1} + C_0\right).$$

2.2.1.2.2. Similarly, solutions corresponding to positive c_1 are

$$\frac{\rho - \sigma}{C_1} = \sin\left(\frac{\rho + \sigma}{C_1} + C_0\right). \tag{29}$$

2.2.2 Let $c = -1$. Equation (26) becomes

$$(\eta - 1)^2\left(\xi\frac{d\eta}{d\xi} - 2\eta\right)\left(\xi\frac{d\eta}{d\xi} + 2\eta\right) = 0. \tag{30}$$

Solutions corresponding to $\eta = 1$ belong to case 1 (constant astigmatism surfaces).

2.2.2.1. The general solution of $\xi(d\eta/d\xi) = 2\eta$ is $\eta = c_1\xi^2$. Substituting from equation (22), we obtain the Riccati equation $\rho' = c_1(\rho - \sigma)^2$.

2.2.2.1.1. For $c_1 > 0$ we get

$$\rho = \sigma - \frac{\tanh(\sqrt{c_1}\sigma + c_2)}{\sqrt{c_1}} \quad \text{or} \quad \rho = \sigma - \frac{\coth(\sqrt{c_1}\sigma + c_2)}{\sqrt{c_1}} \tag{31}$$

according to whether the integration constant is positive or negative.

2.2.2.1.2. Similarly, for $c_1 < 0$ we get

$$\rho = \sigma - \frac{\tan(\sqrt{-c_1}\sigma + c_2)}{\sqrt{-c_1}} \quad \text{or} \quad \rho = \sigma + \frac{\cot(\sqrt{-c_1}\sigma + c_2)}{\sqrt{-c_1}}. \tag{32}$$

2.2.2.2. When solving $\xi(d\eta/d\xi) = -2\eta$, we get (31) and (32) with ρ, σ interchanged.

2.2.3. We are left with the generic case $c_0 \notin \{-1, 1\}$. Equation (26) has the general solution

$$(\eta + c_0 + \sqrt{\eta^2 + 2c_0\eta + 1})(c_0\eta + 1 + \sqrt{\eta^2 + 2c_0\eta + 1}) = c_1\xi^{\pm 2}\eta. \tag{33}$$

If $c_1 = 0$, then $\eta = 0$ in view of $c_0 \notin \{-1, 1\}$, which yields the tubular surfaces $\rho = \text{const}$. Let us, therefore, assume that $c_1 \neq 0$. Upon substituting from (22), equation (33) becomes a first-order ODE, separable in terms of variables (28) and having the elliptic integral

$$\kappa = \int^{\xi} \frac{-c_1 t^{\pm 2} + c_0^2 - 1}{\sqrt{c_1^2 t^{\pm 4} - 2(c_0 + 1)(c_0 + 3)c_1 t^{\pm 2} + (c_0^2 - 1)^2}}\, dt$$

as the general solution. The two cases the '$\pm$' symbol refers to can be converted one into another by the substitution $c_1 \to (c_0^2 - 1)^2/c_1$. Therefore, we can safely choose the sign to be '+', which we do in the following. Moreover, if κ is a solution, then so is $-\kappa$ (as a combination of the $\rho \leftrightarrow \sigma$ switch and a scaling by factor of -1). This is why we often ignore the sign of κ in what follows.

Substituting $t \to s/m$, $m = \sqrt{|c_1/(1 - c_0^2)|}$, we simplify the integral above to

$$\kappa = \frac{1}{m} I_{\pm}(m\xi, c), \qquad I_{\pm}(\xi, c) = \int^{\xi} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}}\, ds, \tag{34}$$

where '$\pm$' refers to the signum of $c_1/(1 - c_0^2)$; in particular, is unrelated to the '$\pm$' sign in (33). The real parameter c is related to c_0 by $c = \pm(c_0 + 3)/(c_0 - 1)$.

Formula (34) describes possible dependences $\rho(\sigma)$ via the substitution $\kappa = \rho + \sigma$, $\xi = \rho - \sigma$. Three independent parameters are involved: m, c and the integration constant (the lower limit of the integral). Obviously, m plays the role of the scaling parameter. The integration constant can be easily identified with the offsetting parameter T from (21).

Each dependence between κ and ξ has a unique representative modulo scaling and offsetting, obtainable by fixing the lower limit of the integral $I_{\pm}(\xi, c)$ in (34). This is straightforward when $c > -1$; we simply redefine $I_{\pm}(\xi, c)$ to be

$$I_{\pm}(\xi, c) = \int_0^{\xi} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}}\, ds. \tag{35}$$

If, however, $c < -1$, then the integrand in (34) is real in three separate intervals $(-\infty, -\sqrt{\gamma_+})$, $(-\sqrt{\gamma_-}, \sqrt{\gamma_-})$ and $(\sqrt{\gamma_+}, \infty)$, where

$$\gamma_{\pm} = -c \pm \sqrt{c^2 - 1} > 0. \tag{36}$$

We choose the representatives $-\tilde{I}_{\pm}(-\xi, c)$, $I_{\pm}(\xi, c)$ and $\tilde{I}_{\pm}(\xi, c)$, respectively, where $I_{\pm}(\xi, c)$ is given by (35) in the interval $-\gamma_- \leqslant \xi \leqslant \gamma_-$, while

$$\tilde{I}_{\pm}(\xi, c) = \int_{\gamma_+}^{\xi} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}}\, ds, \qquad \gamma_+ \leqslant \xi. \tag{37}$$

5. Summary of the solutions

As demonstrated in the preceding section, each integrable class is determined by certain relation between the radii of curvature, which can be subject to rescaling $\rho \to c_1\rho, \sigma \to c_1\sigma$, offsetting $\rho \to \rho + c_0, \sigma \to \sigma + c_0$ and the twist $\rho \leftrightarrow \sigma$.

With the help of proposition 1, we can find the corresponding integrable Gauss equation. To start with, we investigate the generic class determined by formula (34); we fix the scaling for simplicity.

Proposition 4. *Assuming*

$$\rho + \sigma = I_\pm(\rho - \sigma, c), \qquad I_\pm(\xi, c) = \int^\xi \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}}\, ds, \tag{38}$$

the Gauss equation (4) for $\xi = \rho - \sigma$ *reads as*

$$R'\xi_{yy} + R''\xi_y^2 + S'\xi_{xx} + S''\xi_x^2 + T = 0, \tag{39}$$

where

$$R' = \frac{1 + c\xi^2 + \Delta(\xi, c)}{\xi^2 \Delta(\xi, c)}, \qquad S' = \frac{c \mp 1}{2} \frac{\xi^2}{(1 + c\xi^2 + \Delta(\xi, c))\Delta(\xi, c)},$$

$$\Delta(\xi, c) = \sqrt{1 + 2c\xi^2 + \xi^4}, \qquad T = -\frac{1}{\xi}.$$

The metric coefficients u, v *in (1) are*

$$u = \frac{\xi + I_\pm(\xi, c)}{2\xi}\sqrt{1 \mp \xi^2 + \Delta(\xi, c)}, \qquad v = \frac{\xi - I_\pm(\xi, c)}{2\xi}\sqrt{\frac{1 \mp \xi^2 - \Delta(\xi, c)}{2c \pm 2}}.$$

Proof. We parametrize ρ and σ by ξ, i.e. we solve (38) as

$$\rho = \frac{I_\pm(\xi, c) + \xi}{2}, \qquad \sigma = \frac{I_\pm(\xi, c) - \xi}{2}.$$

The general form of the Gauss equation, along with the last term $T = 1/(\sigma - \rho) = -1/\xi$, follows from proposition 1. To find R', S', we compute

$$(\ln R')' = \frac{R''}{R'} = \frac{(\rho - \sigma)\rho'' - 2\rho'^2}{(\rho - \sigma)\rho'} = -\frac{2}{\xi}\frac{c\xi^2 + \xi^4 + \sqrt{1 + 2c\xi^2 + \xi^4}}{1 + 2c\xi^2 + \xi^4},$$

$$(\ln S')' = \frac{S''}{S'} = \frac{(\rho - \sigma)\sigma'' + 2\sigma'^2}{(\rho - \sigma)\sigma'} = -\frac{2}{\xi}\frac{c\xi^2 + \xi^4 - \sqrt{1 + 2c\xi^2 + \xi^4}}{1 + 2c\xi^2 + \xi^4}$$

from (9) under constraint (8). These equations need to be integrated once, which is easy; the integration constants have been chosen to match equations (8) and (9). Finally, from (9) one easily computes the coefficients u, v as $u = \sqrt{R'\rho^2/\rho'}$, $v = \sqrt{-S'\sigma^2/\sigma'}$. □

Apart from the generic class we also obtained a number of special solutions, listed in table 1 (omitting the tubular surfaces). Rows 5b and 6b differ only by translation (offsetting) and can be identified one with another.

The first column contains a determining relation (up to a scaling), while the second harbours the corresponding integrable equation in the compact form (7). Table 2 gives the principal radii of curvature ρ, σ, metric coefficients u, v and the variable z (see table 1) in terms of a suitably chosen parametrizing variable w.

Neither of the special cases is new to differential geometry. Row 1 reflects that, in terms of the curvature line coordinates, minimal surfaces correspond to solutions of the Liouville equation [5, section 351]. Similarly, row 2a reproduces the relation between surfaces of negative constant Gaussian curvature and solutions of the elliptic sinh–Gordon equation. Row 2b does the same for the hyperbolic sine–Gordon equation and surfaces of positive constant Gaussian curvature (or constant mean curvature, by the theorem of Bonnet on parallel surfaces). Nowadays, surfaces of constant mean or Gaussian curvature are undoubtedly the best understood classes of surfaces integrable in the sense of soliton theory (see, e.g., [6, 7, 14, 22, 30, 32] and references therein).

Table 1. Special integrable cases and the associated integrable Gauss equations.

	Relation	Integrable equation
1	$\rho + \sigma = 0$	$z_{xx} + z_{yy} + e^z = 0$
2a	$\rho\sigma = 1$	$z_{xx} + z_{yy} - \sinh z = 0$
2b	$\rho\sigma = -1$	$z_{xx} - z_{yy} + \sin z = 0$
3a	$\rho - \sigma = \sinh(\rho + \sigma)$	$(\tanh z - z)_{xx} + (\coth z - z)_{yy} + \operatorname{csch} 2z = 0$
3b	$\rho - \sigma = \sin(\rho + \sigma)$	$(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$
4	$\rho - \sigma = 1$	$z_{xx} + (1/z)_{yy} + 2 = 0$
5a	$\rho - \sigma = \tanh \rho$	$\frac{1}{4}(\sinh z - z)_{xx} + (\coth \frac{1}{2}z)_{yy} + \coth \frac{1}{2}z = 0$
5b	$\rho - \sigma = \tan \rho$	$\frac{1}{4}(\sin z - z)_{xx} + (\cot \frac{1}{2}z)_{yy} + \cot \frac{1}{2}z = 0$
6a	$\rho - \sigma = \coth \rho$	$\frac{1}{4}(\sinh z + z)_{xx} - (\tanh \frac{1}{2}z)_{yy} + \tanh \frac{1}{2}z = 0$
6b	$\rho - \sigma = -\cot \rho$	$\frac{1}{4}(\sin z + z)_{xx} + (\tan \frac{1}{2}z)_{yy} + \tan \frac{1}{2}z = 0$

Table 2. Special integrable cases. The radii of curvature ρ, σ, the metric coefficients u, v and the unknown z of the integrable Gauss equation in terms of a variable w.

	ρ	σ	u	v	z
1	w	$-w$	$\sqrt{w/2}$	$\sqrt{w/2}$	$-\ln w$
2a	w	$\dfrac{1}{w}$	$\dfrac{w}{\sqrt{w^2-1}}$	$\dfrac{-1}{\sqrt{w^2-1}}$	$2 \operatorname{arctanh} w$
2b	w	$-\dfrac{1}{w}$	$\dfrac{w}{\sqrt{w^2+1}}$	$\dfrac{1}{\sqrt{w^2+1}}$	$2\arctan w$
3a	$\dfrac{w + \sinh w}{2}$	$\dfrac{w - \sinh w}{2}$	$\dfrac{w + \sinh w}{2\sqrt{\cosh w - 1}}$	$\dfrac{w - \sinh w}{2\sqrt{\cosh w + 1}}$	$\dfrac{1}{2}w$
3b	$\dfrac{w + \sin w}{2}$	$\dfrac{w - \sin w}{2}$	$\dfrac{w + \sin w}{2\sqrt{1 - \cos w}}$	$\dfrac{w - \sin w}{2\sqrt{1 + \cos w}}$	$\dfrac{1}{2}w$
4	w	$w - 1$	$\dfrac{w}{e^w}$	$(1 - w)e^w$	e^{2w}
5a	w	$w - \tanh w$	$\dfrac{w}{\sinh w}$	$\sinh w - w \cosh w$	$2w$
5b	w	$w - \tan w$	$\dfrac{w}{\sin w}$	$\sin w - w \cos w$	$2w$
6a	w	$w - \coth w$	$\dfrac{w}{\cosh w}$	$\cosh w - w \sinh w$	$2w$
6b	w	$w + \cot w$	$\dfrac{w}{\cos w}$	$\cos w + w \sin w$	$2w$

It may come as a surprise that the other cases are classical as well. Introduced by Weingarten [39, section 4] ('eine neue Flächenklasse'), surfaces satisfying the relation $\rho - \sigma = \sin(\rho + \sigma)$ (row 4b) are covered in Darboux [13, sections 745, 746, 766, 769, 770] ('une classe nouvelle de surfaces découverte par M Weingarten') and Bianchi [4, section 135], [5, section 245]. Darboux [13, section 746] gave a general solution of an equation equivalent to our $(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$. He also provided a remarkable geometric construction in [13, section 770], further developed by Bianchi [5, section 245]. In a nutshell, the middle evolutes are translation surfaces generated by curves of opposite constant nonzero torsion; conversely the Weingarten surfaces are orthogonal to the osculation planes of the generating curves. Bianchi's research extends to the complementary relation $\rho - \sigma = \sinh(\rho + \sigma)$ (row 3a) as well [5, section 246]. The remaining rows (from 4 to 6b) correspond to involutes of surfaces of constant Gaussian curvature studied by Beltrami [3, chapter 9, section 20]. Row 4 (surfaces of constant astigmatism) has been addressed in part I; we have nothing to add except the Beltrami's work as the earliest reference we know of.

Table 3. Special integrable cases as limits of $I_\pm(\xi, c)$.

	Relation	Limit
1	$\kappa = 0$	$I_\pm(\xi, \infty)$
2a	$\kappa^2 = \xi^2 + 4$	$\lim_{m=\infty} I_\pm(m\xi, 2m^2)/m$
2b	$\kappa^2 = \xi^2 - 4$	$\lim_{m=\infty} I_\pm(m\xi, -2m^2)/m$
3a	$\kappa = \operatorname{arcsinh} \xi$	$\lim_{m=0} I_\pm(m\xi, 1/2m^2)/m$
3b	$\kappa = \arcsin \xi$	$\lim_{m=0} I_\pm(m\xi, -1/2m^2)/m$
4	$\xi = 1$	$\lim_{m=\infty} \tilde{I}_\pm(m\xi, -m^2/2)/m$
5a	$\kappa = -\xi + 2 \operatorname{arctanh} \xi$	$I_+(\xi, -1),\ \lvert\xi\rvert < 1$
5b	$\kappa = -\xi + 2 \arctan \xi$	$I_-(\xi, 1)$
6a	$\kappa = -\xi + 2 \operatorname{arccoth} \xi$	$I_+(\xi, -1),\ \lvert\xi\rvert > 1$
6b	$\kappa = -\xi - 2 \operatorname{arccot} \xi$	$I_-(\xi, 1)$

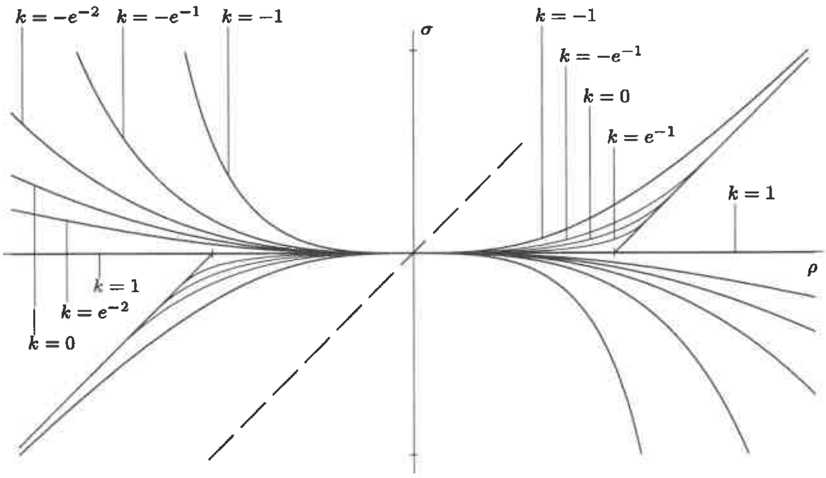

Figure 1. Curvature diagrams $\kappa = \mathcal{I}_B(\xi, k)$ (the left-hand legend) and $\kappa = \mathcal{I}_A(\xi, k)$, $|\xi| < 1$ (the right-hand legend), where $\kappa = \rho + \sigma$, $\xi = \rho - \sigma$. More can be obtained by rescaling and translating along the dashed line $\rho = \sigma$, the axis κ. Here $\mathcal{I}_A(\xi, -1) = -\xi + 2 \arctan \xi$ (row 5b), $\mathcal{I}_A(\xi, 0) = \arcsin \xi$ (row 3b), $\mathcal{I}_A(\xi, 1) = \xi$; $\mathcal{I}_B(\xi, -1) = -\xi + 2 \operatorname{arctanh} \xi$ (row 5a), $\mathcal{I}_B(\xi, 0) = \operatorname{arcsinh} \xi$ (row 3a), $\mathcal{I}_B(\xi, 1) = \xi$. Graphs of $\kappa = \mathcal{I}_A(\xi, k)$ end on the solid lines $|\xi| = 1$.

Table 3 demonstrates how the cases expressible in terms of elementary functions arise as limits of the generic integral (34) for c approaching ± 1 or $\pm\infty$ along a suitable curve in the (c, m) space. The tubular surfaces σ = const, which are omitted, correspond to $\kappa = I_+(\xi, 1) = \xi + \text{const}$.

6. Curvature diagrams

To exemplify the wealth of classes of integrable surfaces, we plot the representative solutions of the governing equation (10) in figures 1 and 2. We call them curvature diagrams, even though the radii of curvature ρ, σ, rather than the curvatures $1/\rho, 1/\sigma$, are plotted, contrary to the customary practice [24, chapter 5]. The benefit is that diagrams can be not only scaled arbitrarily, but also freely translated along the dashed line $\rho = \sigma$; the translation corresponds

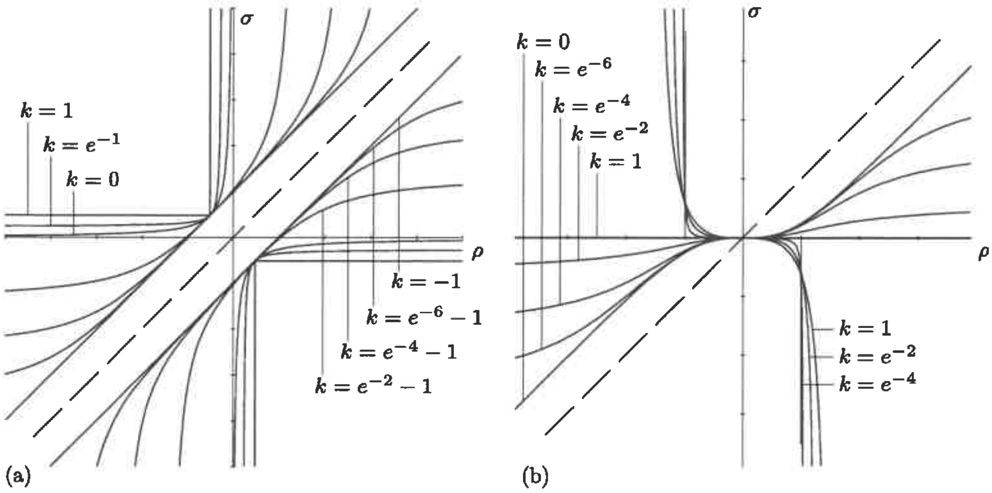

Figure 2. Curvature diagrams (a) $\kappa = k\tilde{\mathcal{I}}_A(\xi/k, k)$, $|\xi| > 1/|k|$; (b) $\kappa = \mathcal{I}_{C+}(\xi, k)$ (the top left-hand legend) and $\kappa = -\mathcal{I}_{C-}(\xi, k)$ (the bottom right-hand legend), where $\kappa = \rho + \sigma$, $\xi = \rho - \sigma$. More can be obtained by rescaling and translating along the dashed line $\rho = \sigma$, the axis κ.
In (a), the line $k = 1$ corresponds to tubular surfaces, $k = 0$ to surfaces of negative constant curvature (row 2b), and $k = -1$ to the constant astigmatism surfaces (row 4). In (b), $\mathcal{I}_{C+}(\xi, 0) = -\xi + 2\arctan\xi$ (row 5b) $\mathcal{I}_{C-}(\xi, 1) = \xi - 2\arctan((\xi - 1)/(\xi + 1))$ (row 5a after reparametrization).

to offsetting. For clarity, we adjusted the offsetting so that the diagrams are symmetric about the origin, i.e. $\rho(\sigma) = -\rho(-\sigma)$.

The diagrams contain plots of functions $\mathcal{I}_A(\xi, k)$, $\mathcal{I}_B(\xi, k)$, $\mathcal{I}_{C\pm}(\xi, k)$ and $k\tilde{\mathcal{I}}_A(\xi/k, k)$. All special cases are explicitly included as limits, except the surfaces of constant positive curvature (row 2a). These could be obtained as the limit of $k\mathcal{I}_B(\xi/k, k)$ as k approaches zero.

The plots have been calculated using the Legendre normal form [16, 33] of the elliptic integrals (35) and (37), which could be of independent interest. As well known, the Legendre normal form depends on the configuration of roots of the quartic polynomial $\Pi = s^4 + 2cs^2 + 1$.

(A) If $c < -1$, then $\Pi = (s^2 - \gamma_+)(s^2 - \gamma_-)$ has four real roots $\sqrt{\gamma_\pm}$ and $-\sqrt{\gamma_\pm}$ given by formula (36). By using the substitution $s = \sqrt{k}r$, where $k = \gamma_-$, we easily obtain the Legendre normal form

$$\frac{1}{\sqrt{k}} I_\pm\left(\xi\sqrt{k}, -\frac{k^2+1}{2k}\right) = \int_0^\xi \frac{1 \pm kr^2}{\sqrt{(1-r^2)(1-k^2r^2)}}\,\mathrm{d}r, \qquad 0 < k < 1.$$

On the right-hand side, we can remove the $\pm$ sign from the numerator by allowing k to range between -1 and 1. For $-1 \leqslant \xi \leqslant 1$, $-1 < k < 1$, we have a unified representative given by $\kappa = \mathcal{I}_A(\xi, k)$, where

$$\mathcal{I}_A(\xi, k) = \int_0^\xi \frac{1 - kr^2}{\sqrt{(1-r^2)(1-k^2r^2)}}\,\mathrm{d}r = \frac{1}{k}E(\xi; k) + \frac{k-1}{k}F(\xi; k)$$

in terms of the Legendre elliptic integrals E, F.

For real ξ such that $|\xi| > 1$, the function $\mathcal{I}_A(\xi, k)$ is complex valued. Yet we obtain a real function for $1/|k| \leqslant \xi$ by choosing the lower limit of the integral to be $1/k$, $-1 < k < 1$.

Thus,

$$\tilde{\mathcal{I}}_A(\xi, k) = \begin{cases} \displaystyle\int_{1/|k|}^{\xi} \frac{1 - kr^2}{\sqrt{(1 - r^2)(1 - k^2r^2)}} \, dr = \mathcal{I}_A(\xi, k) - \mathcal{I}_A\left(\frac{1}{|k|}, k\right), & \xi > \dfrac{1}{|k|}, \\ -\tilde{\mathcal{I}}_A(-\xi, k), & \xi < -\dfrac{1}{|k|}. \end{cases}$$

(B) Similarly, when $c > 1$, then $\gamma_\pm < 0$, the roots $\sqrt{\gamma_\pm}$, $-\sqrt{\gamma_\pm}$ of Π are purely imaginary, and

$$\frac{1}{\sqrt{k}} I_\pm\left(\xi\sqrt{k}, \frac{k^2+1}{2k}\right) = \int_0^{\xi} \frac{1 \pm kr^2}{\sqrt{(1 + r^2)(1 + k^2r^2)}} \, dr, \qquad 0 < k < 1.$$

The two representatives can be unified into $\kappa = \mathcal{I}_B(\xi, k)$, where

$$\mathcal{I}_B(\xi, k) = \int_0^{\xi} \frac{1 - kr^2}{\sqrt{(1 + r^2)(1 + k^2r^2)}} \, dr = \frac{1}{k\mathrm{i}} E(\xi\mathrm{i}; k) + \frac{k - 1}{k\mathrm{i}} F(\xi\mathrm{i}; k)$$

for $-1 < k < 1$.

(C) When $-1 < c < 1$ (four distinct complex roots), we substituted

$$s = \frac{1 + \sqrt{k}r}{1 - \sqrt{k}r}, \qquad 0 < k < 1,$$

to obtain two more representatives $\kappa = \mathcal{I}_{C+}$ and $\kappa = \mathcal{I}_{C-}$, where

$$\mathcal{I}_{C\pm} = \begin{cases} J_{C\pm}(\xi, k) - J_{C\pm}(0, k), & \xi \geqslant 0, \\ -\mathcal{I}_{C\pm}(-\xi, k), & \xi < 0, \end{cases}$$

$$J_{C\pm}(\xi, k) = \frac{\sqrt{1 + 2c\xi^2 + \xi^4}}{1 + \xi} + \frac{2}{(k + 1)\mathrm{i}} E\left(\frac{\xi - 1}{\xi + 1}\frac{\mathrm{i}}{\sqrt{k}}, k\right) + \frac{\varepsilon_\pm}{\mathrm{i}} F\left(\frac{\xi - 1}{\xi + 1}\frac{\mathrm{i}}{\sqrt{k}}, k\right),$$

$$\varepsilon_\pm = \frac{(1 \pm 1)k - 3 \pm 1}{2} = \begin{cases} k - 1, \\ -2, \end{cases}$$

$$c = -\frac{k^2 - 6k + 1}{(k + 1)^2}.$$

7. Normal congruences and their focal surfaces

The fact that the governing equation (10) has the offsetting symmetry (21) is not a pure coincidence. Being invertible, the offsetting transformation $\boldsymbol{r} \mapsto \boldsymbol{r} + T\boldsymbol{n}$ preserves integrability in every reasonable sense of the word. Surfaces related by the offsetting transformation are said to be parallel and either all are integrable or none is. However, parallel surfaces can be alternatively described as normal surfaces to the same line congruence. Consequently, integrability is a property of this congruence and, therefore, must have an expression in terms of congruence invariants.

Normal congruences of Weingarten surfaces, also known as W-congruences, are rather special with regard to properties of their focal surfaces. It is therefore natural to look for characterization of the former in terms of the latter. Naturally, we expect the focal surfaces of integrable W-congruences to be integrable as well.

Recall that a generic surface has two focal surfaces (often considered as two sheets of a single surface),

$$\boldsymbol{r}^{(1)} = \boldsymbol{r} + \sigma\boldsymbol{n}, \qquad \boldsymbol{r}^{(2)} = \boldsymbol{r} + \rho\boldsymbol{n}.$$

each of which is formed by the evolutes of one family of the curvature lines. Focal surfaces can degenerate into a line or even a point. In the case of a Weingarten surface $\boldsymbol{r}$ with fundamental forms (1), one of the focal surfaces degenerates into a line if $\sigma_y = \sigma' w_y = 0$ or $\rho_x = \rho' w_x = 0$; both degenerate into a point if the surface is a sphere (already excluded from consideration); otherwise they are regular surfaces. Therefore, we assume $\rho'\sigma' \neq 0$ in what follows.

To compute the respective first and second fundamental forms $\mathrm{I}^{(i)}$ and $\mathrm{II}^{(i)}$, $i = 1, 2$, we proceed as follows. In view of the Gauss–Mainardi–Codazzi equations (4) and (5), the Gauss–Weingarten (3) equations can be written as

$$\begin{aligned} \boldsymbol{r}_{xx} &= \frac{u_x}{u}\boldsymbol{r}_x + \frac{\sigma\rho' u^2 w_y}{\rho(\rho-\sigma)v^2}\boldsymbol{r}_y + \frac{u^2}{\rho}\boldsymbol{n}, & \boldsymbol{n}_x &= -\frac{1}{\rho}\boldsymbol{r}_x, \\ \boldsymbol{r}_{xy} &= \frac{\sigma\rho' w_y}{\rho(\sigma-\rho)}\boldsymbol{r}_x + \frac{\rho\sigma' w_x}{\sigma(\rho-\sigma)}\boldsymbol{r}_y, & & \\ \boldsymbol{r}_{yy} &= \frac{\rho\sigma' v^2 w_x}{\sigma(\sigma-\rho)u^2}\boldsymbol{r}_x + \frac{v_y}{v}\boldsymbol{r}_y + \frac{v^2}{\sigma}\boldsymbol{n}, & \boldsymbol{n}_y &= -\frac{1}{\sigma}\boldsymbol{r}_y. \end{aligned} \tag{40}$$

One easily finds

$$\begin{aligned} \boldsymbol{r}^{(1)}_x &= \frac{\rho-\sigma}{\rho}\boldsymbol{r}_x + \sigma' w_x\boldsymbol{n}, & \boldsymbol{r}^{(1)}_y &= \sigma' w_y\boldsymbol{n}, & \boldsymbol{n}^{(1)} &= \frac{\boldsymbol{r}_y}{v}, \\ \boldsymbol{r}^{(2)}_x &= \rho' w_x\boldsymbol{n}, & \boldsymbol{r}^{(2)}_y &= \frac{\sigma-\rho}{\sigma}\boldsymbol{r}_y + \rho' w_y\boldsymbol{n}, & \boldsymbol{n}^{(2)} &= \frac{\boldsymbol{r}_x}{u}. \end{aligned}$$

Using equations (40) and (1), we get

$$\mathrm{I}^{(1)} = \frac{(\rho-\sigma)^2u^2}{\rho^2}\mathrm{d}x^2 + \mathrm{d}\sigma^2, \qquad \mathrm{I}^{(2)} = \mathrm{d}\rho^2 + \frac{(\rho-\sigma)^2v^2}{\sigma^2}\mathrm{d}y^2, \tag{41}$$

where $\mathrm{d}\rho = \rho'\,\mathrm{d}w = \rho'(w_x\,\mathrm{d}x + w_y\,\mathrm{d}y)$, $\mathrm{d}\sigma = \sigma'\,\mathrm{d}w = \sigma'(w_x\,\mathrm{d}x + w_y\,\mathrm{d}y)$.

With u, v determined from proposition 1, we can write

$$\mathrm{I}^{(1)} = (f^{(1)}(\sigma)\,\mathrm{d}x)^2 + \mathrm{d}\sigma^2, \qquad \mathrm{I}^{(2)} = (f^{(2)}(\rho)\,\mathrm{d}y)^2 + \mathrm{d}\rho^2.$$

Hence, all focal surfaces $\boldsymbol{r}^{(i)}$ corresponding to a given dependence $\rho(\sigma)$ are isometric. Moreover, the first fundamental forms (41) are typical of surfaces of revolution. These are among the classical results by Weingarten [39].

Omitting details, we further compute the second fundamental forms

$$\mathrm{II}^{(1)} = \frac{\sigma w_y}{v}\left(\frac{\rho' u^2}{\rho^2}\mathrm{d}x^2 - \frac{\sigma' v^2}{\sigma^2}\mathrm{d}y^2\right), \qquad \mathrm{II}^{(2)} = -\frac{\rho w_x}{u}\left(\frac{\rho' u^2}{\rho^2}\mathrm{d}x^2 - \frac{\sigma' v^2}{\sigma^2}\mathrm{d}y^2\right) \tag{42}$$

and note that they are conformally related, which is another way to express Ribaucour's classical result [35] that asymptotic coordinates on $\boldsymbol{r}^{(1)}$ and $\boldsymbol{r}^{(2)}$ correspond. The Gaussian curvatures are

$$K^{(1)} = \frac{\det \mathrm{II}^{(1)}}{\det \mathrm{I}^{(1)}} = -\frac{\rho'}{(\rho-\sigma)^2\sigma'}, \qquad K^{(2)} = \frac{\det \mathrm{II}^{(2)}}{\det \mathrm{I}^{(2)}} = -\frac{\sigma'}{(\rho-\sigma)^2\rho'}. \tag{43}$$

Consequently, the focal surfaces have one and the same sign of the Gaussian curvature, which we denote as ε. We have $\varepsilon = -1$ (both focal surfaces are hyperbolic) if and only if $\mathrm{d}\rho/\mathrm{d}\sigma = \rho'/\sigma' > 0$ (if ρ increases as σ increases), and +1 if $\mathrm{d}\rho/\mathrm{d}\sigma < 0$. The relation

$$K^{(1)}K^{(2)} = \frac{1}{(\rho-\sigma)^4} \tag{44}$$

away of umbilic points is known as the Halphen theorem (see [4, section 129]).

As we have already explained, to every particular relation $\rho(\sigma)$ of curvatures there corresponds an isometry class of focal surfaces, which contains a unique rotational representative (which is the way the classes have been characterized in the classical literature).

However, we believe that a description in terms of metric invariants is more appropriate. It is convenient to choose

$$\kappa^{(i)} = \frac{1}{\sqrt{\varepsilon K^{(i)}}},$$

where $\varepsilon K^{(i)} = |K^{(i)}|$ is the absolute value of the Gaussian curvature of the ith focal surface.

Further, let $\gamma^{(i)}$ be defined by

$$\gamma^{(1)} = \frac{(\rho-\sigma)(\rho''\sigma' - \sigma''\rho') - 2\rho'\sigma'(\rho'-\sigma')}{2\,(-\varepsilon\rho'\sigma')^{3/2}},$$
$$\gamma^{(2)} = \frac{(\rho-\sigma)(\rho''\sigma' - \sigma''\rho') + 2\rho'\sigma'(\rho'-\sigma')}{2\,(-\varepsilon\rho'\sigma')^{3/2}}. \qquad (45)$$

One can directly check that $|\gamma^{(i)}|$ equals the norm of the gradient of $\kappa^{(i)}$ with respect to $\mathrm{I}^{(i)}$,

$$|\gamma^{(i)}| = \|\mathrm{grad}^{(i)}\kappa^{(i)}\|^{(i)} = \sqrt{I^{(i)}(\mathrm{grad}^{(i)}\kappa^{(i)}, \mathrm{grad}^{(i)}\kappa^{(i)})}.$$

Hence, $\gamma^{(i)}$ is a metric invariant of the respective focal surface. It is sometimes more convenient to use invariants

$$G^{(1)} = \frac{[(\rho-\sigma)(\rho''\sigma' - \sigma''\rho') - 2\rho'\sigma'(\rho'-\sigma')]^2}{16\,(\rho'\sigma')^3},$$
$$G^{(2)} = \frac{[(\rho-\sigma)(\rho''\sigma' - \sigma''\rho') + 2\rho'\sigma'(\rho'-\sigma')]^2}{16\,(\rho'\sigma')^3}, \qquad (46)$$

satisfying

$$\gamma^{(i)2} = -4\varepsilon G^{(i)}, \qquad -16\,G^{(i)}K^{(i)3} = I^{(i)}(\mathrm{grad}^{(i)}K^{(i)}, \mathrm{grad}^{(i)}K^{(i)}).$$

Clearly, both $\kappa^{(i)}$ and $G^{(i)}$ are functions of w. Consequently, $G^{(i)}$ can be considered as a function of $\kappa^{(i)}$ unless $\kappa^{(i)}$ is a constant. Our nearest aim is to establish the dependence between $\kappa^{(i)}$ and $G^{(i)}$ in terms of the dependence between ρ and σ.

Proposition 5. *Let the principal radii of curvature ρ, σ of an integrable surface satisfy the generic relation (34). Then the metric invariants $G^{(i)}$ and $\kappa^{(i)}$ satisfy the relations*

$$G^{(i)} = \left(1 \pm \varepsilon\sqrt{\frac{2}{|c \mp 1|}\frac{\kappa^{(i)}}{m}}\right)\left(-1 + \sqrt{\frac{2}{|c \mp 1|}\frac{m}{\kappa^{(i)}}}\right), \quad i = 1, 2. \qquad (47)$$

Furthermore,

$$G^{(1)}G^{(2)} = \left(\frac{c \pm 1}{c \mp 1}\right)^2$$

is constant (hence, so is the product $\gamma^{(1)}\gamma^{(2)}$).

Table 5 lists the product $G^{(1)}G^{(2)}$ and the algebraic relations between $G^{(i)}$ and $\kappa^{(i)}$ in the special cases.

Proof. For the sake of simplicity, we start assuming a fixed scaling, i.e. we depart from formula (38). We routinely compute

$$K^{(1)} = \frac{(1 \pm w^2 + \sqrt{1 + 2cw^2 + w^4})^2}{2(c \mp 1)w^4}, \qquad K^{(2)} = \frac{(1 \pm w^2 - \sqrt{1 + 2cw^2 + w^4})^2}{2(c \mp 1)w^4}.$$

Consequently, $\varepsilon = \mathrm{sgn}(c \mp 1)$, and

$$\kappa^{(1)} = \frac{1 \pm w^2 - \sqrt{1 + 2cw^2 + w^4}}{\sqrt{2|c \mp 1|}}, \qquad \kappa^{(2)} = \frac{1 \pm w^2 + \sqrt{1 + 2cw^2 + w^4}}{\sqrt{2|c \mp 1|}}.$$

Table 4. Special integrable cases. Metric invariants of focal surfaces in terms of w.

	ε	$\kappa^{(1)}$	$\kappa^{(2)}$	$G^{(1)}$	$G^{(2)}$
1	1	$2\lvert w\rvert$	$2\lvert w\rvert$	-1	-1
2a	1	$\left\lvert\frac{1}{w^2}-1\right\rvert$	$\lvert w^2-1\rvert$	$-\frac{1}{w^2}$	$-w^2$
2b	-1	$\frac{1}{w^2}+1$	w^2+1	$\frac{1}{w^2}$	w^2
3a	1	$-1+\cosh w$	$1+\cosh w$	$\frac{1+\cosh w}{1-\cosh w}$	$\frac{1-\cosh w}{1+\cosh w}$
3b	-1	$1-\cos w$	$1+\cos w$	$\frac{1+\cos w}{1-\cos w}$	$\frac{1-\cos w}{1+\cos w}$
4	-1	1	1	0	0
5a	-1	$\tanh^2 w$	1	$\frac{1}{\sinh^2 w\cosh^2 w}$	0
5b	1	$\tan^2 w$	1	$-\frac{1}{\sin^2 w\cos^2 w}$	0
6a	-1	$\coth^2 w$	1	$\frac{1}{\sinh^2 w\cosh^2 w}$	0
6b	1	$\cot^2 w$	1	$-\frac{1}{\sin^2 w\cos^2 w}$	0

Furthermore,

$$G^{(1)} = -\frac{(1 \mp w^2 + \sqrt{1+2cw^2+w^4})^2}{2(c\mp 1)w^2}, \qquad G^{(2)} = -\frac{(1 \mp w^2 - \sqrt{1+2cw^2+w^4})^2}{2(c\mp 1)w^2}.$$

Under the scaling by factor of m, the metric invariants $K^{(i)}$ and $\kappa^{(i)}$ become $K^{(i)}/m^2$ and $m\kappa^{(i)}$, respectively, while $G^{(i)}$ remains invariant. Formulae (47) are then easily checked. Moreover, all three metric invariants are invariant under offsetting (21).

Formulae for $G^{(i)}$ and $\kappa^{(i)}$ in the special cases are given in table 4 along with the sign ε of the Gaussian curvatures. □

Summarizing, focal surfaces of integrable Weingarten surfaces belong to the isometry classes specified in proposition 5.

A natural question is whether is the condition $G^{(1)}G^{(2)} = \text{const}$, or equivalently, $\gamma^{(1)}\gamma^{(2)} = \text{const}$, not only necessary, but also sufficient for condition (10) to hold.

Proposition 6. *Under the condition $\gamma^{(1)} + \gamma^{(2)} \neq 0$, a surface satisfies the governing equation (10) if and only if the product*

$$\gamma^{(1)}\gamma^{(2)} = \pm \left\|\mathrm{grad}^{(1)}\kappa^{(1)}\right\|^{(1)} \left\|\mathrm{grad}^{(2)}\kappa^{(2)}\right\|^{(2)} \tag{48}$$

is constant.

Proof. Assuming the $\rho(\sigma)$ dependence, $\gamma^{(1)} + \gamma^{(2)}$ simplifies to $(\rho-\sigma)\rho''/\sqrt{|\rho'|^3}$ and the product in question to

$$\gamma^{(1)}\gamma^{(2)} = \frac{(\rho'-1)^2}{\varepsilon\rho'} - \frac{(\rho-\sigma)^2\rho''^2}{4\varepsilon\rho'^3}.$$

Factorizing the σ-derivative of this expression as

$$\pm\frac{(\rho-\sigma)^2}{2\varepsilon\rho'^3}\left(\rho''' - \frac{3}{2\rho'}\rho''^2 + \frac{\rho'-1}{\rho-\sigma}\rho'' - 2\frac{(\rho'-1)\rho'(\rho'+1)}{(\rho-\sigma)^2}\right)\rho''$$

and comparing to the governing equation (10) proves the proposition. □

Table 5. Special integrable cases. Relations between metric invariants of focal surfaces.

	ε	$G^{(1)}G^{(2)}$	$G^{(1)}(\kappa^{(1)})$	$G^{(2)}(\kappa^{(2)})$
1	1	-1	-1	-1
2a	1	1	$-1 \pm \kappa^{(1)}$	$-1 \pm \kappa^{(2)}$
2b	-1	1	$-1 + \kappa^{(1)}$	$-1 + \kappa^{(2)}$
3a	1	-1	$-1 - \dfrac{2}{\kappa^{(1)}}$	$-1 - \dfrac{2}{\kappa^{(2)}}$
3b	-1	1	$-1 + \dfrac{2}{\kappa^{(1)}}$	$-1 + \dfrac{2}{\kappa^{(2)}}$
4	-1	0	0	0
5a	-1	0	$\left(\sqrt{\kappa^{(1)}} - \dfrac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0
5b	1	0	$-\left(\sqrt{\kappa^{(1)}} + \dfrac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0
6a	-1	0	$\left(\sqrt{\kappa^{(1)}} - \dfrac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0
6b	1	0	$-\left(\sqrt{\kappa^{(1)}} + \dfrac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0

It follows from the proof that condition (48) also holds when $\rho'' = 0$, i.e. if there is a linear relation between the radii of curvature. As of now, there seems to be no indication towards integrability of the latter class (except when $\rho \pm \sigma = \text{const}$, which satisfies (10) as well).

8. Conclusions and future work

In this work we singled out a class of Weingarten surfaces on the basis of its solitonic integrability. Although special cases were not unknown to nineteenth-century geometers, the overall result appears to be new. We also characterized integrability in terms of metric invariants of the focal surfaces.

For time reasons, many questions had to be left for further research. We do not know the Bäcklund transformation, recursion operator, bi-Hamiltonian structure and other attributes of integrability. We did not provide any solutions to the Gauss equation (39). We do not know what is the true geometric meaning of the spectral parameter. Even the task of computing third-order symmetries of the Gauss equation proved to be very complex.

We have seen in part I that integrability of surfaces of constant astigmatism is attributable to the fact that their focal surfaces are pseudospherical. In the general case, the existence of an integrability-preserving relation to previously known integrable surfaces is an open problem.

Our nearest goals include exploring the induced Bianchi type transformation between surfaces satisfying relation (47) as well as investigating the extended symmetries of the class in the sense of Cieśliński [11, 12].

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On integrability of Weingarten surfaces: a forgotten class

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Abstract
Rediscovered by a systematic search, a forgotten class of integrable surfaces is shown to disprove the Finkel–Wu conjecture. The associated integrable nonlinear partial differential equation

$$z_{yy} + (1/z)_{xx} + 2 = 0$$

possesses a zero curvature representation, a third-order symmetry and a nonlocal transformation to the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$. We leave open the problem of finding a Bäcklund autotransformation and a recursion operator that would produce a local hierarchy.

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1. Introduction

With this paper, we launch a project to classify integrable classes of surfaces. These are classes of surfaces whose Gauss–Mainardi–Codazzi equations are integrable in the sense of soliton theory. Our long-term goals include obtaining lists of integrable classes as complete as computing resources permit, clarifying their mutual relations and identifying known subcases. Our immediate goal is to demonstrate that the task is feasible and worth doing.

The classical geometry of immersed surfaces in the Euclidean space is well known to be closely connected with the modern theory of integrable systems [35]. The Gauss–Weingarten equations of a moving frame Ψ always take the form

$$\Psi_x = A\Psi, \qquad \Psi_y = B\Psi, \tag{1}$$

where A, B are appropriate matrix functions. Integrability conditions of (1) are called the Gauss–Mainardi–Codazzi equations and take the form of a *zero-curvature representation*

$$A_y - B_x + [A, B] = 0. \tag{2}$$

Equation (2) is invariant under a huge group of *gauge transformations*

$$A' = S_x S^{-1} + SAS^{-1}, \qquad B' = S_y S^{-1} + SBS^{-1}, \tag{3}$$

induced by linear transformations $\Psi' = S\Psi$ of the frame. Here S is an invertible functional matrix, which can be restricted to take values in the Lie group G associated with the Lie algebra $\mathfrak{g}$ matrices A, B belong to—typically $\mathfrak{so}(3)$.

The zero-curvature representation (2) is the key ingredient in the soliton theory [15], where matrices A, B are additionally assumed to depend on what is called the *spectral parameter*. The essential requirement for solitonic integrability is that the spectral parameter cannot be removed by means of the gauge transformation (3). Consequently, if the matrices A, B can be modified so that they depend on a nonremovable parameter and still satisfy (2), then the corresponding Gauss–Mainardi–Codazzi equations are considered to be integrable in the sense of soliton theory, and their solutions are known as *integrable* or *soliton surfaces* [40].

Solitonic integrability can appear only when surfaces are subject to a constraint (such as being pseudospherical, etc). For numerous classical and recent examples, see, e.g., [4, 35, 38] (or [16] in the projective setting). Workable tools to classify such constraints include all the general integrability criteria [31], which are, however, not immediately applicable to non-evolutionary systems [30]. Other methods take advantage of the already known non-parametric zero-curvature representation (2), e.g., the method of extended symmetries by Cieśliński *et al* [10–12].

In this paper we employ a recent method due to one of us [29]. Its essence can be summarized as follows: we attempt to extend the given non-parametric zero-curvature representation (a seed) to a power series in terms of the spectral parameter. In the work [29], the relevant computable cohomological obstructions are identified. Two obstacles make this procedure not entirely algorithmic: the parameter-dependent zero-curvature representation could exist in an extension of the Lie algebra $\mathfrak{g}$ and its jet order (the order of derivatives) could exceed that of the seed. If no obstructions are found, various ways exist to incorporate the true nonremovable parameter.

2. Weingarten surfaces

To be of genuine interest in geometry, the determining constraint on integrable surfaces must be invariant with respect to coordinate changes. The general non-differential invariant constraint is a functional relation $f(p, q) = 0$ between the principal curvatures p, q. Such a functional relation is a characteristic of Weingarten surfaces, which have been a topic of continuous interest, especially in global differential geometry [20, 26, 38, 41] and computer graphics [8]. Well known to be integrable is the class of *linear Weingarten surfaces* [13, 35], characterized by a linear relation

$$ak + bh + c = 0, \qquad a, b, c = \text{const} \tag{4}$$

between the Gauss curvature $k = pq$ and the mean curvature $h = \frac{1}{2}(p+q)$ (not to be confused with a linear relation between the principal curvatures [23, 26]). Other integrable classes of Weingarten surfaces that sporadically occur in the literature all have a differential defining relation (e.g., the Hazzidakis equation of the Bonnet surfaces [4, 5, 7]; a harmonicity condition of Schief's [37] generalized linear Weingarten surfaces) or the class is not determined by the functional relation $f(p, q) = 0$ alone (e.g., [9]).

So far, nothing contradicts the conjecture of Finkel [17, conjecture 3.4] and Wu [43] that the only functional relation $f(p, q) = 0$ to determine an integrable class of Weingarten surfaces is the linear relation (4). Supporting arguments include Wu's [43] proof of non-existence of an $\mathfrak{so}(3)$-valued zero-curvature representation depending only on x-derivatives.

Finkel's [17] argument roots in an unsuccessful search for higher order symmetries and a (disputable, see [30, section 2]) conjecture that integrability implies the existence of a local higher order symmetry (actually the infinite hierarchy can be nonlocal, see also [31, section 1.4.4.2]).

Nevertheless, the main result of the present paper asserts that the simple relation

$$\frac{1}{p} - \frac{1}{q} = \text{const} \tag{5}$$

between the main curvatures p, q determines an integrable class of Weingarten surfaces. The associated nonlinear partial differential equation (21) has a parameter-dependent zero-curvature representation (22) (outside the class considered in [43]), a third-order symmetry (24) (missed in [17]), and a recursion operator (25).

Paradoxically enough, surfaces satisfying relation (5) were not unknown to 19th century geometers. In view of their knowledge, our integrability result is not an entirely unexpected one. In fact, Ribaucour [34] established that the corresponding focal surfaces (evolutes) have a constant Gaussian curvature $k < 0$ (are pseudospherical). Conversely, surfaces satisfying equation (5) are involutes of pseudospherical surfaces. Moreover, the classical Bianchi transformation [2] is nothing but the induced correspondence between the two focal pseudospherical surfaces. Ribaucour's theorems are covered in Darboux [13] and early 20th-century monographs, such as [3, 14, 18, 42]. Later they became obsolete and forgotten as the induced Bianchi relation between pseudo-spherical surfaces became superseded by the classical Bäcklund transformation (the history is nicely reviewed by Prus and Sym in [32, section 4]).

The first examples of surfaces satisfying relation (5) also date to the 19th century. Lipschitz [25] derived a four-parametric family in terms of elliptic integrals. A particular subcase, the rotation surface of von Lilienthal [24], is the involute surface of the pseudosphere.

The left-hand side of equation (5) is equal to the difference of the principal radii of curvature at a point. This geometric quantity has a definite physical meaning, being associated with the *interval of Sturm* [39], also known as the *astigmatic interval* or the *amplitude of astigmatism* or simply the *astigmatism* [19]. A mirror or a refracting surface satisfying relation (5) will feature a constant amplitude of astigmatism in the normal directions. In the following, surfaces satisfying condition (5) will be called *surfaces of constant astigmatism.* Accordingly, equation (21) to determine the surfaces of constant astigmatism will be called the *constant astigmatism equation.*

3. Preliminaries

We shall consider surfaces $\mathbf{r}(x, y)$ parametrized by curvature lines. As is well known, the fundamental forms can be written as

$$\begin{aligned} \mathrm{I} &= u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2, \\ \mathrm{II} &= u^2 p\,\mathrm{d}x^2 + v^2 q\,\mathrm{d}y^2, \end{aligned}$$

where p, q are the principal curvatures. Coordinates x, y are unique up to arbitrary changes $x = X(x), y = Y(y)$.

Let $\Psi = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ denote the orthonormal frame, given by $\mathbf{e}_1 = \mathbf{r}_x/u, \mathbf{e}_2 = \mathbf{r}_y/v, \mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$. The Gauss–Weingarten equations

$$\Psi_x = \begin{pmatrix} 0 & -\frac{u_y}{v} & up \\ \frac{u_y}{v} & 0 & 0 \\ -up & 0 & 0 \end{pmatrix} \Psi, \qquad \Psi_y = \begin{pmatrix} 0 & \frac{v_x}{u} & 0 \\ -\frac{v_x}{u} & 0 & vq \\ 0 & -vq & 0 \end{pmatrix} \Psi \tag{6}$$

are easily established. Their integrability conditions are the Gauss equation

$$uu_{yy} + vv_{xx} - \frac{v}{u} u_x v_x - \frac{u}{v} u_y v_y + u^2 v^2 pq = 0 \tag{7}$$

and the Mainardi–Codazzi equations

$$(p - q)u_y + up_y = 0, \qquad (q - p)v_x + vq_x = 0. \tag{8}$$

Consequently, the two $\mathfrak{so}(3)$ matrices occurring in formulae (6) constitute a nonparametric zero-curvature representation of the Gauss–Mainardi–Codazzi system (7) and (8). Because of the isomorphism $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$, the same zero-curvature representation can be alternatively written in terms of 2×2 matrices

$$A_0 = \begin{pmatrix} \frac{\mathrm{i}u_y}{2v} & -\frac{1}{2}up \\ \frac{1}{2}up & -\frac{\mathrm{i}u_y}{2v} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -\frac{\mathrm{i}v_x}{2u} & -\frac{1}{2}\mathrm{i}qv \\ -\frac{1}{2}\mathrm{i}qv & \frac{\mathrm{i}v_x}{2u} \end{pmatrix}. \tag{9}$$

Let us impose a constraint $f(p, q) = 0$. If nontrivial, it can be resolved with respect to one of the curvatures, say

$$q = F(p), \tag{10}$$

which we assume henceforth. Then the Gauss–Mainardi–Codazzi system reduces substantially [8, 17, 43]. In particular, the Mainardi–Codazzi equations (8) have a general solution

$$u = \frac{u_0}{E}, \qquad v = -v_0 E', \qquad q = p - \frac{E}{E'},$$

where $E = E(p)$ is an arbitrary nonconstant function, $E' = \mathrm{d}E/\mathrm{d}p$, and u_0, v_0 are functions of x and y, respectively, removable by reparametrization $x \to \int u_0\, \mathrm{d}x$, $y \to \int v_0\, \mathrm{d}y$. Therefore, we can put $u_0 = -v_0 = 1$ without loss of generality, i.e.,

$$u = \frac{1}{E}, \qquad v = E', \qquad q = p - \frac{E}{E'}. \tag{11}$$

The Gauss equation (7) then becomes

$$p_{yy} = E^3 E'' p_{xx} + 2\frac{E'}{E} p_y^2 + E^2 (EE'')' p_x^2 + EE' p^2 - E^2 p. \tag{12}$$

Summarizing, the Gauss–Mainardi–Codazzi system of Weingarten surfaces reduces to the single equation (12). The classification problem considered in this paper is 'for which choices of the function $E(p)$ is equation (12) integrable?'

By substituting (11) into (9), we easily obtain a nonparametric zero-curvature representation of equation (12),

$$A_0 = \begin{pmatrix} \frac{\mathrm{i}}{2}\frac{p_y}{E^2} & -\frac{1}{2}\frac{p}{E} \\ \frac{1}{2}\frac{p}{E} & -\frac{\mathrm{i}}{2}\frac{p_y}{E^2} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} \frac{\mathrm{i}}{2}EE''p_x & \frac{\mathrm{i}}{2}(E'p - E) \\ \frac{\mathrm{i}}{2}(E'p - E) & -\frac{\mathrm{i}}{2}EE''p_x \end{pmatrix}, \tag{13}$$

which will be the starting point of the calculations to follow.

4. Cohomological criteria

Readers not interested in details of the classification method can skip this section and continue to investigation of surfaces of constant astigmatism in section 5.

We use the formal theory of partial differential equations, which treats coordinates, unknown functions and their derivatives as independent quantities. Equations can be conveniently represented as submanifolds in appropriate jet spaces [6]. All our considerations being local, we let $J^\infty = J^\infty(\mathbb{R}^2, \mathbb{R})$ denote the space of ∞-jets of smooth functions $\mathbb{R}^2 \to \mathbb{R}$. The base $\mathbb{R}^2$ being equipped with coordinates x, y, the natural coordinates along fibres of $J^\infty \to \mathbb{R}^2$ correspond to p and its derivatives. These will be denoted p_I, where I stands for a symmetric multi-index in x, y (including the 'empty' multi-index $\emptyset$ such that $p_\emptyset = p$). The usual total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_I p_{xI}\frac{\partial}{\partial p_I}, \qquad D_y = \frac{\partial}{\partial y} + \sum_I p_{yI}\frac{\partial}{\partial p_I}$$

can be viewed as acting on smooth functions defined on J^∞ (by definition, a smooth function locally depends on a finite number of coordinates).

In J^∞, we consider a submanifold $\mathcal{G}$ determined by equation (12) and all its differential consequences obtained by taking successive total derivatives of both sides of (12). On $\mathcal{G}$, all derivatives of the form p_{Jyy} become expressible in terms of the others. Therefore, derivatives p_I with y occurring no more than twice in I serve as natural coordinates along the fibres of $\mathcal{G} \to \mathbb{R}^2$. Being tangent to (12), the total derivatives admit a restriction to $\mathcal{G}$. We retain the same notation D_x, D_y for the restricted total derivatives.

The essence of the adopted point of view can be summarized as follows: a function f on J^∞ satisfies $f|_\mathcal{G} = 0$ if and only if f is zero as a consequence of equation (12). From now on we assume that all objects (like the matrices A, B) are defined on $\mathcal{G}$. When writing

$$(D_yA - D_xB + [A, B])|_\mathcal{G} = 0 \tag{14}$$

we mean that the zero-curvature condition (2) holds as a consequence of equation (12).

In what follows, characteristic elements [27, 28, 36] play a crucial role. These are non-Abelian analogues of characteristics of conservation laws [6]. For instance, the characteristic element of the initial zero-curvature representation (13) is the $\mathfrak{sl}(2, \mathbb{C})$-matrix

$$C_0 = \begin{pmatrix} \frac{\mathrm{i}}{2}\frac{1}{E^2} & 0 \\ 0 & -\frac{\mathrm{i}}{2}\frac{1}{E^2} \end{pmatrix}.$$

This immediately follows from the fact that

$$D_yA_0 - D_xB_0 + [A_0, B_0] = C_0F,$$

where

$$F = p_{yy} - E^3E''p_{xx} - 2\frac{E'}{E}p_y^2 - E^2(EE'')'p_x^2 - p^2EE' - pE^2,$$

so that the Gauss equation (12) can be written as $F = 0$.

Let $A = A(\lambda), B = B(\lambda)$ be the parametric zero-curvature representation sought, $C = C(\lambda)$ the corresponding characteristic element. Besides (14), they will also satisfy the formula [27]

$$\sum_I (-\hat{D})_I \left(\frac{\partial F}{\partial u_I^k}C\right)\bigg|_\mathcal{G} = 0, \tag{15}$$

with I running over all symmetric multi-indices, including the empty one. Here $\hat{D}_x = D_x - [A, \cdot]$, $\hat{D}_y = D_y - [B, \cdot]$, the other values being obtained by composition, which can be taken in any order since (14) implies that $\hat{D}_x, \hat{D}_y$ commute.

Characteristic elements of gauge equivalent zero-curvature representations are conjugate (similar). This allows us to transform characteristic elements into the normal form with respect to conjugation, namely, the Jordan normal form. Since the matrix C_0 above is diagonal, it follows that for λ sufficiently close to zero the characteristic element $C(\lambda)$ will be also diagonalizable.

However, diagonal matrices have a nontrivial stabilizer $\mathcal{S} \subset$ '$SL(2, \mathbb{C})$ with respect to conjugation, which consists of diagonal matrices

$$\begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix}.$$

Gauge transformations from the group $\mathcal{S}$ (henceforth $\mathcal{S}$-transformations) preserve the characteristic elements $C(\lambda)$. Their gauge action on a general $\mathfrak{sl}(2)$-valued zero-curvature representation A, B is sufficiently simple:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{s_x}{s} + a_{11} & s^2 a_{12} \\ \dfrac{a_{21}}{s^2} & -\dfrac{s_x}{s} - a_{11} \end{pmatrix}$$

and similarly for B. Using $\mathcal{S}$-transformations, one can achieve a unique normal form of matrices A, B as follows: if $a_{12} \neq 0$, then by setting $s = (a_{21}/a_{12})^{1/4}$ we turn A into a symmetric matrix, while in the remaining case $a_{12} = 0$ the zero-curvature representation degenerates to a pair of conservation laws [28]. In other words, having a symmetric component is a normal form of nondegenerate zero-curvature representations with respect to $\mathcal{S}$-transformations.

Turning back to our original problem, we see that B_0 is symmetric, and therefore the nearby matrices $B(\lambda)$ can also be symmetrized by an $\mathcal{S}$-transformation. A simple calculation shows that, by assuming diagonality of $C(\lambda)$ and symmetricity of $B(\lambda)$, we make the system (15) determined, hence solvable (actually, we fix the gauge).

Summarizing, the computation of the zero-curvature representation has been reduced to the solution of the determined system (14) and (15) under a suitable choice of normal forms for C and B. However, this nonlinear system is still quite difficult to solve even with the help of computer algebra. To linearize the system, the work [29] considers Taylor expansions

$$A(\lambda) = \sum_{k=0} A_k \lambda^k, \qquad B(\lambda) = \sum_{k=0} B_k \lambda^k, \qquad C(\lambda) = \sum_{k=0} C_k \lambda^k, \tag{16}$$

with A_0, B_0, C_0 coming from the initial parameterless zero-curvature representation (9). The condition of the zero curvature for $A(\lambda), B(\lambda)$ implies an infinite sequence of conditions of the zero curvature for block triangular matrices

$$A^{[m]} = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_m & \cdots & A_1 & A_0 \end{pmatrix}, \qquad B^{[m]} = \begin{pmatrix} B_0 & 0 & \cdots & 0 \\ B_1 & B_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_m & \cdots & B_1 & B_0 \end{pmatrix}. \tag{17}$$

Characteristic elements $C^{[m]}$ assume the same form. Zero curvature representations $A^{[m]}$, $B^{[m]}$ are to be considered under the gauge group consisting of block triangular matrices

$$S^{[m]} = \begin{pmatrix} E & 0 & \cdots & 0 \\ S_1 & E & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ S_m & \cdots & S_1 & E \end{pmatrix}$$

with unit matrices E in the diagonal positions. By a cohomological argument presented in [29, proposition 1], a nontrivial family $A(\lambda)$, $B(\lambda)$ with analytic dependence on λ has expansions (16) such that A_1 or B_1 is not zero.

Let $(14)^{[m]}$ and $(15)^{[m]}$ denote the system obtained by substituting $A \to A^{[m]}$, $B \to B^{[m]}$ into system (14) and (15), for arbitrary $m > 0$. Observe that systems $(14)^{[m]}$ and $(15)^{[m]}$ are linear in their highest order unknowns A_m, B_m, C_m and can be solved sequentially. Then the applicable cohomological criterion can be summarized as follows.

Proposition 1 [29, proposition 3]. *Let $m > 0$. If $A_1 = B_1 = 0$ for all solutions $A^{[m]}$, $B^{[m]}$ of system $(14)^{[m]}$, $(15)^{[m]}$, then there is no possibility to construct expansions (16) of order m, and consequently, the seed zero curvature representation A_0, B_0 cannot belong to a nontrivial analytic family.*

Finally, to be able to solve system $(14)^{[m]}$ and $(15)^{[m]}$, we need to know the normal forms of matrices $A^{[m]}$, $B^{[m]}$. However, the normal forms for $B(\lambda)$, $C(\lambda)$ established above immediately imply the same normal forms for C_k (diagonal) and B_k (symmetric).

5. Results

In this section, we present the results of computation of the cohomological obstructions in the case of the nonparametric zero-curvature representation (13) of equation (12). As a sub-result we obtain the first few coefficients A_k, B_k of Taylor expansions (16).

As we have seen in the preceding section (proposition 1), the problem reduces to solving the system $(14)^{[m]}$ and $(15)^{[m]}$ of linear differential equations in total derivatives, for increasing values of m. This is only possible under a suitable restriction on the jet order of the unknowns $A_k, B_k, C_k, k > 0$. To start with, we assume dependence on the first-order jets at most. Upon expanding all total derivatives, equations $(14)^{[m]}$ and $(15)^{[m]}$ become a large overdetermined system of linear partial differential equations. As such, the system is solvable by computing the passive (or involutive) form under a suitable (elimination) ranking [33].

Starting with $m = 1$, we checked that nonzero matrices A_1, B_1 depending on second-order derivatives exist for all possible determining relations (10). When incrementing m to 2, nontrivial conditions started to appear, but we also reached the boundaries of our available computing resources. Consequently, our present classification results are still incomplete. Nevertheless, we were able to obtain a passive system of differential equations in several cases. Moreover, in two cases we were able to find A_2, B_2 explicitly. One of them was the linear Weingarten surfaces (4). Their integrability is a well-established fact [35], the associated sine-Gordon equation $\phi_{xy} = \sin\phi$ being a textbook example of integrability. The other class emerged as a solution

$$E = \frac{p}{\mathrm{e}^{1+c/p}}, \qquad c = \text{const} \tag{18}$$

of the ordinary differential equation

$$\frac{E''}{E} - \left(\frac{E'}{E}\right)^2 + \frac{2}{p}\frac{E'}{E} - \frac{1}{p^2} = 0.$$

Henceforth we concentrate on the solution (18). The coefficients u, v, q are easily found from (11) to be

$$u = \frac{\mathrm{e}^{1+c/p}}{p}, \qquad v = \frac{p+c}{p\,\mathrm{e}^{1+c/p}}, \qquad q = \frac{pc}{p+c}.$$

The last equality shows that the condition of constant astigmatism (5) holds with the constant $-1/c$ on the right-hand side. The Gauss equation (12) becomes

$$p_{yy} = \frac{c^2}{\mathrm{e}^{4(1+\frac{c}{p})}} p_{xx} + 2\frac{p+c}{p^2} p_y^2 + 2\frac{c^2(c-p)}{\mathrm{e}^{4(1+\frac{c}{p})} p^2} p_x^2 + \frac{cp^2}{\mathrm{e}^{2(1+\frac{c}{p})}}.$$

In principle, the cohomological method we applied can only prove nonintegrability and only indicate, but not prove, integrability. However, it was easy to guess an ansatz based on the form of A_k and B_k. By solving (14) and (15) we obtained a λ-dependent zero-curvature representation

$$A = \begin{pmatrix} \lambda c\frac{p_x}{p^2} + \sqrt{\lambda^2+\lambda}\,\mathrm{e}^{2(1+\frac{c}{p})}\frac{p_y}{p^2} & \lambda\,\mathrm{e}^{1+2\frac{c}{p}} \\ (\lambda+1)\,\mathrm{e} & -\lambda c\frac{p_x}{p^2} - \sqrt{\lambda^2+\lambda}\,\mathrm{e}^{2(1+\frac{c}{p})}\frac{p_y}{p^2} \end{pmatrix},$$
$$B = \begin{pmatrix} \lambda c\frac{p_y}{p^2} + \sqrt{\lambda^2+\lambda}c^2\,\mathrm{e}^{-2(1+\frac{c}{p})}\frac{p_x}{p^2} & \sqrt{\lambda^2+\lambda}c\,\mathrm{e}^{-1} \\ \sqrt{\lambda^2+\lambda}c\,\mathrm{e}^{-1-2\frac{c}{p}} & -\lambda c\frac{p_y}{p^2} - \sqrt{\lambda^2+\lambda}c^2\,\mathrm{e}^{-2(1+\frac{c}{p})}\frac{p_x}{p^2} \end{pmatrix}, \qquad (19)$$

which reduces to the initial A_0, B_0 given by (13) when $\lambda = -\frac{1}{2}$. The dependence on p_y explains why this class of Weingarten surfaces is missing in Wu's paper [43].

Upon substitution

$$x \to \frac{x}{|c|^{1/4}}, \qquad y \to \frac{y}{|c|^{3/4}}, \qquad p \to \frac{4c}{2\ln z + \ln|c| - 4} \qquad (20)$$

the Gauss equation (12) simplifies to

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0, \qquad (21)$$

and the zero-curvature representation (19) to

$$A = \begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2+\lambda}z_y + \frac{1+2\lambda}{4}\frac{z_x}{z} & (\lambda+1)\sqrt{z} \\ \lambda\sqrt{z} & -\frac{1}{2}\sqrt{\lambda^2+\lambda}z_y - \frac{1+2\lambda}{4}\frac{z_x}{z} \end{pmatrix},$$
$$B = \begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2+\lambda}\frac{z_x}{z^2} + \frac{1+2\lambda}{4}\frac{z_y}{z} & \frac{\sqrt{\lambda^2+\lambda}}{\sqrt{z}} \\ \frac{\sqrt{\lambda^2+\lambda}}{\sqrt{z}} & -\frac{1}{2}\sqrt{\lambda^2+\lambda}\frac{z_x}{z^2} - \frac{1+2\lambda}{4}\frac{z_y}{z} \end{pmatrix}. \qquad (22)$$

Let us remark that one can remove the x-derivatives from A and y-derivatives from B by the gauge transformation (3), albeit at the cost of introducing an exponential dependence on the spectral parameter. In (19) and (22), the corresponding gauge matrix is

$$S = \begin{pmatrix} \mathrm{e}^{-\lambda c/p} & 0 \\ 0 & \mathrm{e}^{\lambda c/p} \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} z^{\lambda/2} & 0 \\ 0 & z^{-\lambda/2} \end{pmatrix},$$

respectively. For instance, the pair (22) becomes

$$A' = \begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2+\lambda}z_y & (\lambda+1)z^{-\lambda} \\ \lambda z^{\lambda+1} & -\frac{1}{2}\sqrt{\lambda^2+\lambda}z_y \end{pmatrix}, \qquad B' = \begin{pmatrix} \frac{1}{2}\sqrt{\lambda^2+\lambda}\frac{z_x}{z^2} & \sqrt{\lambda^2+\lambda}z^{-\lambda-1} \\ \sqrt{\lambda^2+\lambda}z^{\lambda} & -\frac{1}{2}\sqrt{\lambda^2+\lambda}\frac{z_x}{z^2} \end{pmatrix}.$$

Equation (21) has obvious translational symmetries ∂_x, ∂_y, the scaling symmetry $2z\partial_z - x\partial_x + y\partial_y$, and a discrete symmetry

$$x \to y, \qquad y \to x, \qquad z \to \frac{1}{z}. \tag{23}$$

Computation reveals also two third-order symmetries of equation (21). One of them has the generator

$$\begin{aligned} &\frac{z^3}{K^3}(z_{xxx} - zz_{xxy}) - \frac{3}{K^5}z^3(z_x - zz_y)(z_{xx} - zz_{xy})^2 - \frac{2}{K^5}z^5(9z_x - zz_y)z_{xx} \\ &\quad + \frac{1}{2K^5}z^2(9z_x^2 + 4zz_xz_y - z^2z_y^2)(z_x - zz_y)z_{xx} - \frac{2}{K^5}z^3z_x(z_x - zz_y) \\ &\quad \times (4z_x - zz_y)z_{xy} + \frac{4}{K^5}z^6z_xz_{xy} + \frac{3}{K^5}z^4(5z_x - zz_y)z_x^2 - \frac{3}{K^5}z(z_x - zz_y)z_x^4, \end{aligned} \tag{24}$$

where

$$K = \sqrt{(z_x - zz_y)^2 + 4z^3}.$$

The other is obtained by conjugation with the discrete symmetry (23).

Moreover, A Sergyeyev (private communication) succeeded in finding a recursion operator for equation (21) in the usual pseudo-differential form

$$-z_yD_x^{-1} + z_xD_x^{-2}D_y + 2zD_x^{-1}D_y. \tag{25}$$

As far as we could see, the operator generates only nonlocal symmetries. We leave as an open problem to find a recursion operator that would generate the third-order symmetry (24).

Let us conclude this section with some easy geometric observations. First of all, we can put $c = 1$ without loss of generality. This can always be achieved by rescaling the ambient Euclidean metric and, if necessary, changing the orientation.

Now, the symmetries of the constant astigmatism equation (21) have the following geometric interpretation. Translation symmetries are simply reparametrizations of the surface. The scaling symmetry ϕ_ε: $x \to e^{\varepsilon}x$, $y \to e^{-\varepsilon}y$, $z \to e^{-2\varepsilon}z$ takes a given surface $\mathbf{r}(x, y)$ to the parallel surface $\mathbf{r}(x, y) + \varepsilon\,\mathbf{n}(x, y)$. This is not surprising since parallel surfaces obviously have equal astigmatism in the corresponding points. Finally, swapping the orientation is another symmetry, which can be identified with a composition of the discrete symmetry (23) and the rescaling ϕ_1. Hence, the discrete symmetry (23) corresponds to the change of the orientation followed by taking the parallel surface at the unit distance.

6. Relation to pseudospherical surfaces

As already mentioned in section 1, 19th century geometers knew of a simple relation between pseudospheric surfaces and surfaces of constant astigmatism, even though they did not find the latter important enough to be named. In this section we reproduce some of their findings and derive a nonlocal transformation between the constant astigmatism equation (21) and the famous sine-Gordon equation. Again, we put $c = 1$ for simplicity, meaning that the associated focal surfaces will be of Gaussian curvature -1.

The forthcoming calculations are conveniently performed in terms of the variable z given by formula (20) or a new variable w related to z by

$$z = e^{2w}. \tag{26}$$

Then we have

$$u = (w-1)\,e^{w}, \qquad v = \frac{w}{e^{w}}, \qquad p = \frac{1}{w-1}, \qquad q = \frac{1}{w}. \tag{27}$$

and the discrete symmetry (23) becomes simply

$$x \to y, \qquad y \to x, \qquad w \to -w. \tag{28}$$

Given a surface $\mathcal{L}$, recall that its *evolutes* (also known as focal surfaces) are the loci of the principal centres of curvature of $\mathcal{L}$. Obviously, a generic surface $\mathcal{L}$ has two evolutes. They interchange positions under the change of the orientation.

Proposition 2 (Ribaucour [34]). *Evolutes of surfaces of constant astigmatism are pseudospherical surfaces.*

Proof. Let $\mathbf{r}(x, y)$ be a surface parametrized by curvature lines. We use the orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$, where

$$\mathbf{e}_1 = \mathbf{r}_x/u, \qquad \mathbf{e}_2 = \mathbf{r}_y/v, \qquad \mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2.$$

Then the two evolutes $\mathcal{L}'$ and $\mathcal{L}''$ are given by

$$\mathbf{r}' = \mathbf{r} + \frac{\mathbf{n}}{p}, \qquad \mathbf{r}'' = \mathbf{r} + \frac{\mathbf{n}}{q},$$

respectively. An easy calculation using the Gauss–Weingarten formulae (6) shows that

$$\mathbf{r}'_x = -\frac{p_x}{p^2}\mathbf{n}, \qquad \mathbf{r}'_y = -\frac{p_y}{p^2}\mathbf{n} + \left(1 - \frac{q}{p}\right)\mathbf{r}_y,$$

$$\mathbf{r}''_x = -\frac{q_x}{q^2}\mathbf{n} + \left(1 - \frac{p}{q}\right)\mathbf{r}_x, \qquad \mathbf{r}''_y = -\frac{q_y}{q^2}\mathbf{n},$$

the unit normals being

$$\mathbf{n}' = \frac{\mathbf{r}_x}{u}, \qquad \mathbf{n}'' = \frac{\mathbf{r}_y}{v}.$$

Now assume $\mathbf{r}(x, y)$ to be a surface of constant astigmatism. By applying the substitutions (27) we obtain the first fundamental form of the evolutes in terms of w:

$$\mathrm{I}' = (w_x\,\mathrm{d}x + w_y\,\mathrm{d}y)^2 + e^{-2w}\,\mathrm{d}y^2 = \mathrm{d}w^2 + e^{-2w}\,\mathrm{d}y^2,$$
$$\mathrm{I}'' = e^{2w}\,\mathrm{d}x^2 + (w_x\,\mathrm{d}x + w_y\,\mathrm{d}y)^2 = e^{2w}\,\mathrm{d}x^2 + \mathrm{d}w^2.$$

These are the well-known pseudospherical metrics in terms of geodesic coordinates w, y and w, x on the first and the second sheet, respectively. □

For further reference we also compute the second fundamental forms

$$\mathrm{II}' = -e^{w} w_x\,\mathrm{d}x^2 + \frac{w_x}{e^{3w}}\,\mathrm{d}y^2, \qquad \mathrm{II}'' = e^{3w} w_y\,\mathrm{d}x^2 - \frac{w_y}{e^{w}}\,\mathrm{d}y^2.$$

Proposition 2 provides as with a couple of transformations from the constant astigmatism equation (21) to the sine-Gordon equation. To write them explicitly, we need to equip $\mathcal{L}'$ and $\mathcal{L}''$ with the asymptotic coordinates ξ, η, i.e., the fundamental forms have to be

$$\mathrm{I}' = \mathrm{d}\xi^2 + 2\cos\phi'\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{d}\eta^2, \qquad \mathrm{II}' = 2\sin\phi'\,\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$\mathrm{I}'' = \mathrm{d}\xi^2 + 2\cos\phi''\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{d}\eta^2, \qquad \mathrm{II}'' = 2\sin\phi''\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Here ϕ' and ϕ'' are the angles between the coordinate lines on $\mathcal{L}'$ and $\mathcal{L}''$, respectively. Using the previous expression of fundamental forms I′, II′ and I″, II″ in terms of the variable w, we easily see that ξ, η can be obtained by the 'reciprocal transformation' [35]

$$\begin{aligned} \mathrm{d}\xi &= \tfrac{1}{2}\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\, \mathrm{d}x + \tfrac{1}{2}\sqrt{(\mathrm{e}^{-2w} w_x + w_y)^2 + \mathrm{e}^{-2w}}\, \mathrm{d}y, \\ \mathrm{d}\eta &= \tfrac{1}{2}\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\, \mathrm{d}x - \tfrac{1}{2}\sqrt{(\mathrm{e}^{-2w} w_x - w_y)^2 + \mathrm{e}^{-2w}}\, \mathrm{d}y. \end{aligned} \tag{29}$$

These formulae are valid on both sheets and reflect another property established by Ribaucour [34], namely that the asymptotic lines on $\mathcal{L}'$ and $\mathcal{L}''$ correspond.

Then the angle ϕ' associated with the first sheet satisfies

$$\begin{aligned} \cos\phi' &= \frac{w_x^2 - \mathrm{e}^{2w} - \mathrm{e}^{4w} w_y^2}{\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}}, \\ \sin\phi' &= -\frac{2\,\mathrm{e}^{w} w_x}{\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}}, \end{aligned} \tag{30}$$

while the angle ϕ'' associated with the second sheet satisfies a similar set of equations related by the substitution (28).

Proposition 3. *Let $z(x, y)$ be a solution of the constant astigmatism equation (21), let $w = \frac{1}{2}\ln z$. Determine function ϕ' by formula (30) and new coordinates ξ, η by the reciprocal transformation (29). Then $\phi'(\xi, \eta)$ is a solution of the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$.*

Another solution of the sine-Gordon equation can be obtained by combination with the discrete symmetry (28). The other symmetries (translation and scaling) do not lead to essentially new solutions.

Now, it is easy to check that *the evolutes of surfaces of constant astigmatism are related by the classical Bianchi transformation.* Indeed, the corresponding points $\mathbf{r}'$ and $\mathbf{r}''$ have a constant distance equal to $1/p - 1/q$. The corresponding normals $\mathbf{n}' = \mathbf{r}_x/u$ and $\mathbf{n}'' = \mathbf{r}_y/v$ are orthogonal. Finally, being directed along the normal vector $\mathbf{n}$, the line joining the points $\mathbf{r}'$ and $\mathbf{r}''$ is tangent to both surfaces $\mathcal{L}'$ and $\mathcal{L}''$. These three properties characterize the classical Bianchi transformation. The Bianchi transformation is, however, superseded by the classical Bäcklund transformation [1], where the condition on the angle between the normals is relaxed from being right to being constant.

7. Surfaces of constant astigmatism as involutes

In principle, all surfaces of constant astigmatism can be obtained from solutions of the sine-Gordon equation as involute surfaces, see, e.g., Darboux [13, section 802], Bianchi [3, sections 130–150] or Weatherburn [3, chapter 8]. Geodesic nets on pseudospheric surfaces fall into three classes: hyperbolic, parabolic and elliptic [3, section 102]. Of them only the parabolic geodesic nets lead to surfaces of constant astigmatism [3, section 136].

Recall that the sine-Gordon $\phi_{\xi\eta} = \sin\phi$ describes surfaces of the constant curvature -1 in the asymptotic coordinates ξ, η. By definition,

$$\mathrm{I} = \mathrm{d}\xi^2 + 2\cos\phi\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{d}\eta^2, \qquad \mathrm{II} = 2\sin\phi\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

which leads to the Gauss–Weingarten equations

$$\begin{aligned} \mathbf{r}_{\xi\xi} &= \frac{\cos\phi\,\mathbf{r}_\xi - \mathbf{r}_\eta}{\sin\phi}\phi_\xi, \qquad \mathbf{r}_{\xi\eta} = \sin\phi\,\mathbf{n}, \qquad \mathbf{r}_{\eta\eta} = \frac{\cos\phi\,\mathbf{r}_\eta - \mathbf{r}_\xi}{\sin\phi}\phi_\eta, \\ \mathbf{n}_\xi &= \frac{\cos\phi\,\mathbf{r}_\xi - \mathbf{r}_\eta}{\sin\phi}, \qquad \mathbf{n}_\eta = \frac{\cos\phi\,\mathbf{r}_\eta - \mathbf{r}_\xi}{\sin\phi}. \end{aligned} \tag{31}$$

Recall that coordinates X, Y on a pseudospheric surface are called *parabolic geodesic* if the first fundamental form can be written as

$$\mathrm{I} = \mathrm{d}X^2 + \mathrm{e}^{2X}\,\mathrm{d}Y^2.$$

To find the transformation from asymptotic to parabolic geodesic coordinates, observe that $\mathrm{d}\xi^2 + 2\cos\phi\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{d}\eta^2 = \mathrm{d}X^2 + \mathrm{e}^{2X}\,\mathrm{d}Y^2$ is equivalent to the system

$$X_\xi^2 + \mathrm{e}^{2X} Y_\xi = 1, \qquad X_\xi X_\eta + \mathrm{e}^{2X} Y_\xi Y_\eta = \cos\phi, \qquad X_\eta^2 + \mathrm{e}^{2X} Y_\eta = 1.$$

This system can be rewritten as

$$\begin{aligned} X_\xi &= \cos\alpha, \qquad & Y_\xi &= \mathrm{e}^{-X}\sin\alpha, \\ X_\eta &= \cos\beta, \qquad & Y_\eta &= \mathrm{e}^{-X}\sin\beta, \end{aligned} \tag{32}$$

and

$$\phi = \alpha - \beta. \tag{33}$$

In fact, (33) could also be $\phi = \beta - \alpha$, which can be reversed by changing the orientation of the surface. The new unknowns α and β can be identified with the angles between the geodesics and the two asymptotic coordinate lines.

The integrability conditions of system (32) are

$$\beta_\xi = -\sin\alpha, \qquad \alpha_\eta = -\sin\beta, \tag{34}$$

or, in view of relation (33),

$$\beta_\xi = -\sin(\phi + \beta), \qquad \beta_\eta = -\phi_\eta - \sin\beta. \tag{35}$$

These are already compatible by virtue of the sine-Gordon equation. From equations (32) we obtain

$$\mathbf{r}_X = -\frac{\sin\beta}{\sin\phi}\mathbf{r}_\xi + \frac{\sin\alpha}{\sin\phi}\mathbf{r}_\eta, \qquad \mathbf{r}_Y = \left(\frac{\cos\beta}{\sin\phi}\mathbf{r}_\xi + \frac{\cos\alpha}{\sin\phi}\mathbf{r}_\eta\right)\mathrm{e}^X.$$

With respect to a given geodesic net, the involute surface $\tilde{\mathbf{r}}$ is composed of individual involute curves to the geodesics, based on one and the same orthogonal line $Y = \text{const}$. Hence,

$$\tilde{\mathbf{r}} = \mathbf{r} + (a - X)\mathbf{r}_X,$$

where a is an arbitrary constant. With the help of equations (31), the fundamental forms $\tilde{\mathrm{I}}$, $\tilde{\mathrm{II}}$ of the involute surface $\tilde{\mathbf{r}}$ can be routinely computed in asymptotic coordinates. In particular, the unit normal is $\tilde{\mathbf{n}} = \mathbf{r}_X$ and

$$\begin{aligned} \tilde{\mathrm{I}} &= \left(X^2 - X + \tfrac{1}{2}\right)(1 - \cos 2\alpha)\,\mathrm{d}\xi^2 + (2X - 1)(\cos(\alpha+\beta) - \cos\phi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad + \left(X^2 - X + \tfrac{1}{2}\right)(1 - \cos 2\beta)\,\mathrm{d}\eta^2, \\ \tilde{\mathrm{II}} &= \left(X - \tfrac{1}{2}\right)(\cos 2\alpha - 1)\,\mathrm{d}\xi^2 + (\cos(\alpha+\beta) - \cos\phi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad + \left(X - \tfrac{1}{2}\right)(\cos 2\beta - 1)\,\mathrm{d}\eta^2. \end{aligned}$$

Hence, the principal radii of curvature are $X, X - 1$. The Gauss–Mainardi–Codazzi equations of the involute surface hold as a consequence of the sine-Gordon equation, the two equations (32) on X and the system (35) on β.

To obtain the corresponding solution of the constant astigmatism equation (21), we have to reparametrize the involute surfaces by curvature lines. Let x, y denote the new coordinates. We choose $x = Y$ and define y by the compatible system of equations

$$y_\xi = \mathrm{e}^X\sin\alpha, \qquad y_\eta = \mathrm{e}^X\sin\beta. \tag{36}$$

A routine calculation shows that $e^{-2X(x,y)}$ is a solution of the constant astigmatism equation (21). Summarizing, we have the following proposition.

Proposition 4. *Let $\phi(\xi, \eta)$ be a solution of the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$. Let α, β be solutions of the compatible equations*

$$\beta_\xi = -\sin\alpha, \qquad \alpha_\eta = -\sin\beta, \qquad \alpha - \beta = \phi.$$

Determine functions X, x, y by equations

$$\begin{aligned} dX &= \cos\alpha\, d\xi + \cos\beta\, d\eta, \\ dx &= e^{-X}(\sin\alpha\, d\xi + \sin\beta\, d\eta), \\ dy &= e^{X}(\sin\alpha\, d\xi + \sin\beta\, d\eta). \end{aligned}$$

Then the function $e^{-2X(x,y)}$ is a solution of the constant astigmatism equation (21).

Example 1. *Von Lilienthal's surfaces* (involutes of the pseudosphere). Published in 1887, these surfaces seem to have fallen into oblivion. Recall that the pseudosphere is a surface obtained by rotating the tractrix around its asymptote. The meridians are geodesics of the parabolic type and therefore von Lilienthal's surface is obtained by rotating the involute of the tractrix (which itself is the involute of the catenary).

In geodesic coordinates X, Y, the 'upper half' of the pseudosphere has a parametrization

$$\mathbf{r} = \begin{pmatrix} e^{-X}\cos Y \\ e^{-X}\sin Y \\ \operatorname{arcosh} e^{X} - \sqrt{1 - e^{-2X}} \end{pmatrix}, \qquad X > 0,$$

whose first fundamental form is $dX^2 + e^{-2X}\, dY^2$ (differs by the sign of X from the canonical form used in the preceding section). Then

$$\tilde{\mathbf{r}} = \mathbf{r} + (a - X)\,\mathbf{r}_X = \begin{pmatrix} (X - a + 1)\, e^{-X}\cos Y \\ (X - a + 1)\, e^{-X}\sin Y \\ \operatorname{arcosh} e^{X} - (X - a + 1)\sqrt{1 - e^{-2X}} \end{pmatrix}, \qquad X > 0,$$

parametrizes a rotational surface, for every real constant a. The surface is regular for all $a \leqslant 0$. Otherwise it has a cuspidal edge at $X = a$, which is a circle of radius e^{-a}. Another singularity that occurs for every $a > 1$ is the intersection with the rotation axis at $X = a - 1$. Choosing the orientation so that the normal vector is

$$\tilde{\mathbf{n}} = \begin{pmatrix} -e^{-X}\cos Y \\ -e^{-X}\sin Y \\ \sqrt{1 - e^{-2X}} \end{pmatrix}$$

(i.e., $\mathbf{n}$ swaps orientation when crossing either of the singularities), then

$$\tilde{\mathrm{I}} = \frac{(X - a)^2}{e^{2X} - 1}\, dX^2 + \frac{(X - a + 1)^2}{e^{2X}}\, dY^2, \qquad \tilde{\mathrm{II}} = \frac{X - a}{e^{2X} - 1}\, dX^2 + \frac{X - a + 1}{e^{2X}}\, dY^2.$$

and the principal radii of curvature are $X - a$ and $X - a + 1$. The corresponding solution of the constant astigmatism equation (21) is

$$z = \frac{1}{x^2 - e^{2(a-1)}}.$$

Plane sections of von Lilienthal surfaces for various values of the parameter a can be seen in figure 1. Besides the rotation axis, each picture shows the tractrix, which is the plane section

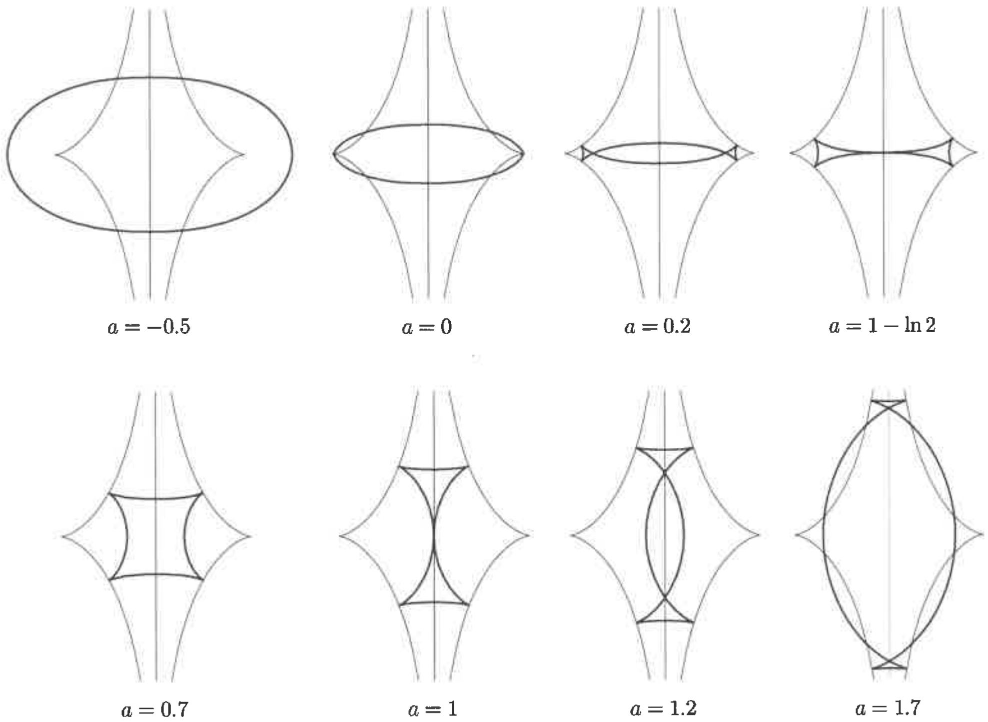

Figure 1. A gallery of von Lilienthal surfaces.

of the pseudosphere, and its involute curve, which is the plane section of the von Lilienthal surface.

We finish this example with a short exploration of the behaviour at the limits of the definition domain. For $X = \infty$ the surface closes up at a point on the rotation axis at the height $a - 1 + \ln 2$, where both principal radii of curvature are infinite (the zero height is that of the cusp of the tractrix). For $X \to 0$ the surface vertically approaches a horizontal circle of diameter $|1 - a|$. Two surfaces $\tilde{\mathbf{r}}(X, Y)$ and $-\tilde{\mathbf{r}}(X, Y)$ can be glued along this circle to form a single surface of constant astigmatism 1. For $a = 1$ both glued surfaces have a cusp here.

8. Conclusions and discussion

Among the still incomplete results of classification of integrable Weingarten surfaces, we have identified a class originally introduced and investigated by 19th-century geometers. The class, which we propose to call surfaces of constant astigmatism, is governed by the equation

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0.$$

For this equation we found an $\mathfrak{sl}(2)$-valued zero-curvature representation depending on a parameter, a third-order symmetry and a nonlocal transformation to the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$. We had to leave aside the problem of finding a Bäcklund transformation as well as a recursion operator producing a hierarchy of local symmetries.

It should be stressed that the classification problem of integrable surfaces is far from being easy. An obvious reason lies in the abundance of integrability-preserving ways to derive one surface from another. Clearly, parallel surfaces, evolutes and involutes of integrable surfaces are integrable. On the differential equation level, the corresponding notion is that of the covering [22]. The integrable classes of surfaces must be either closed with respect to taking derived surfaces or the derivation must map one integrable class into another.

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