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On Li-Yorke sensitivity and other types of chaos in dynamical systems

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I would like to express my sincere gratitude
to management of Mathematical Institute
and my colleagues for support
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Contents

1. Introduction

A dynamical system describes a dependence of the position of a point in some space on the time that is continuous or discrete. The founder of general theory of one and low–dimensional discrete dynamical systems is Alexander Sharkovsky. In 1964 he specified a new total ordering of natural numbers, known today as Sharkovsky's ordering, which describes the co–existence of periodic orbit for continuous interval maps. In the '70s it has turned out that this ordering gives direction from more complex to simpler behavior of systems. One of the most significant traits of dynamical systems is just the existence of chaotic behavior. In 1972 during the 139th meeting of the American Association for the Advancement of Science, Lorenz described the Butterfly Effect in his talk entitled "*Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?*" The Butterfly Effect shows that very small change in initial conditions can create a significant difference in the results. It is also known as the sensitive dependence on initial conditions [22]. For the first time the notion of chaos was introduced by Li and Yorke in 1975 [20]. They showed that not only the existence of periodic point of period 3 implies the existence of periodic points of all periods (which is a special case of Sharkovsky's Theorem), but also in addition it implies the existence of uncountable set whose points never map to any cycle. There was suggested a defining criterion on the existence of chaos for maps on the interval. This notion can be extended to more general metric spaces and now this kind of chaos is known as Li–Yorke chaos. Various alternative definitions of chaos was introduced later. However, it should be noted that unique and universally accepted definition of chaos does not exists currently and it will probably never exist. A summary of chaos theory is given by Li and Ye in [21]. In recent years, the chaotic behavior of dynamical systems is a common concern not only among various branches of mathematics. The chaos is also intensively studied by various branches of science and engineering.

Combining Li-Yorke version of chaos with the notion of sensitivity to initial conditions leads to a definition of Li–Yorke sensitivity that is the main area of interest of this thesis. The thesis is structured as follows. After defining some elementary notions and introducing notations in the next section, we explain some kinds of chaos and provide relationship among them. In the last two sections of the first part we recapitulate the main results contained in the papers concerning the thesis. These three papers form the second part of the thesis. We focus on minimal Li-Yorke sensitive systems. In particular, on relations between Li–Yorke sensitivity and spatio-temporal chaos, and extensions and factors of Li–Yorke sensitive systems.

2. Basic notions

A *topological dynamical system* is a pair (X,T), where X is a non-empty compact metric space with metric ρ and T is a surjective, continuous map from X to itself. If the space X contains only one point, then the unique map on X is the identity map and we say that the dynamical system on X is *trivial*. For any nonnegative integer n, denote by T^n the nth iterate of T. One of the main topics in dynamical system is the behavior of T^n when n goes to infinity. For a point $x \in X$, its *trajectory* under the map T is the sequence $(T^n(x))_{n=0}^{\infty}$.

If Y is a non-empty closed invariant (i.e., $T(Y) \subset Y$) subset of X, we call $(Y, T|_Y)$ a subsystem of (X,T) and we denote it briefly by (Y,T). A system (X,T) is *minimal* if it contains no proper subsystem. Equivalently, (X,T) is minimal if any point $x \in X$ has a dense trajectory in X. A subset $A \subset X$ is *minimal* if (A,T) is a minimal subsystem of (X,T). By the Axiom of choice, any dynamical system contains some minimal set. A point $x \in X$ is *minimal* if it belongs to a minimal set. A dynamical system (X,T) is called *transitive* if for every pair of non-empty open subsets U and V of X, there exists a positive integer n such that $U \cap T^n(V) \neq \emptyset$. A point with dense orbit is called *transitive point*. It is well known that the set of all transitive points in a transitive system (X,T) is a dense G_δ subset of X. Obviously, if X has no isolated point and has a transitive point, then (X,T) is transitive. A system (X,T) is *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive. In [13], it was proved that for weakly mixing system (X,T), the system $(X^n, T \times \ldots \times T)$ is transitive for every positive integer n.

A system (Y,S) is said to be a *factor* of (X,T) if there is a continuous onto map $\pi : X \to Y$ such that

$$S \circ \pi(x) = \pi \circ T(x), \quad \text{for all } x \in X.$$

In this case (X,T) is an *extension* of (Y,S) and π is a *factor map*.

Let X, Y be compact metric spaces. A continuous map $F : X \times Y \to X \times Y$ is called a *skew-product map* if it has the form $F(x,y) = (T(x), G(x,y))$. We say that the map $T : X \to X$ is the *basis map* of F and the maps $G_x : Y \to Y$, $G_x(y) = G(x,y)$, are *fiber maps*. It is obvious that the skew-product system $(X \times Y, F)$ is an extension of (X,T).

3. Chaos in discrete dynamical systems

There are many papers on chaotic behavior. The following list contains only some of the most famous kinds of chaos and the relationships among them.

3.1. **Sensitive dependence on initial conditions and equicontinuity.** For continuous dynamical systems the idea and the significance of sensitive dependence on initial conditions has first been introduced in [22] and [24]. However, firstly the phrase "sensitive dependence on initial conditions" was used in [23] and [15]. For topological dynamical systems the notion of sensitivity in the below form was defined by Auslander and Yorke in [5].

We say that a dynamical system (X,T) has *sensitive dependence on initial conditions* (or more briefly, is *sensitive*) if there exists $\delta > 0$ such that any neighborhood of any point $x \in X$ contains a point $y \in X$ such that $\rho(T^n(x), T^n(y)) > \delta$ for some $n \in \mathbb{N}$.

The opposite of sensitivity is equicontinuity. A dynamical system (X,T) is called *equicontinuous* if the sequence of maps $\{T^n : n \geq 0\}$ is uniformly equicontinuous, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(T^n(x), T^n(y)) < \varepsilon$ for all $n \geq 0$ and all $x, y \in X$ satisfying $\rho(x,y) < \delta$. In [5] (respectively [2]) it is shown the following dichotomy result for minimal (respectively transitive) systems. If (X,T) is minimal, then (X,T) is either sensitive or equicontinuous; and if (X,T) is transitive, then (X,T) is either sensitive or *almost equicontinuous*, i.e., there exists some point $x \in X$ such that for every $\varepsilon > 0$ there is some $\delta > 0$ satisfying for any $y \in X$, $\rho(x,y) < \delta$ implies $\rho(T^n(x), T^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.

It is known that transitive almost equicontinuous (thus also equicontinuous) systems have very simple dynamical behaviors. In [14], it is shown that every such system is uniformly rigid (i.e., there exists a subsequence of $(T_n)_{n=0}^{\infty}$ that converges uniformly to the identity map) and consequently it has zero topological entropy, see below.

3.2. **Li–Yorke chaos.** The definition of Li–Yorke chaos is based on ideas in [20] where pairs of points which are proximal (i.e., the trajectories of x and y are at some times arbitrarily close), but not asymptotic are considered. Accordingly, a pair of points $(x,y) \in X \times X$ is called a *Li–Yorke pair* whenever

$$\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) = 0 \tag{1}$$

and

$$\limsup_{n\to\infty} \rho(T^n(x), T^n(y)) > 0. \tag{2}$$

In the case that a pair $(x,y) \in X^2$ besides the condition (1) satisfy also the following condition

$$\limsup_{n\to\infty} \rho(T^n(x), T^n(y)) > \delta, \tag{3}$$

the pair (x,y) is *δ-Li–Yorke pair*. That is, (x,y) is proximal, but their orbits are frequently (i.e., infinitely many times) at least δ apart.

A dynamical system (X,T) is *Li–Yorke chaotic* if there is an uncountable subset $S \subset X$ such that any two distinct points $x, y \in S$ form a Li–Yorke pair. Moreover, if there exists a number $\delta > 0$ such that any two distinct points $x, y \in S$ form a δ-Li–Yorke pair, we say that the system (X,T) is *δ-Li–Yorke chaotic*. It is known a system, which is Li–Yorke chaotic, but is not δ-Li–Yorke chaotic for any $\delta > 0$ (see [4]). Note that in the case of interval maps, it is not possible.

3.3. **Li–Yorke sensitivity.** A concept of Li–Yorke sensitivity is a combination of sensitivity and δ-Li–Yorke chaos and it links these ideas together. It was introduced by Akin and Kolyada in 2003 [3]. A dynamical system (X, T) is *Li–Yorke sensitive* if there is a number $\delta > 0$ such that any neighborhood of any $x \in X$ contains a point $y \in X$ such that the pair (x, y) is δ-Li–Yorke. In this case, the number δ is called a *constant of sensitivity.*

3.4. **Spatio-temporal chaos.** A weaker notion than Li–Yorke sensitivity is *spatio-temporal chaos*, which means that any neighborhood of any point $x \in X$ contains a point $y \in X$ such that the pair $(x, y) \in X^2$ is Li–Yorke. The concept of spatio-temporal chaos was introduced in [7].

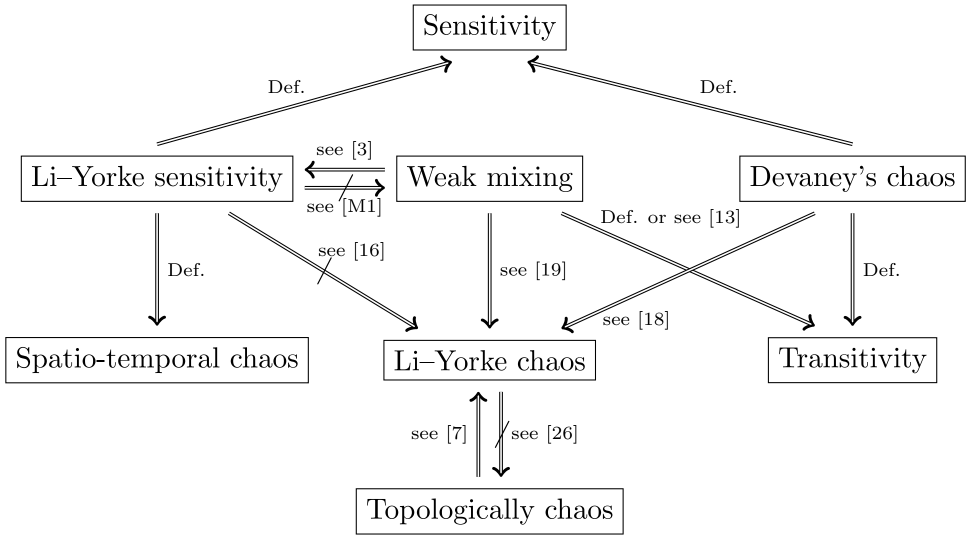
In [3], it was proved that the nonminimal, transitive, almost equicontinuous system such that every minimal point is periodic, is spatio-tempotally chaotic but is not Li–Yorke sensitive.

3.5. **Other forms of chaos.** Other popular forms of chaos are Devaney chaos and positive entropy.

Devaney's definition of chaos emphasized the significance of sensitive dependence on initial condition (see [11]). A dynamical system (X, T) is said to be *chaotic in the sense of Devaney*, if it is transitive, sensitive and the set of periodic points is dense in X. In [6], it is proved that the sensitive dependence is implied by transitivity and density of periodic points.

A quantitative measure of chaos of a dynamical system (X, T) is given by the *topological entropy* $h(T)$. It is a nonnegative extended real number that can be defined in various ways in [1, 9, 12]. Equivalence between the above definitions was proved by Bowen in [10]. A dynamical system (X, T) is called *topologically chaotic* if its topological entropy $h(T)$ is positive.

3.6. **Relationships.** In [19], it was proved that weakly mixing implies Li–Yorke chaos. Akin and Kolyada [3] shown that any nontrivial weakly mixing system is Li–Yorke sensitive and in [M1], there is a minimal Li–Yorke sensitive system which is not weakly mixing. By using an ergodic method, Blanchard et al. proved that positive topological entropy implies Li–Yorke chaos in [7]. On the other hand there are Li–Yorke chaotic systems with zero topological entropy (see e.g. [26]). In [18], it is shown that any Devaney's chaotic system is Li-Yorke chaotic. More recently in [16], it was shown that Li–Yorke sensitivity does not imply Li–Yorke chaos. It is still an open question whether every minimal (or transitive) Li–Yorke sensitive system is Li–Yorke chaotic. So, the interrelations among above mentioned notions are summarized in the following diagram. In this scheme the arrows mean implications. In the case that a relationship is not shown in the diagram, then either it is a corollary that follows by the transitivity of implications or the implication does not hold.



4. Minimal Li–Yorke sensitive systems without weakly mixing factor

For minimal system, in [3] it was introduced five conjectures. Here are three of them.

(C1) In a minimal system, spatio-temporal chaos is equivalent to Li–Yorke sensitivity.

(C2) Every minimal Li–Yorke sensitive system has a nontrivial, weakly mixing factor.

(C3) If $\pi : (X,T) \to (Y,S)$ is a factor map between minimal systems with (Y,S) Li–Yorke sensitive, then (X,T) is Li–Yorke sensitive.

Exactly by the definitions it follows a Li–Yorke sensitive system is spatio-temporally chaotic. In [17] was shown that every minimal spatio-temporally chaotic system (X,T) is *thickly sensitive*, i.e., there exists $\varepsilon > 0$ such that for any non-empty open set $U \subset X$, a set $N_T(U,\varepsilon) = \{n \in \mathbb{N} : \mathrm{diam}(T^n(U)) > \varepsilon\}$ contains arbitrarily long intervals of positive integers. Consequently, such systems are sensitive. Indeed, a system (X,T) is sensitive if and only if there is an $\varepsilon > 0$ with the property that for every open non-empty subset U of X, the set $N_T(U,\varepsilon)$ is non-empty.

The following theorem disproves conjecture (C1). We denote by Q a Cantor ternary set and by $\mathbb{S}$ the unit circle.

Theorem 1. ([M1]) *There is a parametric family $\mathcal{F}_1$ of skew-product homeomorphisms $F : Q \times \mathbb{S} \to Q \times \mathbb{S}$ such that for any $F \in \mathcal{F}_1$, $(Q \times \mathbb{S}, F)$ is*

(i) minimal system,

(ii) spatio-temporally chaotic,

(iii) not Li–Yorke sensitive.

Moreover, $(Q \times \mathbb{S}, F)$ possesses no nontrivial weakly mixing factor for any $F \in \mathcal{F}_1$.

Let $\pi : (X,T) \to (Y,S)$ be a factor map. It is known (e.g. [28]) that if a trajectory of $x \in X$ is dense in X then a trajectory of $\pi(x)$ is dense in Y. In particular, a factor of a transitive system is transitive and a factor of a minimal system is minimal. Obviously, an extension of a transitive or minimal system may not be transitive or minimal.

Every dynamical system (X,T) possesses a *maximal equicontinuous factor* (Y,S), i.e., an equicontinuous factor (Y,S) with a corresponding factor map $\pi : X \to Y$ such that for any equicontinuous (Z,R) and factor map $\phi : (X,T) \to (Z,R)$ there is a factor $\psi : (Y,S) \to (Z,R)$ such that $\phi = \psi \circ \pi$. The maximal equicontinuous factor is unique up to conjugacy.

A weakly mixing system has no non-trivial equicontinuous factor (see e.g. [27]). In [4] is shown that a minimal system (X,T) is weakly mixing if and only if (X,T) has no non-trivial equicontinuous factor. Another situation is in the case of Li–Yorke sensitive systems. In [3] it was proved that the product of Li–Yorke sensitive system with any dynamical system (and thus also with an equicontinuous system) is Li–Yorke sensitive. Consequently, a Li-Yorke sensitive system can have an equicontinuous factor.

The following theorem disproves conjecture (C2). Moreover, the system in theorem 2 is Li–Yorke sensitive, but not weakly mixing.

Theorem 2. ([M1]) *There is a parametric family $\mathcal{F}_2$ of skew-product homeomorphisms $F : Q \times \mathbb{S} \to Q \times \mathbb{S}$ such that for any $F \in \mathcal{F}_2$, $(Q \times \mathbb{S}, F)$ is*

(i) minimal Li–Yorke sensitive system,

(ii) without weakly mixing factor.

Glasner and Weiss in [14] proved that any transitive uniformly rigid system has an extension with no isolated points that is transitive, uniformly rigid and is not sensitive. There exist uniformly rigid systems which are sensitive, e.g. every minimal uniformly rigid weakly mixing system. Hence any system of this type has an almost equicontinuous transitive extension. In the case a feebly open factor map $\pi : (X,T) \to (Y,S)$, i.e., $\pi(U)$ has non-empty interior for any non-empty open set U in X, Akin and Kolyada in [3] proved that an extension of sensitive system is sensitive.

Auslander in [4] showed that any factor map between minimal systems is always feebly open. A system (X,T) is said to be an *almost one-to-one extension* of (Y,S) if the corresponding factor map π is almost one-to-one, i.e., the set of the points $x \in X$ such that $\pi^{-1}(\pi(x)) = \{x\}$ is dense in X. The following theorem disproves conjecture (C3).

Theorem 3. ([M2]) *There is a minimal system which is not Li–Yorke sensitive, but has Li–Yorke sensitive factor. In fact, any system $(Q \times \mathbb{S}, F)$, with $F \in \mathcal{F}_2$, where $\mathcal{F}_2$ is as in Theorem 2, has almost one-to-one minimal extension (X,T) with T being a homeomorphism but not Li–Yorke sensitive.*

In the case when (X,T) is an extension of (Y,S), it is known [8] that a Li–Yorke pair in (Y,S) need not be a projection of a Li–Yorke pair in (X,T). But this in not the case of Theorem 3. In [M2] we showed the following assertion.

Proposition 4. ([M2]) *Let (X,T) be an almost one-to-one extension of a Li–Yorke sensitive minimal system (Y,S) with a factor map π. Then there is an $\delta > 0$ such that, for any*

$y \in Y$, there is a $u \in \pi^{-1}(y)$ such that any neighborhood U of u contains a v satisfying (1) and (3) from the definition of δ-Li–Yorke pair (on page 3) such that $(y, \pi(v))$ is a Li–Yorke pair, too.

5. Minimal extension of weakly mixing systems

Besides hypotheses already mentioned above, Akin and Kolyada in [3] introduced other conjecture.

(C4) Every minimal system with a weakly mixing factor is Li-Yorke sensitive.

Furstenberg in [13] proved that for a minimal distal system (X, T) and a minimal weakly mixing system (Y, S), the product system $(X \times Y, T \times S)$ is minimal. As was mentioned above, this system $(X \times Y, T \times S)$ has to be also Li–Yorke sensitive.

In [M3] it is given a partial solution of this problem. Later Shao and Ye in [25] proved that the conjecture (C4) is true. The following theorem shows that in certain cases, a skew-product map on product of minimal weakly mixing system and finite space is Li–Yorke sensitive.

Theorem 5. ([M3]) *Let (X, T) be a minimal weakly mixing system. Let A be a finite space with discrete topology, $Y = X \times A$ with the maximum metric, and (Y, S) a skew-product extension of (X, T) such that $S(t, a) = (T(t), G_t(a))$, where every fibre map G_t is a bijection of A. Then (Y, S) is Li–Yorke sensitive with a constant of sensitivity ε for any $0 < \varepsilon < \operatorname{diam}(X)$.*

Moreover, the system from the previous theorem is also Li-Yorke chaotic.

Theorem 6. ([M3]) *Let (X, T) be a minimal weakly mixing system and A be a nonempty finite space with discrete topology. Let S be a skew-product map of $X \times A$. Then, for any $0 < \varepsilon < \operatorname{diam}(X)$, S is Li–Yorke chaotic with corresponding constant ε .*

Let $\mathcal{A}$ be a collections of generalized odometers on Cantor-type set (i.e., nowhere dense non-empty compact set without isolated points) Y. The elements of $\mathcal{A}$ are *synchronous* if there is an increasing sequence $m_1, m_2, \dots$ of natural numbers such that for any odometer $(Y, \tau) \in \mathcal{A}$ related to a sequence $p_1, p_2, \dots$ of prime numbers and for any $j \in \mathbb{N}$ there is $l_j \in \mathbb{N}$ with $m_j = p_1 \cdot p_2 \cdot \dots \cdot p_{l_j}$.

Here are generalizations of Theorem 5 for certain types of skew-product maps of $X \times Y$, where a set Y is infinite compact.

Theorem 7. ([M3]) *Let (X, T) be a minimal weakly mixing system, Y a Cantor-type set, $\mathcal{A}$ be a collection of synchronous odometers on Y and $S : X \times Y \to X \times Y$ a skew-product map, $S(x, y) = (T(x), R_x(y))$ such that, for every $x \in X$, R_x is an odometer in the class $\mathcal{A}$, or the identity. Then $(X \times Y)$ is Li–Yorke sensitive.*

A continuum X (i.e., non-empty connected compact metric space) is *unicoherent* if for any two continua A, B with $A \cup B = X$, the set $A \cap B$ is connected.

Theorem 8. ([M3]) *Let (X, T) be a minimal, weakly mixing, not connected, and such that every subcontinuum of X is unicoherent. Let $n > 0$ be an integer and let (Y, S) be an extension of (X, T) such that the set $\{y \in Y : \pi(y) = x\}$ contains n points for every $x \in X$. Then $(X \times Y)$ is Li–Yorke sensitive.*

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Li–Yorke sensitive minimal maps

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Abstract
Let Q be the Cantor middle third set and S the circle, and let $\tau : Q \to Q$ be an adding machine (i.e. odometer). Let $X = Q \times S$ be equipped with (a metric equivalent to) the Euclidean metric. We show that there are continuous triangular maps $F_i : X \to X$, $F_i : (x, y) \mapsto (\tau(x), g_i(x, y))$, $i = 1, 2$, with the following properties.

(i) Both (X, F_1) and (X, F_2) are minimal systems, without weak mixing factors (i.e. neither of them is semiconjugate to a weak mixing system).
(ii) (X, F_1) is spatio-temporally chaotic but not Li–Yorke sensitive.
(iii) (X, F_2) is Li–Yorke sensitive.

This disproves conjectures of Akin and Kolyada (2003 *Nonlinearity* **16** 1421–33.)

Mathematics Subject Classification: 37B05, 37D45, 54H20, 37E30

1. Introduction

In [2] Akin and Kolyada considered, for a *surjective continuous map* $T : X \to X$ of a compact metric space (X, ρ), the notions of chaos which are related to proximality. In particular, they introduced and studied the concept of *Li–Yorke sensitivity*, in brief *LYS*. A map T is *LYS* if there is an $\varepsilon > 0$ with the property that any neighbourhood of any $x \in X$ contains a point y *proximal* to x (i.e. the trajectories of x and y are at some times arbitrarily close), such that the trajectories of x and y are *frequently* (i.e. infinitely many times) at least ε apart. Thus,

$$\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) = 0, \qquad \text{and} \qquad \limsup_{n\to\infty} \rho(T^n(x), T^n(y)) > \varepsilon, \tag{1.1}$$

where T^n denotes the nth iterate of T, for $n = 0, 1, 2, \ldots$. This notion is stronger than sensitivity, since any minimal system which is distal but not equicontinuous is sensitive but not *LYS*. Akin and Kolyada proved, among others, that the weak mixing systems are *LYS*. Recall that T is *sensitive* if, for some $\varepsilon > 0$, the set of pairs x, y whose trajectories are frequently at least ε apart is dense in $X \times X$.

Since only the construction in [2] uses weak mixing to obtain *LYS*, there is a natural question posed by Akin and Kolyada whether every *LYS* minimal system has a nontrivial weak mixing factor. In this paper we use a new method of constructing *LYS* maps, as skew-product (or, triangular) maps $T : (x, y) \mapsto (f(x), g_x(y))$ without nontrivial weak mixing factors. This shows that the answer to the above question is negative. Our proof is based on ideas from [3] (cf also [5]). We construct a parametric family of skew-product minimal maps on the product $X = Q \times S$ of the Cantor set Q and the circle S and then properly choose the parameters.

A weaker notion than *LYS* is that of *spatio-temporal chaos*, or *STC*, which means that any neighbourhood of any point $x \in X$ contains a point y proximal to x but not asymptotic to x, i.e. (1.1) is satisfied with $\varepsilon = 0$. *STC* was introduced in [4]. However, all known *STC* systems which are not *LYS* are nonminimal. Therefore, in [2] there is a question whether, for minimal systems, *LYS* is equivalent to *STC*. We show that again the answer is negative, even for the class of maps without nontrivial weak mixing factors. Our construction uses the same parametric family of skew-product maps as before but with different choice of parameters.

For convenience we recall here some notions used throughout the paper. Points x, y in X are *asymptotic* if, for any $\varepsilon > 0$, their trajectories are at a distance less than ε, except for a finite number of times. A point $x \in X$ is *distal* if the only point in X asymptotic to x is x itself. A system (X, T) is *distal* if any point of X is distal; it is *proximal* if any two points of X are proximal. A set $A \subset X$ is *minimal* if it is closed, nonempty, invariant (i.e. $T(A) = A$) such that no proper subset of A has the same properties. A *system* (X, T) *is minimal* if X is a minimal set or, equivalently, if every point $x \in X$ has a dense orbit $\{T^n(x)\}_{n=0}^{\infty}$.

A system (X, T) is *topologically transitive* if for every pair of open, nonempty subsets $U, V \subset X$ there is a positive integer, n, such that $U \cap T^{-n} \neq \emptyset$. A system (X, T) is *weakly mixing* when the product system $(X \times X, T \times T)$ is transitive. Finally, a system (Y, S) is a *factor* of (X, T) if there is a surjective continuous map $g : X \to Y$ such that

$$T \circ g(x) = g \circ S(x), \qquad \text{for all } x \in X. \tag{1.2}$$

In this case we also say that (X, T) is *semiconjugate* to (Y, S) via homomorphism g.

Our main results, theorems 5.1 and 5.2 given below, can be summarized as follows.

Theorem 1.1. *Neither of the following conjectures stated in* [2] *is true:*

(i) In a minimal system, STC implies LYS.
(ii) If a minimal system is LYS then it has a nontrivial weak mixing factor.

The paper is organized as follows. In the next section we describe a construction of a minimal homeomorphism on the product of the odometer (Q, τ) with the circle S. This map is of triangular type and depends on a sequence $\{\varphi_k^j\}_{k=1}^{\infty}$ of continuous homeomorphisms of S. The maps φ_k^j are just the maps $g_x(\cdot)$ in $F : (x, \cdot) \mapsto (f(x), g_x(\cdot))$. In section 3 we design the system such that the maps φ_k^j, with odd k, are rotations of S. It follows that the set of $x \in Q$ for which $g_x(\cdot)$ is rotation is dense in Q. This causes a rigidity of the system and consequently the system has no nontrivial weak mixing factor (cf lemma 3.1). The family of homeomorphisms φ_k^j with even k are 'free' parameters; in section 4 we prove some general properties of these systems. Finally, in section 5 we specify the parameters so that the resulting system is *STC* but not *LYS* (cf theorem 5.1) and we show that for another specification of the free parameters the system is *LYS*; see lemma 5.3 and theorem 5.2.

Further terminology and notation, if not standard, is explained below in the proper places. For basic facts concerning systems considered in this paper, the reader is referred to standard books such as [1] or to papers dealing with the subject such as [2–5].

2. A class of minimal skew-product maps

Let (X, ρ_X) and (Y, ρ_Y) be compact metric spaces and let ρ denote the max-metric on the product $X \times Y$. Let $\mathcal{T}(X, Y)$ denote the class of skew-product (or triangular) maps of $X \times Y$. Thus, $F \in \mathcal{T}(X, Y)$ if $F : X \times Y \to X \times Y$ is continuous such that $F(x, y) = (f(x), g_x(y))$. The map f is called the *base* for F and g_x is a map from $Y_x = \{x\} \times Y$ to Y so we consider g_x as a map from Y to Y. The set Y_x is the *layer* or *fibre* of $X \times Y$ over x; similarly a layer M_x of arbitrary $M \subset X \times Y$ is defined.

In this paper we consider the class $\mathcal{T}(Q, S)$, where Q is the Cantor set in [0, 1] and S is the circle with radius $R > 0$ (i.e. the set $|z| = R$ in the complex plane). For any family

$$\Phi = \{\varphi_k^j; 0 \leqslant j \leqslant 2^{n_k} - 2\}_{k=1}^{\infty} \tag{2.1}$$

of homeomorphisms of S where $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of positive integers, we define a special map $F_\Phi \in \mathcal{T}(Q, S)$. In the next sections we show that a specification of the sequence Φ will provide counterexamples to some conjectures in [2]. Our construction is inspired by [5] where a special class of triangular maps of $[0, 1] \times [0, 1]$ monotone on the fibres was used to prove some conjectures on minimal sets. However, in this class of triangular maps any minimal set M has empty interior in $[0, 1] \times [0, 1]$ and hence most of the fibres of such M are singletons. Consequently, no such system M can be spatio-temporally chaotic since the odometer is distal. Therefore this family is modified so that the maps are defined on the product $[0, 1] \times S$. Since we are interested in minimal systems, we consider the subspace $X = Q \times S$ (since Q is a minimal set), but all maps considered below on $Q \times S$ can be continuously extended onto $[0, 1] \times S$. We also modify the coding system in order to simplify the calculation. A similar approach was used in [3].

To define F_Φ we recall that any point α of the Cantor set $Q \subset [0, 1]$ can be uniquely represented as

$$\alpha = \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i}, \qquad \text{where } \alpha_i \in \{0, 1\}, \tag{2.2}$$

so the set Q is homeomorphic to the set $\{0, 1\}^N$ of sequences of two symbols equipped with a metric $\rho_Q(\alpha, \beta) = \max\{1/i, \alpha_i \neq \beta_i\}$, for any distinct $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$. Let $f : [0, 1] \to [0, 1]$ be a continuous map defined by

$$f(0) = 2/3, \qquad f(1) = 0, \tag{2.3}$$

$$f(1 - 2/3^k) = 1/3^{k-1}, \quad f(1 - 1/3^k) = 2/3^{k+1} \qquad \text{for } k \leqslant 1 \tag{2.4}$$

and by linearity at intermediate points. The graph of the map f is as follows (figure 1).

The *Adding Machine* (or *odometer*) $\tau : Q \to Q$ is defined as follows (cf, e.g., [5]): for every $\alpha \in \{0, 1\}^N$, $\tau(\alpha) = \alpha + 10\,000\ldots$ where the addition is modulo 2 from left to right. Obviously, τ is continuous on Q and it is well known that (Q, τ) is a minimal system. It is easily seen that $f|_Q(\alpha) = \tau(\alpha)$ for any $\alpha \in \{0, 1\}^N$, where the map f is defined by (2.3) and (2.4).

Any $\alpha \in Q$, $\alpha = \{\alpha_i\}_{i=1}^{\infty}$, can be written as

$$\alpha = \alpha^1\alpha^2\alpha^3, \ldots, \qquad \alpha^j \text{ is the block of } n_j \text{ digits of } \alpha. \tag{2.5}$$

For any finite or infinite sequence $x = x_1x_2\ldots x_k\ldots$ in $\{0, 1\}$ with only finitely many nonzero digits let $e(x) = x_1 + 2x_2 + 2^2x_3 + \cdots + 2^{k-1}x_k + \cdots \in \mathbb{N}$ be the *evaluation* of x. Conversely, for any integer n denote by $\underline{n}$ the infinite sequence x in $\{0, 1\}$ such that $e(x) = n$. Now we let

$$F_\Phi(\alpha, y) = (\tau(\alpha), y) \qquad \text{if } \alpha = 1^\infty. \tag{2.6}$$

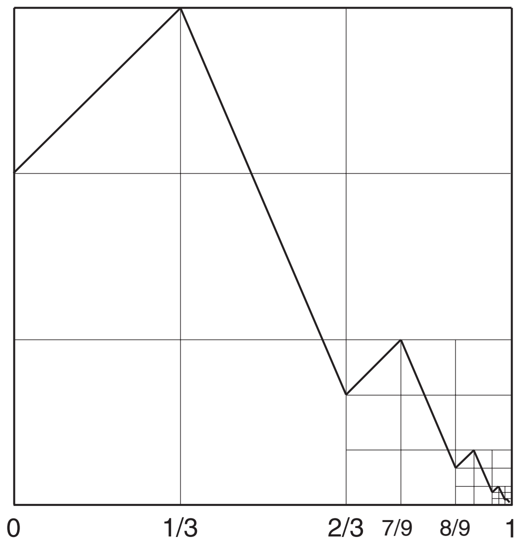

Figure 1. The graph of the map f.

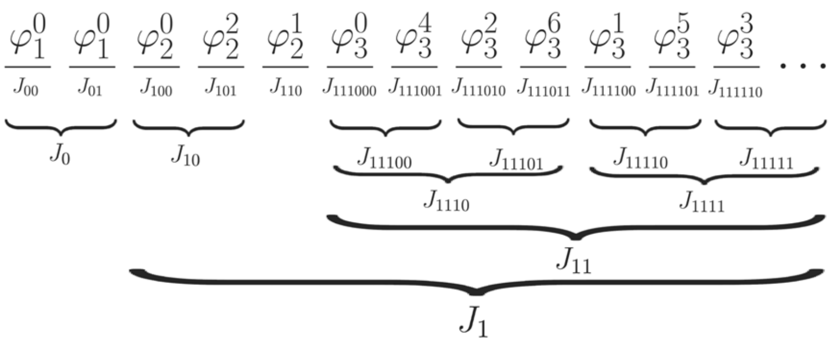

Figure 2. Application of φ.

Otherwise, let α^k be the first block in (2.5) containing at least one zero digit. Then

$$F_\Phi(\alpha, y) = (\tau(\alpha), \varphi_k^{e(\alpha^k)}(y)). \tag{2.7}$$

For example, if we put $n_k = k$ for every $k \in \mathbb{N}$, figure 2 shows which of the maps, φ_k^j, are used as fibre maps in respective portions $J_{\alpha_1\alpha_2\ldots\alpha_n} = [\alpha_1\alpha_2\ldots\alpha_n 0^\infty, \alpha_1\alpha_2\ldots\alpha_n 1^\infty]$ of the Cantor set.

Note that respective portions, $J_{\alpha_1\alpha_2\ldots\alpha_n}$ and $J_{\beta_1\beta_2\ldots\beta_n}$, have the same length $1/3^n$, for all $n \in \mathbb{N}$. So $|J_0| = |J_1| = 1/3$, $|J_{00}| = |J_{01}| = |J_{10}| = |J_{11}| = 1/9$, and so on. For typographic reasons, in the figure the lengths of the portions are distorted.

Finally, let ρ be a metric on $Q \times S$ given by $\rho((\alpha, u), (\beta, v)) = \max\{\rho_Q(\alpha, \beta), \rho_S(u, v)\}$ where $\rho_S(u, v)$ is the Euclidean length of the arc between the points u and v; if possible we always take the shorter arc. It is obvious (cf also [3]) that any F_Φ is continuous if

$$\lim_{k\to\infty} \max_j \|\varphi_k^j - Id\| = 0, \tag{2.8}$$

where $\|\cdot\|$ is the uniform norm.

Now we can prove some general formulae for iterates of F_Φ. Let $\underline{0} \in Q$ be the sequence of zeros and let $y_0 \in S$. Then y_n is (uniquely) defined by

$$(\underline{n}, y_n) = (\tau^n(\underline{0}), y_n) = F_\Phi^n(\underline{0}, y_0). \tag{2.9}$$

In the following we make use of some combinatorial results. To simplify the calculation we assume throughout that, in (2.1),

$$\varphi_k^0 = Id, \qquad k \in \mathbb{N}, \tag{2.10}$$

and

$$\varphi_k^{2^{n_k}-2} \circ \varphi_k^{2^{n_k}-3} \circ \cdots \circ \varphi_k^1 \circ \varphi_k^0 = Id, \qquad k \in \mathbb{N}. \tag{2.11}$$

Then we have the following lemma.

Lemma 2.1. *Assume (2.10) and (2.11). For $s, k \in \mathbb{N}$, $k > 0$, denote $m_k := 2^{n_1+n_2+\cdots+n_k}$ and let $1 \leqslant s < 2^{n_{k+1}}$. Then*

$$y_{m_k} = y_0 \tag{2.12}$$

and

$$y_{s\cdot m_k} = \varphi_{k+1}^{s-1} \circ \varphi_{k+1}^{s-2} \circ \cdots \circ \varphi_{k+1}^1 \circ \varphi_{k+1}^0(y_0). \tag{2.13}$$

Proof. By (2.7),

$$y_{2^{n_1}} = \varphi_2^0 \circ \varphi_1^{2^{n_1}-2} \circ \varphi_1^{2^{n_1}-3} \circ \cdots \circ \varphi_1^1 \circ \varphi_1^0(y_0),$$

hence, by (2.10) and (2.11), $y_{2^{n_1}} = y_0$. Now assume by induction that

$$y_{2^{n_1+n_2+\cdots+n_k}} = y_0.$$

We show that

$$y_{m_{k+1}} = y_{2^{n_1+n_2+\cdots+n_k+n_{k+1}}} = y_0.$$

Denote by β_k the composition of m_k maps, φ_i^t, such that $\beta_k(y_0) = y_{m_j}$. Since the block β_k begins with the map, $\varphi_{n_{k+1}}^0$, which is by (2.10) the identity map, formally we have $\beta_k = \varphi_{n_{k+1}}^0 \circ \beta_k'$, where β_k' is a composition of $m_k - 1$ maps in $\{\varphi_i^t\}$, but, for any $y \in S$, $\beta_k(y) = \beta_k'(y) = y$. Consequently,

$$y_{m_{k+1}} = \varphi_{n_{k+2}}^0 \circ \varphi_{n_{k+1}}^{2^{n_{k+1}}-2} \circ \beta_k' \circ \varphi_{n_{k+1}}^{2^{n_{k+1}}-3} \circ \beta_k' \circ \cdots \circ \beta_k' \circ \varphi_{n_{k+1}}^1 \circ \beta_k' \circ \varphi_{n_{k+1}}^0 \circ \beta_k'(y_0). \tag{2.14}$$

Since β_k' is the identity, (2.14) reduces to

$$y_{2^{n_1+\cdots+n_{k+1}}} = \varphi_{n_{k+2}}^0 \circ \varphi_{n_{k+1}}^{2^{n_{k+1}}-2} \circ \varphi_{n_{k+1}}^{2^{n_{k+1}}-3} \circ \cdots \circ \varphi_{n_{k+1}}^1 \circ \varphi_{n_{k+1}}^0(y_0), \tag{2.15}$$

which by (2.10) and (2.11) amounts to (2.12).

Proof of the second formula is similar. By (2.10)–(2.12), we have

$$y_{s\cdot m_k} = \varphi_{k+1}^{s-1} \circ \beta_k' \circ \varphi_{k+1}^{s-2} \circ \beta_k' \circ \cdots \circ \beta_k' \circ \varphi_{k+1}^1 \circ \beta_k' \circ \varphi_{k+1}^0 \circ \beta_k'(y_0),$$

and via $\beta_k' = Id$ we obtain (2.13). □

3. A subclass of minimal skew-product maps without weak mixing factors

Now we specify the homeomorphisms, φ_k^j, for odd k to obtain minimal systems without nontrivial weak mixing factors for an arbitrary choice of the remaining homeomorphisms φ_k^j (with even k) satisfying (2.8). First of all, to simplify the computation we assume throughout the paper that the circle S has perimeter 3. So, let

$$R = \frac{3}{2\pi}, \quad n_{2k-1} = 3^k, \quad \theta_k = \frac{4\pi}{2^{n_{2k-1}} - 2}, \quad \text{and} \quad r_k = \frac{6}{2^{n_{2k-1}} - 2}, \qquad \text{for } k \geqslant 1. \tag{3.1}$$

For $0 < j \leqslant 2^{n_{2k-1}} - 2$, let φ_{2k-1}^j be the rotation of S with angle θ_k, in the positive direction if $0 < j \leqslant 2^{n_{2k-1}-1} - 1$ and in the opposite direction otherwise. Clearly, these mappings φ_{2k-1}^j satisfy (2.10) and (2.11). In particular,

$$\varphi_{2k-1}^j = (\varphi_{2k-1}^l)^{-1}, \qquad \text{if } 0 < j \leqslant 2^{n_{2k-1}-1} - 1 < l < 2^{n_{2k-1}} - 1, \tag{3.2}$$

and

$$\rho_S(u, \varphi_{2k-1}^j(u)) = r_k, \qquad \text{if } u \in S, \ k \geqslant 1, \quad 0 < j < 2^{n_{2k-1}} - 1. \tag{3.3}$$

Let $\alpha^{2k-1} = a_1 a_2 \ldots a_{2k-1}$ be the first block containing at least one zero digit. It is easy to see that rotation in the positive direction is used if $a_{2k-1} = 0$; see figure 2.

Lemma 3.1. *Assume (2.8), (2.10), (2.11) and (3.1)–(3.3). Then F_Φ is a minimal homeomorphism of $Q \times S$, for any choice of homeomorphisms φ_{2k}^j, $k \in \mathbb{N}$, $0 < j < 2^{n_{2k}} - 1$.*

Proof. Keeping notation from lemma 2.1, by (2.12),

$$\lim_{k\to\infty} \tau^{m_k}(\underline{0}) = \underline{0}, \text{ and } y_{m_k} = y_0, \tag{3.4}$$

whence any point $(\underline{0}, y_0) \in S$ is recurrent. Fix a $k_0 \in \mathbb{N}$ and let $s_0 \in \mathbb{N}$ be any integer such that $p := s_0 r_{k_0} < 1$. Then, by definition of φ_{2k-1}^j, for any $k \geqslant k_0$ there is an s_k such that

$$0 \leqslant s_k < 2^{n_{2k-1}} - 1 \qquad \text{and} \qquad s_k r_k = p. \tag{3.5}$$

By (2.13), (3.3) and (3.5),

$$\lim_{k\to\infty} \tau^{s_k m_k}(\underline{0}) = \underline{0} \qquad \text{and} \qquad y_{s_k m_k} = (\varphi_{2k_0-1}^1)^{s_{k_0}}(y_0). \tag{3.6}$$

Since the set $\{(\varphi_{2k_0-1}^1)^{s_{k_0}}(y_0); 0 < s_{k_0} < 2^{n_{2k_0-1}}\}_{k_0=1}^\infty$ is dense in S, by (3.6) we get

$$S_{\underline{0}} \subset \omega_{F_\Phi}(\underline{0}, y_0), \qquad \text{for any } y_0 \in S.$$

To complete the argument note that $\omega_{F_\Phi}(\underline{0}, y_0)$ is compact and invariant, $F_\Phi^j(S_{\underline{0}}) = S_{\underline{j}}$ and the set $\{S_{\underline{j}}\}_{j=1}^\infty$ is dense in $Q \times S$. □

Lemma 3.2. *Assume (2.8), (2.10), (2.11) and (3.1)–(3.3). Then, for any choice of homeomorphisms $\{\varphi_{2k}^j\}$, $k \in \mathbb{N}$, $0 < j < 2^{n_{2k}} - 1$, $(Q \times S, F_\Phi)$ has no nontrivial weak mixing factor.*

Proof. Let (X, ρ_X) be a compact metric space and let $G : X \to X$ be a surjective, continuous map. Let $\nu : Q \times S \to X$ be (a continuous) action map, so that $\nu \circ F_\Phi = G \circ \nu$. Let P_0, P_1 be the periodic portions of Q of period 2. Thus,

$$\tau(P_0) = P_1, \qquad \tau(P_1) = P_0, \qquad P_0 \cap P_1 = \emptyset, \qquad \text{and} \qquad P_0 \cup P_1 = Q. \tag{3.7}$$

Put $A_0 = \nu(P_0 \times Q)$, $A_1 = \nu(P_1 \times Q)$. Then

$$G(A_0) = G(\nu(P_0 \times S)) = \nu(F_\Phi(P_0 \times S)) = \nu(P_1 \times S) = A_1 \tag{3.8}$$

and similarly $G(A_1) = A_0$. If A_0 and A_1 are disjoint then they are closed, by the continuity of ν, and open since $X = A_0 \cup A_1$, and consequently G is not weak mixing.

So assume $z \in A_0 \cap A_1$. Then there are points $u \in P_0 \times S$, $v \in P_1 \times S$ so that $\nu(u) = \nu(v) = z$. Let U be the orbit of u with respect to the second iterate F_Φ^2 of F_Φ. Then $\nu(U) = Z$, and since $U \subset P_0 \times S$ is dense in $P_0 \times S$, Z is dense in A_0. Similarly, $\nu(V) = Z$ and Z is dense in A_1. It follows that $A_0 = A_1$.

Now let P be a τ-periodic portion of Q of period $m = 2^k$. Hence, $P \times S$ is invariant with respect to $\tilde{F} := F_\Phi^m$. Split P into two periodic portions $\tilde{P}_0$ and $\tilde{P}_1$ of period 2^{k+1} and denote $\tilde{A}_i = \nu(\tilde{P}_i \times S)$, $i = 0, 1$. Similarly as in the first case, we get $\tilde{A}_0 = \tilde{A}_1$. Consequently, since any point $\alpha \in Q$ is the intersection of a nested family of periodic portions of Q, $\nu(S_\alpha) = X$. Thus,

$$\text{if } (X, G) \text{ is weak mixing then } \nu(S_\alpha) = X, \qquad \text{for any } \alpha \in Q. \tag{3.9}$$

We finish the proof by showing that from (3.9) it follows that X is a singleton. Assume first that, for some $\alpha \in Q$, $\nu|_{S_\alpha}$ is a bijection. Then $\nu|_{S_\beta}$ is a bijection for any β in the backward orbit of α.

Indeed, let $\tau^n(\beta) = \alpha$, for some $n > 0$, and let, for some $u_0 \neq v_0$ in S_β, $\nu(u_0) = \nu(v_0)$. Let $\{u_i\}_{i=0}^\infty$ and $\{v_i\}_{i=0}^\infty$ be the F_Φ-trajectories of u_0 and v_0, respectively. Then

$$\nu(u_n) = \nu(F_\Phi^n(u_0)) = G^n(\nu(u_0)) = G^n(\nu(v_0)) = \nu(F_\Phi^n(v_0)) = \nu(v_n),$$

which is impossible since $\nu|_{S_\alpha)}$ is bijective, $u_n \neq v_n$ since F_Φ is bijective, and $u_n, v_n \in S_\alpha$.

So, if $\gamma \in Q$ is the τ-pre-image of α then

$$G = (\nu|_{S_\alpha}) \circ F_\Phi \circ (\nu|_{S_\gamma})^{-1}$$

is bijective. Since any continuous bijection between compact metric spaces is a homeomorphism, X is (homeomorphic to the) circle S and G is a homeomorphism and, as a factor of minimal map, it is also minimal. It is well known [6] that any minimal homeomorphism of the circle is conjugate to an irrational rotation, which is obviously not weak mixing. This is a contradiction and hence (3.9) implies that

$$\nu|_{S_\alpha} \text{ is bijective for no } \alpha \in Q. \tag{3.10}$$

Next we show that

$$\nu(\underline{0}, a) = \nu(\underline{0}, b) \text{ implies } \nu(\underline{0}, a+h) = \nu(\underline{0}, b+h), \qquad \text{for any } a, b \in S, \quad h \in \mathbb{R}, \tag{3.11}$$

where the points $a + h$, $b + h$ are obtained from a, b by rotation of S at angle $\frac{2}{3}\pi h$. Let a_0, b_0 be distinct points in S. Denote $\alpha_0 = (\underline{0}, a_0)$ and $\beta_0 = (\underline{0}, b_0)$ and assume that $\nu(\alpha_0) = \nu(\beta_0)$. For $n \in \mathbb{N}$, define a_n and b_n as the second coordinates of $F^n(\alpha_0)$ and $F^n(\beta_0)$, respectively. Fix an $h \in \mathbb{R}$; without loss of generality we may assume that $h \in (0, 3)$ since the circle S has perimeter 3. By (2.13), (3.2) and (3.3), there is a sequence $\{s_k\}_{k=1}^\infty$ in $\mathbb{N}$ such that

$$\lim_{k\to\infty} a_{p_k} = a_0 + h, \qquad \lim_{k\to\infty} b_{p_k} = b_0 + h, \quad \text{where } p_k = s_k m_{2k}. \tag{3.12}$$

Since $\lim_{k\to\infty} \tau^{p_k}(\underline{0}) = \underline{0}$,

$$\lim_{k\to\infty} \alpha_{p_k} = (\underline{0}, a_0 + h) =: \alpha, \qquad \lim_{k\to\infty} \beta_{p_k} = (\underline{0}, b_0 + h) =: \beta, \tag{3.13}$$

and hence, by the continuity of ν,

$$\rho_X(\nu(\alpha), \nu(\beta)) = \lim_{k\to\infty} \rho_X(\nu(\alpha_{p_k}), \nu(\beta_{p_k})) = 0$$

whence $\nu(\alpha) = \nu(\beta)$. This proves (3.11).

Now we show that

$$\text{if } J \subset S_{\underline{0}} \text{ is an interval then } \nu|_J \text{ is not one-to-one.} \tag{3.14}$$

To do this suppose that $J_0 = [\alpha, \beta]$, with $\alpha \neq \beta$, is a (minimal) interval of $S_{\underline{0}}$ such that $\nu(\alpha) = \nu(\beta)$ and $\nu|_{[\alpha,\beta)}$ is one-to-one mapping. Denote $h = \beta - \alpha$. There is a $k \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{k-1} J_i = S_{\underline{0}}, \qquad \text{where } J_i = [\alpha + ih, \beta + ih]. \tag{3.15}$$

Thus, by (3.9) and (3.11), $\nu(J_0) = X$ and X is (a homeomorphic copy of) the circle S. Let $K_0 = F_\Phi^{-1}(J_0) \subset S_{\underline{1}}$ where $\underline{1}$ denotes the sequence of ones. Then $K_0 = [\alpha_{-1}, \beta_{-1}]$ is an interval since F_Φ is a homeomorphism and $\nu|_{[\alpha_{-1},\beta_{-1})}$ is one-to-one since $G \circ \nu = \nu \circ F_\Phi$. Consequently, $\nu|_{K_0}$ is a homeomorphism. Denote $X_0 = \nu(K_0)$. Then X_0 is a compact interval on the circle X and $G(X_0) = \nu \circ F_\Phi(K_0) = X$. By the minimality of G, $X_0 = X$. Since

$$G(z) = \nu \circ F_\Phi \circ \tilde{\nu}(z), \qquad \text{for any } z \in X, \tag{3.16}$$

where $\tilde{\nu}$ is the inverse of $\nu|_{[\alpha_{-1},\beta_{-1})}$. Then G as a composition of homeomorphisms itself is a homeomorphism of the circle X which is not possible (see above). We have proved that (3.10) implies (3.14).

To finish the proof we show that (3.11) and (3.16) imply that X is a singleton. By (3.16), there are pairs of distinct points $A_k, B_k \in S_{\underline{0}}$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} A_k = \lim_{k\to\infty} B_k \qquad \text{and} \qquad \nu(A_k) = \nu(B_k). \tag{3.17}$$

Let C_k be the shorter of the arcs on $S_{\underline{0}}$ connecting the points A_k, B_k. We may assume that $C_k \supset C_{k+1}$, for any k. By (3.17), $\bigcap_{k=1}^{\infty} C_k = \{c\}$ is a singleton. Since $\nu(C_k) = X$, by (3.11) and (3.9), we obtain

$$X = \bigcap_{k=1}^{\infty} \nu(C_k) = \nu\left(\bigcap_{k=1}^{\infty} C_k\right) = \nu(c),$$

hence X is a singleton. □

4. A subclass of minimal skew-product *STC* maps

In the previous section we specified homeomorphisms $\{\varphi^j_{2k-1}\}$ so that the system has no weak mixing factors, for any choice of the homeomorphisms $\{\varphi^j_{2k}\}$, $k \in \mathbb{N}$, $0 < j < 2^{n_{2k}} - 1$. In this section we let the homeomorphisms $\{\varphi^j_{2k-1}\}$ depend on sequences $\{t_k\}$ and $\{n_{2k}\}$ of positive integers, called 'parameters'. For any proper choice of parameters the resulting system is *STC*, but different choices provide different examples disproving the conjectures stated in the introduction.

Let $\{t_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers tending to 1; we will specify it later. Let $a = a_0 \in S$ be given and let a_1 and a_2 be points in S obtained from a_0 by rotating S in the positive direction at angle $\theta_1 = 2\pi/3$ or $2\theta_1 = 4\pi/3$, respectively. Let S_i be the arc with the endpoints a_i and a_{i+1}, $0 \leqslant i \leqslant 2$. By (3.1), since the length of any S_i is 1, we may represent S_0 as the unit interval $[0, 1]$ and similarly $S_i = [i, i + 1]$, where i is taken to be mod 3. Thus,

$$S_0 \cup S_1 \cup S_2 = S, \qquad \operatorname{diam}(S_i) = 1, \tag{4.1}$$

and any two distinct S_i, S_j have exactly one point in common. Without loss of generality we may consider any φ_{2k}^j on S_i as a map from $I = [0, 1] \to I$. For $k \geqslant 1$ and $y \in I$, let

$$\varphi_{2k}^j(y) = y^{t_k}, \qquad \text{if } 1 \leqslant j < 2^{n_{2k}-1}, \tag{4.2}$$

$$\varphi_{2k}^j(y) = y^{1/t_k}, \qquad \text{if } 2^{n_{2k}-1} \leqslant j < 2^{n_{2k}} - 1. \tag{4.3}$$

Since, by (2.10), $\varphi_{2k}^0 = Id$, then the maps φ_{2k}^j satisfy (2.11). For $j \neq 0$ every φ_{2k}^j has exactly two fixed points, one repulsive and one attractive. For simplicity, denote by φ_{2k} the map from (4.2) and by $\varphi_{2k}^- = (\varphi_{2k})^{-1}$ the map from (4.3). Similarly, by (3.2), let $\varphi_{2k-1} = \varphi_{2k-1}^1$ and $\varphi_{2k-1}^- = (\varphi_{2k-1})^{-1}$.

Let $\underline{x} = x^1x^2 \ldots \in Q$ where any x^i is the block of n_i digits, and $z_0 \in S$. For any $n \geqslant 0$, define z_n by $F_\Phi^n(\underline{x}, z_0) = (\tau^n(\underline{x}), z_n)$.

Lemma 4.1. *Assume (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). Let $\underline{x} \in Q$ and $z_0 \in S$. Then, for any $q \geqslant 1$,*

$$\tau^{m_q - e(x^1x^2\ldots x^q)}(\underline{x}) = 0^{n_1+n_2+\cdots+n_q}\tau(x^{q+1}x^{q+2}\ldots).$$

Proof. It is obvious and we omit it. □

Lemma 4.2. *Assume (2.10),(2.11), (3.1)–(3.3), (4.2) and (4.3). Let $\underline{x} \in Q$ and $z_0 \in S$. Then, for any $q \geqslant 1$,*

$$F_\Phi^{m_q}(0^{n_1+\cdots+n_q}x^{q+1}x^{q+2}\ldots, z_0) = (0^{n_1+\cdots+n_q}\tau(x^{q+1}x^{q+2}\ldots), \varphi_{q+1}^{e(x^{q+1})}(z_0)).$$

Proof. It follows easily by (2.7). □

Lemma 4.3. *Assume (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). Let $\underline{x} \in Q$ and $z_0 \in S$. Then, for any $q \geqslant 1$,*

$$z_{m_q - e(x^1x^2\ldots x^q)} = \varphi_{q+1}^{e(x^{q+1})} \circ \left(\varphi_q^-\right)^{c_q} \circ \left(\varphi_{q-1}^-\right)^{c_{q-1}} \circ \cdots \circ \left(\varphi_1^-\right)^{c_1}(z_0), \tag{4.4}$$

where

$$c_i = 2^{n_i-1} - 1 - |e(x^i) - 2^{n_i-1}|$$

and

$$z_{m_q} = (\varphi_1)^{c_1} \circ (\varphi_2)^{c_2} \circ \cdots \circ \left(\varphi_{q-1}\right)^{c_{q-1}} \circ \left(\varphi_q\right)^{c_q}(z_{m_q - e(x^1x^2\ldots x^q)}). \tag{4.5}$$

Proof. It follows by (3.2), (4.2), (4.3) and lemmas 4.1 and 4.2. □

Lemma 4.4. *Let $\mathcal{F}$ be a finite family of homeomorphisms of a compact metric space (X, ρ_X). Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\psi^{-1} \circ h \circ \psi - Id\| < \varepsilon$, for any $\psi \in \mathcal{F}$ and any map h on X with $\|h - Id\| < \delta$.*

Proof. Given an $\varepsilon > 0$, for any sufficiently small $\delta > 0$, any $\psi \in \mathcal{F}$ and any $x \in X$, $|h(\psi(x)) - \psi(x)| < \delta$ implies $|\psi^{-1} \circ h \circ \psi(x) - x| < \varepsilon$, by the continuity of ψ^{-1}. □

Lemma 4.5. *Assume (2.8), (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). Let $n_2, n_4, \ldots, n_{2(k-1)}$ be a sequence of positive numbers. Then, for any $k \geqslant 1$, there is a $T_k > 1$, depending only on $t_1, t_2, \ldots, t_{k-1}, \ldots$, with the following property: if*

$$1 < t_{k+1} < t_k < T_k, \qquad \text{for any } k \geqslant 1, \tag{4.6}$$

then, for any $\underline{x} \in Q$ and $u, v \in S_{\underline{x}}$,

$$\lim_{k\to\infty} \rho(F_\Phi^{m_{2k-1}}(\underline{x}, u), F_\Phi^{m_{2k-1}}(\underline{x}, v)) = \rho((\underline{x}, u), (\underline{x}, v)). \tag{4.7}$$

Proof. Denote $u = u_0$ and $v = v_0$. Then, by lemma 4.4,

$$u_{m_{2k-1}} = \psi^{-1} \circ \varphi_{2k}^{e(x^{k+1})} \circ \psi(u_0), \tag{4.8}$$

where $\psi = (\varphi_{2k-1}^{-})^{c_{2k-1}} \circ (\varphi_{2k-2}^{-})^{c_{2k-2}} \circ \cdots \circ (\varphi_1^{-})^{c_1}$, or $\psi = (\varphi_1)^{c_1} \circ (\varphi_2)^{c_2} \circ \cdots (\varphi_{2k-2})^{c_{2k-2}} \circ (\varphi_{2k-1})^{c_{2k-1}}$ are in the family, Ψ_{2k-1}, of all possible compositions of the maps, φ_j, φ_j^{-}, of rank $< 2k$ that may occur in (4.4) or (4.5). Put $\varepsilon = 1/k$ and apply lemma 4.4 to Ψ_{2k-1} to obtain a $\delta_k > 0$. Obviously, by (4.2) and (4.3), there is a T_k depending only on Ψ_{2k-1}, such that

$$1 < t_k < T_k \text{ implies } \|\varphi_{k+1}^{j} - Id\| < \delta_k, \qquad 0 \leqslant j < 2^{n_{2k}} - 1. \tag{4.9}$$

For such a choice, by (4.8) and lemma 4.4, $\|u_{m_k} - u_0\| < 1/k$ whence $\lim_{k\to\infty} u_{m_k} = u_0$, and similarly for v_{m_k}. This implies (4.7). □

5. Specifications of Φ

Lemma 5.1. *Assume (2.8), (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). Assume that the sequences $\{t_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ satisfy (4.6) and*

$$\lim_{k\to\infty} (1-\theta_k)^{t_k^{q_k}} = 0, \qquad \text{where } q_k = 2^{n_{2k}-1} - 1, \quad \theta_k \in (0,1)$$
$$\text{and} \qquad \lim_{k\to\infty} \theta_k = 1. \tag{5.1}$$

Then, for any $\alpha \in S_{\underline{x}}$, $\underline{x} \in Q$ there is an arc $A \subset S_{\underline{x}}$ such that any two points from A are proximal.

Proof. It suffices to show that, for any $\alpha := (\underline{x}, u)$, there is a $\beta := (\underline{x}, v) \neq \alpha$ proximal to α. Fix an $\underline{x} \in Q$ and assume without loss of generality that u is in $S_0 = [0, 1]$.

Denote by $\nu_q(z_0)$ the right-hand side of (4.4). Since any of the maps φ_i^k in Φ preserves the distance of the endpoints of any S_n (cf (3.1), (3.3), (4.2) and (4.3)),

$$\rho_S(\nu_q(0), \nu_q(1)) = \rho_S(0,1) = 1, \qquad \text{for any } q \in \mathbb{N}. \tag{5.2}$$

For $y \in (0, 1)$, denote

$$\mu(y) = \liminf_{q\to\infty} \rho_S(\nu_q(y), \nu_q(1)), \qquad M(y) = \limsup_{q\to\infty} \rho_S(\nu_q(y), \nu_q(1)). \tag{5.3}$$

To prove the lemma it suffices to show that

$$\mu(y) = 0 \quad \text{or} \quad M(y) = 1, \qquad \text{for any } y \in (0,1). \tag{5.4}$$

Indeed, by (5.2), $0 \leqslant \mu \leqslant M \leqslant 1$. For $u = 0$ put $v = -\frac{1}{2}$ if $\mu(\frac{1}{2}) = 0$, and let $v = \frac{1}{2}$ if $M(\frac{1}{2}) = 1$. For $u \in (0, 1)$, let $v = 0$ if $\mu(u) = 1$, and $v = 1$ if $M(u) = 0$. To prove (5.4) assume

$$y \in (0,1) \quad \text{and} \quad \delta < \mu(y) \leqslant M(y) < 1 - \delta, \qquad \text{for some } \delta > 0. \tag{5.5}$$

Denote

$$m(q,j,s) := m_q - e(x^1 x^2 \ldots x^q) + s \cdot m_j, \quad (\tau^j(\underline{x}), y_j) := F_{\Phi}^{j}(\underline{x}, y), \qquad q, j, s \in \mathbb{N}. \tag{5.6}$$

By (2.13) and (4.4), for $0 \leqslant k < q$ and $0 < s < 2^{n_{k+1}}$,

$$u_{m(q,k,s)} = \varphi_{k+1}^{s-1} \circ \varphi_{k+1}^{s-2} \circ \cdots \circ \varphi_{k+1}^{1} \circ \varphi_{k+1}^{0} \circ \nu_q(u), \qquad 0_{m(q,k,s)} = \nu_q(0). \tag{5.7}$$

Hence, by (4.2), (5.6), (5.7), (5.5) and (5.1),

$$\rho_S(u_{m(q,2k-1,q_k)}, 0_{m(q,2k-1,q_k)}) \leqslant [\nu_q(u) - \nu_q(0)]^{t_k^{q_k}} \to 0, \qquad \text{if } k, q \to \infty, \quad k < \frac{q}{2}. \tag{5.8}$$

□

Lemma 5.2. *Assume (2.8), (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). Then there are sequences* $\{n_{2k}\}_{k=1}^{\infty}$ *and* $\{t_k\}_{k=1}^{\infty}$ *satisfying (4.6) and (5.1) such that* F_{Φ} *is not LYS.*

Proof. Let $u = \frac{1}{2}$ be the centre of $S_0 = [0, 1]$. It suffices to show that t_k and n_{2k} satisfying (4.6) and (5.1) can be chosen such that, for suitable $\underline{x} \in Q$ and $\delta_0 > 0$,

$$\rho_S(F^n(\alpha), F^n(\beta)) \leqslant \rho_S(\alpha, \beta), \qquad \alpha = (\underline{x}, u - \delta), \quad \beta = (\underline{x}, u + \delta),$$
$$0 < \delta < \delta_0, \quad n \geqslant 1, \tag{5.9}$$

since τ on Q is distal (actually, τ preserves the distances between points in Q) and hence points in different fibres of $Q \times S$ are distal. To do this assume that, instead of (5.1), the stronger condition

$$(1 - r_{k+1})^{p_k} < \frac{1}{2} r_{k+1}, \qquad \text{where } p_k = t_k^{q_k} = t_k^{2^{n_{2k-1}} - 1}, \quad k \geqslant 1, \tag{5.10}$$

is satisfied. Let $\underline{x} = x^1 x^2 \ldots x^k \ldots$ be such that

$$e(x^{2k}) = 2^{n_{2k}} - q_k - 1, \qquad k \geqslant 1. \tag{5.11}$$

Since

$$\left(\frac{1}{2} + \delta\right)^t - \left(\frac{1}{2} - \delta\right)^t \leqslant 2\delta, \qquad t \in \mathbb{R}, \quad 0 \leqslant \delta < \frac{1}{2},$$

the $U_\delta = \{\underline{x}\} \times [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, with $\delta < \frac{1}{2}$, could be mapped by an F_{Φ}^n to a set with greater diameter than 2δ only when the end-points of U_δ are mapped by a rotation of the circle to different sets S_i and S_j. But this is impossible by (5.10) and (5.11). □

Theorem 5.1. *Assume (2.8), (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). If the sequences* $\{t_k\}_{k=1}^{\infty}$ *and* $\{n_k\}_{k=1}^{\infty}$ *satisfy (4.6) and (5.10) then the system* $(Q \times S, F_{\Phi})$ *is STC and not LYS.*

Proof. If $\alpha \neq \beta$ are points in $Q \times S$ then they are not asymptotic. This follows by lemma 4.5 if $\alpha, \beta \in S_{\underline{x}}$, for some $\underline{x}$. Otherwise, α, β have different first coordinates and hence they are distal since Q is a distal system. On the other hand, any point $\alpha \in Q \times S$ is a limit of points proximal to α, by lemma 5.1. Hence, the system is *STC*. Finally, the system is not *LYS*, by lemma 5.2. □

Lemma 5.3. *Assume (2.8), (2.10), (2.11), (3.1)–(3.3), (4.2) and (4.3). Then there are sequences* $\{n_{2k}\}_{k=1}^{\infty}$ *and* $\{t_k\}_{k=1}^{\infty}$ *satisfying (4.6) and (5.1) such that* F_{Φ} *is LYS.*

Proof. By (3.1), $n_{2k-1} = 3^k$ and $r_k = 6/(2^{n_{2k-1}} - 2)$. Define sequences $\{p_k\}_{k=1}^{\infty}$ and $\{\delta_k\}_{k=1}^{\infty}$ by

$$(1 - 2r_{k+1})^{p_k} = \frac{1}{2} \qquad \text{and} \qquad \delta_k = (3r_{k+1})^{1/p_k}. \tag{5.12}$$

Obviously, $p_k < p_{k+1}$, for any $k \geqslant 1$, and $\lim_{k\to\infty} p_k = \infty$. Similarly, $\delta_k > 0$ for any $k \geqslant 1$ and it is easy to verify that

$$\lim_{k\to\infty} \delta_k = 0. \tag{5.13}$$

Indeed, by (5.12),

$$\delta_k = (3r_{k+1})^{1/p_k} = \left(3 \cdot \frac{6}{2^{3^k} - 2}\right)^{1/p_k}.$$

Then (5.13) follows since, for $a_k > 0$, $\lim_{k\to\infty} a_k^{1/k} = a$ if and only if $\lim_{k\to\infty} a_{k+1}/a_k = a$.

Hence there is a unique $t_k > 1$ such that

$$p_0 = 1, \quad p_k = t_k^{2^{n_{2k}-1}-1} p_{k-1} = t_k^{q_k} p_{k-1}, \qquad k \geqslant 1, \tag{5.14}$$

where q_k is given in (5.1). Thus,

$$p_k = t_k^{q_k} \cdot t_{k-1}^{q_{k-1}} \cdots t_2^{q_2} \cdot t_1^{q_1}, \qquad k \geqslant 1. \tag{5.15}$$

It is easy to see that

$$t_k > 1, \quad k \geqslant 1 \qquad \text{and} \qquad \lim_{k\to\infty} t_k = 1. \tag{5.16}$$

Denote $\psi_{2k} = \varphi_{2k}^{q_k} \circ \varphi_{2(k-1)}^{q_{k-1}} \circ \cdots \circ \varphi_4^{q_2} \circ \varphi_2^{q_1}$. Then

$$\psi_{2k}(y) = y^{p_k}, \qquad k \geqslant 1, \quad y \in [0, 1]. \tag{5.17}$$

Let $u, v \in S_i$, for some $i \in \{0, 1, 2\}$ with $u < v$, and let δ_{k_0} be given such that

$$|u - v| \geqslant \delta_{k_0}.$$

Then, by (5.12) and (5.17),

$$|\psi_{2k_0}(u) - \psi_{2k_0}(v)| = |u^{p_{k_0}} - v^{p_{k_0}}| \geqslant |u - v|^{p_{k_0}} \geqslant 3r_{k_0+1}. \tag{5.18}$$

Let $\underline{x} \in Q$, and put $\alpha = (\underline{x}, u)$, $\beta = (\underline{x}, v)$. Since any of the maps φ_{2k-1}^j is rotation it preserves the distance between points. Therefore, by (5.18),

$$|u_{m_{2k_0+1}-e(x^1x^2\ldots x^{2k_0+1})} - v_{m_{2k_0+1}-e(x^1x^2\ldots x^{2k_0+1})}| \geqslant |\psi_{2k_0}(u) - \psi_{2k_0}(v)| \geqslant 3r_{k_0+1}, \tag{5.19}$$

where y_n denotes the second coordinate of $F^n(\underline{x}, y)$. For simplicity, denote $2k_0 + 1 = K$ and $M = m_K - e(x^1x^2 \ldots x^K)$. Then

$$F^M(\alpha) = F^M(\underline{x}, u) = (0^{n_1}0^{n_2} \ldots 0^{n_K}\tau(x^{K+1}x^{K+2} \ldots), u_M), \tag{5.20}$$

and similarly for $F^M(\beta)$. Find $0 < s \leqslant 2^{n_K-1} - 1$ such that $(\varphi_K^1)^s(u_M)$ and $(\varphi_K^1)^s(v_M)$ belong to different intervals $[i, i+1]$ of S, separated by a point, say, j; here $(\varphi_K^1)^s$ denotes the sth iterate of φ_K^1. Thus, $(\varphi_K^1)^s(u_M) < j < (\varphi_K^1)^s(v_M)$. Moreover, let s be such that the distance between $(\varphi_K^1)^s(u_M)$ and j is greater than $2r_{k_0+1}$. By (5.19), this is always possible since φ_K^1 is the rotation which shifts any point of S to the right at distance r_{k_0+1}. By (5.12),

$$\limsup_{n\to\infty} \rho(F_\Phi^n(\alpha), F_\Phi^n(\beta)) > \frac{1}{2}. \tag{5.21}$$

Since α, β are arbitrary distinct points with the same first coordinate, we proved the following: for any $\alpha \in S_{\underline{x}}$ there is a neighbourhood U_α of α in $S_{\underline{x}}$ such that, for any $\beta \in U_\alpha$, (5.21) is satisfied. Since $\lim p_k = \infty$, there exists a sequence $\{\theta_k\}$ satisfying (5.1). Hence, by lemma 5.1 and (5.21), $(Q \times S, F_\Phi)$ is *LYS*. □

Theorem 5.2. *There exists a minimal system which is LYS but has no weak mixing factor.*

Proof. The theorem follows by lemmas 3.1, 3.2, 5.3. □

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Li–Yorke sensitive minimal maps II

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Abstract
In a previous paper (Čiklová 2006 *Nonlinearity* **19** 517–29) a family of minimal, Li–Yorke sensitive dynamical systems (X, T) without weak mixing factors has been constructed, disproving a conjecture by Akin and Kolyada (2003 *Nonlinearity* **16** 1421–33). In this article we show that, in addition, any system in the above-mentioned family has an almost one-to-one minimal extension which fails to be Li–Yorke sensitive. This disproves another conjecture by Akin and Kolyada.

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1. Introduction

Akin and Kolyada [1] introduced and studied the concept of *Li–Yorke sensitivity*, in brief *LYS*, for surjective continuous maps $T : X \to X$ of a compact metric space (X, ρ). A map T is *LYS* if there is an $\varepsilon > 0$ such that any neighbourhood of any $u \in X$ contains a point v satisfying

$$\liminf_{n\to\infty} \rho(T^n(u), T^n(v)) = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \rho(T^n(u), T^n(v)) > \varepsilon, \tag{1.1}$$

where T^n denotes the nth iterate of T. Any such pair u, v is an ε-*Li–Yorke pair*, briefly ε*LY* or *LY* pair. In [1] they also provide five conjectures concerning *LYS*. Two of them are disproved in [4]. In particular, the following is proved.

Theorem 1.1 (see [4]). *There is a parametric family $\mathcal{F}$ of minimal skew-product homeomorphisms $F : X = Q \times \mathbb{S}^1 \to X$, where Q is a Cantor set and $\mathbb{S}^1$ the unit circle, such that for any $F \in \mathcal{F}$, (X, F) is Li–Yorke sensitive and possesses no weak mixing factor.*

Recall that a system (X, T) is a *factor* of a system (Y, S), or (Y, S) is an *extension* of (X, T), if there is a continuous surjective map $\pi : Y \to X$, *a factor map*, such that $\pi \circ S = T \circ \pi$. A system (Y, S) is an *almost one-to-one extension* of (X, T) if X contains a dense subset of points x such that $\pi^{-1}(x)$ is a singleton. A system (X, T) is *weak mixing* if the system

$(X \times X, T \times T)$ is transitive, i.e. contains a point whose trajectory is dense in $X \times X$. A system (X, T) is *minimal* if any point $x \in X$ has a dense trajectory in X. A *skew-product map* $F : X \times Y \to X \times Y$, where X, Y are compact spaces, is any continuous map of the form $(x, y) \mapsto (f(x), g(x, y))$. Points x, y in (X, T) are *distal* if $\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) > 0$, and are *proximal* otherwise. By a *Cantor set* we mean any compact, totally disconnected set without isolated points; any such set is homeomorphic to the middle-third Cantor set on [0, 1]. For more information concerning terminology, the reader is referred to [1] or [4].

In this note we show that the family $\mathcal{F}$ from theorem 1.1 consists of *LYS* maps F such that (X, F) is a factor of a minimal system (Y, S) which is not *LYS*. This disproves another conjecture of Akin and Kolyada, namely that any minimal system with a *LYS* factor must be *LYS* (cf [1 question 3]). Our main result is the following:

Theorem 1.2. *There is a minimal system which is not LYS, but has a LYS factor. In fact, any system* $(Q \times \mathbb{S}^1, F)$*, with* $F \in \mathcal{F}$*, where* $\mathcal{F}$ *is as in theorem 1.1, has almost one-to-one minimal extension* (Y, S) *with* S *being a homeomorphism but not LYS.*

Remark. If (X, T) is a factor of (Y, S) then a *LY* pair in (X, T) need not be a projection of a *LY* pair in (Y, S) [3]. This is not the case of theorem 1.2 since we have the following:

Proposition 1.1. *Let* (Y, S) *be an almost one-to-one extension of a LYS minimal system* (X, T)*, with a factor map* π*. Then there is an* $\varepsilon > 0$ *such that, for any* $x \in X$*, there is a* $u \in \pi^{-1}(x)$ *such that any neighbourhood* U *of* u *contains a* v *satisfying (1.1) such that* $(x, \pi(v))$ *is a Li–Yorke pair, too.*

Consequently, any (not necessarily minimal) almost one-to-one extension of a minimal *LYS* system is 'almost' *LYS*. This shows that theorem 1.2, in some sense, is the best possible result since the factor map π from theorem 1.2 is at most two-to-one, i.e. $\#\pi^{-1}(x)$ is 1 or 2 for any $x \in X$ (see the proof of theorem 1.2).

Proof of proposition 1.1. Let $x \in X$ and, for any $n \in \mathbb{N}$, let $x_n \in X$ be such that $x_n \to x$, and x, x_n is an $\varepsilon' LY$ pair. Since Y is compact, π^{-1} is an upper-semicontinuous set-valued map. Thus, for any neighbourhood V of $\pi^{-1}(x)$, $\pi^{-1}(x_n) \subset V$, whenever n is sufficiently large. Hence, by compactness of Y, there is a $u \in \pi^{-1}(x)$ such that any neighbourhood U of u intersects some $\pi^{-1}(x_n)$. Take $v \in U \cap \pi^{-1}(x_n)$. Since π is continuous, there is an $\varepsilon > 0$ such that the distance of any two points a, b in Y is greater than ε whenever $\pi(a), \pi(b)$ are more than ε' apart. It follows that $\limsup_{i\to\infty} \rho(S^i(u), S^i(v)) > \varepsilon$, where ρ is the metric in Y.

To finish the argument it suffices to show that the points u, v are proximal. Let $z \in X$ be such that $\pi^{-1}(z) = \{w\}$, and let W be a δ-neighbourhood of w. Since π^{-1} is upper semicontinuous there is an open $G \neq \emptyset$ such that $\pi^{-1}(G) \subset W$. Since (X, T) is minimal and x, x_n are proximal, there is an $m \geqslant 0$ such that $T^m(x), T^m(x_n) \in G$. It follows $\rho(S^m(u), S^m(v)) < \delta$. Since δ is arbitrary, u, v are proximal. □

2. Proof of theorem 1.2

Lemma 2.1. *Let* X, R *be compact metric spaces such that* R *is the interval* $I = [0, 1]$*, or the circle* $\mathbb{S}^1$*, equipped with the standard topology. Assume that*

(i) $F : X \times R \to X \times R$ *is a surjective skew-product map,* $(x, y) \mapsto (\tau(x), g_x(y))$*, and* (X, τ) *is distal;*

(ii) Φ *is a countable family of homeomorphisms* $R \to R$ *containing the identity and such that, for any* $x \in X$, $g_x \in \Phi$*;*

(iii) there are points $x_0 \in X$ *and* $y_0 \in R$*, and an open interval* $V \subset R$ *with* y_0 *as one of its endpoints such that no point* $v = (x_0, y_0')$ *with* $y_0' \in V$ *is proximal to* $u = (x_0, y_0)$*.*

Then there is an almost one-to-one extension (Y, S) *of* $(X \times R, F)$ *which is not LYS. If* $(X \times R, F)$ *is minimal then* (Y, S) *is minimal.*

Proof. Let $D \subset R$ be the set of points $\psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_2 \circ \psi_1(y_0)$, for all finite choices of $\psi_j \in \Phi$. Then D is a countable set, $y_0 \in D$, $\psi(D) = D$ for any $\psi \in \Phi$ and hence, since F is a skew-product map, $F(X \times D) = X \times D$. Now we use the standard technique of doubling points in D (see, e.g., [5] for details). Assuming that R is ordered in the usual way, we split any point $a \in D$ into a pair $a^- < a^+$, and compress the rest of R preserving the order to obtain a Cantor set P (which may be considered as a subset of R). Let π_0 be the associated continuous projection of P onto R. Thus, $\pi_0^{-1}(a) = \{a^-, a^+\}$ if $a \in D$, and $\pi_0^{-1}(a)$ is a singleton otherwise. For any $\psi \in \Phi$ there is a unique homeomorphism $\widetilde{\psi} : P \to P$ such that $\pi_0 \circ \widetilde{\psi} = \psi \circ \pi_0$. Let $\widetilde{\Phi}$ be the family of all $\widetilde{\psi}$, $\psi \in \Phi$.

To finish the argument let $Y = X \times P$, and let $S(x, y) = (\tau(x), \widetilde{g}_x(y))$, where $\widetilde{g}_x = \widetilde{\psi}$ provided $g_x = \psi$. Let $\pi : Y \to X \times R$ be the factor map given by $\pi(x, y) = (x, \pi_0(y))$. Obviously, π is a factor map whence $(X \times R, F)$ is a factor of (Y, S). Moreover, (Y, S) is not *LYS*. Assume that, e.g., y_0 is the right-hand endpoint of V, and let $\widetilde{V} = \pi_0^{-1}(V) \setminus \{y_0^+\}$. Then $G = X \times \widetilde{V}$ is an (open) neighbourhood of $u = (x_0, y_0^-)$. However, by the definition of V, no $v = (x, y) \in G$ is proximal to u. This is clear if $x \neq x_0$ since (X, τ) is distal. If $x = x_0$ then the statement follows by the definition of V. Consequently, points u, v cannot satisfy (1.1). The statement concerning minimality of (Y, S) is obvious. □

To prove theorem 1.2 it suffices to show that any map $F \in \mathcal{F}$ satisfies the hypothesis of lemma 2.1. First we give a brief description of $\mathcal{F}$. For details see [4]; the general form of the parametric family was introduced in [2]. Let $Q \subset I$ be the Cantor set represented as the space $Q = \{0, 1\}^{\mathbb{N}}$ of sequences of two symbols, with the topology of pointwise convergence, and the mapping $\tau : Q \to Q$ given by $x \mapsto x + 1000\cdots$, *the adding machine* or *odometer*. Then τ is a distal homeomorphism of Q.

Any map $F \in \mathcal{F}$ has the form $(x, y) \mapsto (\tau(x), g_x(y))$ where, for any x, g_x is a continuous homeomorphism of the unit circle $\mathbb{S}^1$. It is associated with a sequence $\nu = \{n_k\}_{k=1}^{\infty}$ of positive integers, and a family $\Phi = \{\varphi_k^j; 0 \leqslant j < 2^{n_k}\}_{k=1}^{\infty}$ of homeomorphisms of $\mathbb{S}^1$; any map φ_k^j is a map of *rank* k.

For any (finite or infinite) sequence $x = x_1 x_2 \ldots$, $x_i \in \{0, 1\}$, with finitely many nonzero terms define the *evaluation* $e(x)$ of x as

$$e(x) = x_1 + 2x_2 + 2^2 x_3 + \cdots + 2^{k-1} x_k \cdots.$$

The map g_x is defined as follows. If $x = 1^\infty$ then $g_x = \mathrm{Id}$, the identity. If $x \neq 1^\infty$, write x in blocks as $x = x^1 x^2 x^3 \cdots$ where x^j is the block of n_j digits of x, and let x^k be the first block containing at least one zero digit. Then $g_x = \varphi_k^j$ where $j = e(x^k)$. From the above description it is obvious that any map $F \in \mathcal{F}$ satisfies hypotheses (i) and (ii) of lemma 2.1. A special choice of ν and Φ implies that F is minimal and *LYS*, see theorem 5.2, or lemmas 3.1 and 5.3 in [4]. Hence to prove theorem 1.2 it suffices to show that there are x_0, y_0 and V satisfying hypothesis (iii) of lemma 2.1. This is given in the next lemma 2.2.

To simplify the notation we may assume that $\mathbb{S}^1$ is a positively oriented circle with perimeter 3, composed of three closed arcs S_0, S_1, S_2, each of length 1, with endpoints $a_0 < a_1 < a_2 < a_3 = a_0$ so that S_j is the arc with endpoints $a_j < a_{j+1 (\mathrm{mod}\, 3)}$. For $a, b \in \mathbb{S}^1$

denote by $\rho(a,b)$ the length of arc connecting a with b in the positive direction; thus, ρ is not a metric since $\rho(a_0,a_1)=1\neq\rho(a_1,a_0)=2$. Denote by 0^j and 1^j the block of j zeros or ones, respectively.

Lemma 2.2. *Keeping the notation as above, let*

$$x_0 = 0^{n_1+n_2-1}10^{n_3+n_4-1}1\cdots0^{n_{2k-1}+n_{2k}-1}1\cdots,\quad y_0=a_0 \qquad \text{and} \qquad V=(a_2,a_0).$$

Then, for any $z_0\in V$, $\liminf_{n\to\infty}\rho(F^n(x_0,y_0),F^n(x_0,z_0))>0$, i.e. (x_0,y_0) and (x_0,z_0) are distal points.

Proof. We need more information on the family Φ from [4]. Maps in Φ are defined using a sequence $\theta_1,\zeta_2,\theta_3,\zeta_4,\theta_5,\zeta_6,\ldots$ of auxiliary maps of $\mathbb{S}^1$, where θ_k are rotations of $\mathbb{S}^1$ in the positive direction by a small angle so that $\theta_k(y)>y$ with $\theta_k(y)\approx y$, for any $y\in\mathbb{S}^1$ [4]. Any ζ_k acts on any S_j as $y\mapsto y^{t_k}$ on $[0,1]$ (with 0 and 1 replaced by a_j and a_{j+1}, respectively), with $t_k>1$. Thus, $\zeta_k(y)>y$, for any $y\in\mathbb{S}^1$, $y\neq a_0,a_1,a_2$, while a_0,a_1 and a_2 are fixed points. The maps φ_k^j, for $j,k\in\mathbb{N}$, $0\leqslant j<2^{n_k}$, are defined by the following rule:

$$k\text{ odd}\Rightarrow\varphi_k^0=\mathrm{Id},\quad \varphi_k^j=\theta_k \text{ if } 1\leqslant j<2^{n_k-1}\quad\text{and}\quad \varphi_k^j=\theta_k^{-1}\text{ if } 2^{n_k-1}\leqslant j<2^{n_k}-1, \tag{2.2}$$

$$k\text{ even}\Rightarrow\varphi_k^0=\mathrm{Id},\quad \varphi_k^j=\zeta_k \text{ if } 1\leqslant j<2^{n_k-1}\quad\text{and}\quad \varphi_k^j=\zeta_k^{-1}\text{ if } 2^{n_k-1}\leqslant j<2^{n_k}-1. \tag{2.3}$$

For any $k\in\mathbb{N}$, let y_k,z_k be given by $F^k(x_0,y_0)=(x_k,y_k)$ and $F^k(x_0,z_0)=(x_k,z_k)$, where $x_k=\tau^k(x_0)$. Since any φ_k^j is an orientation preserving homeomorphism, we always have $z_k<y_k$ which actually means that $\rho(z_k,y_k)<\rho(y_k,z_k)$. To prove the lemma it suffices to show that

$$\rho(z_k,y_k)\geqslant\rho(z_0,y_0),\qquad \text{for any } k\in\mathbb{N}. \tag{2.4}$$

Let $w_k\in\{y_k,z_k\}$. Then $w_k=\Psi_k(w_0)$ where Ψ_k is the formal composition $\psi_k\circ\psi_{k-1}\circ\cdots\circ\psi_2\circ\psi_1$ of k maps θ_s,ζ_t, their inverses, or the identities. The map ψ_1 is applied as the first one, and ψ_k as the last one; their order is given by (2.2) and (2.3). It follows (see [4] for more details) that

$$\begin{aligned}\Psi_{2^{n_1+n_2}} &= \underbrace{\zeta_2\circ\cdots\circ\alpha_1\circ\zeta_2\circ\alpha_1\circ\mathrm{Id}\circ\alpha_1}_{2^{n_2-1}-1\text{ maps }\zeta_2,\ 2^{n_2-1}\text{ blocks }\alpha_1}\circ\underbrace{\mathrm{Id}\circ\alpha_1\circ\zeta_2^{-1}\circ\cdots\circ\alpha_1\circ\zeta_2^{-1}\circ\alpha_1}_{2^{n_2-1}-1\text{ maps }\zeta_2^{-1},\ 2^{n_2-1}\text{ blocks }\alpha_1}\\ &=:\beta_2\circ\alpha_2,\end{aligned} \tag{2.5}$$

where the block $\alpha_1=\mathrm{Id}$ is a formal composition of $2^{n_1}-1$ maps of rank 1, i.e. rotations and identities, and any of α_2 and β_2 is a formal composition of $2^{n_1+n_2-1}$ maps of rank 1 or 2. Similarly,

$$\begin{aligned}\Psi_{2^{n_1+n_2+n_3}} &= \underbrace{\alpha_2\circ\gamma_2'\circ\beta_2\cdots\circ\beta_2\circ\gamma_2'\circ\beta_2\circ\gamma_2'}_{2^{n_3-1}-1\text{ blocks }\gamma_2',\ 2^{n_3-1}\text{ blocks }\beta_2}\circ\underbrace{\beta_2\circ\gamma_2\circ\cdots\circ\gamma_2\circ\beta_2\circ\gamma_2\circ\beta_2\circ\alpha_2}_{2^{n_3-1}-1\text{ blocks }\gamma_2,\ 2^{n_3-1}\text{ blocks }\beta_2}\\ &=:\beta_3\circ\alpha_3,\end{aligned} \tag{2.6}$$

where

$$\gamma_2=\underbrace{\theta_3\circ\alpha_1\circ\zeta_2^{-1}\circ\cdots\circ\alpha_1\circ\zeta_2^{-1}\circ\alpha_1}_{2^{n_2-1}-1\text{ maps }\zeta_2^{-1},\ 2^{n_2-1}\text{ blocks }\alpha_1},\qquad \gamma_2'=\underbrace{\theta_3^{-1}\circ\alpha_1\circ\zeta_2^{-1}\circ\cdots\circ\alpha_1\circ\zeta_2^{-1}\circ\alpha_1}_{2^{n_2-1}-1\text{ maps }\zeta_2^{-1},\ 2^{n_2-1}\text{ blocks }\alpha_1}. \tag{2.7}$$

Obviously, any of α_3 and β_3 is a composition of $2^{n_1+n_2+n_3-1}$ maps of rank $\leqslant 3$, and, by (2.5)–(2.7),

$$\alpha_2 \circ \beta_2 = \beta_2 \circ \alpha_2 = \mathrm{Id}, \quad \gamma_2 \circ \beta_2 = \theta_3, \quad \gamma_2' \circ \beta_2 = \theta_3^{-1} \quad \text{and} \quad \beta_3 \circ \alpha_3 = \mathrm{Id}. \tag{2.8}$$

Since rotations and identities do not change $\delta(i) := \rho(z_i, y_i)$ we have $\delta(i+1) = \delta(i)$, whenever the ith map in (2.6) is not ζ_k or its inverse. In particular, the equality holds if the ith map is in any of the blocks α_1. Any map ζ_2^{-1} in the block α_2 is applied to y_i, z_i in the situation when $y_i = y_0 (= a_0)$ is a fixed point of ζ_2^{-1} which repulses z_i so that $\delta(i+1) > \delta(i)$. After application of all maps in α_2 we have $\delta(2^{n_1+n_2-1}) = \zeta_2^{-m}(z_0)$ where $m = 2^{n_2-1} - 1$ is the number of ζ_2^{-1} in α_2, see (2.5). Then, successive application of the maps ζ_2 in β_2 decreases $\delta(i)$ back to $\delta(0)$. This proves (2.4) for $k \leqslant 2^{n_1+n_2}$.

Application of the maps in the first block γ_2 in (2.7) again is similar as for α_2, except for the last map which is not the identity, but the rotation θ_3. So, the first map ζ_2 in the next block (the second block β_2 in α_3) is applied in the situation when y_i is not its fixed point. It follows that to prove (2.4) for $j < 2^{n_1+n_2+n_3}$ it suffices to show that

$$\rho(\zeta_2^k \circ \theta_3 \circ \zeta_2^{-k}(z_i), \zeta_2^k \circ \theta_3 \circ \zeta_2^{-k}(y_i)) \geqslant \rho(z_i, y_i), \qquad \text{for any } i, k \geqslant 0. \tag{2.9}$$

Denote $\widetilde{y_i} = \theta_3 \circ \zeta_2^{-k}(y_i)$, and similarly with $\widetilde{z_i}$, and consider two cases.

If the points $\widetilde{z_i} < \widetilde{y_i}$ are separated by one of the points a_0, a_1, a_2 on $\mathbb{S}^1$ (i.e. $\widetilde{z_i} < a_j < \widetilde{y_i}$, $0 \leqslant j \leqslant 2$), let $h = \rho(a_j, \widetilde{y_i})$. Then, by the definition of ζ_2,

$$\begin{aligned}\rho(\zeta_2^k(\widetilde{z_i}), \zeta_2^k(\widetilde{y_i})) &= (\rho(\widetilde{z_i}, \widetilde{y_i}) - h)^{k\cdot t_2} + 1 - (1-h)^{k\cdot t_2} =: \lambda(h) > (\rho(\widetilde{z_i}, \widetilde{y_i}))^{k\cdot t_2} \\ &\equiv \lambda(0)\end{aligned} \tag{2.10}$$

since $t_2 > 1$ whence $\lambda(h)$ is an increasing function.

If the points are not separated by an a_j then we may assume $a_0 < \widetilde{y_i} < \widetilde{z_i} < a_1$ and, letting $h = \rho(a_1, \widetilde{y_i})$, we have

$$\rho(\zeta_2^k(\widetilde{z_i}), \zeta_2^k(\widetilde{y_i})) = (\rho(\widetilde{z_i}, \widetilde{y_i}) + h)^{k\cdot t_2} - (h)^{k\cdot t_2} =: \nu(h) > (\rho(\widetilde{z_i}, \widetilde{y_i}))^{k\cdot t_2} \equiv \nu(0) \tag{2.11}$$

since $\nu(h)$ is an increasing function. So, we have proved (2.4) for $j < 2^{n_1+n_2+n_3}$.

The argument now follows by induction, using the above approach and formulae similar to (2.10) and (2.11), since, by (2.2) and (2.6), for odd k, $\Psi_{2^{n_1+n_2+\cdots+n_k}} = \beta_k \circ \alpha_k$, where α_k and β_k are obtained when in (2.6), index 2 is replaced by $k-1$. Blocks γ_k, γ_k' are obtained from β_k using θ_{k+1} and θ_{k+1}^{-1} instead of θ_3 and θ_3^{-1}, respectively. □

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Li–Yorke sensitive and weak mixing dynamical systems

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ABSTRACT

Akin and Kolyada in 2003 [E. Akin, S. Kolyada, *Li–Yorke sensitivity*, Nonlinearity 16 (2003), pp. 1421–1433] introduced the notion of Li–Yorke sensitivity. They proved that every weak mixing system (X, T), where X is a compact metric space and T a continuous map of X is Li–Yorke sensitive. An example of Li–Yorke sensitive system without weak mixing factors was given in [M. Čiklová, *Li–Yorke sensitive minimal maps*, Nonlinearity 19 (2006), pp. 517–529] (see also [M. Čiklová-Mlíchová, *Li–Yorke sensitive minimal maps II*, Nonlinearity 22 (2009), pp. 1569–1573]). In their paper, Akin and Kolyada conjectured that every minimal system with a weak mixing factor, is Li–Yorke sensitive. We provide arguments supporting this conjecture though the proof seems to be difficult.

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1. Introduction

A *topological dynamical system* (X, T) is a compact metric space (X, ρ) endowed with a continuous surjective map $T : X \to X$. Denote by T^n the nth iterate of T, $n \geq 0$. Points $x, y \in X$ are *proximal*, or *δ-asymptotic* (with $\delta \geq 0$), or *distal* if $\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) = 0$, $\limsup_{n\to\infty} \rho(T^n(x), T^n(y)) \leq \delta$, or $\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) > 0$, respectively; instead of 0-asymptotic we say *asymptotic*. A system (X, T) is *distal* if all pairs of points $x \neq y$ are distal. A map $T : X \to X$ is *Li–Yorke sensitive*, briefly *LYS* or *LYS$_\varepsilon$*, if there is an $\varepsilon > 0$ with the property that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is *proximal* but not *ε-asymptotic*, i.e. if

$$\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) = 0, \quad \text{and} \quad \limsup_{n\to\infty} \rho(T^n(x), T^n(y)) > \varepsilon. \tag{1}$$

Every pair $(x, y) \in X \times X$ satisfying (1) is an *ε-Li–Yorke pair*. A set $S \subseteq X$ such that any points $x \neq y$ in S satisfy (1) is an (ε-)*scrambled set*. A map T is *Li–Yorke chaotic*, briefly *LYC* or *LYC$_\varepsilon$* if it has an uncountable ε-scrambled set, for some $\varepsilon > 0$.

A system (X, T) is *transitive* if for every pair of open, nonempty subsets $U, V \subset X$ there is a positive integer n such that $U \cap T^{-n}(V) \neq \emptyset$, or equivalently, if there is a *transitive point*, i.e. a point $x \in X$ having a dense orbit $\{T^n(x)\}_{n=0}^{\infty}$; it is *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive; it is *minimal* if every point $x \in X$ is transitive. Finally, a system (Y, S) is a *factor* of (X, T) if there is a surjective continuous map $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. In this case we say that (X, T) is an *extension* of (Y, S). A *skew-product*

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system is a system $(X \times Y, F)$ where X, Y are compact metric spaces, and F a continuous map such that $F(x,y) = (f(x), g_x(y))$, for every $x \in X, y \in Y$. Other notions will be defined later, or can be found in [1] or in related papers listed in references.

The notion of *LYS* was introduced and studied by Akin and Kolyada [2]. It turns out that such systems are related to weak mixing systems. For example, every nontrivial weak mixing system is *LYS*. Therefore, Akin and Kolyada stated in [2] five conjectures concerning *LYS* systems. Three of them were disproved in [5] and [6]. In particular, it was proved that a minimal *LYS* system need not have a nontrivial weak mixing factor, and that a minimal system with a nontrivial *LYS* factor need not be *LYS*. The remaining two open problems are the following:

(P1) Is every minimal system with a nontrivial weak mixing factor *LYS*?
(P2) Does *LYS* imply *LYC*?

Both problems seem to be difficult but, in contrast to the preceding ones, it seems that the answer is in both cases positive. In this paper we give partial solutions. Recall that the only known result related to (P1) is the following

Theorem 1.1 (See [2,3]): *If (X, T) is minimal weak mixing and (Y, S) minimal and distal then $(X \times Y, T \times S)$ is minimal and LYS.*

We generalize it to some skew-product extensions of the original system in Theorems 2.5, 3.1 which represent the main results of this paper. Then, in Theorem 4.4 we show that the restriction to skew-product extensions is not too limiting. Finally, our last result, Theorem 5.1, essentially diminishes the possible class of systems which may not satisfy (P2). For convenience, we recall several known results which will be of use in the next sections.

Lemma 1.2 (See [8]): *If (X, T) is a minimal system then, for every open set $G \subseteq X$ there is an open set $H \subseteq X$ such that $H \subseteq T(G) \subseteq \overline{H}$.*

Lemma 1.3 (See [7]): *Let X be a complete separable metric space without isolated points. If $R \subseteq X \times X$ is a symmetric relation with the property that for each $x \in X$, $R(x) = \{y \in X; (x,y) \in R\}$ contains a dense G_δ subset, then there is a dense uncountable set $D \subseteq X$ such that $D \times D \setminus \Delta \subset R$, where Δ is the set of pairs (x,x), $x \in X$.*

The following is a topological version of the Fubini Theorem.

Lemma 1.4 (See [1] or [7]): *Let R be a relation on a complete separable metric space X which contains a dense G_δ subset of $X \times X$. Then there is a dense G_δ set $A \subseteq X$ such that for each $x \in A$, there exists a dense G_δ set $X_x \subseteq X$ with $\{(x,y); x \in A, y \in X_x\} \subseteq R$.*

Lemma 1.5 (See [7] and [2]): *If (X, T) is LYS then, for some $\delta > 0$, the set of δ-asymptotic pairs is a first category subset of $X \times X$.*

2. Minimal finite-type skew-product extensions of weak mixing systems

Lemma 2.1: *If (X, T) is minimal weak mixing, then for every $x \in X$ the set* $\mathrm{Tran}(x) \subset X$ *of points y such that (x, y) is a transitive point with respect to $T \times T$, is a dense G_δ set.*

Proof: Let $x_0 \in X$ be given. The set $\mathrm{Tran}(x_0)$ of points $y \in X$ such that (x_0, y) is a transitive point of $T \times T$, is a G_δ set since it is the intersection of two G_δ sets, the set of transitive points $(x, y) \in X \times X$, and $\{x_0\} \times X$. So it suffices to show that $\mathrm{Tran}(x_0)$ is dense in X. Let $\{G_n\}_{n\geq 1}$ be a base of open sets for $X \times X$ of the form $G_n = I_n \times J_n$, where I_n, J_n are open sets. Let $U_0 \subset X$ be nonempty open. By induction, there are nonempty open sets $U_0 \supset U_1 \supset U_2 \supset \cdots$, and a sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\overline{U}_j \subset U_{j-1},\ T^{n_j}(x_0) \in I_j \quad \text{and} \quad T^{n_j}(U_j) \subset J_j,\ j \in \mathbb{N}. \tag{2}$$

Indeed, since T is minimal, there is a $k_1 > 0$ such that, for every j, there is an s, $0 \leq s < k_1$, with $T^{j+s}(x_0) \in I_1$. Since T is weak mixing, the set $N(U_0, J_1)$ of times i such that $T^i(U_0) \cap J_1 \neq \emptyset$, contains arbitrarily long blocks of successive integers. It follows that there is an n_1 such that $T^{n_1}(x_0) \in I_1$ and $T^{n_1}(U_0) \cap J_1 \neq \emptyset$. Since T is minimal and U_0 open, $T^{n_1}(U_0)$ has nonempty interior (see Lemma 1.2) and hence $T^{n_1}(U_0) \cap J_1$ contains a nonempty open set H. It suffices to take for U_1 a nonempty open set such that $\overline{U}_1 \subset T^{-n_1}(H)$. Thus, we have n_1 and U_1 satisfying (2) (for $j = 1$). Next we apply the above process with U_0 replaced by U_1, G_1 by G_2, obtaining $U_2 \subset U_1$ and $n_2 > n_1$, etc. This proves (2).

To finish the argument put $Y = \bigcap_{j\geq 1} U_j = \bigcap_{j\geq 1} \overline{U}_j$. Then Y is a nonempty G_δ set and, by (2), for every $y \in Y \subset U_0$, (x_0, y) is a transitive point. □

Let (X, T) be a minimal weak mixing topological dynamical system. Let A be a finite space with discrete topology, $Y = X \times A$ with the max-metric, and (Y, S) a skew-product extension of (X, T) such that $S(t, a) = (T(t), G_t(a))$, where every fibre map G_t is a bijection of A. The following terminology and notation will be useful. For $\xi > 0$ let $\Delta_\xi \subseteq X \times X$ be the set of pairs (x, y) such that $\rho(x, y) < \xi$. For every $z = (x, y) \in X \times X$ and $i \in \mathbb{N}$, denote by $(T \times T)^i(z) = (x_i, y_i)$ the ith iterate of z, with $x_0 := x$, $y_0 := y$. Let $g_0 = h_0 = Id$, the identity and, for $i > 0$ let $g_i = G_{x_{i-1}} \circ G_{x_{i-2}} \circ \cdots \circ G_{x_0}$, similarly let h_i be the composition of $i - 1$ corresponding maps G_{y_j}, and let $c_i := (g_i, h_i)$. For $\eta > 0$ let $N = N(x, y, \eta) = \{i \in \mathbb{N};\ (x_i, y_i) \in \Delta_\eta\}$. The sequence $\{c_i\}_{i\in N}$ is the *η-characteristic sequence* of (x, y). Let $j_0 < j_1 < \cdots$ be the numbers in N. A finite string $c_{j_0}, c_{j_1}, \ldots, c_{j_{k-1}}$ is an *η-saturated chain for (x, y) of length k* if the string contains all members of the η-characteristic sequence of (x, y); we denote it as $M(x, y, \eta)$, and we let $C(x, y, \eta)$ denote the set of elements in $M(x, y, \eta)$. Notice that we do not determine uniquely the length of a saturated string: if $M(x, y, \eta) = \{c_{j_0}, \ldots, c_{j_{k-1}}\}$ then $\{c_{j_0}, \ldots, c_{j_{k-1}}, c_{j_k}\}$ is also saturated string. When dealing with an another pair, (x', y'), we use primes to distinguish the related symbols like $x'_i, y'_i, j'_i, c'_{j'_i}, k'$, etc. By the continuity there is an $\varepsilon > 0$ such that $u, v \in \Delta_\varepsilon$ implies $G_u = G_v$. Again by continuity, for every saturated chain $M(x, y, \eta)$ there is an open neighborhood $U(x, y, \eta)$ of (x, y) such that every pair $(x', y') \in U(x, y, \eta)$, $\{(x'_i, y'_i)\}_{0\leq i\leq j_{k-1}}$ ε-traces $\{(x_i, y_i)\}_{0\leq i\leq j_{k-1}}$ so that the distance between (x_i, y_i) and (x'_i, y'_i) is less than η and $(x'_i, y'_i) \in \Delta_\eta$ for $i \in \{j_0, j_1, j_2, \ldots j_{k-1}\}$. In particular, $M(x, y, \eta) = M(x', y', \eta)$.

Saturated strings $M(x, y, \eta), M(x', y', \eta')$ with $\eta' \leq \eta$ of two *transitive* pairs (x, y) and (x', y') of length k and k', respectively, can be joined in a single chain of (x, y) in the following sense. Since (x, y) is transitive, there is an $n \geq j_{k-1}$ such that $(x_n, y_n) \in U(x', y', \eta')$. It follows that $n = j_s$ for some $s \geq k - 1$, and the trajectory $\{(x_{n+i}, y_{n+i})\}$ traces the trajectory $\{(x'_i, y'_i)\}$ for $i \in [0, j'_{k'-1}]$, remaining within distance $\eta' \leq \eta$ for

$i = j'_l, 0 \le l \le k' - 1$. The resulting string is obtained from $M(x, y, \eta)$ and we denote it as $M(x, y, \eta) * M(x', y', \eta')$. Its length is $s + k'$. Thus, we have the following

Lemma 2.2: *Let (X, T) be a minimal weak mixing topological dynamical system. Let A be a finite space with discrete topology, $Y = X \times A$ with the max-metric, and (Y, S) a skew-product extension of (X, T) such that $S(t, a) = (T(t), G_t(a))$, where every fibre map G_t is a bijection of A. Let $\eta \ge \eta' > 0$, and $M(x, y, \eta), M(x', y', \eta')$ be saturated strings of transitive pairs, of length k and k', respectively. Then*

- *(i) The string $M(x, y, \eta) * M(x', y', \eta')$ need not be saturated for η or η', but it is obtained from $M(x, y, \eta)$ of sufficiently hight length by omitting some elements;*
- *(ii) $C(x, y, \eta) \supseteq C(x', y', \eta') \circ c_{j_s}$, where j_s is specified above (c_{j_s} is the element "connecting"both saturated strings), and $\{f, g\} \circ h$ means $\{f \circ h, g \circ h\}$.*

Lemma 2.3: *Let (X, T) be a minimal weak mixing topological dynamical system. Let A be a finite space with discrete topology, $Y = X \times A$ with the max-metric, and (Y, S) a skew-product extension of (X, T) such that $S(t, a) = (T(t), G_t(a))$, where every fibre map G_t is a bijection of A. Let $(x, y), (x', y')$ be transitive pairs. Then*

- (i) *$\#C(x, y, \eta) = \#C(x', y', \eta)$ for any $\eta > 0$;*
- (ii) *there is a $\xi > 0$ such that if $(x, y), (x', y') \in \Delta_\xi$ and $0 < \eta, \eta' < \xi$ then $\#C(x, y, \eta) = \#C(x', y', \eta')$.*

***Proof*:**

- (i) Assume $m := \#C(x, y, \eta) < m' := \#C(x', y', \eta)$. Then $\#C(x', y', \eta) = m'$ and, by Lemma 2.2(ii), $C(x, y, \eta)$ contains m' distinct elements, a contradiction.
- (ii) We may assume $\eta' < \eta$. Then obviously $\#C(x, y, \eta') \le \#C(x, y, \eta)$. Since a collection of permutations of A is finite, there is a $\xi > 0$, and $m_0 \ge 1$ such that $\#C(x, y, \eta) = m_0$ whenever $\eta < \xi$. To finish apply (i).

□

Lemma 2.4: *Let ξ be as in Lemma 2.3, and $(x, y) \in \Delta_\xi$ be a transitive pair, and $\eta' < \eta := \xi$. If $c_{j_s} \in C(x, y, \eta)$ is the element connecting the strings $M(x, y, \eta)$ and $M(x, y, \eta')$, then $C(x, y, \eta') \circ c_{j_s}$ contains the identity map (Id, Id).*

***Proof*:** Let x, y be as in the hypothesis. By Lemma 2.2, $M(x, y, \eta) * M(x, y, \eta')$ contains the string $M(x, y, \eta') \circ c_{j_s}$. But $M(x, y, \eta') \circ c_{j_s}$ must contain the identity. To see this note that, by definition, the first member c_0 of the η-characteristic sequence is the identity. Since $\rho(x, y) < \eta$ ($= \xi$, c_0 is also the first member of $M(x, y, \eta)$, i.e. $j_0 = 0$. By Lemma 2.3 (ii), $C(x, y, \eta)$ has the same cardinality as $C(x, y, \eta')$ hence as $C(x, y, \eta') \circ c_{j_s}$, since c_{j_s} is a bijection. Thus $C(x, y, \eta') \circ c_{j_s}$ contains the identity map (Id, Id). □

Theorem 2.5: *Let (X, T) be a minimal weak mixing topological dynamical system. Let A be a finite space with discrete topology, $Y = X \times A$ with the max-metric, and (Y, S) a skew-product extension of (X, T) such that $S(t, a) = (T(t), G_t(a))$, where every fibre map G_t is a bijection of A. Then (Y, S) is LYS_ε for any $0 < \varepsilon < \mathrm{diam}(X)$.*

***Proof*:** Let $x \in X$ and U be an arbitrary neighborhood of x. Assume that $\xi > 0$ is as in Lemma 2.3. By Lemma 2.1 there is a point $y \in U$ such that $(x, y) \in \Delta_\xi$ is transitive with respect to $T \times T$. Since Y is equipped with the max-metric it suffices to prove that if $a \in A$ then, for every $0 < \eta' < \xi$, $((x,a)(y,a))$ is an η'-proximal pair. This follows by Lemma 2.4. □

Theorem 2.6: *Let (X,T) be minimal weak mixing, and $A \neq \emptyset$ a finite metric space. Let S be a skew-product map of $X \times A$. Then, for every $0 < \varepsilon < \mathrm{diam}(X)$, S is LYC$_\varepsilon$.*

***Proof*:** Let $V \subset X$ be an open set such that $V \times V \subset \Delta_\xi$, where $\xi > 0$ is as in Lemma 2.3. By the proof of Theorem 2.5, for arbitrary $a \in A$ and a transitive pair $(x,y) \in \Delta_\xi$ with respect to $T \times T$, a pair $\big((x,a),(y,a)\big)$ is ε-Li–Yorke in $X \times A$. To finish the argument, fix an $a \in A$, and let $Y := V \times \{a\}$. Denote by R the set of ε-Li–Yorke pairs (x,y) in $Y \times Y$. By Lemma 1.3 there is an uncountable dense scrambled set $D_a \subset Y$ such that every distinct points in D_a form an ε-Li–Yorke pair. □

3. Infinite type skew-product extensions of weak mixing systems

Theorem 2.5 and hence, Theorem 2.6 can be generalized to certain types of skew-product maps of $X \times A$, where (X,T) is minimal weak mixing, and A is infinite compact. As a sample we provide the following. Recall that an *adding machine* or *odometer* related to a sequence $p_1, p_2, \dots$ of primes is a system (X,τ), where $X = \prod_{j\geq 1} X_j$, $X_j = \{0,1,\dots,p_j - 1\}$, and $\tau(x_1x_2x_3\dots) = x_1x_2x_3\dots + 1000\dots$, when adding is modulo p_j at the jth position from the left to the right, see, e.g. [4]. Obviously X is a Cantor-type set. Let $\mathcal{A}$ be a collection of odometers. We say that elements of $\mathcal{A}$ are *synchronous* if there are naturals number $m_1 < m_2 < \dots$ such that for any odometer $(X,\tau) \in \mathcal{A}$ related to a sequence $p_1, p_2, \dots$ of primes and for any $j \in \mathbb{N}$ there is $l_j \in \mathbb{N}$ with $m_j = p_1 \cdot p_2 \cdot \dots \cdot p_{l_j}$.

Theorem 3.1: *Let (X,T) be minimal weak mixing, Y a Cantor-type set, $\mathcal{A}$ be a collection of synchronous odometers on Y and $S : X \times Y \to X \times Y$ a skew-product map, $S(x,y) = (T(x), R_x(y))$ such that, for every $x \in X$, R_x is an odometer in the class $\mathcal{A}$, or the identity. Then $(X \times Y, S)$ is LYS.*

***Proof*:** It follows by the following Lemma 3.2, and from Theorem 2.5 and Lemma 2.1. □

Lemma 3.2: *Let X, Y and S be as in Theorem 3.1. Then for every $\delta > 0$ there is an $m > 0$ such that, for every $x \in X$, $y \in Y$ and $k \in \mathbb{N}$, $|y - R_x^{km}(y)| < \delta$.*

***Proof*:** It suffices to show that, for every $\delta > 0$ and for every $x \in X$, there is a decomposition of Y into clopen portions $Y_1, Y_2, \dots, Y_m$ forming an R_x-periodic orbit, such that the diameter of every Y_j is less than δ. Assume the contrary. Then there is an increasing sequence $m_1 < m_2 < \cdots$ of positive integers, a sequence $x_1, x_2, \dots$ in X, and a sequence $Y_1, Y_2, \dots$ of clopen portions of Y such that, for every j, $\mathrm{diam}(Y_j) \geq \delta$ and Y_j is a periodic portion with respect to R_{x_j} of period m_j. Taking a subsequence if necessary, we may assume that $\lim_{j\to\infty} x_j = x_0$, and $\lim_{j\to\infty} Y_j = Y_0$. Then Y_0 is a compact portion of Y with "infinite"period, i.e. $R^n_{x_0}(Y_0)$ is disjoint from Y_0, for every $n > 0$, contrary to the assumption that R_{x_0} is an odometer, or the identity. □

4. Finite-type extensions and skew product systems

Here we show that the assumption in Theorem 2.5, that the corresponding map is a skew-product map on $Y \times \{1, 2, \dots, n\}$ is not too restrictive since, for certain but not all types of minimal weak mixing systems every n to one extension is a skew-product map, see Theorem 4.4. On the other hand not every finite type extension of a minimal weak mixing system is (conjugate to) a skew-product map. Recall (see, e.g. [9] for details) that a *continuum* is a nonempty connected compact metric space. A continuum X is *unicoherent* if for every two continua A, B with $A \cup B = X$ the set $A \cap B$ is connected.

Lemma 4.1: *Let (X, ρ) be a compact metric space such that every connected component of X is nowhere dense in X. Then, for every $\delta > 0$ there is a finite decomposition $X_1 \cup X_2 \cup \cdots \cup X_m$ of X into disjoint compact subsets such that, for every j, X_j is a subset of the δ-neighborhood of a connected component of X.*

***Proof*:** Let X_δ be the union of connected components of X with diameter $\geq \delta$. Then X_δ is a closed set. Indeed, let $x_n \in X_\delta$ be such that $\lim_{n\to\infty} x_n = x$. Then there are connected components $K_n \subseteq X_\delta$ such that $x_n \in K_n$, for every n. Since the set of nonempty compact subsets of X, with the Hausdorff metric ρ_H, is a compact set, we may assume that there is a compact set $K \subset X$ such that $\lim_{n\to\infty} \rho_H(K_n, K) = 0$. Since $x \in K$, it suffices to show that K is a connected component of X with $K \subset X_\delta$. Obviously, $\operatorname{diam}(K) \geq \delta$. To show that K is connected, assume the contrary. Then there are disjoint closed sets G, H with $K = G \cup H$ such that $K \cap G \neq \emptyset \neq H \cap K$. Let G', H' be disjoint closed neighborhoods of G and H, respectively. Then, for every sufficiently large n, $K_n \subseteq G' \cup H'$, $K_n \cap G' \neq \emptyset \neq K_n \cap H'$ which is a contradiction. Thus, X_δ is a closed set.

Next we show that for every connected component $K \subset X_\delta$ there is a compact neighborhood $U(K) = U$ of K such that $X \setminus U$ is compact, and U is contained in the open δ-neighborhood V of K. Since K is a component, for every $x \in X \setminus V$ there is a decomposition of X into disjoint compact sets G_x, H_x such that G_x is a neighborhood of x and $K \subseteq H_x$. Since $X \setminus V$ is compact, there is a finite cover $G_{x_1} \cup G_{x_2} \cup \cdots \cup G_{x_k} = X \setminus V$. Take $U(K) = H_{x_1} \cap \cdots \cap H_{x_k}$.

To finish the proof it suffices to take a finite cover $W = W_1 \cup W_2 \cup \cdots \cup W_s$ of X_δ consisting of disjoint compact sets such that every W_j is a set $U(K_j)$ with $K_j \subseteq X_\delta$ a connected component. Then $X \setminus W$ is a compact set which can be divided into finitely disjoint compact sets with diam $\leq \delta$. □

We say that a set A in a metric space is *δ-separated* if $\rho(u, v) \geq \delta$ for every distinct $u, v \in A$.

Lemma 4.2: *Let X, Y be compact metric spaces, $n > 0$ an integer, and $\pi : X \to Y$ a continuous map such that, for every $y \in Y$, $\#\pi^{-1}(y) = n$. Assume that (Y, S) is a minimal system. Then there is a $\delta_0 > 0$ such that every set $\pi^{-1}(y)$ is δ_0-separated.*

***Proof*:** For $\delta > 0$ let Y_δ be the set of $y \in Y$ such that $\pi^{-1}(y)$ is δ-separated. Then Y_δ is a compact set. Indeed, let $y_j \in Y_\delta$ such that $\lim_{j\to\infty} y_j = y_0$. Since the space of nonempty compact subsets of X, equipped with the Hausdorff metric ρ_H, is a compact space there is a subsequence $j_1 < j_2 < \cdots$ and a set $A \subset X$ such that $\lim_{k\to\infty} \rho_H(\pi^{-1}(y_{j_k}), A) = 0$. By the continuity, $\pi(A) = y_0$ and $\#A = n$. Hence $A = \pi^{-1}(y_0)$ is δ-separated, i.e. $y_0 \in Y_\delta$. Since every $\pi^{-1}(y)$ is finite, $\bigcup_{j>0} Y_{1/j} = Y$. By the Baire category theorem there is a

$k > 0$ such that $Y_{1/k}$ has nonempty interior. Since Y is minimal, there is an $m > 0$ such that $\bigcup_{0 \le j \le m} S^{-j}(Y_{1/k}) = Y$. By the continuity of S there is a $\delta_0 > 0$ such that $S^{-j}(Y_{1/k}) \subset Y_{\delta_0}$, $0 \le j \le m$. Consequently, $Y = Y_{\delta_0}$. □

Lemma 4.3: *Let (Y, S) be a factor of (X, T), with factor map $\pi : X \to Y$. Assume that (Y, S) is minimal, weak mixing, not connected, and such that every subcontinuum of Y is unicoherent. Finally, let $n > 0$ be an integer such that, for every $y \in Y$, $\#\pi^{-1}(y) = n$. Then (X, T) is conjugate to a skew-product map $F : Y \times N \to Y \times N$, where $N = \{1, 2, \dots, n\}$.*

***Proof*:** By Lemma 4.2 there is a $\delta_0 > 0$ such that $\pi^{-1}(y)$ is δ_0-separated, for every $y \in Y$. Let $0 < \eta < \delta_0/3$ be such that, for every $u, v \in X$, $\rho(u, v) < 2\eta$ implies $\rho(T(u), T(v)) < \delta_0/3$. Since (Y, S) is weak mixing and not connected every connected component of (Y, ρ) is nowhere dense. By Lemma 4.1, there is a finite decomposition $Y_1 \cup Y_2 \cup \cdots \cup Y_m$ of Y into disjoint compact sets such that Y_j is contained in the η-neighborhood of a connected component P_k of Y. Let $p_k \in P_k$ and let U_0 be a compact η-neighborhood of p_k. Define a continuous map $\psi_k : \pi^{-1}(U_0 \cap P_k) \to (U_0 \cap P_k) \times N$. If $Y_k \subset U_0$ extend ψ_k continuously onto $\pi^{-1}(Y_k)$; by the choice of η this extension is uniquely determined by ψ_k restricted to $\pi^{-1}(U_0 \cap P_k)$. Otherwise take U_1 the compact η-neighborhood of U_0 and extend ψ_k continuously to a map $\pi^{-1}(U_1 \cap P_k) \to (U_1 \cap P_k) \times N$, etc. Since P_k is compact after finite number of steps ψ_k is continuously extended onto $\pi^{-1}(P_k)$ such that $\psi_k(\pi^{-1}(P_k)) = P_k \times N$. Since P_k is unicoherent continuum, this extension is uniquely determined by ψ_k on $\pi^{-1}(U_0 \cap P_k)$. Finally, by the choice of η, ψ_k can be continuously (and uniquely) extended onto $\pi ?^{-1}(Y_k)$. To finish the argument take $\psi = \psi_1 \cup \cdots \cup \psi_m$ which is a continuous bijective map $X \to Y \times N$, and take $F = \psi \circ T \circ \psi^{-1}$. □

Theorem 4.4: *Let (Y, S) be minimal, weak mixing, not connected, and such that every subcontinuum of Y is unicoherent. Let $n > 0$ be an integer, and let (X, T) be an extension of (Y, S) such that $\#\pi^{-1}(y) = n$ for every $y \in Y$. Then (X, T) is LYS.*

***Proof*:** It follows by Lemma 4.3 and Theorem 2.5. □

5. Li–Yorke sensitivity and Li–Yorke chaos

In [2] there is a problem whether *LYS* implies *LYC*; the converse implication obviously is not true. Here we show that under some additional conditions, the answer is positive. This significantly restricts the class of systems (X, T) for which the implication need not hold. To simplify the argument, we will use the following notation. Given a system (X, T) denote by *Dist* the set of distal pairs $(x, y) \in X \times X$, and by $Asym_\varepsilon$ the set of ε-asymptotic pairs (x, y) in $X \times X$.

Theorem 5.1: *Let (X, T) be LYS. Assume there is a non-empty open set $H \subset X$ such that $(H \times H) \cap Dist$ has empty interior or (equivalently) that $(H \times H) \cap Dist$ is a set of the first Baire category. Then there is an $\varepsilon > 0$ such that (X, T) is LYC_ε.*

***Proof*:** By Lemma 1.5 there is an $\varepsilon > 0$ such that T is LYS_ε, and $Asym_\varepsilon$ is a first category set. It is easy to see that *Dist* and $Asym_\varepsilon$ are F_σ sets hence, by the Baire category theorem, $(H \times H) \cap Dist$ is of the first category if and only if it has the empty interior. Assume $(H \times H) \cap Dist$ is a first category set and put $L = X \times X \setminus (Dist \cup Asym_\varepsilon)$. Then L is a G_δ set dense in $H \times H$. By Lemmas 1.4 and 1.3, there is an uncountable set $D \subset H$ such that $D \times D \setminus \Delta \subset L$. Obviously, L is the set of ε-Li–Yorke pairs in $X \times X$. Hence, D is an ε-scrambled set for (X, T) and hence, T is LYC_ε. □

Remark 1: For a minimal (X, T) which is both LYS and LYC, the set $Dist$ can be very large. In [5] there is an example of such a system, even without a weak mixing factor such that the set $Dist$ contains an open dense subset of $X \times X$.

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