SLEZSKÁ UNIVERZITA V OPAVĚ
MATEMATICKÝ ÚSTAV V OPAVĚ

Invarianty v teorii relativity a kalibrační teorii

Habilitační práce

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2019

SILESIAN UNIVERSITY IN OPAVA
MATHEMATICAL INSTITUTE IN OPAVA

Invariants in Relativity and Gauge Theory

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2019

Contents

0 Introduction

In mathematical physics, one of the goals in the study of *classical field theory* (essentially, a variational partial differential equation (PDE)) with *gauge symmetries* (a.k.a *gauge theories*) is a precise and rigorous construction of the corresponding *reduced phase space* [6, 12, 22, 23]: the *space of solutions*, endowed with the *canonical Poisson structure*, and *quotiented* by the gauge transformations. A large number of technical problems stands in the way, including but not limited to describing the solutions of a non-linear PDE as an infinite dimensional space with some kind of smooth structure, specifying a sufficiently regular class of functions on this space on which the Poisson structure is well-defined, explicitly describing the structure of the quotient, or alternatively the structure of the functions invariant under gauge symmetries. The functions in the latter class are referred to as *gauge-invariant observables* (or *invariants*).

The scope of this Habilitation Thesis is to address some purely geometric problems that arise in the study of gauge-invariant observables and can be attacked using tools from the theory of *differential invariants* [30] and the theory of *formal integrability* of *overdetermined PDEs* [31, 35]. The main focus is on General Relativity (GR) as a non-trivial representative example, but the perspective is such that the tools used would also apply to other gauge theories, of which Electrodynamics, Yang-Mills, Chern-Simons, Supergravity and many other models used in fundamental theoretical physics and geometry [12], are prominent examples. The papers collected in this Thesis consist of [25], [27], [26], [10], [28], [29], all of which have been published with the exception of [29], which has been submitted for publication to *Communications in Mathematical Physics*.

In the remainder of the Introduction, we give a brief summary of relevant geometric notions (Section 0.1) and summarize the main problems addressed and results obtained in the above papers, grouped by theme in Sections 0.2, 0.3 and 0.4.

Acknowledgments. I would like to thank many colleagues with whom over the years I have had fruitful and helpful interactions that have influenced the contents of this Thesis: Glenn Barnich, Marco Benini, Romeo Brunetti, Mauro Carfora, Claudio Dappiaggi, Chris Fewster, Markus Fröb, Alfonso García-Parrado, Joseph Krasil'shchik, Boris Kruglikov, Benjamin Lang, Wouter von Limbek, Valentin Lychagin, Valter Moretti, Pedro Lauridsen Ribeiro, Vladimir Rubtsov, Miguel Sánchez, Ko Sanders, Werner Seiler, Alexander Schenkel, Urs Schreiber, Yuko Urakawa, Bob Wald, Alexander Verbovetsky, Willie Wong, Michał Wrochna, Jochen Zahn. A special thanks goes to Artur Sergyeyev and Michal Marvan for their encouragement during the preparation of the Thesis.

0.1 Geometric background

We will be mostly working with smooth finite dimensional manifolds and smooth functions between them, where smooth means C^∞ differentiable. Any references to infinite dimensional manifolds may be made precise by reference to the theory from [32].

0.1.1 Jets and PDEs

The main references for this section can be taken as [31, 2, 35].

Given a smooth bundle $B \to M$, the corresponding *k-jet bundle* $J^k B \to M$ is a smooth bundle, where a point in the fiber over $x \in M$ can be identified with an equivalence class of sections $\Gamma(B) \ni \beta \colon M \to B$ whose Taylor series at x agree up to order k in a fixed coordinate system (a coordinate invariant condition). Setting $k = \infty$, we get the *(infinite) jet bundle* whose fibers are infinite dimensional (Fréchet manifolds modelled on $\mathbb{R}^\infty$). The jet bundles are natural with respect to smooth bundle maps $f \colon B \to B'$ over M, which induce the bundle maps $j^k f \colon J^k B \to J^k B$, and are equipped with natural obvious projections $J^k B \to J^l B$ for $k \geq l$, with $J^0 B = B$. The topology and smooth structure of $J^\infty B$ are such that any smooth function on it is locally (on some open neighborhood of any given point $j^\infty_x \beta \in J^\infty B$) a function of only finitely many coordinates, or equivalently is a pullback from a corresponding locally defined function on $J^k B$ with k possibly depending on the given point $j^\infty_x \beta$ and the neighborhood.

When $V \to M$ is a vector bundle, then $J^\infty V \to M$ is naturally also a vector bundle. But in the general case, $J^{k+1} B \to J^k B$ is only an affine bundle modeled on the vector bundle $T_V B \otimes_B S^{k+1} T^* M \to B$ (pulled back to $J^k B$, of course), where T_V denotes the vertical tangent bundle and $S^\bullet$ denotes the symmetric tensor product.

For any section $\beta \in \Gamma(B)$, the *jet extended* section $j^k \beta \colon M \to J^k B$ assigns $x \mapsto j^k_x \beta$. The jet extension $j^\infty \colon \Gamma(B) \to \Gamma(J^\infty B)$ defines a universal differential operator, in the sense that for any (non-linear) differential operator $D \colon \Gamma(B) \to \Gamma(B')$, there exists a unique bundle map denoted (slightly abusing notation) by $D \colon J^\infty B \to B'$ such that $D[\beta] = D(j^\infty \beta)$. For a linear differential operator $L \colon \Gamma(V) \to \Gamma(V')$, the corresponding $L \colon J^\infty V \to V'$ is also linear. The composition $D_2 \circ D_1$ of two differential operators, say $D_1 \colon \Gamma(B) \to \Gamma(B')$ and $D_2 \colon \Gamma(B') \to \Gamma(B'')$, is again a differential operator. Let us denote the corresponding jet bundle map by $D_2 \circ D_1 \colon J^\infty B \to B''$. It is not strictly speaking a composition of the corresponding bundle maps $D_1 \colon J^\infty B \to B'$ and $D_2 \colon J^\infty B' \to B''$, but the latter two maps are anyway not directly composable. Hence there should be no confusion in this notation.

Jet extended sections $j^\infty \beta \in \Gamma(J^\infty B)$ are also called *holonomic*, to distinguish the from those sections $\beta^\infty \Gamma(J^\infty B)$ for which $\beta^\infty \neq j^\infty \beta$ for any $\beta \in \Gamma(B)$. The linear span of the tangent spaces of all holonomic sections define a distribution $\mathcal{C} \subset T J^\infty B$ known as the *Cartan* (alternately *contact*) distribution. Any maximal integral leaf of the Cartan distribution is necessarily the graph of some section.

There are two ways to define PDEs, via differential operators and via submanifolds. Given a compound bundle $V \to B \to M$, where $V \to B$ is a vector bundle, and a differential operator $E \colon J^\infty B \to V$ that covers the identity morphism $B \to B$, the *equation* $E[\beta] = 0_V$, where $0_V \in \Gamma(V \to B)$ is the zero section, defines the *PDE "submanifold"* $\mathcal{E} = \{\beta^\infty \in J^\infty B \mid E(\beta^\infty) = 0_V\}$. The condition $j^\infty \beta \subset \mathcal{E}$ is then equivalent to the equation $E[\beta] = E(j^\infty \beta) = 0_V$. Of course, as defined, $\mathcal{E}$ need not be a submanifold (level sets of smooth functions may have singularities). But when $\mathcal{E} \subset J^\infty B$ is a submanifold and itself a smooth bundle $\mathcal{E} \to M$, and in addition the Jacobian of $E[\beta]$ is of maximal rank on $\mathcal{E}$, the corresponding equation $E[\beta] = 0_V$ is called *regular*. When $\mathcal{E}$ is not a submanifold (it could even be an arbitrary subset), the condition $j^\infty\beta \subset \mathcal{E}$ is called a *partial differential relation* [20]. The same PDE submanifold $\mathcal{E} \subset J^\infty B$ can be specified by more than one differential operator E, or specified independently from any such differential operator. Hence the PDE submanifold description is in a way a more intrinsic way to define a PDE, as opposed to using a specific equation.

On the other hand, the equation approach has some advantages too. It is well-known that just because $\beta^\infty_x \in \mathcal{E}_x$, there is no guarantee that there exists an actual solution section $\beta \in \Gamma(B)$ (even locally), such that $j^k\beta(x) = \beta^\infty_x$ and $j^\infty\beta(M) \subset \mathcal{E}$. An obvious set of necessary conditions on β^∞_x are the differential consequences $p^\infty E(\beta^\infty_x) = j^\infty 0_V = 0_{J^\infty V}$, where on the left-hand side the *infinite prolongation* of E is defined to satisfy $(p^\infty E)[\beta] = j^\infty[E[\beta]]$. The corresponding PDE submanifold is $\mathcal{E}^\infty = \{\beta^\infty \in J^\infty B \mid p^\infty E(\beta^\infty) = 0_{J^\infty V}\}$ is the *infinite prolongation* of E. When $\mathcal{E}^\infty$ is itself regular, the corresponding PDE is said to be *formally integrable.* The prolongation from $\mathcal{E}$ to $\mathcal{E}^\infty$ can be carried out without a presenting differential operator E, but is more cumbersome to describe. When considering a vector bundle $V \to M$, instead of a general smooth bundle $B \to M$, all the compound bundles collapse to bundles over M.

A vector field $\xi^\infty \in \mathfrak{X}(J^\infty B) = \Gamma(TJ^\infty B \to J^\infty B)$ is *vertical* if it is tangent to the fibers of $J^\infty B \to M$. A general bundle does not have a natural notion of a horizontal vector field, but on the infinite jet bundle we define *horizontal* vector fields to be those $\eta \in \mathfrak{X}(J^\infty B)$ that are everywhere tangent to the Cartan distribution $\mathcal{C} \subset TJ^\infty B$. Any vector field on $J^\infty B$ then uniquely decomposes into its horizontal and vertical parts. Any vector field that preserves the Cartan distribution is called a *$\mathcal{C}$-vector field* or a *contact vector field.* It is a non-trivial but basic result of jet bundle theory, that the horizontal (a.k.a *total horizontal*) and vertical (a.k.a *evolutionary*) parts of a $\mathcal{C}$-vector field satisfy special properties. Namely, any total horizontal vector field is a linear combination of horizontal lifts of vector field on M, while an evolutionary vector field is the prolongation of a differential operator $J^\infty B \to T_V B$, called the *generator.* That is, if $\xi\colon J^\infty B \to T_V B$ is the *generator* then its evolutionary *prolongation* ξ^∞ acts as $\mathcal{L}_{\xi^\infty} j^\infty\beta = j^\infty\xi[\beta]$, with every evolutionary vector field being of that form.

Given a PDE submanifold $\mathcal{E} \subset J^\infty B$, a *(local) symmetry* of $\mathcal{E}$ is a $\mathcal{C}$-vector field on $J^\infty B$ that is tangent to $\mathcal{E}$. It so happens that a total horizontal vector field is *always* a symmetry of *any* PDE submanifold. Hence local symmetries come in equivalence classes, where two $\mathcal{C}$-vector fields are considered equivalent if they differ by a total horizontal vector field. Obviously, any such equivalence class is uniquely represented by an evolutionary vector field. Thus, while looking for local symmetries of PDEs, it is sufficient to concentrate on evolutionary vector fields. If $\xi\colon J^\infty B \to T_V B$ generates an evolutionary symmetry of the PDE presented by $E[\beta] = 0$, then the fact that it is a symmetry on an open neighborhood of the graph of $j^\infty\beta_0(M)$ for some solution of $E[\beta_0] = 0_V$ is equivalently detected by the condition

$$\dot{E}[\beta; \xi[\beta]] = L[\beta; E[\beta]], \tag{1}$$

where $\dot{E}\colon J^\infty(B \times_B \times T_V B) \to V$ is the so-called *universal linearization* of the operator E (linear in its second argument) defined by the identity

$$E[\beta + \varepsilon\dot{\beta}] = E[\beta] + \varepsilon\dot{E}[\beta; \dot{\beta}] + O(\varepsilon^2) \tag{2}$$

for small ε (at least informally), and $L\colon J^\infty(B \times_B V) \to V$ is just some differential operator also linear in its second argument (though L might be well-defined only on some open neighborhood of the graph $j^\infty\beta_0(M)$, with possibly different L's defined for different solutions β_0). Since the property of being a symmetry is determined by the behavior of an evolutionary vector field ξ^∞ only on the PDE submanifold $\mathcal{E}$, its behavior away from $\mathcal{E}$ could be arbitrary. It could even happen that ξ^∞ vanishes when restricted to $\mathcal{E}$ even though it is not zero elsewhere, in which case we say that ξ^∞ is a trivial symmetry.

A *variational PDE* is a PDE that can be presented by an equation derived from an *action* or *variational principle*, which ostensibly is a (non-linear) functional

$$S = \int_M \tilde{L}[\beta], \tag{3}$$

where $\tilde{L}\colon J^\infty B \to B^*(\Lambda^n M)$ is a ($n = \dim M$)-form valued differential operator called the *Lagrangian (density)*, where we denoted by B^* the pullback of a bundle from M to B. That is, the corresponding PDE is the condition on β to be a critical point of S. One must be careful with interpreting the integral over M in the above formula when M is not compact, since the integral may not converge for arbitrary β. However, when everything is well-defined, the critical point condition ends up being a PDE $\mathrm{E}(\tilde{L})[\beta] = 0$ that is derived by purely local operations from the Lagrangian density $\tilde{L}[\beta]$, the so-called *Euler-Lagrange equation* of $\tilde{L}$. Hence, the Euler-Lagrange equations can be well-defined even if the above integral over M is not well-defined for every β. The Euler-Lagrange equations $\mathrm{E}(\tilde{L})\colon J^\infty B \to T_V^* B \otimes_B B^*(\Lambda^n M)$ are uniquely and invariantly defined by the identity,

$$\mathcal{L}_{\xi^\infty} \tilde{L}[\beta] = \xi[\beta] \cdot \mathrm{E}(\tilde{L})[\beta] + \mathrm{d}W[\beta; \xi[\beta]] \tag{4}$$

for some differential operator $W\colon J^\infty(B \times_M T_V B) \to B^*(\Lambda^{n-1} M)$ linear in its second argument, which must hold for any evolutionary vector field ξ^∞. On the other hand, it is easiest to write down in explicit adapted coordinates. Namely, let (x^i, u^a) be an adapted chart on $B \to M$ and $(x^i, u^a, u^a_i, u^a_{ij}, u^a_{ijk}, \ldots)$ be an induced adapted chart on $J^\infty B$, such that, e.g., $u^a_{ij}(j^\infty \beta(x)) = \partial_i \partial_j u^a(\beta(x))$. Then for $\tilde{L}[\beta](x) = \left(L(x^i, u^a, u^a_i, u^a_{ij}, \ldots)\, \mathrm{d}^n x\right)[\beta]$, the Euler Lagrange equations are

$$\eta \cdot \mathrm{E}(\tilde{L}) = \eta^a \left(\frac{\partial L}{\partial u^a} + (-\partial_i) \frac{\partial L}{\partial u^a_i} + (-\partial_i)(-\partial_j) \frac{\partial L}{\partial u^a_{ij}} + \cdots \right) \mathrm{d}^n x, \tag{5}$$

where η is now used just as a $T_V B$-valued dummy argument. To summarize, a PDE is *variational* if it is presented by the Euler-Lagrange equations $\mathrm{E}(\tilde{L})[\beta] = 0$ for some Lagrangian density $\tilde{L}$.

It is well-known that an *exact* (a.k.a *trivial* or *total divergence*) *Lagrangian*, one of the form $\tilde{L}[\beta] = \mathrm{d}\tilde{K}[\beta]$ for some $\tilde{K}\colon J^\infty B \to B^*(\Lambda^{n-1} M)$, has vanishing Euler-Lagrange equations $\mathrm{E}(\mathrm{d}\tilde{K}) = 0$. In that sense, two Lagrangian densities are equivalent if they differ by an exact one. In this sense, one naturally defines a *(local) variational symmetry* as an evolutionary vector field ξ^∞ such that

$$\mathcal{L}_{\xi^\infty} \tilde{L}[\beta] = \mathrm{d}\tilde{N}[\beta; \xi[\beta]] \tag{6}$$

for some differential operator $\tilde{N}\colon J^\infty(B \times_B T_V B) \to B^*(\Lambda^{n-1} M)$ linear in its second argument. A non-trivial but basic result of local variational calculus is that any variational symmetry of $\tilde{L}$ is a symmetry of the PDE presented by $\mathrm{E}(\tilde{L})[\beta] = 0$, which is due to the following identity implied by (6):

$$\dot{\mathrm{E}}(\tilde{L})[\beta; \xi[\beta]] = -\dot{\xi}^* \left[\beta; \mathrm{E}(\tilde{L})[\beta]\right], \tag{7}$$

where $*$ denotes the formal adjoint of $\dot{\xi}[\beta; \dot{\beta}]$ with respect to its linear second argument. For any linear differential operator $Q\colon J^\infty V \to W$ between vector bundles $V, W \to M$, its *formal adjoint* is the differential operator $Q^*\colon J^\infty \tilde{W}^* \to \tilde{V}^*$ between the densitized dual bundles uniquely defined by the identity

$$\omega \cdot Q[\phi] - Q^*[\omega] \cdot \phi = \mathrm{d}U[\omega, \phi], \tag{8}$$

for some bilinear differential operator $U\colon J^\infty(\tilde{W}^* \otimes_M V) \to \Lambda^{n-1} M$, where of course the *densitized* dual of a vector bundle $V \to M$ is the vector bundle $\tilde{V}^* = V^* \otimes_M \Lambda^n M \to M$. In local adapted coordinates, $(x, u^a, u^a_i, \ldots)$ on $J^\infty V \to M$ and $(x, v_A, v_{A,i}, \ldots)$ on $\tilde{W}^* \to M$, the formal adjoint of

$$\omega \cdot Q[\phi] = \omega_A \left(Q^A_a u^a + Q^{A,i}_a u^a_i + Q^{A,ij}_a u^a_{ij} + \cdots \right) (j^\infty \phi) \tag{9}$$

is given by

$$Q^*[\omega] \cdot \phi = \phi^a \cdot \left(Q^A_a v_A(j^\infty \omega) + (-\partial_i)[Q^{A,i}_a v_A(j^\infty \omega)] + (-\partial_i)[-\partial_j)(Q^{A,ij}_a v_A(j^\infty \omega)] + \cdots \right). \tag{10}$$

Clearly, Q^* is obtained from Q by local operations. This means that the definition of Q^* easily extends to any differential operator depending on multiple arguments, as long as the adjoint is taken with respect to a specific linear argument, as for instance in (7).

Two important properties of the formal adjoint are that it is an involution, $Q^{**} = Q$ (after taking some obvious canonical isomorphisms between the bundles on which these operators act), and that it reverses the order of composition of linear differential operators, namely $(Q \circ R)^* = R^* \circ Q^*$.

The formal adjoint also has a crucial relationship with the Euler-Lagrange operator $\mathrm{E}(\tilde{L})[\beta]$. This expression is not linear, so it does not make sense to consider its adjoint. On the other hand, the linearization $\dot{\mathrm{E}}(\tilde{L})[\beta;\dot{\beta}]$ is linear in its second argument, so its adjoint can be taken. It is a non-trivial but basic property of the linearized Euler-Lagrange operator that it is self-adjoint,

$$\dot{\mathrm{E}}(\tilde{L})^*[\beta;\dot{\beta}] = \dot{\mathrm{E}}(\tilde{L})[\beta;\dot{\beta}]. \tag{11}$$

Let us conclude this section with

Definition 1. *A* classical field theory *on a manifold M and bundle $B \to M$ is a variational PDE presented by $\mathrm{E}(\tilde{L})[\beta] = 0$ for a Lagrangian density $\tilde{L}$. The manifold M is then referred to as the* spacetime*, the bundle $B \to M$ as the* field bundle*, its sections $\beta \in \Gamma(B)$ as* fields *(or* field configurations*) and the variational PDE itself as the* equations of motion (EOM)*.*

It is worth observing that, when linearizing using $\beta \mapsto \beta + \varepsilon\dot{\beta}$ about a *solution* β of $\mathrm{E}(\tilde{L})[\beta] = 0$, the linearized EOM become the Euler-Lagrange equations of the quadratic part $\tilde{L}^{(2)}_\beta[\dot{\beta}] = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\tilde{L}[\beta+\varepsilon\dot{\beta}]\Big|_{\varepsilon=0}$ of the Lagrangian density expanded about β. That is,

$$\dot{\mathrm{E}}(\tilde{L})[\beta;\dot{\beta}] = \mathrm{E}(\tilde{L}^{(2)}_\beta)[\dot{\beta}]. \tag{12}$$

So a linearized variational PDE is still a variational PDE.

0.1.2 Gauge symmetry

The main references for this section can be taken as [4, 41].

In theoretical physics, the notion of a *gauge symmetry* is rather broad and refers to a family of variational symmetries locally parametrized by free functions. To formalize this, we start with a classical field theory with field bundle $B \to M$, Lagrangian density $\tilde{L}$ and a vector bundle $P \to B$, the bundle of *gauge parameters*. A differential operator $\gamma\colon J^\infty(B \times_B P) \to T_V B$ linear in its second argument is a *gauge symmetry generator* if the evolutionary vector field $\xi[\beta;\rho]^\infty \in \mathfrak{X}(J^\infty B)$ prolonging the generator ξ is a variational symmetry of $\tilde{L}$ for an arbitrary section $\rho \in \Gamma(P \to B)$. Alternatively, there exists a differential operator $\tilde{K}\colon J^\infty(B \times_B P) \to B^*(\Lambda^{n-1}M)$ linear in the second argument such that

$$\gamma[\beta;\rho]\cdot \mathrm{E}(\tilde{L})[\beta] = \mathrm{d}\tilde{K}[\beta;\rho]. \tag{13}$$

The family of symmetries generated by acting with γ on $\rho \in \Gamma(P \to B)$ is called a *gauge symmetry* or *(infinitesimal) gauge transformation*, the same term is also used for any element of such a family. If $\gamma[\beta;\rho]$ happens to vanish at x when $j^\infty_x\beta$ is restricted to the PDE submanifold $\mathcal{E}$ of the equations of motion, for any $x \in M$ and $\rho \in \Gamma(P \to B)$, we say that γ is a *trivial gauge symmetry.*

Definition 2. *A* gauge theory *is a classical field theory with non-trivial gauge symmetries.*

Example 1 (Maxwell electrodynamics)*.* The fields of the theory are 1-forms on a Lorentzian manifold (M,g), $B = \Lambda^1 M$, whose sections we denote by $A \in \Gamma(\Lambda^1 M)$. The Lagrangian is $\tilde{L}[A] = -\frac{1}{4}F_{ab}[A]F^{ab}[A]\,\mathrm{d}_g x$, with $F_{ab}[A] = (\mathrm{d}A)_{ab}$, the indices raised and lowered by the metric g_{ab} and $\mathrm{d}_g x$ is the g-volume form on M. The EOM are $\mathrm{E}(\tilde{L})_b[A] = \nabla^a F_{ab}[A]\,\mathrm{d}_g x$, while all infinitesimal gauge symmetries are of the form $\gamma[A;\rho] = \mathrm{d}\rho$, with the gauge parameters ρ being scalars, $P = \Lambda^0 M$.

Example 2 (Yang-Mills (YM) theory)*.* Consider a Lie group G and a G-principal bundle $\mathcal{P} \to M$ over a Lorentzian manifold (M,g). Let $\mathfrak{g}$ be the Lie algebra of G and denote by $\mathfrak{g}_\mathcal{P} \to M$ the associated bundle for the adjoint representation of G on $\mathfrak{g}$. For convenience, fix a reference connection on the G-principle bundle $\mathcal{P} \to M$ and denote the induced connection on $\mathfrak{g}_\mathcal{P} \to M$ by $D_0\colon \Gamma(\mathfrak{g}_\mathcal{P}) \to \Gamma(\mathfrak{g}_\mathcal{P} \otimes_M \Lambda^1 M)$. Any other G-equivariant connection on $\mathfrak{g}_\mathcal{P}$ can be parametrized as $Du = D_0 u + [A,u]$, for any $u \in \Gamma(\mathfrak{g}_\mathcal{P})$ and some $A \in \Gamma(\mathfrak{g}_\mathcal{P} \otimes_M \Lambda^1)$.

Let the fields of the theory be G-equivariant connections D on $\mathfrak{g}_\mathcal{P} \to M$, parametrized as above by sections A of $B = \mathfrak{g}_\mathcal{P} \otimes_M \Lambda^1 M$. The curvature of D is $F[A] = D_0 \wedge A + \frac{1}{2}[A \wedge A]$ is a $\mathfrak{g}_\mathcal{P}$-valued 2-form, where $\wedge$ acts on the 1-form parts of A and $[-,-]$ acts on the $\mathfrak{g}_\mathcal{P}$ parts of A. The Lagrangian is $\tilde{L}[A] = -\frac{1}{4}\langle F_{ab}[A], F^{ab}[A]\rangle\, \mathrm{d}_g x$, where $\langle -,-\rangle$ is the pointwise bilinear form on $\mathfrak{g}_{\mathcal{P}}$ induced by the Killing form on $\mathfrak{g}$, the indices raised and lowered by the metric g_{ab} and $\mathrm{d}_g x$ is the g-volume form on M. The EOM are $\mathrm{E}(\tilde{L})_b[A] = D^a F_{ab}[A]$. All infinitesimal gauge symmetries are of the form $\gamma[A;\rho] = D\rho$, with gauge parameters ρ being sections of the bundle $P = \mathfrak{g}_{\mathcal{P}}$.

Example 3 (General Relativity (GR)). Let the field bundle $B = \mathring{S}^2 T^*M$ be the bundle of *Lorentz signature* metrics (distinguished by the $(\mathring{-})$ notation) on a manifold M, and denote its sections by $g \in \Gamma(\mathring{S}^2 T^*M)$. If $R_{abcd}[g]$ is the Riemann tensor of g, then $R_{ac}[g] = R_{abcd}[g]g^{bd}$ is the Ricci tensor, and $R[g] = R_{ac}[g]g^{ac}$ is the Ricci scalar. The Lagrangian of the theory is $\tilde{L}[g] = R[g]\,\mathrm{d}_g x$, where $\mathrm{d}_g x$ is the g-volume form on M. The EOM $\mathrm{E}(\tilde{L})_{ab}[g] = (R_{ab}[g] - \frac{1}{2}R[g]g_{ab})\,\mathrm{d}_g x$ are the *Einstein equations*. All infinitesimal gauge symmetries are of the form $\gamma_{ab}[g;\rho] = K^g_{ab}[\rho]$, where the right-hand side is given by the *Killing operator*

$$K^g_{ab}[\rho] = \mathcal{L}_\rho g_{ab} = \nabla_a \rho_b + \nabla_b \rho_a, \tag{14}$$

with gauge parameters ρ being vector fields, sections of the bundle $P = TM$.

Consider a universal linearization of (13) in the direction $\beta \mapsto \beta + \varepsilon\dot{\beta}$:

$$\gamma[\beta;\rho]\cdot\dot{\mathrm{E}}(\tilde{L})[\beta;\dot{\beta}] = -\dot{\gamma}[\beta;\dot{\beta},\rho]\cdot\mathrm{E}(\tilde{L})[\beta] + \mathrm{d}\dot{\tilde{K}}[\beta;\dot{\beta},\rho]. \tag{15}$$

For a solution β, satisfying $\mathcal{E}(\tilde{L})[\beta] = 0$, using the fact that the linearized variational equations are still variational, we arrive at the identity

$$\gamma_\beta[\dot{\rho}]\cdot\mathrm{E}(\tilde{L}^{(2)}_\beta)[\dot{\beta}] = \mathrm{d}\tilde{K}_\beta[\dot{\beta},\dot{\rho}] \tag{16}$$

for arbitrary $\dot{\beta} \in \Gamma(\beta^* T_V B)$ and $\dot{\rho} \in \Gamma(\beta^* P)$, where we have set $\gamma_\beta[\beta^*\rho] = \gamma[\beta;\rho]$ and $\tilde{K}_\beta[\dot{\beta},\beta^*\rho] = \dot{\tilde{K}}[\beta;\dot{\beta},\rho]$, which are now differential operators linear in each of their arguments. In other words, the linearization of a gauge theory inherits *field independent* gauge symmetries. Thus, owing to the identity (7), this field independent gauge symmetry is a symmetry "on the nose," meaning that

$$\mathrm{E}(\tilde{L}^{(2)}_\beta)[\gamma_\beta[\dot{\rho}]] = 0, \tag{17}$$

with the right-hand side exactly vanishing for any $\dot{\rho} \in \Gamma(\beta^* P)$ (in general it need only be proportional to the EOM). These observations motivate the following

Definition 3. *A* linear gauge theory *is a linear classical field theory with a quadratic Lagrangian and non-trivial field-independent gauge symmetries.*

0.1.3 Compatibility operators and complexes

The main references for this section can be taken as [19, 37, 35].

Motivated by Definition 3, let us consider a linear field bundle $V \to M$ (alternative notation to $B \to M$, since V is a vector bundle), a quadratic Lagrangian $\tilde{L}$, and the corresponding linear EOM $E[\phi] = \mathrm{E}(\tilde{L})[\beta] = 0$. Supposing that it is a linear gauge theory means that there is also the gauge parameter vector bundle $P \to M$ and a linear differential operator $\gamma\colon P \to V$ generating the gauge symmetries. While we have previously made a distinction between symmetries of the EOM and symmetries of the Lagrangian, the difference disappears for field-independent gauge symmetries:

$$\begin{aligned}
\mathcal{L}_{\gamma[\rho]^\infty}\tilde{L}[\phi] &= \gamma[\rho]\cdot E[\phi] + \mathrm{d}\tilde{K}[\phi,\rho] \\
&= E^*[\gamma[\rho]]\cdot\phi + \mathrm{d}(W_E[\gamma[\rho],\phi] + \tilde{K}[\phi,\rho]) \\
&= E[\gamma[\rho]]\cdot\phi + \mathrm{d}(W_E[\gamma[\rho],\phi] + \tilde{K}[\phi,\rho]) \\
&= \mathrm{d}(W_E[\gamma[\rho],\phi] + \tilde{K}[\phi,\rho])
\end{aligned} \tag{18}$$

where the last equality follows from the self-adjointness $E^* = E$. In other words, while we already knew that a field-independent gauge symmetry satisfies $E\circ\gamma = 0$, now we also know that the converse is true as well. Hence, from now on, for linear gauge theories, we can forget about the

Lagrangian, and define field-independent gauge symmetries as linear operators $\phi = \gamma[\rho]$ annihilated by $E[\phi]$, i.e., $E \circ \gamma = 0$.

Other than $\gamma \neq 0$, we have not yet imposed any conditions on the gauge generator γ. It is important to note that for the same EOM operator $E[\phi]$ there may exist multiple unequal gauge generators. Let us distinguish certain gauge generators as *universal* or *complete* by the following property. We call a gauge generator $\gamma\colon J^\infty P \to V$ *universal* when any other gauge generator $\gamma'\colon P' \to V$ can be factored through γ. More precisely, for any such γ', there should exist another linear differential operator $\delta\colon J^\infty P' \to P$ such that $\gamma' = \gamma \circ \delta$.

As mentioned at the beginning of the introduction, we are eventually interested in quotienting the solutions of the EOM with respect to all gauge symmetries. That is, we need to consider equivalence classes of fields under the equivalence relation $\phi \sim \phi + \gamma[\rho]$ for any $\rho \in \Gamma(P)$, with γ being a universal gauge generator. These are also known as *physical equivalence classes* (also *gauge equivalence class*).

A natural question to ask is the following: given two fields $\phi, \chi \in \Gamma(V)$, do they belong to the same or different equivalence classes? This question can be partially answered with the help of a differential operator $F\colon J^\infty V \to W$ that satisfies $F \circ \gamma = 0$. In the literature on overdetermined PDEs, F is called a *compatibility operator* for γ, while in the study of linear gauge theories, F is called an *local linear gauge-invariant observable* or a *local gauge-invariant field combination*. Given such an F, we can definitely decide that ϕ and χ belong to different physical equivalence classes whenever $F[\phi] \neq F[\chi]$. On the other hand, the equality $F[\phi] = F[\chi]$ or $F[\phi - \chi] = 0$ is not by itself enough to conclude that $\phi - \chi = \gamma[\rho]$ for some $\rho \in \Gamma(P)$, without more information about F. After all, so far the zero operator $F = 0$ still fits all of our definitions. The desired condition on F is that $F[\phi] = 0$ implies $\phi = \gamma[\rho]$ for some ρ (at least locally on open neighborhoods in M). In that case, it is said that the composition of F and γ is *locally exact*.

Unfortunately, local exactness may be a rather subtle property and its verification or disproof could involve a significant amount of analysis. On the other hand, under certain conditions (which are frequently verified in practice), we can deduce local exactness from a more geometric property, dual to the universality property for gauge generators. Namely, we say that F is *universal* or *complete* as a compatibility operator for γ when for any other compatibility operator $F'\colon J^\infty V \to W'$, there exists another differential operator $G\colon J^\infty W \to W'$ such that $F' = G \circ F$, that is, F' necessarily factors through F. While the condition of being a universal compatibility operator is easier to analyze geometrically than local exactness, under some conditions it actually implies local exactness (see [27] for references).

Consider a sequence of linear differential operators $K_i\colon J^\infty W_i \to W_{i+1}$, $i \geq 0$, such that $K_{i+1} \circ K_i = 0$. Such a sequence is called a *complex* and can be visually represented as

$$\bullet \xrightarrow{K_0} \bullet \xrightarrow{K_1} \bullet \xrightarrow{K_2} \bullet \xrightarrow{K_3} \cdots . \tag{19}$$

Note that we can consistently omit the function spaces or jet bundles on which these operators act, as the domain or codomain of a given linear differential operator can be thought to be implicitly part of its definition. The complex is said to *terminate* at $i = k$ if $K_{\bullet \geq k} = 0$. The following stronger definition is quite useful.

Definition 4. *A complex of linear differential operators K_i is said to be a* universal *(or* complete*)* compatibility complex *for K_0 if each of the K_{i+1} operators, $i \geq 0$, is a universal compatibility operator for K_i.*

Often the attribute universal (or complete) is dropped and the term *compatibility complex* is used to mean the same thing (when no confusion is possible).

It is easy to see that setting $K_0 = \gamma$ and $K_1 = F$, with $W_0 = P$, $W_1 = V$ and $W_2 = W$, defines the beginning of a complex, since $K_1 \circ K_0 = F \circ \gamma = 0$, while the universality of F as a compatibility operator for γ makes it also into the beginning of the universal compatibility complex for γ.

As we have seen, given a gauge generator γ, it is natural to try to find a complete set of corresponding local linear gauge-invariant field combinations, which is the same as finding a universal compatibility operator for γ. It is equally natural to further ask to find the full compatibility complex of γ. We will return to this question later in this Thesis (see Problem 1).

0.2 Local functionals and invariants

This section covers the papers [25] and [29].

Consider a classical gauge theory theory (Definition 1) on a field bundle $B \to M$, with Lagrangian $\tilde{L}$, EOM $\mathrm{E}(\tilde{L})$ and gauge symmetry generator $\gamma[\beta; \rho]$, with $\rho \in \Gamma(P \to B)$ being sections of the gauge parameter bundle. As discussed at the top of the Introduction, we would eventually like to construct its *phase space* $\mathcal{S}$. As a space, it consists of the space of solutions

$$\mathcal{S} = \{\beta \in \Gamma(B) \mid \mathrm{E}(\tilde{L})[\beta] = 0\}, \tag{20}$$

which under favorable hypotheses can be endowed with a smooth structure of a Fréchet manifold. The family of gauge symmetries generated by $\gamma[\beta; \rho]$ can be prolonged to vector fields on $J^\infty B$ that are tangent to $\mathcal{S}$, which generate the *gauge symmetry distribution* $\mathcal{G} \subset T\mathcal{S}$ when linearly spanned over all possible gauge symmetry generators γ and all gauge parameter sections $\beta \in \Gamma(P \to B)$. Leaves of this distributions are *gauge orbits*.

At any (non-singular) point $\beta \in \mathcal{S}$, the tangent space $T_\beta \mathcal{S}$ can be identified with the solution space of the gauge theory linearized at β. As described in Sections 0.1.1 and 0.1.2, the linearized theory is defined on the field bundle $\beta^* T_V B \to M$, whose sections we denote by $\dot{\beta} \in \Gamma(\beta^* T_V B)$, with Lagrangian $\tilde{L}^{(2)}_\beta$ and field-independent gauge symmetry generator $\gamma_\beta[\dot{\rho}]$, whose gauge parameters are sections $\dot{\rho} \in \Gamma(\beta^* P)$. The linear span of $\gamma_\beta[\dot{\rho}]$ over $\dot{\rho}$ then coincides with the gauge symmetry distribution $\mathcal{G}_\beta \subset T_\beta \mathcal{S}$ at the background solution. As a result, we can think of linear gauge theories (Definition 3) as models for the tangent space $T_\beta \mathcal{S}$ at a regular point. At singular points of $\mathcal{S}$ (in the neighborhood of which $\mathcal{S}$ fails to have the structure of a manifold) some of these statements may fail, so these cases have to be studied separately [24].

The gauge orbit space $\bar{\mathcal{S}} = \mathcal{S}/\mathcal{G}$, endowed with a so-called *canonical Poisson structure*, which happens under favorable conditions, is the ultimately desired *reduced phase space*. This Poisson structure may not be defined on the unreduced solution space $\mathcal{S}$, which in the worst case carries only a gauge-invariant pre-symplectic structure (also called *canonical* or *Hamiltonian*). Under favorable conditions, this pre-symplectic structure can be extended to compatible symplectic and Poisson structures, via what is commonly called *gauge-fixing*, which are also gauge-compatible and descend to the desired Poisson structure on $\bar{\mathcal{S}}$. Incidentally, since pre-symplectic, symplectic and Poisson structures only need information about the tangent spaces $T\mathcal{S}$ and $T\bar{\mathcal{S}}$ (as well as the corresponding cotangent spaces), it is sufficient to construct them pointwise over $\mathcal{S}$, that is, for the linear gauge theories modelling these tangent spaces in $T\mathcal{S}$. I considered the geometric details of these constructions, with special attention to the Poisson structure in [22, 23].

There are multiple functional analytic details that may modify this basic definition. For instance, for fields, instead of C^∞ smooth sections of $B \to M$, we might consider also sections of finite C^k smoothness or some Sobolev regularity, which would naturally give rise to Banach manifolds. Also, when the spacetime M is non-compact, we may also restrict the asymptotic behavior of the fields in the neighborhood of infinity. All the same comments apply to gauge parameters, which are sections of $P \to B$, hence modifying the gauge symmetry distribution. We shall not consider such details here and will stick to the smooth setting described in the previous paragraph.

Understandably, the structure of the gauge orbit space $\bar{\mathcal{S}} = \mathcal{S}/\mathcal{G}$ may be complicated and singular in any number of ways. Its construction can be simplified (in the sense of being replaced by a simpler problem), while still giving us a lot of useful information. For instance, instead of constructing the quotient space itself, we may be happy to just describe some "sufficiently large" and "nice" class of functions on $\mathcal{S}$ that are gauge-invariant (or constant on the gauge orbits). Furthermore, given a smooth gauge-invariant function $\Phi \in C^\infty(\mathcal{S})$ (frequently, Φ is referred to as a *functional*, since it is a function on a space of sections, which are themselves functions) and a solution $\beta \in \mathcal{S}$, the differential $\mathrm{d}\Phi$ must annihilate the gauge symmetry distribution $\mathcal{G}_\beta \subset T_\beta \mathcal{S}$. Hence $\mathrm{d}\Phi_\beta$ is a *linear gauge-invariant functional* for the linear gauge theory modeling the tangent space $T_\beta \mathcal{S}$. In fact, one way to describe the desired class of invariant functions Φ on $\mathcal{S}$ is to specify the behavior of their differentials $\mathrm{d}\Phi_\beta$ pointwise over $\beta \in \mathcal{S}$ (at least the subset of the regular points of $\mathcal{S}$).

0.2.1 Linear invariants

In the case of linear gauge theories, which as we have seen serve as a simpler model of the full non-linear case, there is a commonly accepted notion of both "sufficiently large" and "nice" attributes of gauge-invariant functionals, they are subsumed by the respective technical terms *gauge orbit separating* and *local functional*. Of course, in this case we restrict ourselves to linear functionals.

Definition 5. *Consider a linear gauge theory defined on the field space $V \to M$, with $\dim M = n$, with field-independent gauge generator $\gamma\colon J^\infty P \to V$, and with EOM solution space $\mathcal{T} \subset \Gamma(V)$. A linear functional Ψ on $\Gamma(V)$ is* local *if it is of the form*

$$\Psi(\phi) = \int_M \alpha \cdot \psi, \tag{21}$$

where $\alpha \in \Gamma_c(\tilde{V}^*)$ is a smooth compactly supported section of the densitized dual $\tilde{V}^* = V^* \otimes_M \Lambda^n M$. The functional Ψ is gauge-invariant *when $\Psi(\gamma[\rho]) = 0$ for every $\rho \in \Gamma(P)$. A set of gauge-invariant linear functionals $\{\Phi_i\}$ on $\Gamma(V)$ is* gauge orbit separating *if the condition $\forall i\colon \Phi_i(\phi) = 0$ implies that $\phi = \gamma[\rho]$ for some $\rho \in \Gamma(P)$.*

A linear functional on $\mathcal{T}$ is local *(*gauge-invariant*) if it is the restriction of a local (gauge-invariant) functional on $\Gamma(V)$.*

Clearly, if a set of functionals $\{\Phi_i\}$ separates gauge orbits on $\Gamma(V)$, then their restrictions also separate the orbits on $\mathcal{T}$. We have formulated the above notions of *local* and *gauge-invariant* functionals even without reference to the solution space $\mathcal{T}$ (that is, *off-shell*, rather than *on-shell*). The reason is that these definitions are easier to state and the distinction with purely on-shell definitions is immaterial for our purposes [22, 23, 5]

An obvious way of producing local gauge-invariant functionals is to use *smearing functions* $\alpha \in \Gamma_c(\tilde{V}^*)$ satisfying $\gamma^*[\alpha] = 0$. In fact the fundamental lemma of the calculus of variations implies that all local gauge-invariant functionals are of that form. If a complete compatibility operator $F\colon J^\infty V \to W$ is known, then it is easy to generate such smearing functions by the formula $\alpha = F^*[\omega]$ for any $\omega \in \Gamma_c(\tilde{W}^*)$, since differential operators preserve supports and the identity $F \circ \gamma = 0$ implies also $\gamma^* \circ F^* = 0$. This observation provides an extra motivation for constructing a complete compatibility operator F for the gauge generator γ.

This is certainly a way to generate a large set of gauge-invariant observables, one for each $\omega \in \Gamma_c(\tilde{W}^*)$. However, it is not obvious whether this set really does separate the gauge orbits in $\Gamma(V)$. This issue has to be investigated on a case by case basis [23, 5]. The conditions for orbit separation end up being cohomological, related to the cohomologies of the compatibility complex for γ or its formal adjoint complex. This observation provides an extra motivation for constructing the full compatibility complex (Definition 4) for the gauge generator γ.

It is important to note that each tangent space $T_\beta\mathcal{S}$ of the phase space of a non-linear gauge theory potentially models a different linear gauge theory. Thus, the construction of a compatibility complex for the linearized gauge generator $\gamma_\beta[\dot\rho]$ for some background solution $\beta \in \mathcal{S}$ does not automatically translate to the same construction by for a different background solution $\beta' \in \mathcal{S}$. Given the importance of the examples of General Relativity and Yang-Mills gauge theories, it would be already important to solve this problem for these two theories, linearized about different non-trivial background solutions. Unfortunately, until my work in [29] only very few results of this type have been available in the literature. This leads us to the following concrete geometric problem.

Problem 1. Consider the Killing operator $K^g\colon J^\infty(TM) \to S^2T^*M$ on a pseudo-Riemannian geometry (M, g), or a covariant derivative $D\colon J^\infty\mathfrak{g}_\mathcal{P} \to \mathfrak{g}_\mathcal{P} \otimes_M \Lambda^1 M$ on Lie algebra $\mathfrak{g}$-valued forms, where $\mathfrak{g}_\mathcal{P} \to M$ is the bundle associated to a G-principal bundle $\mathcal{P} \to M$ by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. **Construct a full compatibility operator for the operators K^g or D in some cases of interest.**

I first formulated this problem precisely in [22, 23, 27] and finally gave a reasonable solution in [29]. The solution is based on a combination of methods from the theory of formal integrability of overdetermined PDEs [19, 37, 35] and from homological algebra [42]. The key observation is that both the Killing K^g and the connection D operators are overdetermined opeartors of *finite type*.

Definition 6. *Let $V \to M$ and $W \to M$ be vector bundles and $K\colon J^\infty V \to W$ a linear differential operator. The PDE $K[v] = 0$ (or just the operator K) is said to be of* finite type *when locally there exists an integer $N < \infty$, a vector bundle morphism $\kappa\colon J^N V \to J^{N+1} V$ and a linear differential operator $\lambda\colon \Gamma(W) \to \Gamma(J^{N+1} V)$ such that $j^{N+1} v - \kappa(j^N v) = \lambda[K[v]]$ for any $v \in \Gamma(V)$. If in addition locally the dimension of the solution space of $K[v] = 0$ is finite and constant, the equation is said to be* regular.

It is well-known that a PDE that is of *regular finite type* is equivalent in a suitable sense to the parallel section equation for a flat connection (see for instance Rmk.2.3.3, Rmk.2.3.6, and Ex.2.3.17 of [35]). The amount of work needed to reduce a particular PDE of this type to the form of a flat connection is roughly equivalent to the amount of work needed to explicitly prolong the equation to *involutive form*, which is the central problem in the theory of formal integrability.

My solution starts from the observation that a flat connection $\mathbb{D}$ defined on an auxiliary vector bundle $U \to M$ has a well-known full compatibility complex, namely the *twisted de Rham complex* $\mathbb{D} \wedge (-)\colon J^\infty(U \otimes_M \Lambda^i M) \to U \otimes_M \Lambda^{i+1} M$. The other crucial observation is that the equivalence of a regular finite type equation $K[v] = 0$ with $\mathbb{D}u = 0$ allows one to lift the full compatibility complex from $\mathbb{D}$ to K. The final result is the following theorem:

Theorem 7. *Given a regular finite type PDE $K[v] = 0$ on a vector bundle $V \to M$, together with an explicit equivalence of this PDE to $\mathbb{D}u = 0$, where $\mathbb{D}$ is a flat connection a vector bundle $U \to M$. There exists an explicit construction of a full compatibility complex for the operator K.*

After formulating this general result, I applied it to produce full compatibility complexes for the Killing operator K^g in two specific cases of practical importance in General Relativity: FLRW cosmological spacetimes in n-dimensions ($n \geq 3$) [29, Sec.3.2], and Schwarschild-Tangherlini spherically symmetric black hole spacetimes in n-dimensions ($n \geq 4$) [29, Sec.3.3]. Both results are new and original to [29].

Using the template of the above calculations and results, one could apply the same method to other geometries of interest in General Relativity. For instance, the case of the Kerr rotating black hole is currently under investigation [1, 3].

Also, since the G-principal bundle connection operator D is also of finite type, the same methods could be easily applied to that case as well. A Yang-Mills instanton field configuration [36] could be an interesting case to consider in detail.

0.2.2 Non-linear invariants

Consider now again a non-linear gauge theory defined on the field bundle $B \to M$, with solution space $\mathcal{S} \subset \Gamma(B)$. Since $\mathcal{S}$ is no longer is a linear space, we must now consider non-linear functionals. Unfortunately, now the situation may be more complicated. But, motivated by Definition 5 for linear gauge theories, for non-linear gauge theories, the following definitions have been used [8, 15, 22, 25]:

Definition 8. *A* partial functional *$\Phi \in C^\infty(\mathcal{U})$ is a smooth function on an open subset $\mathcal{U} \subset \Gamma(B)$ (in a reasonable topology, to be specified as part of the problem). A partial functional Φ is* gauge-invariant *if it is annihilated by every Lie derivative $\mathcal{L}_\xi \Psi|_{\mathcal{U}} = 0$ with respect to a vector field ξ on $\Gamma(B)$ that is induced by a gauge symmetry generator $\gamma[\beta; \rho]$. A partial functional $\Psi \in C^\infty(\mathcal{U})$ is* local *if its functional derivative $\frac{\delta\Psi}{\delta\beta(x)}$ has compact support on M for every $\beta \in \mathcal{U}$ (though possibly depending on β), and the support of every higher functional derivative $\frac{\delta^k\Phi}{\delta\beta(x_1)\cdots\delta\beta(x_k)}$ is both compact and is contained within the total diagonal $x_1 = \cdots = x_k$.*

A local *(or* gauge-invariant*) partial functional on $\mathcal{S} \subset \Gamma(B)$ is the restriction of a local (or gauge-invariant) functional on $\Gamma(B)$.*

For a long time it was a matter of some controversy whether local gauge-invariant functionals for the particular example of General Relativity (considered as a non-linear classical gauge theory) exist at all. The easiest way to generate non-linear local functionals is by the following generalization of formula (21)

$$\Phi(\beta) = \int_M \alpha[\beta] \tag{22}$$

where $\alpha\colon J^\infty B \to \Lambda^n M$ is a differential operator such that $\operatorname{supp}\alpha[\beta]$ is contained in a fixed compact subset of M for every $\beta \in \Gamma(B)$. In fact, roughly until the recent works [15, 8, 22, 25], formula (22) together with these support restrictions on $\alpha[\beta]$ has been the standard notion of *non-linear local functional*. On the other hand, the above formula and restriction on $\alpha[\beta]$ are well-known to be in tension with the gauge-invariance (meaning *diffeomorphism invariance* in the case of GR) of Φ (see the Introduction to [25] for a brief survey and historical references). Moreover, the existence of a sufficiently large family of local gauge-invariant observables to separate gauge orbits on the phase space of GR was certainly in question. This tension lead to the following long standing geometric problem.

Problem 2. Consider the phase space $\mathcal{S}$ of General Relativity on a manifold M, with the gauge symmetry distribution $\mathcal{G} \subset T\mathcal{S}$ generated by infinitesimal diffeomorphisms. **Does there exist a suitable notion of locality, generalizing (22), such that there exist gauge-invariant local functionals on $\mathcal{S}$? If so, are there sufficiently many of them to separate the gauge orbits in $\mathcal{S}$?**

In [25] I addressed this problem by introducing Definition 8 as the new sufficiently general notion of local functional. This definition generalized the rough one based purely on formula (22) and previous more precise definitions given in [15, 8]. The main innovations in Definition 8 consisted of allowing the support of the functional derivative $\frac{\delta\Phi}{\delta\beta(x)}$ to depend on β (while remaining compact), and also of allowing partial functions (defined only on some open $\mathcal{U} \subset \mathcal{S}$). This generalization is reasonable because allowing the support of $\frac{\delta\Phi}{\delta\beta(x)}$ allows many more functionals Φ to be admissible, yet retaining the main benefits of having the convergence of any integrals involving the functional derivatives of Φ. When partial functionals are allowed, restricting to open neighborhoods of particular solutions $\beta \in \mathcal{S}$, there is no difference between partial and globally defined functionals. On the other hand, if we can construct a sufficiently rich family of gauge-invariant functionals on elements of an open cover $\{\mathcal{U}_i\}$ of $\mathcal{S}$ (or at least some dense subset of $\mathcal{S}$), we leave open the possibility that globally defined gauge-invariant functionals could still be constructed by some kind of glueing procedure.

Under this relaxed definition of locality, I then proved the following

Theorem 9. *Consider the phase space $\mathcal{S}$ of General Relativity on a manifold M, consisting of globally hyperbolic Lorentzian metrics g solving the Einstein equations. There exists an open gauge-invariant subset $\mathcal{U} \subset \mathcal{S}$ of* sufficiently generic *metrics. For any two metrics $g, g' \in \mathcal{U}$, there exists a joint gauge-invariant open neighborhood $\mathcal{U}_{g,g'} \subset \mathcal{U}$ of both g and g' and a local gauge-invariant functional $\Phi \in C^\infty(\mathcal{U}_{g,g'})$ that separates the gauge orbits of g and g'.*

The method of proof relied on some results from the theory of scalar differential invariants [30] of (pseudo-)Riemannian metrics. In particular, it is known that scalar differential invariants allow the local distinction on an open subset of sufficiently generic metrics [11]. Basically, there exists an integer N and an invariant differential operator $\mathcal{R}\colon J^\infty(S^2T^*M) \to \mathbb{R}^N$ such that any two sufficiently generic metrics g and g' on M belong to the same gauge orbit (i.e., are isometric) iff graphs $\mathcal{R}[g](M)$ and $\mathcal{R}[g'](M)$ coincide in $\mathbb{R}^N$. Then, it is sufficient to observe that, for any two non-isometric metrics g and g', there must exist an n-form $\alpha \in \Omega^n(\mathbb{R}^N)$ such that $\operatorname{supp}\alpha \cap \mathcal{R}[g](M)$ and $\operatorname{supp}\alpha \cap \mathcal{R}[g'](M)$ are both compact, but only exactly one of $\int_{\mathcal{R}[g](M)} \alpha$ and $\int_{\mathcal{R}[g'](M)} \alpha$ is non-zero. Then it can be proven that the formula

$$\Phi(h) = \int_M (\mathcal{R}[h])^* \alpha \tag{23}$$

is well-defined on at least a joint gauge-invariant open neighborhood $\mathcal{U}_{g,g'} \ni g, g'$ (here we must use the so-called *strong Whitney topology* on the space of smooth metrics [33]), where it defines a gauge-invariant local observable. Then, by construction, exactly one of $\Phi(g)$ and $\Phi(g')$ is non-zero.

Similar basic results exist also for Yang-Mills theory [34], where scalar invariants based on the curvature of a G-principal bundle connection operator D can be used to distinguish the local gauge equivalence classes for sufficiently generic connections. Hence an analogous result to Theorem 9 can be easily proven for Yang-Mills theory as well.

Note that the above results in GR refer to the open subset $\mathcal{U} \subset \mathcal{S}$ of sufficiently generic field configurations. It remains an open (and rather technical) question whether this open set can be dense in $\mathcal{S}$ and under what conditions. It is not hard so hard to deduce that $\mathcal{U} = \mathcal{V} \cap \mathcal{S}$, where $\mathcal{V} \subset \Gamma(\mathring{S}^2 T^* M)$ is a corresponding *dense* subset of all Lorentzian metrics by appealing to a *jet transversality theorem* [40]. However, it is not immediately clear why the intersection $\mathcal{V} \cap \mathcal{S}$ should be dense in $\mathcal{S}$.

The final result that we reproduce from [25] links in a satisfactory way linear and non-linear gauge-invariant functionals.

Theorem 10. *Consider a non-linear local gauge-invariant partial functional Φ (Definition 8) defined on an open $\mathcal{U} \subset \mathcal{S}$. Let $g \in \mathcal{U}$ and let $\dot{\Phi}_g(h)$ be the linearization of Φ at g, defined by $\Phi(g + \varepsilon h) = \Phi(g) + \varepsilon \dot{\Phi}_g(h) + O(\varepsilon^2)$. Then there exists a linear local gauge-invariant functional Ψ on $T_g\mathcal{S}$ (Definition 5), such that $\dot{\Phi}_g(h) = \Psi(h)$.*

0.3 Hodge-like structure and cohomology

This section covers the papers [26] and [27].

In the lead up to the statement of Problem 1, from the results of previous works [22, 23, 5], we motivated the importance of the construction of a full compatibility complex K_i, $i \geq 0$, for the gauge generator $K_0 = \gamma$ of a linear gauge theory. In particular, once the full compatibility complex is known, the cohomological information associated to it can be quite useful. For instance, the obstruction to the separation of gauge orbits by the class of local gauge-invariant observables defined using K_1 in Section 0.2.1 is essentially cohomological, as shown in [22, 23, 5]. Also, the canonical Poisson structure (specifically given by the so-called *Peierls formula* or *Peierls bracket*) on the reduced phase space $\bar{\mathcal{S}} = \mathcal{S}/\mathcal{G}$ of the linear gauge theory may or may not be *degenerate* (recall that a Poisson structure on a linear vector space is a skew-symmetric bilinear form $\{-,-\}\colon \bar{\mathcal{S}} \otimes \bar{\mathcal{S}} \to \mathbb{R}$, so that its *degeneracy* is in the sense of this bilinear form being degenerate). The degeneracy subspace of the canonical Poisson bracket has relations to physical phenomena like *super-selection sectors*, *topological charges*, and *Aharonov-Bohm type effects*. In [22, 23], I proved that the dimension of the degeneracy of subspaces of the canonical Poisson structure on $\bar{\mathcal{S}}$ is bounded from above by the some cohomological dimensions associated to the compatibility complex K_i.

Let us denote by $H^\bullet(K_\bullet)$ the cohomology of the complex $K_i\colon \Gamma(W_i) \to \Gamma(W_{i+1})$. There are a few variations on this basic definition. For instance, denote by $H^\bullet(K^*_\bullet)$ be the cohomology of the formal adjoint complex $K^*_{i-1}\colon \Gamma(\tilde{W}^*_i) \to \Gamma(\tilde{W}^*_{i-1})$. Other variations include $H^\bullet_{(*)}(K_\bullet)$ and $H^\bullet_{(*)}(K^*_\bullet)$, where $(*) \in \{\varnothing, c, +, -, fc, pc, sc, tc\}$ are different support restrictions indicating that we should use sections $\Gamma_{(*)}(-)$ with restricted supports, rather than arbitrary smooth sections $\Gamma(-)$, to define the cohomology. The various support restrictions of interest are c—compact, $+/-$—retarded/advanced, pc/fc—past/future compact, sc/tc—spacelike/timelike compact. Here, the attributes *retarded*, *advanced*, *future*, *past*, *spacelike* and *timelike* refer to a presumed *causal order* [17, 22] defined on our spacetime manifold M.

The causal order should be such that it is compatible with the natural structure of the EOM $\mathrm{E}(\tilde{L})[\phi] = 0$ of the gauge theory (which may become manifest only up *gauge-fixing*), which is encoded in the principal symbol of the $\mathrm{E}(\tilde{L})$ operator. The compatibility should be such that there should be unique solutions of the inhomogeneous equation

$$\mathrm{E}(\tilde{L})[\phi] + (\text{gauge-fixing})[\phi] = \alpha \tag{24}$$

with compact $\operatorname{supp}\alpha$ when $\operatorname{supp}\phi$ is restricted to be either $+$—*retarded* (extending into the future of $\operatorname{supp}\alpha$) or $-$—*advanced* (extending into the past $\operatorname{supp}\alpha$). On the other hand, a subset $S \subset M$ is *spacelike compact* when it is a union $S = S_+ + S_-$ where S_+ is retarded and S_- is advanced with respect to some compact subset of M. The c, fc, pc and tc support restrictions are respectively geometrically dual to the $\varnothing$, $+$, $-$ and sc restrictions, ensuring that the spaces of sections $\Gamma_{\varnothing/+/-/sc}(V)$ and $\Gamma_{c/fc/pc/tc}(\tilde{V}^*)$ have a well-defined and non-degenerate natural pairing defined by the formula $\langle \alpha, \phi \rangle = \int_M \alpha \cdot \phi$. When the principal symbol $\mathrm{E}(\tilde{L})$ is determined by a Lorentzian metric g on our spacetime (M, g). Then this causal order coincides with the usual notion of causality in General Relativity [17] and any hyperbolic PDE whose principal symbol coincides with that of the wave operator $\Box_g$ is compatible with this causal order.

This leads us to the following concrete geometric

Problem 3. Consider a compatibility complex K_i $(i \geq 0)$, defined on a spacetime M with a causal order. **Determine the following cohomologies with and without causal support restrictions: $H^\bullet_{(*)}(K_\bullet)$ and $H^\bullet_{(*)}(K^*_\bullet)$, for $(*) \in \{\varnothing, c, +, -, pc, fc, sc, tc\}$.**

My papers [26, 27] were dedicated precisely to the above problem. Two hypotheses ended being being crucial to obtain the strongest form of the results: (a) the *finite type* of K_0 (Definition 6) and (b) the existence of a *Hodge-like structure* on the complex K_i. By a *Hodge-like structure*, I mean the existence of the following diagram, where the arrows denote differential operators between the bundles on which the K_i complex is defined:

$$\begin{tikzcd} 0 \arrow[r] & \bullet \arrow[r, "K_0"] \arrow[d, "\Box_0"] & \bullet \arrow[r, "K_1"] \arrow[d, "\Box_1"] \arrow[dl, dashed, "\delta_0"] & \cdots \arrow[r, "K_i"] \arrow[dl, dashed, "\delta_1"] & \bullet \arrow[r] \arrow[d, "\Box_{i+1}"] \arrow[dl, dashed, "\delta_i"] & \cdots \\ 0 \arrow[r] & \bullet \arrow[r, "K_0"] & \bullet \arrow[r, "K_1"] & \cdots \arrow[r, "K_i"] & \bullet \arrow[r] & \cdots \end{tikzcd} , \tag{25}$$

where the solid arrows commute,

$$K_i \Box_i = \Box_{i+1} K_{i+1}, \tag{26}$$

with the vertical arrows hence constituting a *morphism of complexes* (a.k.a a *chain* (or *cochain*) *map*), while the dashed arrows constitute a (*chain* or *cochain*) *homotopy* from the complex to itself, inducing the vertical operators by the formula

$$\Box_i = \delta_i K_i + K_{i-1}\delta_{i-1}. \tag{27}$$

The reason I refer to the above operators and identities as *Hodge-like structure* is that it is highly reminiscent of the Hodge theory for the de Rham complex [18], where we have a Riemannian metric (M, g), $K_p = \mathrm{d}$ (the exterior derivative on p-forms), $\delta_p = \mathrm{div}_g$ is the g-divergence on $(p+1)$-forms, and $\Box_p = \Delta = d\delta + \delta d$ is the *Hodge-de Rham Laplacian* on p-forms. The main result of this original Hodge theory is the unique representation of cohomology classes of M by harmonic forms, when (M, g) is a compact Riemannian manifold, where the above structure is crucial.

In our cases of interest, we expect the $\Box_i$ operators not to be elliptic (like the Hodge-de Rham Laplacian), but rather hyperbolic and compatible with a causal structure on M. In that case, we can still put the existence of a Hodge-like structure to good use by appealing to the *snake lemma* of homological algebra [42]. In particular, the main result from [26] can be stated as follows:

Theorem 11. *Consider a complex K_i on a manifold M with a causal structure, and suppose that there exists a Hodge-like structure (25) on it, where the $\Box_i$ are hyperbolic and compatible with the causal structure. Then, all the cohomologies with unrestricted, compact and causally restricted supports mentioned in Problem 3 can be computed (using direct sums, shifted degrees and linear duals) from just $H^\bullet(K_\bullet)$ and $H^\bullet(K^*_\bullet)$.*

The paper [27] is largely a review article about the structure and properties of the so-called *Calabi complex*, with some novel perspectives, which is a full compatibility complex for the Killing operator on a (pseudo-)Riemannian geometry (M, g) of constant curvature, which means that locally the geometry is characterized by a single constant λ and the Riemann curvature identity

$$R_{abcd} - \lambda(g_{ac}g_{bd} - g_{ad}g_{bc}) = 0. \tag{28}$$

Until the new results of my work [29], the Calabi complex was one of the very few cases where the full compatibility complex of the Killing operator was known. So I used it in [27] as a test case to collect various results that would be useful for working with compatibility complexes in general.

The novel perspective in [27] included the following: (a) Explicit expressions for all the differential operators K_i in the Calabi complex, expressed in the usual tensorial language. Such expressions were difficult to find in the existing literature. In particular, Calabi's original paper [9] used a moving frame formalism. (b) An explicit description of the Hodge-like structure on the Calabi complex, as needed to apply Theorem 11. (c) The identification of the adjoint Calabi complex with the full compatibility operator for the *Killing-Yano* operator on $(n-2)$-forms.

Notable general results reviewed in [27] include the following. For the following results, it is useful to recall the notion of *sheaf* and *sheaf cohomology* as used in differential geometry [7].

Proposition 12. *If a complex K_i $(i \geq 0)$ on M is* locally exact*, then $H^\bullet(K_\bullet) \cong H^\bullet(\mathscr{K})$, where the right-hand side denotes the* sheaf cohomology *of the* sheaf *$\mathscr{K}$ of local solution to the equation $K_0[\phi] = 0$.*

When $K_0 = \mathrm{d}$ is the exterior derivative on 0-forms. Then the sheaf $\mathscr{K}$ is the sheaf of *locally constant functions*. The isomorphism in Proposition 12 is the well-known isomorphism between the de Rham cohomology and the cohomology of the sheaf of locally constant functions (which is independently known to be a homotopy invariant, coinciding with the singular cohomology of M) [7].

When K_0 is the Killing operator on a pseudo-Riemannian manifold (M, g), the sheaf $\mathscr{K}$ is the sheaf of local solutions to the Killing equation (basically, local *infinitesimal isometries* of (M, g)), which could be called the *Killing sheaf* of (M, g). Recall that there could of course be more local solutions to the Killing equation than global solutions. For instance, on a flat compact torus, local (infinitesimal) translations can be extended to global ones, but local (infinitesimal) rotations cannot be extended to global ones. However, on a simply connected flat space (like the universal cover of a flat torus), there every local (infinitesimal) isometry can be extended to a global one.

In general, when K_0 is of regular finite type (Definition 6), the sheaf $\mathscr{K}$ has finite dimensional *stalks*, with this dimension being locally constant. This kind of sheaf is called *locally constant* and the computation of the sheaf cohomology of a locally constant sheaf essentially becomes a problem in algebraic topology. Some practical methods of computing sheaf cohomology in the locally constant case are reviewd in [27]. In some cases, the computation could in principle be reduced to finite dimensional linear algebra, giving an effective way, when coupled with Proposition 12, of computing the cohomologies $H^\bullet(K_\bullet)$.

When K_0 is of finite type but is not regular, then the sheaf $\mathscr{K}$ will no longer be locally constant, but instead only *constructible* [21]. Of course, the computation of the sheaf cohomology of a constructible sheaf is expected to be more complicated.

The finite type condition also enters crucially into the following result (reviewed in [27, Sec.3.2]):

Proposition 13. *Consider a compatibility complex K_i $(i \geq 0)$ on M. If the initial operator K_0 is of* finite type *(Definition 6), the complex K_i is* locally exact*.*

Note that Theorem 11 reduces all the cohomology computations to those of $H^\bullet(K_\bullet)$ and $H^\bullet(K^*_\bullet)$. We have now see that $H^\bullet(K_\bullet)$ may be effectively evaluated by algebro-topological methods when K_0 is of regular finite type. If the complex K_i actually terminates at say K_p (meaning that $K_{p<\bullet} = 0$) and the formal adjoint K^*_p is of regular finite type, then the adjoint complex K^*_i can be considered as a (renumbered) compatibility complex for the operator K^*_p, so that similar statements can be made about $H^\bullet(K^*_\bullet)$. It is actually an immediate consequence of the results of [29] that there exists a $p > 0$ such that these conditions on K^*_p are automatically satisfied, whenever K_0 is of regular finite type.

Theorem 14. *If K_i $(i \geq 0)$ is a compatibility complex for a regular finite type operator K_0, then this complex terminates and its adjoint complex K^*_i is also a (renumbered) compatibility complex for a regular finite type opretator.*

0.4 IDEAL characterization

This section covers the papers [10] and [28].

As was mentioned in Section 0.2.2, in General Relativity (or in pseudo-Riemannian geometry in non-definite signature) there exist metrics that cannot be locally distinguished from non-isometric metrics by the values of local scalar curvature invariants (these are some of the metrics that fail the "sufficiently generic" condition of Section 0.2.2). One example is of course flat Minkowski space, for which the Riemann tensor $R_{abcd} = 0$ is identically zero. However, there exists a non-trivial family of non-flat (where R_{abcd} is not identically zero, hence not even locally isometric to Minkowski space) metrics that nonetheless have only vanishing scalar curvature invariants. These metrics constitute the so-called *VSI (vanishing scalar invariants) class* [11].

Hence, the gauge orbit of Minkowski space cannot be separated from the gauge orbit of any other VSI spacetime by the local gauge-invariant observables that we referred to in Theorem 9. However, we can imagine a gauge theory, where, besides the metric g, we also have other scalar or tensor fields, say ϕ. Then, supposing that the field configurations of ϕ permitted by the EOM on top of Minkowski and other VSI spacetimes allow us to define at least two linearly independent vector fields $u^a[g, \phi]$ and $v^a[g, \phi]$, then the scalar contraction

$$R_{abcd}[g] u^a[g, \phi] v^b[g, \phi] u^c[g, \phi] v^d[g, \phi] \tag{29}$$

could serve as the operator $\mathcal{R}$ used to construct local gauge-invariant observables according to formula (23). Such a functional would definitely vanish on Minkowski space, but might not vanish on some non-flat VSI spacetimes, depending of course on the values of the extra fields ϕ. Thus, the contraction of the Riemann tensor with non-purely metric tensor fields has the potential to allow the construction of local gauge-invariant functionals that are capable of separating (g, ϕ) from (g', ϕ), where g is the Minkowski metric, while g' is a non-flat VSI metric.

The above discussion motivates the following concrete geometric definition and problem statement.

Definition 15. *Consider a pseudo-Riemannian reference metric (M, g_0) and a set of tensor valued differential operators $\{T_i[g]\}$ on a general (M, g), built covariantly from the metric g, its Riemann tensor $R_{abcd}[g]$, and its covariant derivatives $\nabla_{e_1} \cdots \nabla_{e_k} R_{abcd}[g]$. We call the set $\{T_i[g]\}$ an* IDEAL characterization *of the reference geometry (M, g_0) when the vanishing $\forall i \colon T_i[g] = 0$ on a neighborhood of $x \in M$ implies that there exists a (potentially smaller) neighborhood of x that is isometric to an open subset of (M, g_0). The tensors $T_i[g]$ may be referred to as the* IDEAL tensors*.*

Problem 4. Consider a pseudo-Riemannian geometry (M, g_0) that is of interest in General Relativity for mathematical or theoretical reasons. **Determine an explicit IDEAL characterization of the reference geometry (M, g_0), or determine that it does not exist.**

IDEAL (Intrinsic, Deductive, Explicit, and ALgorithmic) characterization [13, 14] is in a way an alternative to the more widely known Cartan-Karlhede method [38], which is based on the introduction of an adapted frame. However, the introduction of a frame, in addition to the metric g and its curvature tensors, may not be practically or conceptually desired. On the other hand, IDEAL characterization works only with the metric itself and quantities covariantly obtained from it.

Despite their obvious geometric interest, the investigation of IDEAL characterizations of interesting spacetimes has been confined to the papers of only a few authors. Nonetheless, these authors managed to obtain IDEAL characterizations for the important cases of the Schwarzschild and Kerr black holes in $n = 4$ dimensions [13, 14]. But few other ones are known. A more extensive list of references to works on this topic can be found in the Introductions to [10, 28].

My papers [10, 28] were aimed specifically at filling this gap in the literature. Their main new results can be simply stated as follows.

Theorem 16. *If an n-dimensional Lorentzian manifold (M, g_0) is a cosmological spacetime (belonging to a* regular FLRW *or* inflationary *class) or a spherically symmetric black hole (belonging to the* generalized Schwarzschild-Tangherlini *class), then it possesses an IDEAL characterization.*

In each case, the relevant IDEAL tensors have been given explicitly.

The construction strategy that was used is roughly the following. Suppose that it is already known that the spacetime geometry (M, g_0) can be locally characterized by the existence of some tensor fields $\{u, v, \ldots\}$ satisfying some set of covariant differential and algebraic equations $\{C_i[u, v, \ldots] = 0\}$. Suppose also that there exist tensors $\{U[g], V[g], \ldots\}$ covariantly constructed from the metric alone, such that $U[g_0] = u$, $V[g_0] = v$, $\ldots$, from the previous supposition. Then defining $\{T_i[g] = C_i[U[g], V[g], \ldots]\}$ will give an IDEAL characterization of (M, g_0).

The IDEAL tensors have a curious connection to the problem of constructing a complete compatibility operator for the Killing operator on (M, g_0), which was discussed in Problem 1. The key observation is the following result referred to as the *Stewart-Walker lemma* [39] in the GR literature.

Proposition 17. *Consider a pseudo-Riemannian geometry (M, g_0). Let $T[g]$ be a tensor-valued differential operator constructed covariantly out of the metric g, and define its linearization $\dot{T}[h]$ about g_0 by the identity $T[g_0 + \varepsilon h] = T[g_0] + \varepsilon \dot{T}[h] + O(\varepsilon^2)$. Then, if $T[g_0]$ vanishes or is a linear combination of products of Kronecker δ's with constant coefficients, then $\dot{T} \circ K = 0$, where K is the Killing operator on (M, g_0).*

The proof is an immediate consequence of the elementary identity $\mathcal{L}_v T[g] = \dot{T}[\mathcal{L}_v g]$, while $\mathcal{L}_v g_0 = K[v]$.

Thus, the linearizations of the ideal tensors $\{\dot{T}_i[h]\}$ immediately give a set of operators obeying $\dot{T}_i \circ K = 0$. Now, by their definition, we know that the IDEAL tensors $\{T_i[g]\}$ jointly vanish *only* on those metrics lying on the gauge orbit (diffeomorphism orbit, in this case) of g_0. Thus, it stands to reason that the joint kernel of $\dot{T}_i$ consists *only* of the image of $K[v]$ spanned over all vector fields $v \in \Gamma(TM)$. In other words the $\dot{T}_i$ would constitute the components of a complete compatibility operator for K. This heuristic observation is essentially correct, except for the possibility that some of the components of $T_i[g]$ vanish at higher than linear order as $g \to g_0$, in which case the joint kernel of the $\dot{T}_i$ may be strictly larger. Thus, the operators $\dot{T}_i$ constitute the components of a good, geometrically motivated candidate for a complete compatibility operator for K on (M, g_0), but its completeness should strictly speaking be checked independently, for instance using the results of [29]. For example, the completeness of the linearizations of the FLRW IDEAL tensors from [10] does hold. This was first checked in [16], though by ad hoc methods. A similar check for generalized Schwarzschild-Tangherlini geometries (thus comparing the results from [28] and [29]) is yet to be done.

One of the conceptual advantages of Cartan-Karlhede characterizations method [38] is that they has been proven to exist for a rather general class of metrics (where the main requirement is that the ranks of certain curvature tensors remain locally constant). On the other hand, at the moment, it is not clear for which class of metrics do IDEAL characterizations exist. Neither do we know of an example where an IDEAL characterization is known not to exist. These questions are certainly worth investigating in more detail.

References

[1] S. Aksteiner and T. Bäckdahl, "All local gauge invariants for perturbations of the Kerr spacetime," *Physical Review Letters* **121** (2018) 051104, arXiv:1803.05341.

[2] Ian M. Anderson. The variational bicomplex. 1989.

[3] L. Andersson, S. Aksteiner, T. Bäckdahl, I. Khavkine, and B. Whiting, "Compatibility complex for black hole spacetimes," 2019. In preparation.

[4] G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in gauge theories," *Physics Reports,* **338** (2000) 439–569, arXiv:hep-th/0002245.

[5] M. Benini, "Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies," *Journal of Mathematical Physics* **57** (2016) 053502, arXiv:1401.7563.

[6] E. Binz, J. Śniatycki, and H. Fischer. *Geometry of Classical Fields*, volume 154 of *Notas de Matemática*. North-Holland, 1988.

[7] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, vol. 82 of *Graduate Texts in Mathematics*. Springer, New York, 1982.

[8] R. Brunetti, K. Fredenhagen, and P. L. Ribeiro, "Algebraic structure of classical field theory I: Kinematics and linearized dynamics for real scalar fields." 2012. arXiv:1209.2148.

[9] E. Calabi, "On compact, Riemannian manifolds with constant curvature. I," in *Differential Geometry*, C. B. Allendoerfer, ed., vol. 3 of *Proceedings of Symposia in Pure Mathematics*, pp. 155–180. AMS, Providence, RI, 1961.

[10] G. Canepa, C. Dappiaggi, and I. Khavkine, "IDEAL characterization of isometry classes of FLRW and inflationary spacetimes," *Classical and Quantum Gravity* **35** (2018) 035013, arXiv:1704.05542.

[11] A. Coley, R. Milson, V. Pravda, and A. Pravdová, "Vanishing scalar invariant spacetimes in higher dimensions," *Classical and Quantum Gravity* **21** (2004) 5519, arXiv:gr-qc/0410070.

[12] P. Deligne and D. S. Freed, Classical field theory, in *Quantum fields and strings: a course for mathematicians*, eds. P. Deligne, D. Kazhdan, P. Etingof, J. W. Morgan, D. S. Freed, D. R. Morrison, L. C. Jeffrey and E. Witten (AMS, Providence, RI, 1999), Providence, RI, pp. 137–225.

[13] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of the Schwarzschild metric," *Classical and Quantum Gravity* **15** (1998) 1323–1330.

[14] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of the Kerr metric," *Classical and Quantum Gravity* **26** (2009) 075013, `arXiv:0812.3310`.

[15] K. Fredenhagen and K. Rejzner, "Batalin-Vilkovisky Formalism in the Functional Approach to Classical Field Theory," *Communications in Mathematical Physics* **314** (2012) 93–127, `arXiv:1101.5112`.

[16] M. B. Fröb, T.-P. Hack, and I. Khavkine, "Approaches to linear local gauge-invariant observables in inflationary cosmologies," *Classical and Quantum Gravity* **35** (2018) 115002, `arXiv:1801.02632`.

[17] A. García-Parrado and J. M. Senovilla, "Causal structures and causal boundaries," *Classical and Quantum Gravity* **22** (2017) R1, `arXiv:gr-qc/0501069`.

[18] S. I. Goldberg, *Curvature and homology*. Dover, Mineola, NY, 1998.

[19] H. Goldschmidt, "Existence theorems for analytic linear partial differential equations," *The Annals of Mathematics* **86** no. 2, (1967) 246–270. `http://dx.doi.org/10.2307/1970689`.

[20] M. Gromov. *Partial Differential Relations*, volume 9 of *Ergebnisse Der Mathematik Und Ihrer Grenzgebiete*. Springer, 2010.

[21] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, vol. 292 of *Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer, 1990.

[22] I. Khavkine, "Characteristics, conal geometry and causality in locally covariant field theory." 2012. `arXiv:1211.1914`.

[23] I. Khavkine, "Covariant phase space, constraints, gauge and the Peierls formula," *International Journal of Modern Physics A* **29** (2014) 1430009, `arXiv:1402.1282`.

[24] I. Khavkine, "Topology, rigid cosymmetries and linearization instability in higher gauge theories," *Annales Henri Poincaré* **16** (2015) 255–288, `arXiv:1303.2406`.

[25] I. Khavkine, "Local and gauge invariant observables in gravity," *Classical and Quantum Gravity* **32** (2015) 185019, `arXiv:1503.03754`.

[26] I. Khavkine, "Cohomology with causally restricted supports," *Annales Henri Poincaré* **17** (2016) 3577, `arXiv:1404.1932`.

[27] I. Khavkine, "The Calabi complex and Killing sheaf cohomology," *Journal of Geometry and Physics* **113** (2017) 131–169, `arXiv:1409.7212`.

[28] I. Khavkine, "IDEAL characterization of higher dimensional spherically symmetric black holes," *Classical and Quantum Gravity* (2018, online first), doi:10.1088/1361-6382/aafcf1 `arXiv:1807.09699`.

[29] I. Khavkine, "Compatibility complexes of overdetermined PDEs of finite type, with applications to the Killing equation," 2018. `arXiv:1805.03751`.

[30] I. Kolař, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*. Electronic Library of Mathematics. Springer, 1993.

[31] I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Differential Equations*. Gordon and Beach, New York, 1986.

[32] A. Kriegl and P. W. Michor. *The Convenient Setting of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1997.

[33] D. E. Lerner, "The space of Lorentz metrics," *Communications in Mathematical Physics* **32** (1973) 19–38.

[34] G. Rudolph, M. Schmidt, and I. P. Volobuev, "On the gauge orbit space stratification: a review," *Journal of Physics A: Mathematical and General* **35** (2002) R1, arXiv:hep-th/0203027.

[35] W. M. Seiler, *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*, vol. 24 of *Algorithms and Computation in Mathematics.* Springer, 2010.

[36] M. A. Shifman (ed.), *Instantons in Gauge theories.* World Scientific, Singapore, 1994.

[37] D. C. Spencer, "Overdetermined systems of linear partial differential equations," *Bulletin of the American Mathematical Society* **75** (1969) 179–240.

[38] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations.* Cambridge University Press, Cambridge, 2003.

[39] J. M. Stewart and M. Walker, "Perturbations of Space-Times in general relativity," *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **341** (1974) 49–74.

[40] R. Thom, "Un lemme sur les applications différentiables," *Boletín de la Sociedad Matemática Mexicana (2)* **2** (1956) 59–71.

[41] L. Vitagliano, "Secondary calculus and the covariant phase space," *Journal of Geometry and Physics* **59** (2010) 426–447, arXiv:0809.4164.

[42] C. A. Weibel, *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.

IOP Publishing

Classical and Quantum Gravity

Class. Quantum Grav. **32** (2015) 185019 (19pp)

doi:10.1088/0264-9381/32/18/185019

Local and gauge invariant observables in gravity

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Received 24 April 2015, revised 15 July 2015
Accepted for publication 22 July 2015
Published 2 September 2015

Abstract

It is well known that general relativity (GR) does not possess any non-trivial local (in a precise standard sense) and diffeomorphism invariant observable. We propose a generalized notion of local observables, which retain the most important properties that follow from the standard definition of locality, yet is flexible enough to admit a large class of diffeomorphism invariant observables in GR. The generalization comes at a small price—that the domain of definition of a generalized local observable may not cover the entire phase space of GR and two such observables may have distinct domains. However, the subset of metrics on which generalized local observables can be defined is in a sense generic (its open interior is non-empty in the Whitney strong topology). Moreover, generalized local gauge invariant observables are sufficient to separate diffeomorphism orbits on this admissible subset of the phase space. Connecting the construction with the notion of differential invariants gives a general scheme for defining generalized local gauge invariant observables in arbitrary gauge theories, which happens to agree with well-known results for Maxwell and Yang–Mills theories.

Keywords: local observables, differential invariants, QFT in curved spacetime

1. Introduction

The goal of this note is to outline a connection between the theory of differential invariants and local observables in gauge theories, in the sense of classical and quantum field theory. The main example we will treat is gravity, or more precisely general relativity (GR) possibly coupled to matter fields, which is a gauge theory with diffeomorphisms as the group of gauge transformations. The differential invariants in this case are essentially scalars that can be tensorially constructed from the Riemann curvature tensor and its covariant derivatives. The core idea of the connection to local observables appeared already in the proposal of Bergmann and Komar [2, 3]. However, it seems, that the idea has never been taken to the logical conclusion that we intend to sketch below.

Consider the theory of a, say scalar, field ϕ on an n-dimensional spacetime manifold M. The prototypical example of a local observable in this theory is a smeared field

$$\phi(f) = \int_M \phi(x)f(x), \tag{1}$$

where the smearing test function $f \in \Omega^n(M)$ is C^∞ with compact support. Those last two properties are key to making $\phi(f)$ a useful observable. Classically, an observable $F: \Phi \mapsto F(\Phi)$ is a map from field configurations to real numbers. A smeared field acts as $\phi(f): \Phi \mapsto \int_M \Phi(x)f(x)$. The compactness of the support of f makes sure that this integral converges for an arbitrary field configuration, so that $\phi(f)$ has a large domain of definition on the phase space of the theory (on all of it, in this case). The smoothness of f makes sure that the Poisson bracket

$$\{\phi(f), \phi(g)\} = \int_{M\times M} f(x)E(x, y)g(y), \tag{2}$$

where $E(x, y)$ is the distributional kernel of the Peierls formula and $\phi(g)$ is a similar smeared field, is well defined as a distributional integral. Compact support also helps with the convergence of the Poisson bracket integral. Quantum mechanically, the field $\phi(x)$ is promoted to an operator valued distribution. The smoothness of the smearing function f is then essential to get an honest (though unbounded) operator corresponding to $\phi(f)$. The expectation values of products of smeared fields like

$$\langle\phi(f)\phi(g)\rangle = \int_{M\times M} \langle\phi(x)\phi(y)\rangle f(x)g(y), \tag{3}$$

are also distributional integrals with respect to the 2-point singular kernel $\langle\phi(x)\phi(y)\rangle$. Thus, the smoothness of f and g are again necessary to make sure that this integral is locally well-defined (UV finite), with their compact support ensuring its global convergence (IR finiteness). In short, we say that the smoothness of test functions, like f, *diffuses the UV singularities* of local fields, like $\phi(x)$, and their compact support *IR regularizes them*.

An immediate generalization is the notion of a *multilocal observable*, which is given by a formula of the form

$$\int_{M^m} \phi(x_1)\cdots\phi(x_m)f(x_1, \ldots, x_m), \tag{4}$$

where the smearing test function $f \in \Omega^{mn}(M^m)$ is C^∞ with compact support. It should be noted that the Poisson bracket of two local observables, as defined by equation (2), is in general no longer a local observable. Rather, as in the example of $\phi^2(f) = \int_M \phi^2(x)f(x)$, it is (almost) bilocal (multilocal with $l = 2$),

$$\left\{\phi^2(f), \phi^2(g)\right\} = \int_{M\times M} 2\phi(x)f(x)\, E(x, y)\, 2\phi(y)g(y), \tag{5}$$

with the caveat that the smearing function $f(x)E(x, y)g(y)$ is a distribution and could be non-smooth. Thus, another natural generalization that invites itself is that of multilocal observables with distributional smearing, though the identification of the class of distributions that can be consistently allowed becomes rather technical. We mention these generalizations only for completeness, with the remainder of this note concentrating on local observables with smooth smearings. However, we do briefly come back to multilocal observables in sections 5 and 6.

In the case when $\phi(x)$ is a local field in a gauge theory, another important property demanded of a local observable like $\phi(f)$ is *gauge invariance*. That is, the value of $\phi(f)$ (numerical value classically and operatorial value quantum mechanically) stays invariant under the action of gauge transformations. Any physically meaningful quantity may only be represented by a gauge invariant observable. It is common knowledge that, in gravitational theories, the set of local gauge invariant observables is trivial (see for instance [16] or [7], for a clear discussion). Such a statement can of course be made once a suitably precise notion of locality and gauge invariance is given, as we do in section 2. On the other hand, a slight relaxation of that standard notion of locality, which we propose in section 3, opens the door to the introduction in section 4 of a large class of gravitational observables that are gauge invariant (thanks to the use of differential invariants), diffuse UV singularities and are IR regularizing. Finally, we address the computation of Poisson brackets between generalized local gauge invariant gravitational observables in section 5. Ultimately, we propose to treat this generalized notion as the true definition of local observables.

In the rest of the note we discuss only classical observables. Comments on how the constructions outlined below impact perturbative quantum field theory are left for the discussion in section 6, where we also mention various limitations and open problems of our proposal.

We finish this section with a brief historical remark. The idea of constructing observables in gravitational theories based on differential invariants (curvature scalars) first appeared clearly in the works of Bergmann and Komar [2, 3]. Unfortunately, they never published a computation of Poisson brackets for such observables. Such computations appeared first in the work of DeWitt [12], who used the Peierls bracket formalism. Since then, related ideas have appeared sporadically in the literature, more recently referred to as relational observables [34]. Some ideas in spirit similar to those presented below can also be found in [16] and [7], with the latter following-up a slightly different line of ideas that attempted to expand the notion of local obsrvables by modifying the notion of gauge invariance [15, 31].

2. Standard local observables in field theories

Let us briefly set up the geometric formalism of classical field theory. We will mostly follow the references [21, 23], with [6, 8, 15, 18] being complementary sources. We take M to be an oriented n-dimensional smooth manifold. Usually one endows M with a Lorentzian metric, but we are working at a level of generality where that is not necessary. Take a vector bundle $F \to M$, the *field bundle*, and denote its sections as $\Phi: M \to F$, a *field configuration*. In more generality, $F \to M$ could be a more general smooth bundle, but we will stick to the vector bundle case for simplicity.

By $\pi^k: J^kF \to M$, for $k = 0, 1, \ldots, \infty$, we denote the bundle of k-jets of the field bundle $F \to M$. Jets[1] naturally and geometrically capture information about higher derivatives of sections of $F \to M$ over a point of M. Given a k-jet, throwing away all the information about order-k derivatives gives a $(k-1)$-jet. In other words, we have natural bundle projections $\pi^k_{k-1}: J^kF \to J^{k-1}F$ over M, until we get $J^0F = F$. Any section $\Phi: M \to F$ can be naturally augmented with the information about its derivatives (its jet) at every point of M, thus defining the *k-jet extension* section $j^k\Phi: M \to J^kF$. To be more concrete, consider a fiber-adapted local coordinate system (x^i, ϕ^a) on F. It induces an adapted local coordinate system (x^i, ϕ^a_I) on J^kF over that on F, where $I = \varnothing, i, ij, \ldots$ ranges across all possible multi-indices.

[1] Jets are a standard construction in differential geometry. An introduction to jets, operations on them and their applications to differential equations can be found in [28]. See also the relevant appendices to [21, 23].

The coordinate system is adapted in the sense that the following identity holds for any field section Φ:

$$\phi^a_{i_1\cdots i_l}\left(j^k\Phi(x)\right) = \partial_{i_1}\cdots\partial_{i_l}\phi^a(\Phi(x)). \tag{6}$$

Next, we introduce the *field configuration space* $\mathscr{C} = \Gamma(F)$, consisting of smooth sections of the vector bundle $F \to M$. It is an infinite dimensional vector space. It is convenient to endow it with the Whitney weak topology, which gives it the structure of a Fréchet space [20, 24]. Unfortunately the Whitney weak topology is too coarse for some of our purposes (its fundamental neighborhoods do not control the behavior of sections toward the open ends of non-compact manifolds), so we will mostly make use of the Whitney strong topology (see the discussion in section 3). Further, the equations of motion of the field theory (e.g., Klein–Gordon equation for a scalar field or Einstein's equations for the gravitational field) select the subspace of solutions, $\mathscr{P} \subset \mathscr{C}$, which we refer to as the (*covariant*) *phase space*. For nonlinear equations, $\mathscr{P}$ is in general not a linear subspace of $\mathscr{C}$; however, we will presume that $\mathscr{P}$ has a well-defined Fréchet manifold structure induced by its inclusion as a submanifold of the Fréchet space $\mathscr{C}$. We are ultimately interested in the *algebra of observables* $C^\infty(\mathscr{P})$. However, it is often more convenient to discuss elements of $C^\infty(\mathscr{P})$ as images of elements of $C^\infty(\mathscr{C})$ under the projection induced by the inclusion $\mathscr{P} \subset \mathscr{C}$. We make the simplifying assumption that this inclusion is sufficiently regular for the projection to be surjective. Then, strictly speaking, observables correspond to equivalence classes of elements of $C^\infty(\mathscr{C})$. However, we will not need to make use of this distinction below and may also refer to elements of $C^\infty(\mathscr{C})$ as *observables*, or alternatively as *functionals*.

On $\mathscr{C}$, we can define a special class of functions called *local functionals* (*or observables*) with the help of *horizontal forms* on J^kF. Horizontal forms, whose space we denote as $\Omega^{p,0}(F, k) \subset \Omega^p(J^kF)$, are generated as linear combinations from the pullback $(\pi^k)^*\Omega^p(M)$ of forms on the spacetime with coefficients from $C^\infty(J^kF)$, meaning they are of the form $\alpha_{i_1\cdots i_k}(x^i, \phi^a_I)\mathrm{d}x^{i_1}\cdots\mathrm{d}x^{i_k}$. Of course, elements of $\Omega^{p,0}(F, k)$ can be pulled back to $\Omega^{p,0}(F, l)$ along the natural jet projections $J^lF \to J^kF$ for any $l > k$. It is convenient to take the *increasing union* (or *direct limit*) $\Omega^{p,0}(F) = \oplus_{k=0}^\infty\ \Omega^{p,0}(F, k)/\sim$, where the equivalence relation identifies a form in $\Omega^{p,0}(F, k)$ with its pullback to any higher jet bundle, so that we do not need to worry about the order k when it is not necessary. We call elements of $\Omega^{n,0}(F)$ *horizontal densities*. For any form $\alpha \in \Omega^p(J^kF)$, we define its *spacetime support* as the closure of the projection of its support onto M, $\mathrm{supp}_M\alpha = \overline{\pi^k\mathrm{supp}\,\alpha}$.

It is helpful to note that any form $\beta \in \Omega^p(J^kF)$ can be projected to a horizontal form $\mathsf{h}[\beta] = \alpha \in \Omega^{p,0}(F, k+1)$, where the map acts as $\mathsf{h}[\mathrm{d}x^i] = \mathrm{d}x^i$ and $\mathsf{h}[\mathrm{d}\phi^a_I] = \phi^a_{Ii}\mathrm{d}x^i$ on coordinate forms, extends linearly and respects the wedge product. Another convenient operator to define is the Euler–Lagrange derivative δ_{EL} of a horizontal density $\alpha \in \Omega^{n,0}(F)$. Locally, we define $\delta_{\mathrm{EL}}[\alpha]$ by the following identity on M:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(j^k(\Phi + t\Psi)\right)^*\alpha(x) = \left(j^k\Phi\right)^*\delta_{\mathrm{EL}}[\alpha]_a(x)\Psi^a(x) + \mathrm{d}\xi[\Phi;\Psi], \tag{7}$$

where each $\delta_{\mathrm{EL}}[\alpha]_a \in \Omega^{n,0}(F)$ and ξ is some differential operator that depends linearly on its second argument. Globally, $\delta_{\mathrm{EL}}[\alpha]$ is a horizontal density valued in the dual bundle $F^* \to M$. By the usual methods of variational calculus, this relation makes $\delta_{\mathrm{EL}}[\alpha]$ unique and well-defined. All of these constructions, and more, naturally live in the context of the *variational bicomplex* [28], of which we shall not need to make further use in this note.

To any horizontal density $\alpha \in \Omega^{n,0}(F)$ with compact spacetime support, we can associate a functional

$$A[\Phi] = \int_M \left(j^k\Phi\right)^* \alpha. \tag{8}$$

If, in local adapted coordinates, we have $\alpha = \tilde{\alpha}(x^i, \phi^a, \phi^a_i, \phi^a_{ij}, \dots)\mathrm{d}^n x$, then

$$A[\Phi] = \int_M \tilde{\alpha}\Big(x^i, \phi^a(\Phi(x)), \partial_i\phi^a(\Phi(x)), \partial_i\partial_j\phi^a(\Phi(x)), \dots\Big)\mathrm{d}^n x. \tag{9}$$

It is straightforward to verify that, by the compact spacetime support condition, the above integral converges for an arbitrary field configuration $\Phi \in \mathscr{C}$ and in fact $A \in C(\mathscr{C})$. Of course, we would like A to be not only continuous, but also in some sense smooth on the infinite dimensional manifold $\mathscr{C}$. It is in fact possible to make use of an infinite dimensional calculus on Fréchet manifolds such that $A \in C^\infty(\mathscr{C})$ [8, 15, 24]. We will not enter into such details, and simply declare functions like A to be in $C^\infty(\mathscr{C})$. The class of functions on $\mathscr{C}$ defined by an equation like (8) will be referred to as *local functionals*.

On the other hand, given an element $A \in C^\infty(\mathscr{C})$, we can define a notion of spacetime support that can be attributed directly to A. If A is local and comes from a horizontal density α, there will of course be a relation between these two notions of support. More precisely, we define [6 equation (5.22)]

$$\begin{aligned}\operatorname{supp} A = \{x \in M \mid \forall \text{ open } U \ni x \quad \exists\, \Phi, \Psi \in \mathscr{C}:\\ \operatorname{supp}\Psi \subseteq U \text{ and } A[\Phi + \Psi] \neq A[\Phi]\},\end{aligned} \tag{10}$$

which is always closed. In words, for any point $y \in M$ outside $\operatorname{supp} A$, there is a sufficiently small neighborhood $V \ni y$ so that any perturbation Ψ of the argument of $A[\Phi]$ with $\operatorname{supp}\Psi \subseteq V$ must leave the numerical value of A unchanged, that is, $A[\Phi + \Psi] = A[\Phi]$. In other words, $A[\Phi]$ does not depend on the value of Φ in some neighborhood of y.

As mentioned above, we can give a precise relation between the spacetime support of a horizontal density and that of the corresponding local functional. Recall the Euler–Lagrange derivative $\delta_{\mathrm{EL}}[\alpha]$ of a horizontal density α defined by equation (7). Since $\delta_{\mathrm{EL}}[\alpha]$ is not strictly speaking a horizontal density, we extend to it the notion of spacetime support so that $\operatorname{supp}_M\delta_{\mathrm{EL}}[\alpha]$ is the union of the spacetime supports $\operatorname{supp}_M\delta_{\mathrm{EL}}[\alpha]_a$ of its components.

Lemma 2.1. *Let $\alpha \in \Omega^{n,0}(F)$ be a horizontal density with compact spacetime support and $A[\Phi] = \int_M (j^k\Phi)^*\alpha$. Then*

$$\operatorname{supp}_M\delta_{\mathrm{EL}}[\alpha] \subseteq \operatorname{supp} A \subseteq \operatorname{supp}_M\alpha. \tag{11}$$

Proof. The second inclusion is trivial, because $(j^k(\Phi + \Psi))^*\alpha = (j^k\Phi)^*\alpha$ whenever $\operatorname{supp}\Psi$ is outside of $\operatorname{supp}_M\alpha$, since the restriction of both sides of the equality to $\operatorname{supp}\Psi$ is simply zero. The rest, namely $\operatorname{supp} A \subseteq \operatorname{supp}_M\alpha$, follows from the defining equation (10).

On the other hand, suppose that $p \in \operatorname{supp}\delta_{\mathrm{EL}}[\alpha] \subseteq J^kF$. Then, we can always find a section $\Phi \in \mathscr{C}$ such that $j^k\Phi(x) = p$, where $x = \pi^k(p) \in \operatorname{supp}_M\delta_{\mathrm{EL}}[\alpha]$. Since by construction $(j^k\Phi)^*\delta_{\mathrm{EL}}[\alpha](x) \neq 0$, for each open $U \ni x$ there must exist a (without loss of generality compactly supported) $\Psi \in \mathscr{C}$ with $\operatorname{supp}\Psi \subseteq U$ such that

$$\int_M \left(j^k\Phi\right)^* \delta_{\mathrm{EL}}[\alpha]_a(x)\Psi^a(x) \neq 0. \tag{12}$$

Therefore, by continuity in t, the formula in equation (7) tells us that there must exist a $t \neq 0$, however small, such that $A[\Phi + t\Psi] \neq A[\Phi]$. That concludes the proof that $\operatorname{supp}_M\delta_{\mathrm{EL}}[\alpha] \subseteq \operatorname{supp} A$. □

3. Generalized local observables

A precise notion of a local functional on the space $\mathscr{C}$ of field configurations on a field bundle $F \to M$ was given in section 2. This notion is plenty sufficient to identify a rich set of observables in the usual relativistic field theories, including gauge theories like Maxwell electrodynamics and Yang–Mills theory, but notably excluding gravitational theories like GR or GR with matter fields. The reason gravitational theories are different is because, as will be discussed in section 4, the intersection between the space of local functionals and gauge invariant functionals on $\mathscr{C}$ is trivial (it consists only of constant functions). On the other hand, we can relax the above notion of locality in a precise way, without sacrificing much in the way of the physical motivation that lead to it, such that the new class of generalized local functionals does admit a rich set of gauge invariant observables even in gravitational theories. We discuss this precise generalized notion of locality below and leave the applications to gravitational theories to section 4.

The two main properties of local functionals that we would like to relax are the (a) global domain of definition and (b) field independent compactness of support. We explain both of these properties and how they could be relaxed below.

Any element $A \in C^\infty(\mathscr{C})$, by definition, gives a well-defined value $A[\Phi]$ for any $\Phi \in \mathscr{C}$. That is, the domain of definition of A is all of $\mathscr{C}$ (it is *global*). Imagine, on the other hand, that A is defined only on a subset $\mathscr{U} \subseteq \mathscr{C}$. Could then A still play the role of a physically meaningful observable? The answer is a qualified *yes*, provided $\mathscr{U}$ is sufficiently large, for example an open set. Such a restriction may be necessary if, for instance, we have precise control only over solutions that are not too distant from a reference solution[2], some $\Phi \in \mathscr{U}$. At the classical level, having A and B defined on an open neighborhood $\mathscr{U} \ni \Phi$ is sufficient to compute their Poisson brackets[3] at Φ because that involves only local, differential operations. Perturbative quantum field theory (QFT) about Φ will also not be sensitive to anything outside an arbitrary neighborhood. Eventually, a non-perturbative formulation of a QFT would likely require observables to be globally defined. However, even then, we are likely to be interested in quantum states that (e.g., in a phase space formulation of quantum theory) would assign negligible weight to solutions outside a neighborhood $\mathscr{U}$ of some reference solution Φ. To accommodate such an eventual situation, we could globalize the domain of definition of $A \in C^\infty(\mathscr{C})$ by extending it in an arbitrary, though controlled, way to all of $\mathscr{C}$ using standard geometric tools, like the Tietze extension and Steenrod–Wockel approximation theorems [37].

Given that we would like the domain $\mathscr{U} \subset \mathscr{C}$ of a generalized local functional to be open, it is important to reflect on the topology that we use on $\mathscr{C}$. Technical details on various topologies on function spaces can be found in the references [20, 24]. It was stated in the Introduction that it is conventional to endow $\mathscr{C} = \Gamma(F)$ with the *Whitney weak topology*, whose open sets are generated by those of the form

$$\mathscr{U}^k_{K,U} = \left\{ \Phi \in \Gamma(F) \,|\, j^k\Phi(K) \subseteq U \right\}, \tag{13}$$

where $k \geqslant 0$, $K \subseteq M$ is compact and $U \subseteq J^kF$ is open. The big disadvantage of the weak topology is that its neighborhoods cannot control the behavior of a section outside of a

[2] An example of this kind is the celebrated result of Christodoulou and Klainerman [9] on the stability of Minkowski space in GR. Their result essentially constructs an open neighborhood $\mathscr{U}$ of the Minkowski metric on the phase space $\mathscr{P}$ of GR on $\mathbb{R}^4$ with asymptotically flat boundary conditions. On the other hand, we still have very little information about $\mathscr{P}$ outside that neighborhood.

[3] Strictly speaking, Poisson brackets are expected to be defined only upon restriction to the phase space $\mathscr{P} \subset \mathscr{C}$. However, it is sometimes possible to lift Poisson brackets even to $\mathscr{C}$. This will be discussed in more detail in section 5.

compact subset of the spacetime M, as we will need to do in the sequel. However, except in some cases when boundaries are present, the spacetimes that are of physical interest are non-compact. For example, any globally hyperbolic spacetime must be of the form $M \cong \mathbb{R} \times \Sigma$. An alternative topology is the *Whitney strong topology*, whose open sets are generated by those of the form

$$\mathscr{U}_U^k = \left\{ \Phi \in \Gamma(F) | j^k \Phi(M) \subseteq U \right\}, \tag{14}$$

where $k \geqslant 0$ and $U \subseteq J^k F$ is open. The big disadvantage of the strong topology is that it is incompatible with the structure of a topological vector space on $\mathscr{C}$ (multiplication by scalars fails to be continuous), let alone a Fréchet or any other kind of manifold structure. Note, though, that since our manifolds can be exhausted by compact sets, any open set in the strong topology is at worst a G_δ set in the weak topology (a countable intersection of open sets). Fortunately, there are many intermediate topologies between the weak and the strong that both allow a Fréchet structure and control the behavior of sections on all of M. One example is a variation on the strong topology that allows only those open $U \subseteq J^k F$ that have 'uniform' vertical size over M with respect to some connection, such as one induced by an auxiliary Riemannian metric. Another possibility is to add a compactifying boundary to M and restrict our attention only to those sections that extend in some nice way to the boundary[4], then using the weak topology on that subspace with respect to the compactified spacetime M. However, it does not seem that there is an *a priori* canonical choice of such an intermediate topology and that the choice must be made in a way that is compatible with the behavior of solutions of the equations of motion of the theory. Note that a similar discussion, and in a related context, can be found in section 5.2.1 of [21].

Being pragmatic, we stick to the Whitney strong topology for the remainder of this note, despite its drawbacks. The working hypothesis is that the results that will be found in the sequel, and the methods used to obtain them, will naturally generalize to the appropriate choice of intermediate topology.

Next, having taken the liberty of considering functionals that are defined only on open subsets $\mathscr{U} \subseteq \mathscr{C}$, let us consider the difference between the spacetime supports of a functional $A \in C^\infty(\mathscr{C})$ and its restriction $A|_\mathscr{U} \in C^\infty(\mathscr{U})$. We can reasonably define $\operatorname{supp} A|_\mathscr{U}$ by replacing $\mathscr{C}$ with $\mathscr{U}$ in the definition (10). The logical quantifiers are arranged such that $\operatorname{supp} A|_\mathscr{U} \subseteq \operatorname{supp} A$. In fact, we can define the even finer notion of *spacetime support at* Φ *with respect to* $\mathscr{U}$ given by

$$\begin{aligned} \operatorname{supp}_\Phi A|_\mathscr{U} = \{x \in M | \ \forall \ \text{ open } U \ni x \ \ \exists \ (\Phi + \Psi) \in \mathscr{U}: \\ \operatorname{supp} \Psi \subseteq U \text{ and } A[\Phi + \Psi] \neq A[\Phi]\}, \end{aligned} \tag{15}$$

which is also always closed. A further refinement is the notion of *spacetime support at* Φ given by

$$\operatorname{supp}_\Phi A = \bigcap_\mathscr{U} \operatorname{supp}_\Phi A|_\mathscr{U}, \tag{16}$$

with the intersection taken over all open neighborhoods $\mathscr{U} \ni \Phi$ such that A is defined on $\mathscr{U}$. The distinction is that while $\operatorname{supp}_\Phi A|_\mathscr{U}$ depends on the domain $\mathscr{U}$, $\operatorname{supp}_\Phi A$ only depends on the germ of A at Φ. Then $\bigcup_{\Phi \in \mathscr{U}} \operatorname{supp}_\Phi A \subseteq \operatorname{supp} A|_\mathscr{U}$ and $\operatorname{supp} A|_\mathscr{U} = \bigcup_{\Phi \in \mathscr{U}} \operatorname{supp}_\Phi A|_\mathscr{U}$. So, clearly, $\operatorname{supp} A|_\mathscr{U}$ may fail to be compact, even if each individual $\operatorname{supp}_\Phi A|_\mathscr{U}$ or $\operatorname{supp}_\Phi A$ is.

[4] Perhaps the simplest implementation of this idea is to consider a piece of a globally hyperbolic spacetime that is bounded by two compact Cauchy surfaces as a compact spacetime in its own right with the future and past Cauchy surfaces as its boundaries.

For $A|_{\mathscr{U}}$ to be IR regularizing, as discussed in the Introduction, it suffices that the spacetime supports $\mathrm{supp}_\Phi A$ be compact for each $\Phi \in \mathscr{U}$. Thus, the much stronger condition of compact $\mathrm{supp}\, A|_{\mathscr{U}}$ for an observable $A|_{\mathscr{U}}$, while obviously sufficient for IR regularity, is not necessary. Such a relaxation of the requirements on the field-dependent spacetime support of observables was previously considered in [21 section 5.3.5] (see also [32]).

At the level of local functionals, we can relax the notion of locality given in section 2 in the following way. Let $\Phi \in \mathscr{C}$ be a field configuration and $\alpha \in \Omega^{n,0}(F)$ be a horizontal density such that the intersection $j^k\Phi(M) \cap \mathrm{supp}\,\alpha \subseteq J^k F$ is compact. Then we call the functional

$$A[\Psi] = \int_M \left(j^k\Psi\right)^* \alpha \tag{17}$$

a *generalized local functional* (*or observable*) *at* Φ. The following result makes precise the way in which the properties of the functional A fit with the preceding discussion.

Theorem 3.1. *With Φ and α as above, there exists an open $\mathscr{U} \subseteq \mathscr{C}$ (in the strong topology) with $\Phi \in \mathscr{U}$ such that, for all $\Psi \in \mathscr{U}$, the integral in (17) is convergent and both* $\mathrm{supp}_\Psi A|_{\mathscr{U}}$ *and* $\mathrm{supp}_\Psi A$ *are compact.*

Proof. Pick a compact neighborhood Q of $K = \pi^k(j^k\Phi(M) \cap \mathrm{supp}\,\alpha)$ and an open neighborhood $U \subset J^k F$ of $j^k\Phi(M\backslash Q)$ that does not intersect $\mathrm{supp}\,\alpha$. Let $\mathscr{U} \subseteq \mathscr{C}$ be the set of all sections $\Psi\colon M \to F$ such that $j^k\Psi(M\backslash Q) \subset U$. Clearly, $\Phi \in \mathscr{U}$ and, by the definition of the Whitney strong topology, $\mathscr{U}$ is open. By construction, for any $\Psi \in \mathscr{U}$, we have $(j^k\Psi)^*\alpha = 0$ on $M\backslash Q$. This means that $\mathrm{supp}\,[(j^k\Psi)^*\alpha] \subseteq Q$ and is itself compact (by virtue of being a closed subset of a compact set) and hence the integral defining $A[\Psi]$ is convergent. Finally, from the definition of $\mathscr{U}$, any point $x \in M\backslash Q$ has a neighborhood $V \subseteq M\backslash Q$ such that any Δ that has $\mathrm{supp}\,\Delta \subseteq V$ and with $\Psi + \Delta \in \mathscr{U}$ and must satisfy $j^k(\Psi + \Delta)^*\alpha = 0$ on V and hence $A[\Psi + \Delta] = A[\Psi]$. Therefore $\mathrm{supp}_\Psi A|_{\mathscr{U}} \subseteq Q$ and hence is itself compact. Its subset $\mathrm{supp}_\Psi A \subseteq \mathrm{supp}_\Psi A|_{\mathscr{U}}$ is closed and hence also compact. □

4. Gauge invariance and local observables in gravitational theories

GR is the theory of a Lorentzian metric field G, so that $F = S^2 T^*M$, with the equation of motion (Einstein equation) specified by the Einstein–Hilbert Lagrangian, $\mathcal{L}[G] = R[G]\,\mathrm{vol}_G$, where $R[G]$ is the Ricci scalar and vol_G is the metric volume form. This Lagrangian also determines the gauge symmetries of the theory, which consist of diffeomorphisms of M acting by pullback on metrics, $G \mapsto \chi^*G$ for a diffeomorphism $\chi\colon M \to M$. Thus, the physical (or reduced) phase space of GR is the quotient $\bar{\mathscr{P}} = \mathscr{P}/\mathscr{G}$, where $\mathscr{P} \subset \mathscr{C}$ is the set of solutions of Einstein equations (usually also taken to be globally hyperbolic) and $\mathscr{G}$ is the group of gauge transformations (diffeomorphisms of M). The observables that we are really interested in are those that constitute the algebra $C^\infty(\bar{\mathscr{P}})$. As before, it is convenient to use the quotient map $\mathscr{P} \to \bar{\mathscr{P}}$ to identify $C^\infty(\bar{\mathscr{P}}) \subset C^\infty(\mathscr{P})$ with those observables that are invariant under the action of the group $\mathscr{G}$ of gauge transformations. We refer to any element $A \in C^\infty(\mathscr{P})$, or $C^\infty(\mathscr{C})$, as a *gauge invariant observable* (*or functional*) if it is left invariant by the action of $\mathscr{G}$, that is, $A[\chi^*G] = A[G]$ for any diffeomorphism $\chi\colon M \to M$. For our purposes, a *gravitational theory* is a field theory that involves a metric tensor G (though possibly other fields as well) and has the diffeomorphism group as the group $\mathscr{G}$ of gauge transformations. Clearly, GR is the representative example of a gravitational theory, but GR coupled to matter fields also falls into the same category. We will only consider pure GR below, but the discussion will also apply to more general gravitational theories.

It is a well-known folk result that GR does not have any *local* and *gauge invariant observables* in the standard sense of locality discussed in section 2. However, the main observation of this note is that there in fact do exist local and gauge invariant observables in the generalized sense discussed in section 3. The non-existence argument is pretty straightforward. Let α be a horizontal density on k-jets with $\operatorname{supp}_M\alpha$ compact and hence $A[G] = \int_M (j^k G)^*\alpha$ a local observable. A diffeomorphism $\chi\colon M \to M$ acts on it as

$$(\chi A)[G] = A\left[\chi^* G\right] = \int_M \left(j^k\left(\chi^* G\right)\right)^*\alpha = \int_M \left(j^k G\right)^*\left[\left(p^k\chi^*\right)^*\alpha\right], \tag{18}$$

where $p^k\chi^*\colon J^kF \to J^kF$ is the natural k-jet prolongation of the pullback action of a diffeomorphism on metrics $\chi^*\colon F \to F$. Clearly, the spacetime support of α transforms as $\operatorname{supp}_M[(p^k\chi^*)^*\alpha] = \chi(\operatorname{supp}_M\alpha)$. Thus, by lemma 2.1, the $\operatorname{supp}\chi A$ moves around on M under the action of diffeomorphisms. So, since we can choose χ such that $\operatorname{supp} A$ and $\operatorname{supp}\chi A$ do not coincide, the functionals A and χA themselves cannot coincide. In particular, no observable A can be gauge invariant if its spacetime support is different from M (diffeomorphisms act on M transitively). Spacetime manifolds of physical interest are never compact, hence no local observable (with, by definition from section 2, compact spacetime support) can be gauge invariant. Colloquially, this is phrased as follows: *gauge transformations of gravitational theories move spacetime points*. This property is in contrast with gauge theories of Maxwell or Yang–Mills type, where gauge transformations leave intact the spacetime support of observables, thus allowing local observables to be gauge invariant.

We now give an explicit example of a functional that is both gauge invariant and local in the generalized sense. Subsequently, we will outline a general method for constructing more examples of a similar kind. Let us restrict for the moment the dimension $\dim M = 4$. We will construct a horizontal density $\alpha \in \Omega^{4,0}(F)$ on J^3F. Let $W_{abcd} = W_{abcd}[G]$ and $\varepsilon_{abcd} = \varepsilon_{abcd}[G]$ denote respectively the Weyl and Levi-Civita tensors of the metric G. Then, define the dual Weyl tensor $W^{*}_{ab}{}^{cd} = W_{abc'd'}\varepsilon^{c'd'cd}$ and also the following curvature scalars

$$\begin{aligned} b^1 &= W_{ab}{}^{cd}W_{cd}{}^{ab}, & b^3 &= W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}, \\ b^2 &= W_{ab}{}^{cd}W^{*}_{cd}{}^{ab}, & b^4 &= W_{ab}{}^{cd}W_{cd}{}^{ef}W^{*}_{ef}{}^{ab}. \end{aligned} \tag{19}$$

We have essentially defined maps $b = (b^1, b^2, b^3, b^4)\colon J^kF \to \mathbb{R}^4$, for any $k \geqslant 2$. We will also use the notation (b^i) for the standard global coordinates on this target $\mathbb{R}^4$. It is sufficient for us to take $k = 3$ because we then want to define the horizontal density $\beta = \mathsf{h}[\mathrm{d}b^1 \wedge \mathrm{d}b^2 \wedge \mathrm{d}b^3 \wedge \mathrm{d}b^4] \in \Omega^{4,0}(F, 3) \subset \Omega^{4,0}(F)$. Choose a point $r \in \mathbb{R}^4$, and a function $f \in C^\infty(\mathbb{R}^4)$ with compact support, such that $r \in \operatorname{supp} f$ but $\operatorname{supp} f$ does not intersect any of the planes $b^i = 0$. Finally, we define the desired horizontal density $\alpha = f(b)\beta \in \Omega^{4,0}(F)$, which gives rise to the functional

$$A[G] = \int_M \left(j^k G\right)^*\alpha = \int_M \left(j^k G\right)^*\left(f\left(b^1, b^2, b^3, b^4\right)\mathsf{h}\left[\mathrm{d}b^1 \wedge \mathrm{d}b^2 \wedge \mathrm{d}b^3 \wedge \mathrm{d}b^4\right]\right). \tag{20}$$

By construction, α satisfies two important properties. First, there is a non-empty open set $\mathscr{U} \subseteq \mathscr{C}$ (in the strong topology) such that the form $(j^kG)^*\alpha$ is smooth and has compact support on M for any $G \in \mathscr{U}$. Thus, $A[G]$ is well-defined on $\mathscr{U}$ and hence constitutes a generalized local observable in the sense of section 3. The existence of such a domain $\mathscr{U}$ follows from a general result that will be discussed in theorem 4.2 (see also the comments thereafter). Second, $A[G]$ is invariant under the action of diffeomorphisms. That is, $(p^k\chi^*)^*\alpha = \alpha$ for any diffeomorphism $\chi\colon M \to M$, which implies $A[\chi^*G] = A[G]$ for any G on which the defining integral converges. The last invariance identity has to be used with a little bit of care, in that it only makes sense when both G and χ^*G belong to $\mathscr{U}$, the domain of definition of A. Since, *a priori* $\mathscr{U}$ is not guaranteed to be itself diffeomorphism invariant, that condition may not be satisfied for an arbitrary $G \in \mathscr{U}$. One way to get around this issue is to, very reasonably, declare A to be invariant under diffeomorphisms if $A[\chi^*G] = A[G]$ whenever both G, $\chi^*G \in \mathscr{U}$. Another way is to simply enlarge the domain to $\mathscr{U}' \supseteq \mathscr{U}$ to the smallest diffeomorphism invariant domain that contains $\mathscr{U}$. Clearly, if A is well defined on $\mathscr{U}$ it is also well defined on $\mathscr{U}'$. A note of caution for the second approach: while $\operatorname{supp}_\Phi A$, for any $\Phi \in \mathscr{U}$, is not altered by extending A from $\mathscr{U}$ to $\mathscr{U}'$, the inclusion $\operatorname{supp}_\Phi A|_{\mathscr{U}} \subseteq \operatorname{supp}_\Phi A|_{\mathscr{U}'}$ may be strict, with $\operatorname{supp}_\Phi A|_{\mathscr{U}'}$ possibly failing to be compact even if $\operatorname{supp}_\Phi A|_{\mathscr{U}}$ is.

In other words $A|_{\mathscr{U}} \in C^\infty(\mathscr{U})$ is a local and gauge invariant observable in the generalized sense of section 3.

The idea of using the curvature scalars b^i to define observables in pure gravity goes back to the proposal of Komar and Bergmann [2, 3]. However, these authors, as well as many subsequent ones who came back to this idea (see [34] and references therein), intended to use b^i as independent coordinates and simply express all other fields in terms of them. However, the resulting observables were often too singular in the sense discussed in the Introduction, since they would correspond to something like replacing our test function f with a δ-distribution. On the other hand, our addition of the integral and the smooth compactly supported function f and the definition (20) provides the diffusion of UV singularities and the IR regularization, again discussed in the Introduction, that are needed in the contexts of QFT and classical Poisson structure.

The key ingredients in the above construction were that we could choose the horizontal density $\alpha \in \Omega^{n,0}(F)$ to be invariant under the prolonged action of diffeomorphisms on J^kF and that we could choose such an α to have support on J^kF that intersects compactly the image of the prolongation $j^kG(M) \subset J^kF$ of a certain metric G. Natural questions arise. Are there more local and gauge invariant observables that could be defined in the same way? Are there sufficiently many such observables to separate points[5] on the physical phase space $\bar{\mathscr{P}}$ of GR?

The general mathematical context in which the answers must be sought is known as *differential invariant theory* [25, 26, 30]. Classical invariant theory is concerned with identifying functions on a $\mathscr{G}$-space (a space with an action of a group $\mathscr{G}$) that are invariant under the $\mathscr{G}$-action—these are the usual *invariants*. On the other hand, differential invariant theory is concerned with fiber preserving group actions (more generally *pseudogroup* or *groupoid* actions) on the total space of a bundle, like our field vector bundle $F \to M$ and the actions induced on $J^kF \to M$ by prolongation. Then, *differential invariants* (*of order* k) are functions on $J^kF \to M$ that are invariant under the group action. For our purposes, the field bundle of metrics is $F = S^2T^*M$ and the group is $\mathscr{G} = \mathrm{Diff}(M)$, consisting of diffeomorphisms $\chi\colon M \to M$, and acting by pullback $\chi^*\colon F \to F$. Differential invariants are then precisely the so-called *curvature scalars*, that is, scalar functions tensorially constructed out of the metric, the Riemann curvature tensors and its covariant derivatives. For example, the b^i defined in equation (19) are differential invariants of order $k = 2$. There are two ways of looking at

[5] A set of functions *separates* the points of a space if, for each pair of points, there exists at least one function that takes on different values at these points.

differential invariants: *algebraically* and *geometrically*. Most structural results are proven from the algebraic perspective. On the other hand, it is easier to see from the geometric perspective how to construct local gauge invariant observables similar to the example of equation (20).

The main structural algebraic result that we would like to mention is the so-called *Lie–Tresse* theorem, which dates back to the end of the 19th century, but was established in its global form only rather recently (see [25] and the references therein). This theorem is an analog of the *finite generation* results, originally due to Hilbert, in classical invariant theory [29]. For differential invariants, in addition to algebraic operations, we also need to allow differentiation to generate differential invariants of arbitrary orders from finite data. Before stating the result, let us recall the geometric formulation of differential equations in terms of jets; see [23 apx. B] for more details and references. A differential equation of order $m \geqslant 0$ is usually specified in *equational form*, $P[\psi] = 0$, where $P\colon \Gamma(F) \to \Gamma(E)$ is a possibly non-linear differential operator of order m that takes sections of a bundle $F \to M$ as arguments and output sections of some other vector bundle $E \to M$. Essentially equivalently, we can specify a differential equation of order m as submanifold $\mathcal{E} \subseteq J^m F$. Roughly, the set of all m-jets that satisfy $P = 0$ constitutes the subset $\mathcal{E}$ and inversely, any bundle map $P\colon J^m F \to E$ that is zero only on $\mathcal{E} \subseteq J^m E$ defines the corresponding differential operator. A particular example could be $\mathcal{E} = J^m F$, which corresponds to the trivial equation $0 = 0$. A differential equation $\mathcal{E} \subseteq J^m F$ has natural prolongations $\mathcal{E}^{(k)} \subseteq J^k F$ for all $k \geqslant m$, which corresponds to taking into account all equations of the form $\partial_{i_1} \cdots \partial_{i_{k-m}} P[\psi] = 0$ implied by $P[\psi] = 0$. The following result is a rough restatement (sufficient for the purposes of this note) of the precise results of theorems 1 and 2 of [25].

Proposition 4.1 (**Lie–Tresse**). *Consider a differential equation $\mathcal{E} \subseteq J^k F$ with gauge symmetry,*[6] *defined on a field bundle $F \to M$, with the action of gauge symmetries naturally prolongued to $J^k F \to M$. Assume that the equation and the gauge symmetry action satisfies a specific global algebro-geometric regularity condition (which is in fact satisfied by GR with diffeomorphisms as gauge symmetries). Then, there exists a finite order $l \geqslant 0$, a finite number of differential invariants (those left invariant by gauge transformations) I_j on $J^l F$, and a finite number of invariant differential operators D_i (such an operator acting on an invariant yields another invariant) such that any polynomial differential invariant of an arbitrary order $k \geqslant 0$ can be expressed as a polynomial in the generators I_j, possibly repeatedly differentiated by the D_i. Finally, for arbitrary order $k \geqslant 0$, the differential invariants separate the orbits of the gauge symmetry on a dense open subset $\mathring{\mathcal{E}}^{(k)} \subseteq \mathcal{E}^{(k)}$ consisting of generic orbits.*

More geometrically, we can look at differential invariants as follows. Consider the quotient spaces[7] $\mathcal{M}^k = J^k F/\mathrm{Diff}(M)$, known as the *moduli spaces of k-jets of metrics on M* [17], with the projections denoted by $\mu_k\colon J^k F \to \mathcal{M}^k$. Clearly, differential invariants are precisely the smooth functions on $J^k F$ that come from the pullback of continuous functions on $\mathcal{M}^k$, those that belong to $C^\infty(J^k F) \cap \mu_k^*[C(\mathcal{M}^k)]$. If $\mathcal{M}^k$ were a manifold, it would be sufficient to consider $C^\infty(\mathcal{M}^k)$ instead of $C(\mathcal{M}^k)$. However, while $\mathcal{M}^k$ is well-defined as a topological space, it is only a manifold on a dense open subset [17], say $\mathring{\mathcal{M}}^k \subset \mathcal{M}^k$. Outside $\mathring{\mathcal{M}}^k$, $\mathcal{M}^k$ contains orbifold-type singularities, which correspond to jets of metrics admitting non-trivial isometries. A further complication is that $\mathcal{M}^k$ is in general not Hausdorff. This means that there exist jets of metrics that cannot be distinguished by continuous scalar

[6] In the language of [25], this means that the equation is invariant under a pseudogroup action.

[7] At the moment, we are not making a notational distinction, but we are really only interested in the subset of $J^k F$ corresponding to the jets of Lorentzian metrics, thus excluding degenerate metrics and metrics of other signatures.

curvature invariants alone. This phenomenon is particular to Lorentzian (and other pseudo-Riemannian) metrics and is absent when consideration is restricted to only Riemannian metrics. The failure of the Hausdorff property can be traced back to the non-compactness of the orthogonal group in Lorentzian signature [17].

To connect the algebraic and geometric points of view, consider Einstein's equations prolonged to an arbitrary order $k \geqslant 2$ and represented as a submanifold $\mathcal{E}^{(k)} \subseteq J^k F$. Clearly, $\mathcal{E}^{(k)}$ is invariant under diffeomorphisms and so projects to $\mu_k \colon \mathcal{E}^{(k)} \to \mathcal{R}^k \subseteq \mathcal{M}^k$, with $\mathring{\mathcal{R}}^k = \mathcal{R}^k \cap \mathring{\mathcal{M}}^k$ a submanifold of $\mathring{\mathcal{M}}^k$. The polynomial differential invariants mentioned in proposition 4.1 are then functions on $\mathcal{R}^k$ and in fact separate the points of $\mathring{\mathcal{R}}^k$ and, by the Stone–Weierstrass theorem, generate $C^\infty(\mathring{\mathcal{R}}^k)$ by limits uniformly converging on compact sets.

Now we come to the main observation that prompted this note. The connection between differential invariants and local observables in the generalized sense of section 3 is most clearly seen with the help of the manifold $\mathring{\mathcal{M}}^k$. Namely, consider an n-form $\beta \in \Omega^n(\mathring{\mathcal{M}}^k)$ with compact support and the horizontal density $\alpha \in \Omega^{n,0}(F)$ obtained by the horizontal projection of the pullback of β, $\alpha = \mathsf{h}[\mu_k^* \beta]$. Letting $\mathscr{U} \subset \mathscr{C}$ be the subset of all metrics $G \colon M \to F$ such that $j^k G(M) \cap \operatorname{supp} \alpha$ is compact, we can define a local and gauge invariant observable with domain of definition $\mathscr{U}$ by the usual formula

$$A[G] = \int_M \left(j^k G\right)^* \alpha. \tag{21}$$

It is clearly gauge invariant, since by construction $(p^k \chi^*)^* \alpha = \alpha$. Further, it is clearly local in the generalized sense of section 3, provided that $\mathscr{U}$ is open and non-empty. These properties do hold because of the following.

Theorem 4.2. *Given a non-empty compact set $K \subset \mathring{\mathcal{M}}^k$, there exists a metric $G \in \mathscr{C}$ such that $(\mu_k \circ j^k G)^{-1}(K) \subseteq M$ is non-empty and compact. Further, such a metric G has an open neighborhood $\mathscr{U} \subset \mathscr{C}$ (in the strong topology) such that $\mu_k \circ j^k H(M) \cap K$ is compact for each $H \in \mathscr{U}$.*

Proof. First, we deal with the statement about existence. Let us ignore for the moment issues that might arise from non-trivial topology of M and assume that $M \cong \mathbb{R}^n$, with some fixed global coordinate system. Let $\eta \colon M \to F$ be the standard Minkowski metric in those global coordinates. Take a point $r \in K \subset \mathring{\mathcal{M}}^k$, a point $x \in M$ and an open neighborhood $U \subset M$ of x with compact closure. By construction, there is a jet $p \in J^k_x F$ such that $\mu_k(p) = r$. Consider the closed set $Q = (M \backslash U) \cup \{x\}$. Define $G^k_Q \colon Q \to J^k F$ so that $G^k_Q(x) = p \in J^k_x F$ and $G^k_Q(y) = j^k_y \eta \in J^k_y F$ for any $y \neq x$. By the Whitney extension theorem [24 section 22], there exists a metric $G \in \mathscr{C}$ such that $j^k G(x)|_Q = G^k_Q$, which we can choose to be everywhere Lorentzian (non-degenerate). Thus, $\mu_k \circ j^k G$ and K have at least the point r in common. On the other hand, by construction, the pre-images $(\mu_k \circ j^k G)^{-1}(K) \subset (\mu_k \circ j^k G)^{-1}(\mathring{\mathcal{M}}^k) \subset M$ must be contained in $\bar{U}$, which is compact. Hence,the pre-image of K must be compact, since it is closed and contained in $\bar{U}$. The same argument can be adapted without much difficulty to the case when M has a more complicated topology.

Second,we deal with the statement about an open neighborhood of $G \in \mathscr{C}$,which was constructed above. The following argument echos the proof of theorem 3.1. We will define $\mathscr{U} = \{H \in \mathscr{C} | j^k H(M) \subset U\}$, for some to be determined open neighborhood $U \subset F$ of

$j^k G(M)$. Obviously $G \in \mathscr{U}$ and $\mathscr{U}$ would be open in the strong topology. We build U as the pre-image of an open set $V \subseteq M \times \mathcal{M}^k$ with respect to the map $(\pi^k, \mu_k): J^k F \to M \times \mathcal{M}^k$. If V is an open neighborhood of the graph of $\mu_k \circ j^k G: M \to \mathcal{M}^k$, then U is an open neighborhood of $j^k G(M)$. The way we constructed G above, the intersection I of the set $M \times K$ with the graph of $\mu_k \circ j^k G$ is compact. Take an open neighborhood V' with a compact closure of I and let $V = M \times (\mathcal{M}^k \setminus K) \cup V'$. Thus, if $H \in \mathscr{U}$, the intersection of the graph of $\mu_k \circ j^k H$ with $M \times K$ must be confined to V', which has compact closure, and hence be compact. The last statement is equivalent to the pre-image $(\mu_k \circ j^k H)^{-1}(K) \subset M$ being compact, which concludes the proof. $\square$

Note that a direct application of the above theorem to the compact supp f appearing in the definition of the functional $A[G]$ given by equation (20), interpreted as a subset of $\mathring{\mathcal{M}}^2$, establishes the claimed existence of a non-empty open domain $\mathscr{U} \subseteq \mathscr{C}$, making $A|_{\mathscr{U}}$ a generalized local observable.

While we have concentrated on the case of gravitational theories, whose group of gauge transformations consists of diffeomorphisms, this method of defining gauge invariant local observables happens to reproduce the set of local observables for theories without gauge symmetries (the group of gauge symmetries is trivial) and those with gauge theories with gauge transformations that do not *move points*. Examples of the latter include the Maxwell and Yang–Mills theories. In the Maxwell theory, the basic differential invariant is the field strength. In the Yang–Mills case, the basic differential invariants are the compositions of the Lie algbra valued curvature forms composed with invariant polynomials on the Lie algebra. Smearing these basic invariants (or derivatives thereof) with compactly supported test functions reproduces the well-known standard local and gauge invariant observables in these theories [1].

We conclude this section by coming back to this natural question: are there enough local and gauge invariant observables in GR to separate the points of $\mathscr{C}$? In a sense, the answer is no, because we have already discussed above the fact that certain metrics cannot be distinguished by local curvature scalars. Further, some metrics may be resistant to belonging to the domain of definition $\mathscr{U}$ of any generalized local observable $A|_{\mathscr{U}}$. This may happen when M is non-compact and a metric G possesses a region $U \subseteq M$ such that nearly isometric copies of $G|_U$ repeat infinitely often throughout M (a kind of almost periodic property). There is essentially no obstacle to engineering a gauge invariant local density α on $J^k F$ such that $(j^k G)^* \alpha$ has compact support in U, but it will likely also have support within any region nearly isometric to $G|_U$, thus making the integral over M ill defined. However, these are the only obstacles. We need to introduce a natural but somewhat technical condition on metrics that avoids these difficulties.

First, we say that a map $\nu: M \to N$ is *image proper*[8] if there exists an open set $N_0 \subseteq N$ such that $\nu(M) \subseteq N_0$ and $\nu: M \to N_0$ is *proper* (the pre-image of any compact set is compact). Any proper map is image proper, since we can just choose $N_0 = N$. On the other hand, any embedding is image proper, even if it is not proper, with any tubular neighborhood fulfilling the role of N_0. Let us say that two metrics $G_1, G_2 \in \mathscr{C}$ can be *distinguished by curvature scalars* if there exists a $k \geqslant 0$ such that $\gamma_i = \mu_k \circ j^k G_i: M \to \mathcal{M}^k$ are image proper and the images $\gamma_1(M) \cap \mathring{\mathcal{M}}^k$ and $\gamma_2(M) \cap \mathring{\mathcal{M}}^k$ do not coincide as subsets of $\mathring{\mathcal{M}}^k$.

[8] See [20 exr.2.4.13], where this concept is used but not named.

Theorem 4.3. *For any two metrics $G_1, G_2 \in \mathscr{C}$ that can be distinguished by curvature scalars, there exists a local functional $A[G]$ defined on a domain $\mathscr{U} \subseteq \mathscr{C}$ (open in the strong topology) such that both $G_1, G_2 \in \mathscr{U}$ and $A[G_1] \neq A[G_2]$.*

Proof. By hypothesis, there is a $k \geqslant 0$ and a point $r \in \mathring{\mathcal{M}}^k$ such that, say, $r \in \gamma_1(M) = \mu_k \circ j^k G_1(M)$ but $r \notin \gamma_2(M) = \mu_k \circ j^k G_2(M)$. Take a $\beta \in \Omega^n(\mathring{\mathcal{M}}^k)$ with compact support such that $r \in \operatorname{supp} \beta$ but $\gamma_2(M) \cap \operatorname{supp} \beta = \varnothing$. Let $A[G] = \int_M (\mu_k \circ j^k G)^* \beta$. Since the map γ_1 is image proper, we can always choose β so that supp β is small enough to have compact intersection with $\gamma_1(M)$ and so that $A[G_1] \neq 0$. On the other hand, by construction, $A[G_2] = 0$. Finally, since both $\gamma_i(M)$ have compact intersection with supp β (one of the intersections being empty), by theorem 4.2, there exist (in the strong topology) open neighborhoods $\mathscr{U}_1$ and $\mathscr{U}_2$ of G_1 and G_2, respectively, such that $\mu_k \circ j^k G(M) \cap \operatorname{supp} \beta$ is also compact for each $G \in \mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$. Clearly, $A[G]$ is well defined on $\mathscr{U}$ and $G_1, G_2 \in \mathscr{U}$. □

5. Linearization and Poisson brackets

Once a class of gauge invariant observables has been defined, as was done in section 4, we would like to compute Poisson brackets between them. In general, neither the product nor the Poisson bracket of two local observables is a local observable (instead it is *bilocal*, with distributional smearing in case of the Poisson bracket) and the same is true for local observables in the generalized sense. It is an important and non-trivial question to decide on a minimal physically reasonable class of observables that is closed both under multiplication and Poisson brackets. The answer is essentially a class of multilocal observables with distributional smearings, which satisfy a certain *microlocal spectral condition*, which is discussed in more detail in [6, 8]. Below, we shall not be concerned with these details and instead content ourselves with a gauge invariant formula for the Poisson bracket of two local and gauge invariant observables.

As discussed extensively in [21, 23], what is usually known as the canonical Poisson bracket on the physical phase space $\bar{\mathscr{P}}$ can be equivalently expressed using the so-called *Peierls formula* (or *Peierls bracket*). The Peierls formula actually defines a Poisson bracket not only on $C^\infty(\bar{\mathscr{P}})$, but also extends it to $C^\infty(\mathscr{P})$ and even $C^\infty(\mathscr{C})$. This extension is not unique and is influenced, for instance, by the choice of gauge fixing. However, the restriction of the formula to $C^\infty(\bar{\mathscr{P}})$ is unique. The computation of the value of the Poisson bracket $\{A, B\}[G]$ of arbitrary observables A and B at a particular point (or gauge equivalence class of field configurations) $G \in \bar{\mathscr{P}}$ of a nonlinear field theory reduces to the computation of the Poisson bracket of linear observables $\dot{A}_G$ and $\dot{B}_G$ in the linear theory obtained by linearization about G. Consider the linearized perturbation H of the metric G. The relation between nonlinear observables and linearized observables is

$$A[G + \lambda H] = A[G] + \lambda \dot{A}_G[H] + O\left(\lambda^2\right). \tag{22}$$

In the case of a local observable $A[G] = \int_M (j^k G)^* \alpha$, the linearized observable is also local, $\dot{A}_G[H] = \int_M \dot{\alpha}[H]$, where $\dot{\alpha}$ is a density-valued differential operator defined by

$$\left(j^k(G+\lambda H)\right)^*\alpha = \left(j^k G\right)^*\alpha + \lambda\dot{\alpha}_G[H] + O\left(\lambda^2\right). \tag{23}$$

We can define similarly $B[G] = \int_M (j^k G)^*\beta$ and $\dot{B}_G[H] = \int_M \dot{\beta}_G[H]$.

It is also useful to consider the formal adjoint differential operators $\dot{\alpha}_G^*$ and $\dot{\beta}_G^*$ defined by the existence of form-valued bidifferential operators W_α and W_β such that

$$f\dot{\alpha}_G[H] - \dot{\alpha}_G^*[f]\cdot H = \mathrm{d}W_\alpha[f, H] \quad \text{and} \quad f\dot{\beta}_G[H] - \dot{\beta}_G^*[f]\cdot H = \mathrm{d}W_\beta[f, H] \tag{24}$$

for arbitrary $f \in C^\infty(M)$ and $H \in \Gamma(F)$, with the adjoint operators valued in the densitized dual bundle $\tilde{F}^* = F^* \otimes \Lambda^n M$. Let $\mathscr{U} \subseteq \bar{\mathscr{P}}$ be a common domain on which A and B are defined and let $G \in \mathscr{U}$. Then, by the generalized locality property, $\dot{\alpha}_G[H]$ and $\dot{\beta}_G[H]$ have compact support for arbitrary H. It is then not hard to see that all of $\dot{\alpha}_G^*[1]$, $W_\alpha[1, H]$, $\dot{\beta}_G^*[1]$ and $W_\beta[1, H]$ will also have compact support for arbitrary H. Therefore, an application of Stokes' lemma gives us the identities

$$\dot{A}_G[H] = \int_M \dot{\alpha}_G^*[1]\cdot H \quad \text{and} \quad \dot{B}_G[H] = \int_M \dot{\beta}_G^*[1]\cdot H. \tag{25}$$

The Peierls formula for the Poisson bracket of observables of the form given in equation (25) was considered explicitly in [23 section 4.4] (see also [13] and [18 Ex. 3.8]) and is given by the formula

$$\{A, B\}[G] = \left\{\dot{A}_G, \dot{B}_G\right\}_G = \int_{M\times M} \dot{\alpha}_G^*[1](x)\cdot E_G(x, y)\cdot \dot{\beta}_G^*[1](y), \tag{26}$$

where $E_G(x, y) = E_G^+(x, y) - E_G^-(x, y)$, with $E_G^\pm(x, y)$ being the integral kernels of the retarded and advanced Green functions of the so-called *Lichnerowicz operator* (which is a hyperbolic differential operator obtained from a de Donder gauge fixing of the linearized Einstein equations) of the background metric G.

The result is gauge invariant, that is, $\{A, B\}[\chi^*G] = \{A, B\}[G]$ for a diffeomorphism $\chi^*: M \to M$, essentially by construction. More explicitly, since each of the elements in the formula is invariantly constructed from the metric G, the following identities hold: $\dot{\alpha}_{\chi^*G}^* = \chi^*\dot{\alpha}_G^*$, $\dot{\beta}_{\chi^*G}^* = \chi^*\dot{\beta}_G^*$ and $E_{\chi^*G}(x, y) = (\chi, \chi)^*E_G(x, y)$, where $(\chi, \chi)^*: M \times M \to M \times M$ is defined in the obvious way. Combining these identities with formula (26) explicitly shows that $\{A, B\}$ is a gauge invariant (though now distributional bilocal, instead of local) observable.

It is also worth examining whether the linearized observable $\dot{A}_G[H]$ fits the criteria of being a gauge invariant observable for linearized gravity on the background G. The answer is of course yes, as follows from the identity $\mathcal{L}_v\alpha[G] = \dot{\alpha}[\mathcal{L}_v G]$, where $\mathcal{L}_v$ is the Lie derivative with respect to a vector field v, which is the linearized version of the invariance property $\chi^*\alpha[G] = \alpha[\chi^*G]$, and the Cartan magic formula $\mathcal{L}_v\alpha[G] = \mathrm{d}(\iota_v\alpha[G])$ for top-degree forms. For convenience, let us also define the differential operator $K_G[v] = \mathcal{L}_v G$, which we will call the *Killing operator*. The gauge invariance condition for $\dot{A}_G[H]$ in linearized gravity consists in the requirement that $\dot{A}_G[K_G[v]] = 0$ for any vector field v. This follows from the preceding identities:

$$\dot{A}_G\left[K_G[v]\right] = \int_M \dot{\alpha}_G\left[\mathcal{L}_v G\right] = \int_M \mathcal{L}_v\alpha[G] = \int_M \mathrm{d}\left(\iota_v\alpha[G]\right) = 0, \tag{27}$$

where the last equality follows from the fact that $\iota_v\alpha[G]$ has compact support by the locality hypothesis. Thus, $\dot{A}_G[H]$ is a linear, local and gauge invariant observable in linearized gravity.

Let us recall the notion of linear, local and gauge invariant observable from [13] (also [23 section 4.4], [18 Ex. 3.8]), which is an observable of the form

$$C[H] = \int_M \gamma \cdot H, \tag{28}$$

with a compactly supported section $\gamma: M \to \tilde{F}^*$ that satisfies the condition $K_G^*[\gamma] = 0$, where K_G^* is the formal adjoint of the Killing operator K_G. More explicitly, there exists a form-valued bidifferential operator W_K such that $\gamma \cdot K_G[v] - K_G^*[\gamma] \cdot v = \mathrm{d}W_K[\gamma, v]$ for any vector field v and any section $\gamma: M \to \tilde{F}^*$; K_G^* is equivalent to the divergence of a symmetric 2-tensor.

Proposition 5.1. *Given the linearized observable $\dot{A}_G[H]$, as discussed above, there always exists a local observable $C[H]$ in linearized gravity of the form (28) such that $\dot{A}_G[H] = C[H]$.*

Proof. For this result to hold, it is clearly sufficient that there exists a compactly supported section $\gamma: M \to \tilde{F}^*$, satisfying $K_G^*[\gamma] = 0$, and a form-valued linear differential operator $\mu[H]$, with compact support for arbitrary argument $H: M \to F$, such that $\dot{\alpha}_G[H] = \gamma \cdot H + \mathrm{d}\mu[H]$. We shall construct such γ and $\mu[H]$ explicitly.

Recall the identity $\dot{\alpha}_G[H] = \dot{\alpha}_G^*[1] \cdot H + \mathrm{d}W_\alpha[1, H]$. We set $\mu[H] = W_\alpha[1, H]$ and $\gamma = \dot{\alpha}_G^*[1]$. It remains to show that $K_G^*[\dot{\alpha}_G^*[1]] = 0$. Note that, from the gauge invariance of $\dot{A}_G[H]$ discussed earlier, we already know that

$$\begin{aligned} \dot{\alpha}_G^*[1] \cdot K_G[v] &= \dot{\alpha}_G\big[K_G[v]\big] + \mathrm{d}W_\alpha\big[1, K_G[v]\big] \\ &= \mathrm{d}\Big(\iota_v \alpha[G] + W_\alpha\big[1, K_G[v]\big]\Big) \end{aligned} \tag{29}$$

for an arbitrary vector field v. On the other hand, we also have the equality

$$\dot{\alpha}_G^*[1] \cdot K_G[v] = K_G^*\big[\dot{\alpha}_G^*[1]\big] \cdot v + \mathrm{d}W_K\big[\dot{\alpha}_G^*[1], v\big]. \tag{30}$$

The final tool that we need to invoke is the well-known fact [28 theorem 4.7] that, for any top-degree form valued linear differential operator $\psi[v]$, in any decomposition of the form $\psi[v] = \phi \cdot v + \mathrm{d}\xi[v]$ the coefficients ϕ and the term $\mathrm{d}\xi[v]$ are unique (in particular $\phi = \delta_{\mathrm{EL}}[\psi[v]]$ is the Euler–Lagrange derivative of $\psi[v]$). Thus, comparing equations (29) and (30), we find that $K_G^*[\dot{\alpha}_G^*[1]] = 0$, as was desired. □

6. Discussion

In this note, we have discussed the notion of local observables in field theory, advocating that the standard notion of locality (section 2) should be relaxed in a well-defined way (section 3). We have argued that the two motivating properties of local observables, diffusion of UV singularities and IR regularization, still hold for generalized local observables in the sense defined in section 3.

A small price to pay is that a generalized local observable may be naturally defined as functions only on an open subset[9] of the full phase space of the field theory. Classically, it is

[9] A related mathematical phenomenon occurs in complex and algebraic geometry. Certain complex and algebraic varieties have very few globally defined functions. By restricting to open subsets, many more functions can be considered, that otherwise developed singularities if extended to the entire space. Such partially defined functions are studied in the theory of *sheaves*. We have not developed this analogy in detail because there is not yet a clear application of sheaf theory in this context, other than as a concise terminology.

not a problem to restrict one's attention to an open subset of the full phase space. If needed, such an observable may be extended to the full phase space by appealing to basic results in differential topology. We have shown that linearization about a specific point of the configuration space gives a gauge invariant observable for linearized gravity on the corresponding background, irrespective of how large is the neighborhood of the linearization point on which the observable can be defined. That is of course the expected result for the linearization of an observable invariant under full nonlinear gauge transformations. We expect the same behavior at any order of perturbation theory; the truncated expansion of the observable should be invariant under perturbative gauge transformations truncated at the same order, which is sufficient for the purposes of perturbative quantization.

It is well-known that gravitational theories do not admit any non-trivial local observables that are also gauge-invariant. Hence, it is a significant advantage of the new definition that the class of generalized local observables in gravitational theories does admit a large number of observables that are gauge invariant (section 4). We have given a typical example of one such observable, motivated by an old proposal of Komar and Bergmann [2, 3]. In fact, such gauge invariant observables are sufficient to separate the gauge orbits on a large open subset of the phase space (theorems 4.2 and 4.3). The main technical tool in the construction of these gauge invariant observables is the theory of differential invariants, which in the literature on GR are also known as curvature scalars or curvature invariants.

Unfortunately, the large open subset of the phase space mentioned above specifically excludes solutions that have a high degree of symmetry. Some of these symmetric solutions can be of great physical importance, at least in GR, with examples like Minkowski or Schwarzschild or de Sitter spacetimes. The reason for the exclusion is that observables based on curvature scalars are incapable of separating certain inequivalent gauge equivalence classes of solutions. At the geometric level, the same phenomenon manifests itself in the fact that the moduli space of Lorentzian metrics (the quotient of jets of Lorentzian metrics by the action of diffeomorphisms) is not Hausdorff [17]. A well-known example is that all curvature scalars vanish both on flat Minkowski spacetime as well on non-flat null pp-wave spacetimes (nonlinear wave gravitational wave solutions) [10, 11, 19]. This is problematic if one would like to connect perturbative theory about Minkowski space with nonlinear local observables of the kind discussed above. In principle, it is known that there exist non-scalar differential invariants that are capable of locally distinguishing non-isometric Lorentzian metrics (see the Cartan–Karlhede algorithm discussed in [33 chapter 9] and references therein). At this point it remains an open problem to be investigated whether these more refined differential invariants could be used to construct local (or perhaps multilocal) observables that are capable of separating all gauge orbits on the phase space of GR and other gravitational theories.

In section 5, we showed that generalized local and gauge invariant observables have gauge invariant Poisson brackets using the Peierls formula. However, Poisson brackets of local observables are in general no longer local. At best they could be described as multilocal with distributional smearings. Such observables have been previously discussed in the literature [6, 8], with careful attention paid to the class of distributions that can be consistently allowed to construct an algebra of multilocal observables closed under Poisson brackets. The added complication in gravitational theories, as is evident from the Peierls formula, is that in order to preserve gauge invariance we must allow distributional smearings themselves to depend on the metric and possibly other dynamical fields. Thus, another important avenue for investigation is the generalization of multilocal observables to allow for field-dependent distributional smearings.

It might be argued that the local and gauge invariant observables that we have introduced in this note are of a *relational* kind (see [34] and references therein). However, they do not automatically come with a phenomenological interpretation. That is, given a particular observable of this kind, it may not be immediately clear what kind of experimental protocol would be modeled by it (this issue is discussed clearly in [22]). On the other hand, there is some existing literature that has considered relational observables in linearized and perturbative gravity with clearer phenomenological interpretations, but ran into UV divergences in explicit computations [4, 5, 14, 22, 27, 35, 36, 38, 39]. Perhaps replacing the overly singular proposed observables in these references with regularized versions written as local and gauge invariant observables would yield a double benefit: provide certain local observables with phenomenological interpretations, diffuse UV singularities in explicit computations. As a further step, it would be most interesting to identify local gauge invariant observables that would model some aspects of the data collected by cosmological observations, such as the cosmic microwave background temperature fluctuations and its polarization.

It should also be mentioned that another attempt [16] to write down relational observables (though without clear phenomenological interpretations) using curvature scalars ran into IR divergences in explicit computations. On the other hand, our local observables are designed to be IR regularizing and might give better results in similar computations.

Acknowledgments

The author would like to thank Valentin Lychagin for helpful discussions on the topic of differential invariants. Thanks also to Jochen Zahn for comments on an earlier version of the manuscript. The kind hospitality of the Department of Mathematics at the University of York, where part of the manuscript was completed, is also acknowledged.

References

[1] Barnich G, Brandt F and Henneaux M 2000 Local BRST cohomology in gauge theories *Phys. Rep.* **338** 439–569
[2] Bergmann P and Komar A 1960 Poisson brackets between locally defined observables in general relativity *Phys. Rev. Lett.* **4** 432–3
[3] Bergmann P G 1961 Observables in general relativity *Rev. Mod. Phys.* **33** 510–4
[4] Bonga B and Khavkine I 2014 Quantum astrometric observables: II. Time delay in linearized quantum gravity *Phys. Rev.* D **89** 024039
[5] Borgman J and Ford L H 2004 Effects of stress tensor fluctuations upon focusing *Phys. Rev.* D **70** 064032
[6] Brunetti R and Fredenhagen K 2009 Quantum field theory on curved backgrounds *Quantum Field Theory on Curved Spacetimes: Concepts and Method* (*Lecture Notes in Physics* vol 786) ed C Bär and K Fredenhagen (Berlin: Springer)
[7] Brunetti R, Fredenhagen K and Rejzner K 2013 Quantum gravity from the point of view of locally covariant quantum field theory http://arxiv.org/abs/1306.1058v4
[8] Brunetti R, Fredenhagen K and Ribeiro P L 2012 Algebraic structure of classical field theory: I. Kinematics and linearized dynamics for real scalar fields http://arxiv.org/abs/1209.2148
[9] Christodoulou D and Klainerman S 1993 The global nonlinear stability of the Minkowski space *Princeton Mathematical Series* vol 41 (Princeton, NJ: Princeton University Press)
[10] Coley A, Hervik S and Pelavas N 2009 Spacetimes characterized by their scalar curvature invariants *Class. Quantum Grav.* **26** 025013
[11] Coley A, Hervik S and Pelavas N 2010 Lorentzian manifolds and scalar curvature invariants *Class. Quantum Grav.* **27** 102001
[12] DeWitt B S 1963 The quantization of geometry *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley) ch 8 pp 266–381
[13] Fewster C J and Hunt D S 2013 Quantization of linearized gravity in cosmological vacuum spacetimes *Rev. Math. Phys.* **25** 1330003

[14] Ford L H 1995 Gravitons and light cone fluctuations *Phys. Rev.* D **51** 1692
[15] Fredenhagen K and Rejzner K 2012 Batalin–Vilkovisky formalism in the functional approach to classical field theory *Commun. Math. Phys.* **314** 93–127
[16] Giddings S, Marolf D and Hartle J 2006 Observables in effective gravity *Phys. Rev.* D **74** 064018
[17] Gordillo A, Navarro J and Sancho J B 2009 Moduli spaces for finite-order jets of Riemannian metrics *Differ. Geom. Appl.* **28** 672–88
[18] Hack T-P and Schenkel A 2013 Linear bosonic and fermionic quantum gauge theories on curved spacetimes *Gen. Relativ. Gravit.* **45** 877–910
[19] Hervik S and Coley A 2010 Curvature operators and scalar curvature invariants *Class. Quantum Grav.* **27** 095014
[20] Hirsch M W 1976 Differential topology (*Graduate Texts in Mathematics* vol 33) (Berlin: Springer)
[21] Khavkine I 2012 Characteristics, conal geometry and causality in locally covariant field theory http://arxiv.org/abs/1211.1914
[22] Khavkine I 2012 Quantum astrometric observables: time delay in classical and quantum gravity *Phys. Rev.* D **85** 124014
[23] Khavkine I 2014 Covariant phase space, constraints, gauge and the Peierls formula *Int. J. Mod. Phys.* A **29** 1430009
[24] Kriegl A and Michor P W 1997 The convenient setting of global analysis *Mathematical Surveys and Monographs* vol 53 (Providence, RI: American Mathematical Society)
[25] Kruglikov B and Lychagin V 2015 Global Lie–Tresse theorem http://arxiv.org/abs/1111.5480v3
[26] Krupka D and Janyška J 1990 *Lectures on Differential Invariants* (Brno: Universita J.E. Purkyně)
[27] Ohlmeyer S 1997 The measurement of length in linear quantum gravity *PhD Thesis* DESY, Hamburg http://unith.desy.de/research/aqft/doctoral_theses/
[28] Olver P J 1993 Applications of lie groups to differential equations *Graduate Texts in Mathematics* vol 107 (New York: Springer) 2nd edn
[29] Olver P J 1999 Classical Invariant Theory *London Mathematical Society Student Texts* vol 44 2nd edn (Cambridge: Cambridge University Press)
[30] Olver P J and Pohjanpelto J 2009 Differential invariant algebras of Lie pseudo-groups *Adv. Math.* **222** 1746–92
[31] Rejzner K 2011 Batalin–Vilkovisky formalism in locally covariant field theory *PhD Thesis* Hamburg arXiv:1111.5130
[32] Sharapov A A 2014 Peierls brackets in non-Lagrangian field theory *Int. J. Mod. Phys.* A **29** 1450157
[33] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)
[34] Tambornino J 2012 Relational observables in gravity: a review *Symmetry Integr. Geom.: Methods Appl.* **8** 017
[35] Thompson R T and Ford L H 2006 Spectral line broadening and angular blurring due to spacetime geometry fluctuations *Phys. Rev.* D **74** 024012
[36] Tsamis N C and Woodard R P 1992 Physical Green's functions in quantum gravity *Ann. Phys., NY* **215** 96–155
[37] Wockel C 2009 A generalisation of Steenrod's approximation theorem *Archivum Mathematicum* pp 9–104 (Brno) arXiv:math/0610252
[38] Woodard R P 1984 Invariant formulation of and radiative corrections in quantum gravity *PhD Thesis* Harvard University, Cambridge, MA
[39] Yu H and Ford L H 1999 Light-cone fluctuations in flat spacetimes with nontrivial topology *Phys. Rev.* D **60** 084023

Journal of Geometry and Physics 113 (2017) 131–169

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

The Calabi complex and Killing sheaf cohomology

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ARTICLE INFO

Article history:
Received 25 April 2016
Accepted 17 June 2016
Available online 28 June 2016

Keywords:
Pseudo-Riemannian geometry
Constant curvature
Differential complex
Sheaf cohomology
Linearized gravity

ABSTRACT

It has recently been noticed that the degeneracies of the Poisson bracket of linearized gravity on constant curvature Lorentzian manifold can be described in terms of the cohomologies of a certain complex of differential operators. This complex was first introduced by Calabi and its cohomology is known to be isomorphic to that of the (locally constant) sheaf of Killing vectors. We review the structure of the Calabi complex in a novel way, with explicit calculations based on representation theory of $\mathrm{GL}(n)$, and also some tools for studying its cohomology in terms of locally constant sheaves. We also conjecture how these tools would adapt to linearized gravity on other backgrounds and to other gauge theories. The presentation includes explicit formulas for the differential operators in the Calabi complex, arguments for its local exactness, discussion of generalized Poincaré duality, methods of computing the cohomology of locally constant sheaves, and example calculations of Killing sheaf cohomologies of some black hole and cosmological Lorentzian manifolds.

1. Introduction

The Calabi complex is a differential complex that was introduced in by E. Calabi in 1961 [1]. It has an extended pre-history, though. One way to characterize it is as a canonical formal compatibility complex (the *second Spencer sequence*) of the Killing equation on (pseudo-)Riemannian manifolds of constant curvature. The solutions of the Killing equation are (possibly only locally defined) infinitesimal isometries. In the special context of classical linear elasticity theory, the same operator also maps between the displacement and strain fields [2–4]. It is well known that for flat spaces (zero curvature) a complete set of formal compatibility conditions for the Killing equation is given by the linearized Riemann curvature operator, also known as the Saint-Venant compatibility operator in the context of elasticity [2–4]. Subsequent compatibility conditions are furnished by the Bianchi identities. Thus, it would also be reasonable to refer to it as the Killing–Riemann–Bianchi complex.

Calabi's interest in the eponymous complex stemmed from the isomorphism between the cohomology of its global sections and the cohomology of the sheaf of Killing vectors. Given a fine resolution of a sheaf, like one provided by a locally exact sequence of differential operators on sections of vector bundles, the general machinery of homological algebra implies that the sheaf cohomology is in fact isomorphic to the cohomology of the complex of global sections of its local resolution, with the resolution of the sheaf of locally constant functions by the de Rham complex of differential forms on a manifold being the canonical example. The bulk of Calabi's original article was in fact spent proving that the hypotheses needed for applying this general result actually hold, thus providing a way to represent the cohomology of the Killing sheaf. It is the latter object that was of intrinsic interest, as it was in subsequent works by others [5–7], motivated by the well known interpretations of its lowest cohomology groups: in degree-0 as the Lie algebra of global isometries and in degree-1 as the space of non-trivial infinitesimal deformations of the metric under the constant curvature restriction. Later, the Calabi complex was also seen as

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http://dx.doi.org/10.1016/j.geomphys.2016.06.009
0393-0440/

a non-trivial example of a formally exact compatibility complex [8–11,3] constructed for the Killing operator by the methods of the formal theory of partial differential equations developed by the school of D.C. Spencer [12–16].

More recently, the Calabi complex resurfaced in mathematical physics, in the context of the (pre-)symplectic and Poisson structure of relativistic classical field theories. In [17,18], the author has shown that the degeneracy subspaces of the naturally defined pre-symplectic 2-form and Poisson bivector on the infinite dimensional phase space of relativistic classical field theories with possible constraints and gauge invariance are controlled by the cohomology of some differential complexes. In the case of Maxwell-like theories [18, Sec. 4.2], this role is played by the de Rham complex, while in the case of linearized gravity [18, Sec. 4.4], this role is played by the formal compatibility complex of the Killing operator. In other words, for linearization backgrounds of constant curvature (important examples include Minkowski and de Sitter spaces, as well as quotients thereof), this is precisely the Calabi complex. When the linearization background is merely locally symmetric, rather than of constant curvature, the right complex to use is a slightly different one that was constructed by Gasqui and Goldschmidt [9,10]. However, a discussion of the latter is beyond the scope of this work. The construction of similar complexes adapted to other background geometries appears to be an open problem. The degeneracy subspace of the Poisson bivector of a classical field theory is of importance because it translates almost directly into violations of a (strict) notion of locality of the corresponding quantum field theory, a subject that has recently been under intense investigation [19–26].

The goal of this paper is to exploit the connection between the Calabi complex and Killing sheaf cohomologies, in a direction opposite the original one of Calabi, for the purpose of obtaining results relevant to the above mentioned applications in mathematical physics. More precisely, we consider the computation of certain sheaf cohomologies much simpler than constructing quotient spaces of kernels of complicated differential operators. Thus, the ability to equate the Calabi cohomology groups, which for us are of primary interest, with Killing sheaf cohomology groups is a significant technical simplification. Along the way, we collect some relevant facts about the Calabi complex that are either difficult or impossible to find in the existing literature, along with other little known tools from the theory of differential complexes [27] needed to prove the desired equivalence and to introduce cohomologies with compact supports. It is our hope that this treatment of the Calabi complex could serve as a model for the treatment of other differential complexes that are of importance in mathematical physics.

In Section 2, we discuss the explicit form of the Calabi complex on any constant curvature pseudo-Riemannian manifold. The tensor bundles and differential operators between them are defined using notation and identities from the representation theory of the general linear group, which are reviewed in Appendix A.1. The differential cochain homotopy operators defined in Section 2.2 and Appendix A.5, and the relation of the formal adjoint Calabi complex to the Killing–Yano operator presented in Section 2.3 are likely new. Then, Section 3 recalls some general notions from sheaf cohomology, with emphasis on locally constant sheaves. It also covers the relation between the Calabi cohomology, with various supports, and the cohomologies of the sheaf of Killing vectors and the sheaf of Killing–Yano tensors. In Section 4 we discuss several methods for effectively computing the cohomologies of the Killing sheaf, also outside the constant curvature context. An important application of the above results is described in Section 5, which uses the Calabi cohomology to determine the degeneracy subspaces of presymplectic and Poisson structures of linearized gravity on constant curvature backgrounds. This application, and its generalizations, constitute the main motivation for this work. Finally, Section 6 concludes with a discussion of the presented results and of how they could be generalized to other differential complexes of interest in the mathematical theory of classical and quantum gauge field theories in physics.

It should be emphasized that the Killing sheaf cohomology can be identified with the cohomology of the Calabi complex only on pseudo-Riemannian spaces of constant curvature, where the latter complex is actually defined. The Killing sheaf itself has a wider domain of definition. In terms of applications to linearized gravity, the differential complexes that are to replace the Calabi complex on other background geometries are still expected to have isomorphic cohomology to that of the Killing sheaf. So, from that perspective, the Calabi complex is a particular case study and the Killing sheaf is an object of more permanent value.

2. The Calabi complex

In Sections 2.1 and 2.2, we shall explicitly describe the Calabi complex as a complex of differential operators between tensor bundles on a pseudo-Riemannian manifold (M, g). Furthermore, we will explicitly list a corresponding sequence of differential operators that constitute a cochain homotopy from the Calabi complex to itself. The cochain maps induced by the homotopy operators will have the same principal symbol as the tensor Laplacian $\nabla_a \nabla^a$ induced by the Levi-Civita connection of the metric tensor g, though will differ from it by lower order terms. This geometric structure is very similar to that of the Hodge theory of the de Rham complex on a Riemannian manifold. This structure is used in the later Section 3.2 to show the complex's local exactness. Finally, in Section 2.3, we will describe the formal adjoint Calabi complex, with the formal adjoint cochain maps and homotopies playing roles analogous to the original ones. It turns out that, just as the Calabi complex resolves the sheaf of Killing vectors on (M, g), its formal adjoint complex resolves the sheaf of rank-$(n-2)$ Killing–Yano tensors.

A non-negligible amount of work [5,9,8,10,3,11], though certainly not a large one, has been done on this differential complex since the original work [1] of Calabi in 1961. Its original presentation was in terms of Cartan's moving frame formalism and much of the subsequent work did not put a strong emphasis on explicit formulas. Thus, it is a little difficult to find its presentation in terms of standard covariant derivatives on tensor bundles in the existing literature. We give such

Table 1
The table below lists the tensor bundles of the Calabi complex, the corresponding irreducible $GL(n)$ representations (labeled by Young diagrams), and their fiber ranks, for $\dim M = n$. The rank is given by the famous *hook formula*, which is discussed in Appendix A.1.

Bundle	Young diagram	Fiber rank
$C_0M \cong T^*M$	$\ydiagram{1}$	n
$C_1M \cong S^2M$	$\ydiagram{2}$	$\frac{n(n+1)}{2}$
$C_2M \cong RM$	$\ydiagram{2,2}$	$\frac{n^2(n^2-1)}{12}$
$C_3M \cong BM$	$\ydiagram{2,2,1}$	$\frac{n^2(n^2-1)(n-2)}{24}$
C_lM	$\begin{ytableau} 1 & \\ 2 & \\ \vdots \\ l \end{ytableau}$	$\frac{n^2(n^2-1)(n-2)\cdots(n-l+1)}{2(l+1)l(l-2)!}$

formulas below, together with a complete sequence of cochain homotopy operators from the complex to itself and their corresponding cochain maps. These formulas are apparently new, as their role was played by a more generic, but somewhat less natural, construction applicable to general elliptic complexes in [1,5,9,8]. The advantage of our version is the connection of the homotopy and cochain maps with the equations of linearized gravity and coincidence, in low degrees, with other well known related operators, which include the Killing, linearized Riemann, Bianchi, de Donder and Ricci trace operators. One could also argue that our resulting homotopy and cochain maps are simpler, because they never exceed second differential order (in contrast to fourth differential order). Furthermore, we find that the tensor bundles that constitute the nodes of the complex are best described as having fibers that carry irreducible representations of $GL(n)$, where n is the dimension of the base manifold; moreover, the principal symbols of the differential operators in the complex are $GL(n)$ equivariant maps. Hence they are independent of the background metric, which is no longer true for subleading terms. This observation appears to have escaped the attention of earlier works, thus requiring some seemingly ad-hoc constructions [1]. A notable exception is Eastwood [3], who also identified the principal symbol complex as an instance of the general notion of BGG resolutions [28] in representation theory. Taking advantage of this connection with representation theory, we explicitly describe the tensor bundles of the complex and the equivariant principal symbol maps between them in terms of Young diagrams.

2.1. Tensor bundles and Young symmetrizers

As was mentioned in the Introduction, it is convenient to describe various tensor bundles involved in the Calabi complex, as well as various maps between them, in terms of irreducible representations (*irreps*) of group $GL(n)$, where $n = \dim M$ is the dimension of the base manifold M. Irreps of $GL(n)$ are concisely presented using Young diagrams and corresponding Young tensor symmetrizers. An excellent reference on this topic is the book [29], where we refer the reader for complete details. For an uninitiated reader, we have briefly summarized the relevant concepts and formulas in Appendix A.1. For the expert reader, it is recommended to skim the same appendix for the particulars of our notation.

Given a base manifold M of dimension $n = \dim M$, we can construct tensor bundles over M whose fibers carry irreducible representations of $GL(n)$. Indeed, we will consider Young symmetrized sub-bundles $Y^d T^*M$ of the bundle of covariant k-tensors $(T^*)^{\otimes k}M$, where d is a Young diagram type with k cells.

The Calabi complex, to be introduced in the next section, is a complex of differential operators between certain tensor bundles over M. Let us denote the corresponding sequence of vector bundles by C_lM. More precisely,

$$C_0 = T^*, \qquad C_1 = Y^{(2)}T^*, \qquad C_2 = Y^{(2,2)}T^*, \qquad C_l = Y^{(2,2,1^{l-2})}T^* \quad (l > 2). \tag{1}$$

Note that the bundle C_1M corresponds to symmetric 2-tensors, which we will also denote S^2M. Also, as mentioned in the preceding section, since the bundle C_2M corresponds to 4-tensors with the algebraic symmetries of the Riemann tensor, we will also denote it RM. And the bundle C_3M, also denoted BM, corresponds to 5-tensors with symmetries of the image of the Bianchi operator applied to a section or RM. The index l essentially counts the number of rows in the corresponding Young diagram. So, for $l > n$, the number of rows exceeds the base dimension and the C_lM bundles become trivial. These tensor bundles, the corresponding Young diagrams and their fiber ranks are illustrated in Table 1.

2.2. Differential operators

Below, given any n-dimensional pseudo-Riemannian manifold (M, g) of constant curvature k (normalized so that the Ricci scalar curvature[1] is equal to k), we give explicit formulas for the differential operators, constituting the Calabi complex, as well as formulas for the differential operators that constitute a cochain homotopy from the complex to itself and the corresponding induced cochain maps. In Calabi's original work [1], the corresponding differential operators were constructed using an orthogonal coframe formalism. Thus, it has been difficult to find explicit formulas for these operators in the tensor formalism that is more prevalent in the physics literature on relativity. The cochain homotopy operators and the induced cochain maps coincide, in low degrees, with differential operators well known in the relativity literature. However, their explicit form in all degrees appears to be new. Furthermore, we explicitly demonstrate all the identities between these differential operators that lead to their homological algebra interpretations. We use a mixture of elementary arguments, as well as equivariance and standard $\mathrm{GL}(n)$-representation theoretic identities, unlike Calabi's original proofs [1] that relied on a somewhat ad hoc algebraic constructions, and unlike the derivation of Gasqui and Goldschmidt [9,10] that relied on the sophisticated theory of Spencer sequences.

First, we define a number of differential operators that will be convenient for our purposes. For homogeneous differential operators with constant coefficients, the operator is completely determined by the principal symbol. In general that is not the case, yet the presence of a preferred connection on tensor bundles (the g-compatible Levi-Civita connection) still allows us to specify operators by their principal symbols: the covariant derivative is applied to a tensor k-times, the derivative indices are full symmetrized, and the principal symbol is applied to the result.

The principal symbol of a kth order differential operator between two Young symmetrized bundles $\mathrm{Y}T^*$ and $\mathrm{Y}'T^*$ is a linear map between them that depends polynomially on a covector $p \in T^*$. If the operator (or just its principal symbol) is $\mathrm{GL}(n)$-equivariant, then the principal symbol actually corresponds to an intertwiner between the $\mathrm{Y}^{(k)} \otimes \mathrm{Y}$ and Y' representations. Such an intertwiner is non-zero only if Y' appears in the irrep decomposition of the tensor product. Moreover if Y' appears with single multiplicity, the intertwiner (and hence the principal) symbol is determined uniquely up to a scalar factor. It is an old result due to Pieri [29] that, in fact, the decomposition of the product of $\mathrm{Y}^{(k)} \otimes \mathrm{Y}$ into irreps has only single multiplicities. Not all principal symbols of importance to us are equivariant. The main source of the lack of equivariance is the dependence on the metric g. However, if the metric itself is also allowed to transform, the principal symbol becomes equivariant again. For instance, if the operator is equivariant in this way and depends linearly on the metric in covariant form, it corresponds to an intertwiner between the representations $\mathrm{Y}^{(2)} \otimes \mathrm{Y}^{(k)} \otimes \mathrm{Y}$ and Y'. Because of the presence of a double tensor product, Pieri's rule does not always apply, so sometimes more information is necessary to specify the desired intertwiner unambiguously. As a rule, these ambiguities will be resolved by giving explicit formulas.

Observe that the all tensor fields defined in Section 2.1 correspond to Young diagrams with at most two columns. We shall refer to the columns as *left* and *right*. Let d_L and d_R, the *left* and *right exterior differentials*, be differential operators that increase by one the number of boxes in the, respectively, left or right column. They have equivariant principal symbols. We also define several operators whose principal symbols involve the metric. Two operators of order 0 are the *trace* tr and the *metric exterior product* $(g \odot -)$, respectively, decreasing (contracting indices between the two columns) or increasing (multiplying by g and symmetrizing) by one the number of boxes in each column. Two operators of order 1 are *left* and *right codifferentials* δ_L and δ_R, which decrease (taking a covariant divergence and resymmetrizing, if necessary) by one the number of boxes in, respectively, the left or right column. Finally, we have the *tensor Laplacian* $\square$, a differential operator of order 2 that does not alter the Young symmetry. Explicit formulas for these operators, along with proofs that they respect the corresponding Young symmetries, are given in Appendices A.2, A.4 and A.5.

The differential operators constituting the Calabi complex, as well as cochain self-homotopy and the induced cochain self-maps fit into the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_0 \arrow[r, "B_1"] \arrow[d, "P_0"'] & C_1 \arrow[r, "B_2"] \arrow[d, "P_1"'] \arrow[dl, dashed, "E_1"'] & C_2 \arrow[r, "B_3"] \arrow[d, "P_2"'] \arrow[dl, dashed, "E_2"'] & C_3 \cdots \arrow[dl, dashed, "E_3"'] \arrow[d, "P_3"'] \arrow[r, "B_n"] & C_n \arrow[r] \arrow[d, "P_n"'] \arrow[dl, dashed, "E_n"'] & 0 \\
0 \arrow[r] & C_0 \arrow[r, "B_1"] & C_1 \arrow[r, "B_2"] & C_2 \arrow[r, "B_3"] & C_3 \cdots \arrow[r, "B_n"] & C_n \arrow[r] & 0
\end{tikzcd} , \tag{2}$$

where for simplicity we have used the symbol C_l to stand for the space of sections $\Gamma(C_l M)$. The operators B_l constitute a complex, because $B_{l+1} \circ B_l = 0$. The solid arrows in the diagram commute, $P_{l+1} \circ B_{l+1} = B_l \circ P_l$, so that the P_l are cochain maps from the complex to itself. These cochain maps, $P_l = E_{l+1} \circ B_{l+1} + B_l \circ E_l$, are induced by the homotopy operators E_n, which appear as dashed arrows. Below, we give explicit formulas for each of these operators, discuss these identities, and relate them to well known differential operators from the literature on relativity. We follow the notational conventions of Appendices A.1 and A.2. In particular, we use : to separate fully antisymmetric tensor index groups belonging to different columns of the Young diagram, which characterizes the symmetry type of a given tensor. However, for simplicity, we also

[1] We follow [30] for conventions regarding the definitions of curvature tensors and scalars.

write $g_{ab} = g_{a:b}$ and $h_{ab} = h_{a:b}$.

$$
\begin{aligned}
B_1[v]_{a:b} &= K[v]_{a:b} = \nabla_a v_b + \nabla_b v_a, &(3)\\
B_2[h]_{ab:cd} &= -2\tilde{R}[h]_{ab:cd} \\
&= \left(\nabla_{(a}\nabla_{c)} h_{bd} - \nabla_{(b}\nabla_{c)} h_{ad} - \nabla_{(a}\nabla_{d)} h_{bc} + \nabla_{(b}\nabla_{d)} h_{ac}\right) + \frac{k}{n(n-1)}(g \odot h)_{ab:cd}, &(4)\\
B_3[r]_{abc:de} &= \bar{B}[r]_{abc:de} = \mathrm{d}_L[r]_{abc:de} = 3\nabla_{[a} r_{bc]:de} \\
&= \nabla_a r_{bc:de} + \nabla_b r_{ca:de} + \nabla_c r_{ab:de}, &(5)\\
B_4[b]_{abcd:ef} &= \mathrm{d}_L[b]_{abcd:ef} = 4\nabla_{[a} b_{bcd]:ef} \\
&= \nabla_a b_{bcd:ef} - \nabla_b b_{cda:ef} - \nabla_c b_{dab:ef} - \nabla_d b_{abc:ef}, &(6)\\
B_l[b]_{a_1\cdots a_l:bc} &= \mathrm{d}_L[b]_{a_1\cdots a_l:bc} = l\nabla_{[a_1} b_{a_2\cdots a_l]:bc} \quad (l \geq 3). &(7)
\end{aligned}
$$

Note that we have introduced some suggestive alternative notations for operators B_l of low rank. In particular, $B_1 = K$ is the *Killing* operator. Then, $B_2 = -2\tilde{R}$ is the linearized *corrected Riemann* curvature operator,[2] where $R[g+\lambda h] - \bar{R}[g+\lambda h] = \lambda\tilde{R}[h] + O(\lambda^2)$, with $\bar{R}[g]_{ab:cd} = \frac{k}{n(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad})$, cf. Eq. (124) in Appendix A.4. The precise relation with the linearized Riemann tensor operator is

$$\dot{R}[h] = -\frac{1}{2}B_2[h] + k\frac{2}{n(n-1)}(g \odot h). \tag{8}$$

Note that the identity $R[g] - \bar{R}[g] = 0$ holds precisely when the metric g has constant curvature k. Finally, $B_3 = \bar{B}$ is the background *Bianchi* operator, which also happens to coincide with the left exterior differential d_L. It satisfies the well known Bianchi identity $\bar{B}[R[g]] = 0$. The operators B_l for $l > 3$, which we may call *higher Bianchi* operators, do not appear to have been studied in the literature on relativity. So, as mentioned in the Introduction, the Calabi complex might also be legitimately referred to as the Killing–Riemann–Bianchi complex.

Now we give mostly elementary arguments for the composition identities $B_{l+1} \circ B_l = 0$. Recall that if v is a vector field (identified with a section of $C_0M \cong T^*M$ using the metric), then the Lie derivative of the metric along v is given by the Killing operator, $\mathcal{L}_v g = K[v]$. Now, suppose that $T[g]$ is any tensor field covariantly constructed out of the metric and its derivatives. Consider its linearization $T[g+\lambda h] = T[g] + \lambda\dot{T}[h] + O(\lambda^2)$. The linearization $\dot{T}$ annihilates the Killing operator if $T[g] = 0$ [32]. This fact follows from the fact that $T[g]$ is itself a tensor field, so that

$$\mathcal{L}_v T[g] = \dot{T}[\mathcal{L}_v g] = \dot{T} \circ K[v]. \tag{9}$$

Letting $T[g]_{ab:cd} = R[g]_{ab:cd} - \bar{R}[g]_{ab:cd}$ we obtain the identity $B_2 \circ B_1 = -2\tilde{R} \circ K = 0$, since $T[g] = 0$ by reason of g being of constant curvature equal to k. Further, note that, since the metric is covariantly constant, $\nabla g = 0$, it is trivial to check that $\bar{B}[\bar{R}[g]] = 0$, for any g. Combining this observation with the Bianchi identity, we find that $\bar{B}[R[g] - \bar{R}[g]] = 0$, for any g. Making the dependence of $\bar{B} = \bar{B}_g$ on g explicit, the linearization of this identity gives

$$\bar{B}_{g+\lambda h}[R[g+\lambda h] - \bar{R}[g+\lambda h]] = \bar{B}_g[R[g] - \bar{R}[g]] + \lambda(\bar{B}[\tilde{R}[h]] + \dot{\bar{B}}[h, R[g] - \bar{R}[g]]) + O(\lambda^2) = 0, \tag{10}$$

where $\bar{B}_{g+\lambda h}[T] = \bar{B}[T] + \lambda\dot{\bar{B}}[h, T] + O(\lambda^2)$. At first order in λ, we obtain the desired identity $B_3 \circ B_2 = -2\bar{B} \circ \tilde{R} = 0$. The remaining identities, $B_{l+1} \circ B_l = \mathrm{d}_L^2 = 0$ for $l > 2$, follow from abstract representation theoretic reasons, described in more detail in Appendices A.4 and A.5.

$$
\begin{aligned}
E_1[h]_a &= D[h]_a = \nabla^b h_{ab} - \frac{1}{2}\nabla_a h, &(11)\\
E_2[r]_{a:b} &= \mathrm{tr}[r]_{a:b} = r_{ac:b}{}^{c}, &(12)\\
E_3[b]_{ab:cd} &= \nabla^e b_{eab:cd} + \frac{1}{2}\nabla^e(b_{cab:de} - b_{dab:ce}) - \frac{1}{2}(\nabla_c b_{abe:d}{}^{e} - \nabla_d b_{abe:c}{}^{e}) - \frac{1}{2}(\nabla_a b_{cbe:d}{}^{e} - \nabla_a b_{dbe:c}{}^{e} \\
&\quad + \nabla_b b_{ace:d}{}^{e} - \nabla_b b_{ade:c}{}^{e}), &(13)\\
E_4[b]_{abc:de} &= \nabla^f b_{fabc:de} + \frac{1}{3}\nabla^f(b_{dabc:ef} - b_{eabc:df}) + \frac{1}{3}(\nabla_d b_{abcf:e}{}^{f} - \nabla_e b_{abcf:d}{}^{f}) \\
&\quad + \frac{1}{6}(\nabla_a b_{dbcf:e}{}^{f} - \nabla_a b_{ebcf:d}{}^{f} + \nabla_b b_{adcf:e}{}^{f} - \nabla_b b_{aecf:d}{}^{f} + \nabla_c b_{abdf:e}{}^{f} - \nabla_c b_{abef:d}{}^{f}), &(14)
\end{aligned}
$$

[2] The same corrected curvature tensor can be obtained by linearizing the mixed form $R[g]_{ab}{}^{cd}$ of the Riemann tensor and then lowering all indices with the background metric. This linearized mixed Riemann tensor was previously used to isolate the gauge invariant metric perturbations on de Sitter space in [31]. That the linearized corrected Riemann tensor annihilates the Killing operator also follows from the classical analysis in [32], which noted that the linearization of any tensor built only out of the metric and vanishing on the background spacetime is invariant under linearized diffeomorphisms.

$$E_5[b]_{abcd:ef} = \nabla^i b_{iabcd:ef} + \frac{1}{4}\nabla^i(b_{eabcd:fi} - b_{fabcd:ei}) - \frac{1}{4}(\nabla_e b_{abcdi:f}{}^i - \nabla_f b_{abcdi:e}{}^i)$$
$$- \frac{1}{12}(\nabla_{\{e\}} b_{\{abcd\}i:f}{}^i - \nabla_{\{f\}} b_{\{abcd\}i:e}{}^i), \tag{15}$$
$$E_{l+1}[b]_{a_1\cdots a_l:bc} = (\delta_L[b] - (-1)^l l^{-1} \mathrm{d}_R \circ \operatorname{tr}[b])_{a_1\cdots a_l:bc} \quad (l \geq 2). \tag{16}$$

The notation used in the formula for E_5 is defined in Appendix A.1. Note that $E_1 = D$ is the *de Donder* operator, used as a linearized gauge fixing condition in the literature on relativity. Also, if $R[g]$ is the Riemann tensor of the metric g, then $E_2[R[g]] = \operatorname{tr}[R[g]]$ is the corresponding Ricci tensor. The higher homotopy operators E_l for $l > 2$ do not seem to have previously appeared in the literature on relativity.

$$P_0[v]_a = \Box v_a + k\frac{1}{n}v_a, \tag{17}$$
$$P_1[h]_{ab} = \Box h_{ab} - k\frac{2}{n(n-1)}h_{ab} + 2k\frac{g_{ab}\operatorname{tr}[h]}{n(n-1)}, \tag{18}$$
$$P_2[r]_{ab:cd} = \Box r_{ab:cd} - k\frac{2}{n}r_{ab:cd} + 2k\frac{(g \odot \operatorname{tr}[r])_{ab:cd}}{n(n-1)}, \tag{19}$$
$$P_3[b]_{abc:de} = \Box b_{abc:de} - k\frac{(3n-7)}{n(n-1)}b_{abc:de} - 2k\frac{(g \odot \operatorname{tr}[b])_{abc:de}}{n(n-1)}, \tag{20}$$
$$P_4[b]_{abcd:ef} = \Box b_{abcd:ef} - k\frac{(4n-14)}{n(n-1)}b_{abcd:ef} + 2k\frac{(g \odot \operatorname{tr}[b])_{abcd:ef}}{n(n-1)}, \tag{21}$$
$$P_l[b]_{a_1\cdots a_l:bc} = \Box b_{a_1\cdots a_l:bc} - k\frac{(ln - l^2 + 2)}{n(n-1)}b_{a_1\cdots a_l:bc} + (-)^l 2k\frac{(g \odot \operatorname{tr}[b])_{a_1\cdots a_l:bc}}{n(n-1)} \quad (l \geq 3). \tag{22}$$

Note our notation $\Box = \nabla^a\nabla_a$ for the tensor Laplacian, which is also known as the d'Alembertian in Lorentzian signature. The operator $P_0 = D \circ K$ gives the wave-like residual gauge condition such that the perturbation $h = K[v]$ satisfies the de Donder gauge condition $D[h] = 0$ in linearized gravity. The operator $P_1 = \operatorname{tr}\circ(-2\tilde{R}) + K \circ D$ is the wave-like operator of the linearized Einstein equations for gravitational perturbations h in de Donder gauge $D[h] = 0$. These two operators are well known and can be found (or their close analogs can) for instance in [30, Sec. 7.5] and more they appeared in [22,25,33]. The higher cochain maps and the corresponding identities appear to be new. Though, the identity $P_2 = E_3 \circ \bar{B} - 2\tilde{R} \circ E_2$ is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the *Penrose wave equation*. For linearized fields, a related equation is sometimes known as the *Lichnerowicz Laplacian*. For more details, see references [34], [35, Sec. 1.3], [36, Sec. 7.1], [37, Exr. 15.2], [38, Eq. 35].

Remark 1. It is worth noting that we refer to the operators P_l as wave-like because the principal symbol of P_l has the same principal symbol as the tensor Laplacian $\Box = \nabla^a\nabla_a$, on Lorentzian manifolds also known as the d'Alembertian or wave operator, which is a hyperbolic differential operator. Note that the principal symbol of P_l is determined only by the principal symbols of the B_l and E_l. The principal symbols of B_l are metric independent, while those of E_l depend on the metric g of the background pseudo-Riemannian manifold (M, g). However, we are actually free to choose any metric, say g' that is different from g, to construct the cochain homotopy operators, say E'_l. The principal symbol induced cochain maps $P'_l = E'_{l+1} \circ B_{l+1} + B_l \circ E'_l$ will then still only depend on one metric, g', and be equal to the principal symbol of the tensor Laplacian $\Box'$ defined with respect to g'. Thus, if we choose g' to be Riemannian, we can induce cochain homotopies P'_l that are elliptic. The operators P'_l will of course differ from the P_l by terms of lower differential order that would depend on both g and g'. This remark will be very useful in Proposition 9 in the discussion of the local exactness of the Calabi complex.

2.3. Formal adjoint complex

Given a linear differential operator $f\colon \Gamma(E) \to \Gamma(F)$, between vector bundles $E \to M$ and $F \to M$, its *formal adjoint* is a linear differential operator $f^*\colon \Gamma(\tilde{F}^*) \to \Gamma(\tilde{E}^*)$, where we have used the notation for the bundle $\tilde{V}^* \cong V^* \otimes_M \Lambda^n M$ of *dual densities* of a vector bundle $V \to M$, defined as the tensor product of the its linear dual bundle $V^* \to M$ with that of densities $\Lambda^n M \to M$ on the base manifold if dimension $\dim M = n$. The formal adjoint operator is defined to be the unique differential operator such that a *Green formula* holds,

$$\psi \cdot f[\xi] - f^*[\psi] \cdot \xi = \mathrm{d}G[\psi, \xi], \tag{23}$$

for any $\psi \in \Gamma(\tilde{F}^*)$, $\xi \in \Gamma(E)$, and some bilinear bidifferential operator

$$G\colon \Gamma(\tilde{F}^* \times_M E) \to \Gamma(\Lambda^{n-1}M). \tag{24}$$

A formal adjoint operator always exists and is unique [39,40,27].

In the presence of background pseudo-Riemannian metric g on M, we can canonically identify the trivial bundle $\mathbb{R}\times M$ with $\Lambda^n M$, via multiplication by the canonical volume form $\varepsilon_{a_1\cdots a_n}$ with respect to g ($\varepsilon\in\Omega^n(M)$), and also $V\cong V^*$ for any tensor bundle $V\to M$, by lowering and raising indices with g, thus also canonically identifying $V\cong\tilde{V}^*$. Below, we will take formal adjoints with respect to this identification. Recall the identity [41]

$$\varepsilon^{aa_2\cdots a_n}\varepsilon_{ba_2\cdots a_n}=(-1)^s(n-1)!\delta^a_b \tag{25}$$

(where s counts the number of minuses in the signature of the metric g, with $s=1$ for Lorentzian metrics with mostly-plus convention) and define

$$G^a=\frac{(-1)^s}{(n-1)!}\varepsilon^{aa_2\cdots a_n}G_{a_2\cdots a_n} \tag{26}$$

so that $G_{a_2\cdots a_n}=\varepsilon_{aa_2\cdots a_n}G^a$. The right hand side of the formal adjoint Eq. (23) can then be rewritten as

$$(\mathrm{d}G)_{a_1\cdots a_n}=\frac{(-1)^s}{n!}\varepsilon_{a_1\cdots a_n}\varepsilon^{ab_2\cdots b_n}n\nabla_a G_{b_2\cdots b_n}=\varepsilon_{a_1\cdots a_n}\nabla_a G^a, \tag{27}$$

with the whole equation becoming

$$\psi\cdot f[\xi]-f^*[\psi]\cdot\xi=\nabla_a G^a[\psi,\xi], \tag{28}$$

where the dot indicates contraction of indices using the metric g between two tensors of the same index structure.

With this notation, the formal adjoint Calabi complex $(C_\bullet, B^*_\bullet)$ fits into the following diagram:

$$\begin{tikzcd}
0 & C_0 \arrow[l] & C_1 \arrow[l, "B_1^*"'] & C_2 \arrow[l, "B_2^*"'] & C_3 \cdots \arrow[l, "B_3^*"'] & C_n \arrow[l, "B_n^*"'] & 0 \arrow[l] \\
0 & C_0 \arrow[l] \arrow[u, "P_0^*"] \arrow[ur, dashed, "E_1^*"] & C_1 \arrow[l, "B_1^*"] \arrow[u, "P_1^*"] \arrow[ur, dashed, "E_2^*"] & C_2 \arrow[l, "B_2^*"] \arrow[u, "P_2^*"] \arrow[ur, dashed, "E_3^*"] & C_3 \cdots \arrow[l, "B_3^*"] \arrow[u, "P_3^*"] \arrow[ur, dashed, "E_n^*"] & C_n \arrow[l, "B_n^*"] \arrow[u, "P_n^*"] & 0 \arrow[l]
\end{tikzcd}, \tag{29}$$

where we have identified $\tilde{C}_i^*\cong C_i$ using the background metric. Note that all the analogous identities are satisfied, the solid arrows in the diagram commute and the dashed arrows are homotopy operators inducing the vertical cochain maps, $P_i^*=B_{i+1}^*\circ E_{i+1}^*+E_i^*\circ B_i^*$. The main difference is that the $B^*_\bullet$ now decrease the degree index by one instead of decreasing it. The usual numbering convention can be achieved by relabeling, but we shall not do so here, expecting that no confusion will arise.

Recall that the final differential operator B_n of the Calabi complex is

$$B_n[b]_{a_1\cdots a_n:bc}=\mathrm{d}_L[b]_{a_1\cdots a_n:bc}=n\nabla_{[a_1}b_{a_2\cdots a_n]:bc}, \tag{30}$$

where $b\in\Gamma(C_{n-1}M)$. To compute its formal adjoint, let $c\in\Gamma(C_nM)$ and consider first the identity, derived in Appendix A.6,

$$\nabla_a(c^{aa_2\cdots a_n:bc}b_{a_2\cdots a_n:bc})=\frac{1}{n}c^{aa_2\cdots a_n:bc}\mathrm{d}_L[b]_{aa_2\cdots a_n:bc}+\delta_L[c]^{a_2\cdots a_n:bc}b_{a_2\cdots a_n:bc}. \tag{31}$$

Note that the operators d_L and δ_L specifically produce tensors of the appropriate Young type. Therefore, the formal adjoint operator B_n^* is given by the formula

$$B_n^*[c]_{a_2\cdots a_n:bc}=-\frac{1}{n}\delta_L[c]_{a_2\cdots a_n:bc} \tag{32}$$

$$=-\frac{1}{n}\nabla^a c_{aa_2\cdots a_n:bc}-\frac{2}{n(n-1)}\nabla^a c_{[b|a_2\cdots a_n:|c]a}, \tag{33}$$

with the Green form represented by $G^a[c,b]=\frac{1}{n}c^{aa_2\cdots a_n:bc}b_{a_2\cdots a_n:bc}$.

While this operator B_n^* may look unfamiliar, after a further local invertible transformation the equation $B_n^*[c]=0$ becomes equivalent to the well known *rank-$(n-2)$ Killing–Yano equation*. Let us define a rank-$(n-2)$ anti-symmetric tensor $y^{c_3\cdots c_n}$ such that

$$c_{a_1\cdots a_n:bc}=\varepsilon_{a_1\cdots a_n}y^{c_3\cdots c_n}\varepsilon_{bcc_3\cdots c_n}, \tag{34}$$

$$y^{c_3\cdots c_n}=\frac{1}{2(n-2)!(n-1)!}\varepsilon^{a_1\cdots a_n}c_{a_1\cdots a_n:bc}\varepsilon^{bcc_3\cdots c_n}. \tag{35}$$

It is straightforward to check using the hook formula (Appendix A) that the tensor c of Young type $(2,2,1^{n-2})$ has the same number of independent components as the tensor y of Young type (1^{n-2}). To transform the equation satisfied by c into the Killing–Yano equation satisfied by y, we will need the following identities, which follow from the general properties of the ε tensor [41]:

$$\varepsilon^{aa_2\cdots a_n}c_{a'a_2\cdots a_n:bc}\varepsilon^{bcc_3\cdots c_n}=2(n-2)!(n-1)!\delta^a_{a'}y^{c_3\cdots c_n}, \tag{36}$$

$$\varepsilon^{aa_2\cdots a_n}c_{ba_2\cdots a_n:a'c}\varepsilon^{bcc_3\cdots c_n}=(n-1)!^2y^{b_3\cdots b_n}\delta^{[a}_{a'}\delta^{c_3}_{b_3}\cdots\delta^{c_n]}_{b_n}. \tag{37}$$

Contracting one ε tensor with each index group of the equation $B_n^*[c] = 0$ we get

$$0 = \varepsilon^{aa_2\cdots a_n} B_n^*[c]_{a_2\cdots a_n:bc}\,\varepsilon^{bcc_3\cdots c_n} \tag{38}$$

$$= -\frac{1}{n}\nabla^{a'}\varepsilon^{aa_2\cdots a_n} c_{a'a_2\cdots a_n:bc}\,\varepsilon^{bcc_3\cdots c_n} - \frac{2}{n(n-1)}\nabla^{a'}\varepsilon^{aa_2\cdots a_n} c_{ba_2\cdots a_n:ca'}\,\varepsilon^{bcc_3\cdots c_n} \tag{39}$$

$$= -\frac{2}{n}(n-1)!(n-2)!\left(\nabla^a y^{c_3\cdots c_n} - \nabla^{[a} y^{c_3\cdots c_n]}\right). \tag{40}$$

Note that the derivative $\nabla^a y^{c_3\cdots c_n}$ takes values in the tensor bundle of Young type $(1)\otimes(1^{n-2})$. Using the well-known Littlewood–Richardson rules [29,42] this representation decomposes into the direct sum $(1)^{n-1}\oplus(2, 1^{n-3})$. Note that the antisymmetrization of the above equation gives zero. Thus, the independent components of the equation satisfied by y take values in a tensor bundle of Young type $(2, 1^{n-3})$, which has two columns, of lengths $n-2$ (filled with indices belonging to y) and 1 (filled with index belonging to ∇). It is also well-known that this representation can be isolated by antisymmetrizing along the columns and symmetrizing any two indices between the columns. In our case, the antisymmetrization has no effect (y is already antisymmetric) and the symmetrization, after lowering all indices, gives the equation

$$KY[y]_{ac_3\cdots c_n} = \nabla_{(a} y_{c_3)\cdots c_n} = 0, \tag{41}$$

which is none other than the *rank-$(n-2)$ Killing–Young equation*, whose solutions are called *rank-$(n-2)$ Killing–Young* tensors or *Killing $(n-2)$-forms* [43]. We refer to the differential operator KY as the Killing–Young operator. So, in the same sense that the Calabi complex constitutes the compatibility complex of the Killing equation on a constant curvature background, so would the formal adjoint Calabi complex for the rank-$(n-2)$ Killing–Yano equation on the same background, provided that the formal adjoint complex is formally exact. In general, taking formal adjoints does not preserve exactness [44, p. 449]. However, as we shall see later, the Calabi complex is formally homotopy equivalent (Section 3.2) to a twisted de Rham complex (Section 2.4, Corollary 10), whose formal adjoint is another twisted de Rham complex. Since taking formal adjoints does preserve formal homotopy equivalence and twisted de Rham complexes are always formally exact (Proposition 2), this argument shows that the formal adjoint Calabi complex is in fact formally exact.

2.4. *Equations of finite type, twisted de Rham complex*

The Killing and Killing–Yano equations, which lie at the base of the Calabi and its formal adjoint differential complexes, are well known examples of partial differential equations of *finite type* [16,14,11]. That is, in any neighborhood of a point $x \in M$ they admit only a finite dimensional space of solutions. Each solution is fully determined by its value and finitely many derivatives at x. For the Killing and Killing–Yano equations only the first derivatives are required. This is a strong kind of unique continuation. Such equations are called *regular* if the dimension of the solution space in a sufficiently small neighborhood of a point $x \in M$ is independent of x. That number may, however, differ from the dimension of the global solution space, which can be strictly smaller in the presence of topological or geometric obstructions to continuing local solutions to global ones.

Regular equations of finite type have a very simple existence theory. Let $F \to M$ and $E \to M$ be two vector bundles, together with a differential operator $e\colon \Gamma(F) \to \Gamma(E)$ of order l such that the equation $e[\psi] = 0$, for $\psi \in \Gamma(F)$, is finite type and regular. This means that there exists an integer k such that the knowledge of $j^k\psi(x)$ for any $x \in M$ is sufficient to determine the components of all higher jets of ψ at x. Prolongation of the equation to order k (Appendix C) gives the bundle map $p^{k-l}e\colon J^kF \to J^{k-l}E$. By the regularity hypothesis, the map is of constant rank, so its kernel $V = \ker p^{k-l}e \subseteq J^kF$ is a vector bundle over M. Since all higher derivatives of a solution ψ at x are uniquely determined by $j^k\psi(x)$ and $j^k\psi$ only takes values in V, there is a unique n-dimensional hyperplane in $T_{x,v}V$ that is tangent to the graph of a solution ψ such that $j^k\psi(x) = (x, v)$. These hyperplanes define an n-dimensional distribution on the total space of the bundle V and it is straightforward to check that this distribution is involutive (Lie brackets of vector fields valued in the distribution remain valued in the distribution). Thus, by the theorem of Frobenius [45], V is foliated by n-dimensional leaves tangent to the given hyperplane distribution. Locally, these leaves are precisely the graphs of solutions to the equation $e[\psi] = 0$. Thus the rank $\operatorname{rk} V$ is precisely the dimension of the local solution space on any sufficiently small, connected open set in M.

As we have already mentioned, both the Killing and Killing–Yano operators, $K\colon \Gamma(T^*M) \to \Gamma(S^2M)$ and $KY\colon \Gamma(\Lambda^{n-2}M) \to \Gamma(\mathrm{Y}^{(2,1^{n-1})}T^*M)$, define finite type equations. By the virtue of their covariance, they are also regular on any pseudo-Riemannian symmetric space, which includes constant curvature backgrounds. Furthermore, on constant curvature spaces, the dimensions of their local solution spaces are $\operatorname{rk} V_K = \operatorname{rk} V_{KY} = n(n+1)/2$ [43].

The n-dimensional hyperplane distribution on V and the resulting foliation described above can also be described in another way, namely as a flat linear connection on $V \subseteq J^kF$ [46, Sec. 2.1.3]. The connection is linear because the original equation $e[\psi] = 0$ is itself linear. A linear connection on $V \to M$ can alternatively be described by a first order differential operator $D\colon \Gamma(V) \to \Gamma(T^*M \otimes V)$ defined by the property

$$D[\omega j^k\psi] = \mathrm{d}\omega \otimes j^k\psi, \tag{42}$$

for any $\omega \in C^\infty(M)$ and solution $\psi \in \Gamma(F)$ of $e[\psi]$, where its k-jet is treated as a section $j^k\psi\colon M \to V$. That is, a section $\phi \in \Gamma(V) \subseteq \Gamma(J^kF)$ is constant on an open set $U \subseteq M$ iff it coincides with the k-jet of a solution of $e[\psi] = 0$ on U. So, it is clear that the equations $e[\psi] = 0$ and $D[\phi] = 0$ are equivalent (their spaces of local solutions are locally isomorphic). As discussed in Appendix C this means that there exist differential operators f, f', g, g', p and q, which fit into the following diagram (again, for brevity we use the bundle symbols to stand in for their spaces of sections)

$$\begin{tikzcd} F \arrow[r, "e"] \arrow[d, shift left, "f"] & E \arrow[l, bend right, dashed, "p"'] \arrow[d, shift left, "f'"] \\ V \arrow[r, "D"] \arrow[u, shift left, "g"] & T^*M \otimes V \arrow[l, bend left, dashed, "q"] \arrow[u, shift left, "g'"] \end{tikzcd} \tag{43}$$

and satisfy the following identities:

$$D \circ f = f' \circ e, \qquad g \circ f = \mathrm{id} + p \circ e, \tag{44}$$

$$e \circ g = g' \circ D, \qquad f \circ g = \mathrm{id} + q \circ D. \tag{45}$$

We have already seen that on solutions, the map f simply agrees with the k-jet extension operator j^k. Thus, as a differential operator of order k, it can be chosen to be any projection of J^kF to its subspace V. The choice of this projection then determines the differential operator f'. The differential operators g and g' are constructed in similar ways, making sure that f and g are mutual inverses on solutions. The freedom in the choice of f, f', g and g' also determine the operators p and q.

When it comes to a specific case, say the Killing or Killing–Yano equation, its equivalence to a local constancy condition with respect to a connection can be made explicit only once the solutions are themselves explicitly known. Thus this equivalence is mostly of theoretical, though non-negligible, interest.

Having defined the flat vector bundle (V, D) corresponding to a regular equation of finite type, there is a standard procedure to construct a differential complex associated to it. It is called the *twisted de Rham complex* associated to (V, D),

$$0 \longrightarrow V \xrightarrow{D} \Lambda^1 M \otimes V \xrightarrow{D} \Lambda^2 M \otimes V \cdots \xrightarrow{D} \Lambda^n M \otimes V \longrightarrow 0, \tag{46}$$

where D has been extended to a twisted de Rham differential, defined on sections of $\Lambda^k M \otimes V$ by the condition

$$D[\omega \otimes \psi] = \mathrm{d}\omega \otimes \psi + (-1)^k \omega \wedge D\psi, \tag{47}$$

for any $\omega \in \Gamma(\Lambda^k M)$ and $\psi \in \Gamma(V)$, where we recall that $D\psi$ is a section of $T^*M \otimes V = \Lambda^1 M \otimes V$ and apply the wedge product of forms in the obvious way.

Remark 2. Locally (on sufficiently small contractible open sets), this twisted de Rham complex consists $\operatorname{rk} V$ copies of the ordinary de Rham complex. Globally, of course, if the base manifold M is not simply connected, the twisted de Rham complex $(\Lambda^\bullet M \otimes V, D)$ will differ from $\operatorname{rk} V$ copies of the ordinary de Rham complex $(\Lambda^\bullet M, \mathrm{d})$ because of the possible non-trivial bundle structure of $V \to M$ or the non-trivial monodromy D (parallel transport with respect to D along closed loops). The importance of the twisted de Rham complex will become clear in Section 3 where we discuss the connection between the cohomology of differential complexes and sheaf cohomology.

For later convenience, we shall denote the twisted de Rham complexes associated to the Killing and Killing–Yano equations, respectively, by $(\Lambda^\bullet M \otimes V_K, D_K)$ and $(\Lambda^\bullet M \otimes V_{KY}, D_{KY})$.

3. Cohomology of locally constant sheaves

The main reasons for introducing some of the general sheaf and sheaf cohomology machinery below are two fold. First, we have made a connection between the abstract notion of sheaf cohomology and the cohomology of a differential complex. A priori, computing the cohomology of differential complex is a very hard problem, because it involves solving partial differential equations. On the other hand, because of the flexibility of the general machinery of sheaf cohomology, it may be computable in some effective way, for instance, by reducing it to a problem in finite dimensional linear algebra. The canonical example of where this connection can be leveraged is the computation of de Rham cohomology groups of a manifold M using the equivalent (through sheaf theoretic machinery) computation of the simplicial (or cellular) cohomology of a finite triangulation (or cell decomposition) of M. The second reason is that the ideas that have been introduced give us some tools to explicitly show that the cohomologies of two different differential complexes are isomorphic as long as both complexes are *formally exact*, *locally exact* and *resolve* the same sheaf in degree-0 (this terminology is introduced below).

3.1. Locally constant sheaves

Recall from Section 2.4 that a regular linear differential equation of finite type has only a finite dimensional space of local solutions, with this dimension being constant over the base manifold. It so happens that, from an abstract point of view, it is convenient to view these local solutions as a *locally constant sheaf* of vector spaces. A *sheaf* $\mathcal{F}$ of vector spaces on a topological space M [47,48] is an assignment $U \mapsto \mathcal{F}(U)$ of a vector space (of *local sections* over U, $\mathcal{F}(\varnothing) = 0$) to each open $U \subseteq M$ satisfying the following axioms: *(restriction)* for any inclusion of opens $U \subseteq V$ there exist linear restriction maps $\mathcal{F}(V) \to \mathcal{F}(U)$, also written $f \mapsto f|_U$, such that $U \subseteq U$ induces the identity map and $U \subseteq V \subseteq W$ induces $\mathcal{F}(W) \to \mathcal{F}(U)$ in agreement with the composition $\mathcal{F}(W) \to \mathcal{F}(V) \to \mathcal{F}(U)$; *(descent)* any pair of opens U and V induces an exact sequence $0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \times \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$, where the first map is $f \mapsto (f|_U, f|_V)$ and the second one is $(f, g) \mapsto f|_{U\cap V} - g|_{U\cap V}$. We write $\Gamma(\mathcal{F}) = \Gamma(M, \mathcal{F}) = \mathcal{F}(M)$ for the vector space of *global sections* of the sheaf $\mathcal{F}$. A *sheaf* is called *locally constant* when the number $\overline{\dim}\mathcal{F}_x = \max_{U\ni x} \dim \mathcal{F}(U)$, where U ranges over connected open neighborhoods of $x \in M$, is finite and does not depend on x, so we can write $\overline{\dim}\mathcal{F} = \overline{\dim}\mathcal{F}_x$. Since $\dim \mathcal{F}(U)$ can only decrease for larger connected U, for any $x \in M$ there exists a connected neighborhood U of x such the vector spaces of local sections over smaller connected neighborhoods stabilize (the restriction map becomes an isomorphism), so that we can write $\mathcal{F}(U) \cong \bar{F}$ for some fixed vector space $\bar{F}$ that we call the *stalk* of $\mathcal{F}$. Clearly, $\dim \bar{F} = \overline{\dim}\mathcal{F}$. Also, $\mathcal{F}$ is called *constant* when it is locally constant and $\Gamma(\mathcal{F}) \cong \bar{F}$.

Given a vector bundle $F \to M$, the assignment $\mathcal{F}(U) = \Gamma(F, U)$ of local sections of F over each open $U \subseteq M$ defines a sheaf $\mathcal{F}$ on M, called the *sheaf of (germs of) sections* of $F \to M$. Similarly, it is straightforward to check that, given another vector bundle $E \to M$ and a linear differential operator $e\colon \Gamma(F) \to \Gamma(E)$, the sets $\mathcal{S}_e(U) = \{\psi \in \Gamma(F, U) \mid e[\psi] = 0\}$ of solutions of the partial differential equation $e[\psi] = 0$ also define a sheaf $\mathcal{S}_e$ on M, called the *solution sheaf* of $e\colon \Gamma(F) \to \Gamma(E)$. Following the preceding discussion of equations of finite type, it should be clear that solution sheaves $\mathcal{K} = \mathcal{S}_K$ (the *Killing sheaf*) and $\mathcal{KY} = \mathcal{S}_{KY}$ (the *Killing–Yano sheaf*) of the Killing and Killing–Yano equations are locally constant, provided the background pseudo-Riemannian manifold is chosen such that these equations are regular. Another important example is the constant sheaf $\mathbb{R}_M = \mathcal{S}_{\mathrm{d}}$ of locally constant functions, which solve the equation $\mathrm{d}f = 0$, $f \in C^\infty(M)$ and d the de Rham differential.

Sheaves are important because every sheaf $\mathcal{F}$ (of vector spaces) on M automatically comes with an abstract notion of *sheaf cohomology* (vector spaces) $H^p(M, \mathcal{F})$, called the pth or degree-p cohomology of $\mathcal{F}$, or of M with coefficients in $\mathcal{F}$. Moreover, all classical cohomology theories from algebraic topology can be identified with the cohomologies of certain sheaves. Further, some superficially different looking cohomologies theories may be connected through the fact that they are both equivalent to the sheaf cohomology of the same sheaf. In particular, the classical simplicial, cellular, singular, Čech and de Rham cohomologies of a manifold M all coincide [49,47,48] because they are each equivalent to the cohomology of M with coefficients in the sheaf $\mathbb{R}_M$ of locally constant functions.

The intrinsic definition of sheaf cohomology is somewhat involved and not entirely intuitive (unless one is already intimately familiar with Čech cohomology and the notion of local coefficients). Fortunately, the intrinsic definition can be relegated to standard references [47,48] in favor of an equivalent but more practical definition using *acyclic resolutions*. To explain further, we need to introduce some terminology. A *complex* of sheaves of vector spaces

$$\cdots \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \cdots \tag{48}$$

consists of an assignment of linear maps $\mathcal{F}_i(U) \to \mathcal{F}_{i+1}(U)$ to each open $U \subseteq M$, in a way consistent with restriction maps, such that we have a complex of vector spaces of local sections (two successive maps compose to zero)

$$\cdots \longrightarrow \mathcal{F}_i(U) \longrightarrow \mathcal{F}_{i+1}(U) \longrightarrow \cdots \tag{49}$$

for each open $U \subseteq M$. A local section in $\mathcal{F}_i(U)$ that is in the kernel of the corresponding map is called a *cocycle* and a local section in $\mathcal{F}_i(U)$ that is in image of the corresponding map is called a *coboundary*. A sheaf complex is *exact* when, for each $x \in M$, open neighborhood $U \subseteq M$ of x and local section $\alpha \in \mathcal{F}_i(U)$, there exists a possibly smaller and α-dependent open neighborhood $U' \subseteq U$ of x such that $\alpha|_{U'}$ is a coboundary. For a complex of sheaves, like (48), we could define its *cohomology sheaves* $\mathcal{H}^i(\mathcal{F}_\bullet)$ (distinct from *sheaf cohomology*, to be defined later), by starting with the assignment $\mathcal{H}^i(\mathcal{F}_\bullet)(U) = \ker(\mathcal{F}_i(U) \to \mathcal{F}_{i+1}(U))/\operatorname{im}(\mathcal{F}_{i-1}(U) \to \mathcal{F}_i(U))$, which may not produce a sheaf but only a *presheaf*, and applying the *sheafification* construction to it. We will not go into the details of how sheafification turns presheaves into sheaves here, but they can be found in standard references [47,48]. It suffices to point out that given a sheaf complex in non-negative degrees, $0 \to \mathcal{F}_0 \to \mathcal{F}_i \to \cdots$, the vector space $\mathcal{H}^0(\mathcal{F}_\bullet)(U) \subseteq \mathcal{F}_0(U)$ consists of all cocycle local sections. In the sequel, we shall only need to refer to such cohomology sheaves in degree-0. Given a sheaf $\mathcal{F}$, if $\mathcal{F}_i \to \mathcal{F}_{i+1}$ is a complex of sheaves such that $\mathcal{F}_i = 0$ for $i < 0$, $\mathcal{H}^0(\mathcal{F}_\bullet) = \mathcal{F}$, and $\mathcal{H}^i(\mathcal{F}_\bullet) = 0$ for $i > 0$, we call it a *resolution* of the sheaf $\mathcal{F}$.

In the sequel, we shall only consider sheaves of sections of vector bundles or of solution of some linear PDE and only complexes of sheaves where maps between the vector spaces of local sections are induced by restrictions of differential operators, for which the compatibility with restrictions is automatically satisfied.

3.2. Acyclic resolution by a differential complex

The de Rham complex [49] is the canonical example of a complex of sheaves of sections of vector bundles (differential forms on M), with maps induced by differential operators (de Rham differentials). The Poincaré lemma then demonstrates that this complex of sheaves is exact. For simplicity, we shall call a *differential complex* $(F_\bullet, f_\bullet)$ a sequence of vector bundles

$F_i \to M$ and differential operators $f_i\colon \Gamma(F_{i-1}) \to \Gamma(F_i)$ satisfying $f_i \circ f_{i-1} = 0$, while implicitly setting $F_{-1} = 0$ and $f_0 = 0$. Given a differential complex, it is natural to define its cohomology vector spaces to be the cohomology of the cochain complex of global sections, $H^i(F_\bullet, f_\bullet) = H^i(\Gamma(F_\bullet), f_\bullet)$, which we also refer to as the cohomology with *unrestricted supports*. Since differential operators do not increase supports, we can equally consider the cohomology of the differential complex with *compact supports*, defined as $H^i_c(F_\bullet, f_\bullet) = H^i(\Gamma_c(F_\bullet), f_\bullet)$. A differential complex naturally defines a complex $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots$ of sheaves of sections of these bundles, $\mathcal{F}_i(U) = \Gamma(F_i, U)$. A differential complex is said to be *locally exact* if it defines an exact complex of sheaves. Local exactness is a very strong property that is crucial in the relation of the cohomology of a differential complex to sheaf cohomology, which we discuss next.

In general, given a complex of sheaves $\mathcal{F}_i \to \mathcal{F}_{i+1}$, we call it an *injective resolution* of a sheaf $\mathcal{F}$ if it is a *resolution* of $\mathcal{F}$ (namely, $\mathcal{F}_i = 0$ for $i < 0$, it is exact except for $\mathcal{H}^0(\mathcal{F}_\bullet) = \mathcal{F}$), and each $\mathcal{F}_i$ is *injective*. The injectivity condition is somewhat technical. The same can be said for the fact that every sheaf has an injective resolution. So we will not go into them here and defer to standard references instead [47,48]. We will need these notions only for the following definition. The *degree-i sheaf cohomology* vector spaces $H^i(\mathcal{F}) = H^i(M, \mathcal{F})$, also called the *degree-i cohomology of M with coefficients in* $\mathcal{F}$, as the cohomology vector space of the complex of global sections of any injective resolution $\mathcal{F}_i \to \mathcal{F}_{i+1}$ of $\mathcal{F}$, $H^i(\mathcal{F}) = H^i(\Gamma(\mathcal{F}_\bullet))$. It is important to note that sheaf cohomology is well defined. It does not depend on the chosen injective resolution, because the injectivity condition implies the existence of a homotopy equivalence between the complexes of global sections of any two such resolutions, thus forcing their cohomologies to be isomorphic. This is another technical fact that we shall not go into here.

Instead, we make note of yet another technical fact that provides a practical way to compute sheaf cohomology. For that, we need two more definitions. A sheaf $\mathcal{F}$ is called *acyclic* if $H^i(\mathcal{F}) = 0$ for all $i > 0$, though as usual the degree-0 cohomology $H^0(\mathcal{F}) \cong \Gamma(\mathcal{F})$ is isomorphic to the vector space of global sections of $\mathcal{F}$. A sheaf $\mathcal{F}$ on M is called *soft* if for any closed $A \subseteq M$ the restriction maps $\mathcal{F}(M) \to \mathcal{F}(A)$ are surjective, where $\mathcal{F}(A) = \bigcap_{U \subseteq A} \mathcal{F}(U)$ with U ranging over all open sets that contain the closed set A. In other words, given an open $U \subseteq M$ and a closed subset $A \subseteq U$, a local section on U can always be extended to a global one on M without modification on A, but possibly modified on $U \setminus A$. What is really important for us is the following.

Proposition 1. (i) *If $\mathcal{F}$ is a sheaf on M, and $\mathcal{F}_i \to \mathcal{F}_{i+1}$ is a resolution of $\mathcal{F}$ by acyclic sheaves (acyclic resolution), then* $H^i(M, \mathcal{F}) \cong H^i(\Gamma(M, \mathcal{F}_\bullet))$. (ii) *Any soft sheaf on M is acyclic.* (iii) *Given a vector bundle $F \to M$, the sheaf $\mathcal{F}$ of sections of F is soft.*

Proof. Any standard discussion of sheaf cohomology establishes (i) and (ii) [47,48]. On the other hand, (iii) is simply a restatement of the well known Whitney extension theorem for smooth functions [50, Thm. 2.3.6]. □

Note that the complex of sheaves corresponding to a differential complex then automatically consists of acyclic sheaves. The above proposition essentially tells us that, given a resolution of some sheaf $\mathcal{F}$ on a manifold M by a locally exact differential complex $(F_\bullet, f_\bullet)$, the sheaf cohomology of $\mathcal{F}$ and the cohomology of the differential complex will coincide, $H^i(\mathcal{F}) \cong H^i(F_\bullet, f_\bullet)$. This observation will be particularly important later in Corollary 12.

Next, we discuss some conditions ensuring that the cohomologies of two given differential complexes are isomorphic. As we have now seen, local exactness is a very strong and useful property, unfortunately it can be difficult to check in practice. Two weaker notions of exactness exist that are easier to check in practice. To formulate them, we refer to the notions of *jets* and *jet bundles*, together with associated constructions like *prolongations* and *principal symbols*, all briefly recalled in Appendix C. Given a sequence of vector bundles F_i and a complex of linear differential operators $f_i\colon F_{i-1} \to F_i$, each of order k_i, their prolongations define a complex of vector bundle morphisms,

$$\cdots \longrightarrow J^l F_{i-1} \xrightarrow{p^{l_i} f_i} J^{l_i} F_i \xrightarrow{p^{l_{i+1}} f_{i+1}} J^{l_{i+1}} F_{i+1} \longrightarrow \cdots, \tag{50}$$

with $l_i = l - k_i$ and $l_{i+1} = l - k_i - k_{i+1}$, for each sufficiently large l. The differential complex is said to be *formally exact* if the above compositions are exact, as linear bundle maps over M, for any values of l and i for which they are defined. On the other hand, given $(x, p) \in T^*M$, the principal symbols of the differential operators f_i define a complex of linear maps between the fibers of F_i at x,

$$\cdots \longrightarrow F_{i-1,x} \xrightarrow{\sigma_{x,p} f_i} F_{i,x} \xrightarrow{\sigma_{x,p} f_{i+1}} F_{i+1,x} \longrightarrow \cdots. \tag{51}$$

The differential complex is said to be *elliptic* if the above complex is exact for every $(x, p) \in T^*M$, $p \neq 0$. These two weaker notions are distinct [51]. Formal exactness is a good hypothesis for showing that differential operators factor in certain ways. On the other hand, ellipticity is a condition that can be used to prove local exactness, via the method of parametrices and fundamental solutions. However, the general question of determining necessary and sufficient conditions for local exactness for differential complexes is a difficult and still open problem. The main conjecture is sometimes known as *Spencer's conjecture*: a formally exact, elliptic complex is locally exact [14,51,52]. On the other hand, some supplementary sufficient conditions are known for an elliptic complex to be locally exact. A prominent condition of this kind is known as the *δ-estimate* [27, Sec. 1.3.13], which first appeared in the works of Singer, Sweeney and MacKichan [14].

Proposition 2. *The twisted de Rham complex associated to the flat bundle (V, D) defined by a regular differential equation of finite type, defined in Eq. (46), is formally exact, elliptic and locally exact.*

Proof. As noted in Remark 2, the twisted de Rham complex is locally (on sufficiently small contractible open sets) equivalent to $\operatorname{rk} V$ copies of the ordinary de Rham complex. To see the equivalence, it suffices to locally choose a D-flat basis frame for V. Since all of the desired properties, formal exactness, ellipticity and local exactness are purely local, it suffices to check them for the ordinary de Rham complex. It is well known that each of these properties does hold for the de Rham complex, having served as a model example for each. Formal exactness and ellipticity are discussed, for instance, in [14,11,27] and [53, §XIX.4]. On the other hand, local exactness is essentially the content of the Poincaré lemma [49].

There is another way to establish local exactness that bypasses the Poincaré lemma and does not require an explicit local choice of a D-flat basis frame for V. In particular, as discussed for instance in the given references, local exactness and ellipticity are independent of such a choice. Then, local exactness follows provided the initial operator of the complex, the connection operator $D\colon \Gamma(V) \to \Gamma(T^*M \otimes V)$, satisfies the δ-estimate. According to Example 1.3.58 of [27], any linear connection operator satisfies the δ-estimate. Hence, by Theorem 1.3.61 of [27], the twisted de Rham complex is locally exact. $\square$

As is well known in homological algebra, cochain maps and homotopies between them are important concepts, the first because they descend to cohomology, the second because equivalence up to homotopy descends to isomorphism on cohomology. When dealing with differential complexes, it becomes important to distinguish the case where the cochain maps and homotopies are defined by differential operators. The most important notion we will need is that of a *formal homotopy equivalence*. Let $(F_\bullet, f_\bullet)$ and $(G_\bullet, g_\bullet)$ be two differential complexes. They are said to be *formally homotopy equivalent* provided there exist differential operators e_i, h_i, u_i and v_i fitting into the diagram (we use the bundles to stand in for their spaces of sections)

$$\begin{tikzcd}
\cdots \arrow[r] & F_{i-1} \arrow[l, dashed, bend right] \arrow[r, "f_i"'] \arrow[d, shift left, "u_{i-1}"] & F_i \arrow[l, dashed, bend right, "e_i"'] \arrow[r, "f_{i+1}"'] \arrow[d, shift left, "u_i"] & F_{i+1} \arrow[l, dashed, bend right, "e_{i+1}"'] \arrow[r] \arrow[d, shift left, "u_{i+1}"] & \cdots \arrow[l, dashed, bend right] \\
\cdots \arrow[r] & G_{i-1} \arrow[l, dashed, bend left] \arrow[r, "g_i"] \arrow[u, shift left, "v_{i-1}"] & G_i \arrow[l, dashed, bend left, "h_i"] \arrow[r, "g_{i+1}"] \arrow[u, shift left, "v_i"] & G_{i+1} \arrow[l, dashed, bend left, "h_{i+1}"] \arrow[r] \arrow[u, shift left, "v_{i+1}"] & \cdots \arrow[l, dashed, bend left]
\end{tikzcd}, \tag{52}$$

where the squares composed of solid arrows commute (cochain map condition on u_i and v_i) and the dashed arrows are homotopy operators with respect to which u_i and v_i are quasi-inverses, $v_i \circ u_i - \mathrm{id} = e_{i+1} \circ f_{i+1} + f_i \circ e_i$ and $u_i \circ v_i - \mathrm{id} = h_{i+1} \circ g_{i+1} + g_i \circ h_i$.

Lemma 3. *Consider two differential complexes $(F_\bullet, f_\bullet)$ and $(G_\bullet, g_\bullet)$ that start in degree 0, also denote the corresponding complexes of sheaves of sections as $\mathcal{F}_i \to \mathcal{F}_{i+1}$ and $\mathcal{G}_i \to \mathcal{G}_{i+1}$. Suppose that both differential complexes are formally exact, except in degree 0. Further, suppose that the equations $f_1[\phi] = 0$ and $g_1[\gamma] = 0$, with $\phi \in \Gamma(F_0)$ and $\gamma \in \Gamma(G_0)$, are equivalent, or in other words the degree-0 cohomology sheaves are isomorphic to some given sheaf $\mathcal{F} \cong \mathcal{H}^0(\mathcal{F}_\bullet) \cong \mathcal{H}^0(\mathcal{G}_\bullet)$.*

(i) *Then there exists a formal homotopy equivalence between these differential complexes and their cohomologies are isomorphic, both with unrestricted and compact supports (or any other kind of restriction on supports):*

$$H^i(F_\bullet, f_\bullet) \cong H^i(G_\bullet, g_\bullet) \quad \text{and} \quad H^i_c(F_\bullet, f_\bullet) \cong H^i_c(G_\bullet, g_\bullet). \tag{53}$$

(ii) *If one of the differential complexes is locally exact, then both are locally exact and their cohomologies both compute the sheaf cohomology of $\mathcal{F}$:*

$$H^i(M, \mathcal{F}) \cong H^i(F_\bullet, f_\bullet) \cong H^i(G_\bullet, g_\bullet). \tag{54}$$

Proof. (i) Equivalence of the equations $f_1[\phi] = 0$ and $g_1[\gamma] = 0$ means (Appendix C) that there exist differential operators, say $u_0\colon \Gamma(F_0) \to \Gamma(G_0)$ and $v_0\colon \Gamma(G_0) \to \Gamma(F_0)$, such that $v_0 \circ u_0[\phi] = 0$ whenever $f_1[\phi] = 0$ and such that $u_0 \circ v_0[\gamma] = 0$ whenever $g_1[\phi] = 0$. In other words, there exist differential operators $e_1\colon \Gamma(F_1) \to \Gamma(F_0)$ and $h_1\colon \Gamma(G_1) \to \Gamma(G_0)$ such that $v_0 \circ u_0 = e_1 \circ f_1$ and $u_0 \circ v_0 = h_1 \circ g_1$. These differential operators are the initial step in establishing the desired formal homotopy equivalence.

We proceed by a standard induction argument from homological algebra (in fact, a version of this argument proves the independence of sheaf cohomology from the injective resolution used to compute it). Assume that all the desired differential operators have been defined up to e_i, h_i, u_{i-1} and v_{i-1}, which also satisfy the desired identities. We can easily verify the identities

$$(g_i \circ u_{i-1}) \circ f_{i-1} = (g_i \circ g_{i-1}) \circ u_{i-2} = 0, \tag{55}$$
$$(f_i \circ v_{i-1}) \circ g_{i-1} = (f_i \circ f_{i-1}) \circ v_{i-2} = 0, \tag{56}$$

which together with the formal exactness of the compositions $f_i \circ f_{i-1} = 0$ and $g_i \circ g_{i-1} = 0$ imply the factorizations $g_i \circ u_{i-1} = u_i \circ f_i$ and $f_i \circ v_{i-1} = v_i \circ g_i$, for some differential operators $u_i \colon \Gamma(F_i) \to \Gamma(G_i)$ and $v_i \colon \Gamma(G_i) \to \Gamma(F_i)$ (see Appendix C). Further, we can also verify the identities

$$(v_i \circ u_i - \mathrm{id} - f_i \circ e_i) \circ f_i = (v_i \circ g_i) \circ u_{i-1} - f_i - f_i \circ e_i \circ f_i \tag{57}$$

$$= f_i \circ (v_{i-1} \circ u_{i-1}) - f_i - f_i \circ e_i \circ f_i = 0, \tag{58}$$

$$(u_i \circ v_i - \mathrm{id} - g_i \circ h_i) \circ g_i = (u_i \circ f_i) \circ v_{i-1} - g_i - g_i \circ h_i \circ g_i \tag{59}$$

$$= g_i \circ (u_{i-1} \circ v_{i-1}) - g_i - g_i \circ h_i \circ g_i = 0, \tag{60}$$

which again together with formal exactness imply the factorizations $v_i \circ u_i - \mathrm{id} - f_i \circ e_i = e_{i+1} \circ f_{i+1}$ and $u_i \circ v_i - \mathrm{id} - g_i \circ h_i = h_{i+1} \circ g_{i+1}$, for some differential operators $e_{i+1} \colon \Gamma(F_{i+1}) \to \Gamma(F_i)$ and $h_{i+1} \colon \Gamma(G_{i+1}) \to \Gamma(G_i)$. This concludes the inductive step.

Now, let us consider the cohomology of these complexes, $H^i(F_\bullet, f_\bullet) = H^i(\Gamma(F_\bullet), f_\bullet)$ and $H^i(G_\bullet, g_\bullet) = H^i(\Gamma(G_\bullet), g_\bullet)$. As is well known from homological algebra, a homotopy equivalence (of which a formal homotopy equivalence is a special kind) induces an isomorphism in cohomology: $H^i(F_\bullet, f_\bullet) \cong H^i(G_\bullet, g_\bullet)$. However, if the operators implementing the homotopy equivalence are differential operators, as in this case, we can replace unrestricted sections $\Gamma(-)$ by sections with compact supports $\Gamma_c(-)$, so that $H^i_c(F_\bullet, f_\bullet) = H^i(\Gamma_c(F_\bullet), f_\bullet)$ and $H^i_c(G_\bullet, g_\bullet) = H^i(\Gamma_c(G_\bullet), g_\bullet)$. The homotopy equivalence of the resulting complexes still holds because differential operators do not increase supports, and so we still have an isomorphism in cohomology: $H^i_c(F_\bullet, f_\bullet) \cong H^i_c(G_\bullet, g_\bullet)$. Incidentally, instead of compact supports, any other family of supports would do as well.

(ii) By the local exactness hypothesis, both differential complexes provide resolutions of the sheaf $\mathcal{F}$ (which happens to be isomorphic to the solution sheaves $\mathcal{S}_{f_1} = \mathcal{H}^0(\mathcal{F}_\bullet)$ and $\mathcal{S}_{g_1} = \mathcal{H}^0(\mathcal{G}_\bullet)$). Then, by Proposition 1, these resolutions are acyclic and hence the corresponding cohomologies with unrestricted supports compute the sheaf cohomology of $\mathcal{F}$. This concludes the proof. □

3.3. Generalized Poincaré duality

In Section 3.2, we discussed how the cohomology $H^i(F_\bullet, f_\bullet)$ of a differential complex can, under optimal conditions, be equated with the cohomology $H^i(\mathcal{F})$ of the sheaf resolved by $(F_\bullet, f_\bullet)$. However, even under optimal conditions, this connection breaks down if we consider cohomology $H^i_c(F_\bullet, f_\bullet)$ with compact (or some other family of) supports instead of unrestricted ones. What we discuss below is a way to relate cohomology with compact supports to that with unrestricted supports, a kind of Poincaré duality.

For the de Rham complex on a manifold M, $\dim M = n$, a well known formulation of Poincaré duality is the isomorphism $H^p(M) \cong H^{n-p}_c(M)^*$ [49, Rmk. 5.7] between the linear dual of cohomology in degree-p and compactly supported cohomology in degree-$(n-p)$. This isomorphism is induced by the existence of a non-degenerate natural pairing between p-forms and $(n-p)$-forms on M and its non-degenerate descent to cohomology. The goal of this section is to leverage the properties of the Calabi complex and its formal adjoint complex that were discussed in the preceding section to demonstrate a generalized version of Poincaré duality, which effectively computes the cohomology with compact supports in terms of sheaf cohomology.

There are two ways to establish generalized Poincaré duality for a differential complex $(F_\bullet, f_\bullet)$ that would be applicable to the case of the Calabi complex and its formal adjoint. One of them, discussed in Section 3.3.1, relies on the fact that the corresponding complex of sheaves resolves the sheaf of solutions of a regular differential equation of finite type (a locally constant sheaf). This method is somewhat more elementary. The other, discussed in Section 3.3.2, works for any elliptic complex, but requires some results from functional analysis and distribution theory. Either of these results, as will be shown in Section 3.4, can be applied to prove generalized Poincaré duality for the Calabi complex and its formal adjoint complex.

3.3.1. Twisted de Rham complex

First, we will discuss the twisted de Rham complex, as introduced in Section 2.4. The results will then apply to the Calabi complex and its formal adjoint by virtue of Lemma 3. The strategy is straightforward and reproduces the logic of the proofs of the ordinary Poincaré duality, cf. [49, §5], [54, Ch. 11], or [55, Sec. V.4]. First, generalized Poincaré duality is shown to hold on contractible open patches. Then, given a "good cover" of the manifold consisting of such patches, we use a version of the Mayer–Vietoris exact sequence as an inductive step to conclude that generalized Poincaré duality also holds on the entire manifold.

First, recall that we denote the fiber of the vector bundle $V \to M$ by $\bar{V}$. Then, $\bar{V}^*$ is the fiber of the dual vector bundle $V^* \to M$. We are interested in the relation between the cohomology of the twisted de Rham complex $H^i(\Lambda^\bullet M \otimes V, D)$ and the compactly supported cohomology of the formal adjoint complex, which happens to be $(\Lambda^\bullet M \otimes V^*, D)$, where the connection D has been extended to $V^* \to M$ by the rule $\mathrm{d}(\xi \cdot \psi) = (D\xi) \cdot \psi + \xi \cdot (D\psi)$, with $\xi \in \Gamma(V^*)$ and $\psi \in \Gamma(V)$. Presuming that M is oriented, which is a prerequisite for integrating top-degree forms, there is a duality pairing between elements of $\Gamma(\Lambda^p M \otimes V)$ and $\Gamma_c(\Lambda^{n-p} M \otimes V^*)$ given by the formula

$$\langle \xi, \psi \rangle = \int_M \langle \xi \wedge \psi \rangle, \tag{61}$$

where $\langle(\alpha\otimes\xi)\wedge(\beta\otimes\psi)\rangle = (\alpha\wedge\beta)\otimes(\xi\cdot\psi)$. The formal adjoint relation is established (up to signs) for $\xi\in\Gamma(\Lambda^{n-p-1}M\otimes V^*)$ and $\psi\in\Gamma(\Lambda^p M\otimes V)$ by the identity

$$\mathrm{d}\langle\xi\wedge\psi\rangle = \langle(D\xi)\wedge\psi\rangle - (-1)^{n-p}\langle\xi\wedge(D\psi)\rangle. \tag{62}$$

Lemma 4. *Let $U\subseteq M$ be an oriented contractible open set. Then, generalized Poincaré duality holds, $H^p(\Lambda^\bullet M\otimes V|_U, D)\cong H^{n-p}_c(\Lambda^\bullet M\otimes V^*|_U, D)^*$, because all of the cohomology spaces vanish except $H^0(\Lambda^\bullet M\otimes V, D)\cong\bar V$ and $H^n_c(\Lambda^\bullet M\otimes V^*, D)\cong\bar V^*$.*

Proof. As we have already noted in the proof of Proposition 2, a choice of a locally D-flat basis frame for V over $U\subseteq M$ identifies the twisted de Rham complex with $\operatorname{rk} V$ copies of the usual de Rham complex. Since U is contractible, such a choice is always possible. Moreover, the pairing (61) reduces to the usual pairing between forms and compactly supported forms of complementary degrees on an oriented manifold. Thus, we can easily conclude that

$$H^p(\Lambda^\bullet M\otimes V|_U, D) = H^p(U)\otimes\bar V, \tag{63}$$

$$H^{n-p}_c(\Lambda^\bullet M\otimes V^*|_U, D) = H^{n-p}_c(U)\otimes\bar V^*. \tag{64}$$

Recalling that, for contractible U, $H^p(U)=0$ except for $H^0(U)=\mathbb{R}$ and $H^{n-p}_c(U)=0$ except for $H^n_c(U)=\mathbb{R}$, concludes the proof. □

Lemma 5 (*Mayer–Vietoris*)**.** *Consider two open subsets $U, W\subseteq M$. We have the following long exact sequences in cohomology with unrestricted and compact supports, which we shall for brevity denote as $H^i(-) = H^i(\Lambda^\bullet M\otimes V|_-, D)$ and $H^i_c(-) = H^i_c(\Lambda^\bullet M\otimes V^*|_-, D)$:*

$$\begin{tikzcd} 0 \arrow[r] & H^0(U\cup W) \arrow[r] & H^0(U)\oplus H^0(W) \arrow[r] & H^0(U\cap W) \arrow[dll, rounded corners] \\ & H^1(U\cup W) \arrow[r] & H^1(U)\oplus H^1(W) \arrow[r] & H^1(U\cap W) \arrow[r] & \cdots \end{tikzcd} \tag{65}$$

$$\begin{tikzcd} 0 \arrow[r] & H^0_c(U\cap W) \arrow[r] & H^0_c(U)\oplus H^0_c(W) \arrow[r] & H^0_c(U\cup W) \arrow[dll, rounded corners] \\ & H^1_c(U\cap W) \arrow[r] & H^1_c(U)\oplus H^1_c(W) \arrow[r] & H^1_c(U\cup W) \arrow[r] & \cdots . \end{tikzcd} \tag{66}$$

Proof. Both long exact sequences in cohomology follow from short exact sequences of cochain complexes. These short exact sequences, where for brevity we write $\Gamma^i(-) = \Gamma(\Lambda^i M\otimes V|_-)$ and $\Gamma^i_c = \Gamma(\Lambda^i M\otimes V^*|_-)$ are

$$0\longrightarrow\Gamma^i(U\cup W)\longrightarrow\Gamma^i(U)\oplus\Gamma^i(W)\longrightarrow\Gamma^i(U\cap W)\longrightarrow 0, \tag{67}$$

$$0\longrightarrow\Gamma^i_c(U\cap W)\longrightarrow\Gamma^i_c(U)\oplus\Gamma^i_c(W)\longrightarrow\Gamma^i_c(U\cup W)\longrightarrow 0. \tag{68}$$

In the first sequence, the maps are restrictions, $\alpha\mapsto(\alpha|_U, \alpha|_W)$ and $(\alpha,\beta)\mapsto(\alpha|_{U\cap W}-\beta|_{U\cap W})$. The exactness follows from the usual ability to restrict and glue together smooth sections over open regions, also known as their sheaf property. In the second sequence, the maps are extensions by zero, $\alpha\mapsto(\alpha^U_0, \alpha^W_0)$ and $(\alpha,\beta)\mapsto\alpha^{U\cup W}_0-\beta^{U\cup W}_0$. The exactness follows from the existence of a smooth partition of unity adapted to the cover of $U\cup W$ by U and W.

These maps are clearly compatible with the connection differential operator D and so are cochain maps. The general connection between short exact sequences of cochain complexes and long exact sequences in cohomology (Appendix B) gives the desired long exact sequences and concludes the proof. □

Proposition 6. *Given a flat vector bundle (V, D) on an oriented n-dimensional orientable manifold M, the unrestricted cohomology $H^p = H^p(\Lambda^\bullet M\otimes V, D)$ of the associated twisted de Rham complex and the compactly supported cohomology $H^{n-p}_c = H^{n-p}_c(\Lambda^\bullet M\otimes V^*, D)$ of its formal adjoint complex satisfy generalized Poincaré duality:*

$$H^p\cong(H^{n-p}_c)^*. \tag{69}$$

Note the asymmetry of the isomorphism. The reverse identity $(H^p)^*\cong H^{n-p}_c$ also holds when the cohomology vector spaces are finite dimensional, but in general may not when they are infinite dimensional.

Proof. In this proof, we shall use induction over a special kind of open cover of M. An open cover (U_k) of M is called *good* if it is locally finite, every nonempty finite intersection $U_{k_0} \cap \cdots \cap U_{k_m}$ is diffeomorphic to $\mathbb{R}^n$, and it is closed under finite intersections. In particular, each of the U_k is itself diffeomorphic to $\mathbb{R}^n$ and thus contractible. Good covers are known to exist for any manifold [49, Thm.5.1]. Inducing an orientation on each element of the cover from the orientation on M, Lemma 4 establishes the desired duality relation for any U_k and thus the initial step of the inductive argument.

Next, we show, provided the desired duality relation holds on any finite union $U_{k_0} \cup \cdots \cup U_{k_{m-1}}$ of m sets, that it also holds on any finite union $U_{k_0} \cup \cdots \cup U_{k_m}$ of $m+1$ sets as well. Of course, we take all such unions to be oriented in a way compatible with the global orientation on M. Let $U = U_{k_m}$, $W = U_{k_0} \cup \cdots \cup U_{k_{m-1}}$ and notice that both W and $W \cap U$ are finite unions of m sets from the cover (recall that the cover is closed under intersections). The fact that the pairing (61), well defined on a given oriented, open $U \subseteq M$, descends to cohomology means that we always have a mapping $H^p(U) \to H^{n-p}_c(U)^*$, which may or may not be an isomorphism. It is in fact an isomorphism on U and, by the inductive hypothesis, also on W and $W \cap U$. Combining the long exact sequences of Lemma 5 for W and U together with these maps and isomorphisms, we obtain the following diagram (notice the arrow reversal by linear duality in the second row):

$$\begin{tikzcd}
H^p(W \cup U) \arrow[r] \arrow[d, equal] & H^p(W) \oplus H^p(U) \arrow[r] \arrow[d, equal] & H^p(W \cap U) \arrow[r] \arrow[d] & H^{p+1}(W \cup U) \arrow[r] \arrow[d, equal] & H^{p+1}(W) \oplus H^{p+1}(U) \arrow[d, equal] \\
H^{n-p}_c(W \cup U)^* \arrow[r] & H^{n-p}_c(W)^* \oplus H^{n-p}_c(U)^* \arrow[r] & H^{n-p}_c(W \cap U)^* \arrow[r] & H^{n-p-1}_c(W \cup U)^* \arrow[r] & H^{n-p-1}_c(W)^* \oplus H^{n-p-1}_c(U)^*
\end{tikzcd}$$

Thus, by the 5-lemma (Appendix B), the map in the center of the diagram is also an isomorphism and the inductive step is established.

The only problem remaining is that a good cover is not always finite (though it can be chosen to be finite for compact manifolds). There is a way around that, however. Using a similar argument, one can show that the desired duality holds also on disjoint countable unions of finite unions of covering sets. It is at this stage that the asymmetry between the cohomologies with unrestricted and compact supports appears. Then, provided the manifold is second countable, one can choose a much coarser, yet finite, cover (U'_k). The key property of this cover is that each of the non-empty finite intersections $U'_{k_0} \cap \cdots \cap U'_{k_m}$ is itself either a finite union of sets from (U_k) or a disjoint countable union of those. The same 5-lemma argument then shows that the desired generalized Poincaré duality relation $H^p \cong (H^{n-p}_c)^*$ holds on all of M. The technical details of this argument can be found in [55, Sec. V.4]. $\square$

3.3.2. Elliptic complexes and Serre duality

Now we will discuss generic elliptic complexes, of which both the Calabi and the twisted de Rham complexes are special cases. The result is essentially the same, though clearly more general. The arguments are somewhat less elementary and rely on some background in functional analysis and a result originally due to Serre [56]. The Serre duality method also gives some more information. Namely, that the cohomology does not change if we replace smooth functions by distributions with the same supports. Serre's original work was in the context of the Dolbeault complex in the theory of several complex variables. A good exposition of this result in the setting of general elliptic complexes can be found in [27].

At this point it is convenient to recall some basic facts of distribution theory [57–59]. Recall that, for any vector bundle $F \to M$, we can interpret $\Gamma(F)$ and $\Gamma_c(F)$ as locally convex topological vector spaces, with the Whitney weak Fréchet topology for the former and an inductive limit over supports of similar Fréchet topologies for the latter, with the limit topology still locally convex but no longer Fréchet (metrizable). These are the usual topologies used in the theory of distributions. The spaces of *distributional sections* $\Gamma'(F)$ and $\Gamma'_c(F)$ of F, with respectively compact and unrestricted supports, are defined as topological duals endowed with the strong topology (the usual distributional topology), $\Gamma'(F) = \Gamma(\tilde{F}^*)^*$ and $\Gamma'_c(F) = \Gamma_c(\tilde{F}^*)^*$. Recall that $\tilde{F}^* = \Lambda^n M \otimes F^*$ is the densitized dual bundle; the densitized dual of the densitized dual is the original bundle. It so happens that, if we stick with the strong topology for dual spaces, the topological dual of $\Gamma'(F)$ is again $\Gamma(\tilde{F}^*)$ and that of $\Gamma'_c(F)$ is $\Gamma_c(\tilde{F}^*)$. So the spaces of smooth and distributional sections are *reflexive* (with respect to the strong topology). Using the natural pairing

$$\langle \psi, \alpha \rangle = \int_M \psi \cdot \alpha \tag{70}$$

between $\psi \in \Gamma(F)$ and $\alpha \in \Gamma_c(\tilde{F}^*)$, well-defined provided M is oriented, we have the natural inclusions $\Gamma(F) \subset \Gamma'_c(F)$ and $\Gamma_c(F) \subset \Gamma'(F)$. By the *Schwartz kernel theorem*, the continuous maps $G\colon \Gamma_c(F_1) \to \Gamma'_c(F)$ are in bijection with *bidistributions*, elements $G \in \Gamma'_c(F_2 \boxtimes \tilde{F}^*_2)$, where $F_2 \boxtimes \tilde{F}^*_2 \to M \times M$ is the bundle with total space $F_2 \times \tilde{F}^*_1$ and the obvious projection onto its base, by the formula

$$(G\psi)(x) = \int_M G(x, y) \cdot \psi(y). \tag{71}$$

Let $\pi_1(x, y) = y$ and $\pi_2(x, y) = x$ denote the two projections $M \times M \to M$. We say that a bidistribution $G \in \Gamma'_c(F_2 \boxtimes \tilde{F}^*_1)$ is *properly supported* if $\pi_1\colon \operatorname{supp} G \to M$ is a proper map (the preimage of a compact set is compact). Differential operators define properly supported bidistributions, because their support lies on the diagonal of $M \times M$ by the crucial property that differential operators preserve supports. On the other hand, properly supported bidistributions need not preserve supports, though they still map compactly supported sections to compactly supported distributions. The amount by which the support of the image grows depends on the size of the support of the bidistribution in $M \times M$.

Once we have introduced distributional sections, we can extend to them many operators that were previously defined only on smooth functions. For instance, any linear differential operator $f : \Gamma(F) \to \Gamma(E)$ between vector bundles $F \to M$ and $E \to M$ can be extended to act on distributions, $f : \Gamma'(F) \to \Gamma'(E)$ or even $f : \Gamma'_c(F) \to \Gamma'_c(E)$, according to the following formula:

$$\langle f[\alpha], \psi \rangle = -\langle \alpha, f^*[\psi] \rangle, \tag{72}$$

for any $\psi \in \Gamma_c(\tilde{F}^*)$ and $\alpha \in \Gamma'_c(F)$, where f^* is the formal adjoint of f and $\langle -, - \rangle$ is the natural dual pairing between sections and distributions. Since this natural pairing is non-degenerate, it suffices to define f on the larger domain. Any other operator defined on smooth sections for which the above formula applies can also be extended to distributions, possibly with a restriction on their supports.

In particular, the operators of a differential complex $(F_\bullet, f_\bullet)$ can be extended to distributional sections. Then we can consider the cohomology of the complex in distributional sections, $H^i(\Gamma'_c(F_\bullet), f_\bullet)$, which may a priori be different from its cohomology in smooth sections $H^i(F_\bullet, f_\bullet) = H^i(\Gamma(F_\bullet), f_\bullet)$, and similarly with compact supports. Below we shall see some sufficient conditions for the cohomologies in smooth and distributional sections to coincide.

A crucial concept in the general theory of differential complexes is that of a *parametrix* [27, Ch. 2]. Let the vector bundles F_i with differential operators $f_i : \Gamma(F_{i-1}) \to \Gamma(F_i)$ constitute a differential complex $(F_\bullet, f_\bullet)$ on M. Then, a *parametrix* is a sequence of bidistributions $G_i \in \Gamma'_c(F_{i-1} \boxtimes \tilde{F}^*_i)$ such that

$$\mathrm{id}_i - Q_i = G_{i+1} \circ f_{i+1} + f_i \circ G_i, \tag{73}$$

where $\mathrm{id}_i : \Gamma_c(F_i) \to \Gamma_c(F_i)$ is the identity map and $Q_i \in \Gamma(F_{i+1} \boxtimes \tilde{F}^*_i) \subset \Gamma'_c(F_{i+1} \boxtimes \tilde{F}^*_i)$ is a smooth bidistribution. We say that the parametrix is *properly supported* if each G_i is a properly supported bidistribution. Obviously, if each G_i is properly supported, then so is each Q_i.

Proposition 7. *Let $(F_\bullet, f_\bullet)$ be an elliptic complex on an oriented manifold M.* (i) *Then, for any open neighborhood $U \subseteq M \times M$ of the diagonal $M \subset M \times M$, there exists a properly supported parametrix $G_i \in \Gamma'_c(F_{i-1} \boxtimes \tilde{F}^*_i)$ with support* $\operatorname{supp} G_i \subseteq U$. (ii) *Then also, the cohomologies of smooth and distributional sections are isomorphic:*

$$H^i(\Gamma'_c(F_\bullet), f_\bullet) \cong H^i(F_\bullet, f_\bullet) \quad \text{and} \quad H^i(\Gamma'(F_\bullet), f_\bullet) \cong H^i_c(F_\bullet, f_\bullet). \tag{74}$$

Proof. (i) The existence of a parametrix for any elliptic complex follows from Corollary 2.1.11 and Theorem 2.1.12 of [27]. The support of an existing parametrix can be restricted arbitrarily close to the diagonal since G^χ_i, defined by $G^\chi_i[\psi] = \chi G_i[\psi]$, is a parametrix as long as G_i is a parametrix and $\chi \in C^\infty(M \times M)$ is properly supported with $\chi \equiv 1$ on a neighborhood of the diagonal.

(ii) By the defining Eq. (73), they are cochain homotopic to the identity operator, with respect to the cochain homotopy G_i. Further, being by hypothesis smooth and by (i) properly supported, they define smoothing operators, $Q_i : \Gamma'(F_i) \to \Gamma_c(F_i)$ and $Q_i : \Gamma'_c(F_i) \to \Gamma(F_i)$, when extended to distributions. It is then straightforward to see that the Q_i and the inclusions of smooth sections in distributional ones (well defined because M is oriented) constitute a homotopy equivalence between the complexes of smooth $(\Gamma(F_\bullet), f_\bullet)$ and distributional $(\Gamma'_c(F_\bullet), f_\bullet)$ sections, and similarly for compact supports. Thus, as desired, these complexes have isomorphic cohomologies. □

Proposition 8 (*Serre, Tarkhanov*)**.** *Given a differential complex $(F_\bullet, f_\bullet)$, that is not necessarily elliptic, on an oriented manifold M that is countable at infinity (there exists an exhaustion by a countable sequence of compact sets), let $(\tilde{F}^*_\bullet, f^*_\bullet)$ be its formal adjoint complex. The following are algebraic (the topologies may not agree) isomorphisms of vector spaces*

$$H^i(F_\bullet, f_\bullet)^* \cong H^i(\Gamma'(\tilde{F}^*_\bullet), f^*_\bullet), \qquad H^i(F_\bullet, f_\bullet) \cong H^i(\Gamma'(\tilde{F}^*_\bullet), f^*_\bullet)^*, \tag{75}$$

$$H^i_c(F_\bullet, f_\bullet)^* \cong H^i(\Gamma'_c(\tilde{F}^*_\bullet), f^*_\bullet), \qquad H^i_c(F_\bullet, f_\bullet) \cong H^i(\Gamma'_c(\tilde{F}^*_\bullet), f^*_\bullet)^*, \tag{76}$$

where the cohomology vector spaces are endowed with the natural Hausdorff locally convex topology of a quotient of a subspace of the corresponding space of sections (be it smooth or distributional) and the topological duals are taken with the strong topology.

Proof. The original result of Serre [56] appeared in the context of the Dolbeault differential complex in the theory of several complex variables. A detailed discussion and proof of the result for general differential complexes can be found in Sections 5.1.1 and 5.1.2 of [27]. In particular, the desired conclusion can be found in Remark 5.1.9 thereof. Further conditions under which some of the duality isomorphisms are also continuous, and not merely algebraic, can be found there as well. □

Combining the two preceding propositions, it is easy to see that for any elliptic complex (subject to a countability condition on M) we have the Poincaré-Serre duality relation $H^i(F_\bullet, f_\bullet) = H^i_c(\tilde{F}^*_\bullet, f^*_\bullet)^*$.

3.4. The Calabi cohomology and homology

Below, we finally make use of the background information summarized in Sections 3.1–3.3 and its consequences for the Calabi and its formal adjoint complexes, $(C_\bullet, B_\bullet)$ and $(C_\bullet, B^*_\bullet)$, which were introduced in 2. Namely, we make precise the identification between their cohomologies and the sheaf cohomologies of the Killing and Killing–Yano sheaves, $\mathcal{K}$ and $\mathcal{KY}$, introduced in Section 3.1. The hope created by this identification is that the difficult problem of solving systems of differential equations, which appear in these complexes, can be replaced by the equivalent and potentially easier problem of computing sheaf cohomologies. The latter problem is potentially easier because of the many available methods of computing sheaf cohomology. Some of which will be discussed in Section 4.

First, we introduce the basic definitions of Calabi cohomology and homology. Let us denote the cohomology of the Calabi complex (*Calabi cohomology*) on a pseudo-Riemannian manifold (M, g) of constant curvature as

$$HC^i(M, g) = H^i(C_\bullet, B_\bullet) = H^i(\Gamma(C_\bullet), B_\bullet). \tag{77}$$

Let us also denote the cohomology of the formal adjoint Calabi complex with compact supports (*Calabi homology*)

$$HC_i(M, g) = H^i_c(C_\bullet, B^*_\bullet) = H^i(\Gamma_c(C_\bullet), B^*_\bullet). \tag{78}$$

The naming convention will be justified later by the generalized Poincaré duality relation in Corollary 11. Similarly, we define the cohomology of the Calabi complex with compact supports (*Calabi cohomology with compact supports*) as

$$HC^i_c(M, g) = H^i_c(C_\bullet, B_\bullet) = H^i(\Gamma_c(C_\bullet), B_\bullet) \tag{79}$$

and the cohomology of the formal adjoint Calabi complex (*locally finite Calabi homology*) as

$$HC^{lf}_i(M, g) = H^i(C_\bullet, B^*_\bullet) = H^i(\Gamma(C_\bullet), B^*_\bullet). \tag{80}$$

The following proposition is the main technical tool that we use to establish all other results in this section.

Proposition 9. *Consider a pseudo-Riemannian manifold (M, g) of constant curvature and dimension n. The corresponding Calabi complex $(C_\bullet, B_\bullet)$ is elliptic, formally exact and locally exact (except in degree 0). The same is true for its formal adjoint complex $(C_\bullet, B^*_\bullet)$ (except in degree n).*

Proof. In principle, we would need quite a bit of machinery for a full proof. Instead, we give a sketch of the main ideas and refer to the literature for technical details. The Calabi complex is actually an instance of a *second Spencer sequence* construction [15,16,14,11] applied to the Killing operator $B_1 = K$. This fact is demonstrated in the papers [9,10,8]. These papers make use of the general construction and properties of the differential complex constituting a second Spencer sequence demonstrated in [15,16]. In fact, the resulting differential complex gives a formally exact compatibility complex for the Killing operator, which is also an elliptic complex. This holds since the Killing operator K is itself elliptic (has injective symbol, which follows from the property of being of finite type, cf. Section 2.4) and formally integrable (contains all of its integrability conditions) on a constant curvature background.

A more elementary argument for ellipticity can be made on representation theoretic grounds (Appendix A.1). The fibers of the tensor bundles $C_i M$ carry irreducible representations of $GL(n)$. Further, as mentioned in Remark 1, the principal symbols of the differential operators B_i are all $GL(n)$-equivariant maps $\sigma B_i \colon Y^{(k_i)} T^* \otimes C_{i-1} \to C_i$ or equivalently $\sigma_p B_i \colon C_{i-1} \to C_i$, for $p \in T^*$. By Schur's lemma, the symbol map σB_i is then an isomorphism when restricted to an irreducible summand of the tensor product representation. The well-known Littlewood–Richardson rules [29,42] for tensor products of $GL(n)$ representations then show that the C_i irreps have been chosen precisely such that the symbol sequence $\sigma_p B_i$ is exact for $p \neq 0$. This representation-theoretic line of argument is a special case of the construction of what are known as BGG resolutions [28].

Finally, local exactness (except in degree 0) can be established by checking, for the Killing operator, a sufficient condition known as the *δ-estimate* [27, Sec. 1.3.13]. Equivalently, we can simply invoke Proposition 2, since, being of finite type, the Killing operator is equivalent to a flat covariant operator (Section 2.4).

A more elementary proof of local exactness was given in the original article by Calabi [1]. He relied on the well known local exactness of the de Rham complex and its relation to the simplified form of the complex in the flat (zero curvature) case. The non-zero curvature case was handled by embedding it in a flat space and then restricting and extending the relevant sheaves with respect to this embedding. Unfortunately, unlike the more sophisticated argument above, this simpler argument is unlikely to generalize, when the Calabi complex is replaced by a more general one.

To finish the proof, we note that the properties of formal exactness and ellipticity are obviously preserved by taking formal adjoints, so that they apply equally well to the formal adjoint Calabi complex $(C_\bullet, B^*_\bullet)$. The formal adjoint complex then serves as the formally exact compatibility complex for the Killing–Yano operator $B^*_n = KY$, which is also regular and of finite type on constant curvature backgrounds, as discussed in Section 2.4. Thus, repeating the same arguments as above establishes local exactness (except this time in degree n) for the adjoint complex as well. □

Corollary 10. *There is a formal homotopy equivalence between the Calabi complex* $(C_\bullet, B_\bullet)$ *and the twisted de Rham complex* $(\Lambda^\bullet M \otimes V_K, D_K)$ *resolving the Killing sheaf,* $\mathcal{H}^0(C_\bullet, B_\bullet) = \mathcal{K}$*. The same is true (up to a trivial renumbering) of the formal adjoint complex and the twisted de Rham complex* $(\Lambda^\bullet M \otimes V_{KY}, D_{KY})$ *resolving the Killing–Yano sheaf,* $\mathcal{H}^n(C_\bullet, B_\bullet^*) = \mathcal{KY}$*.*

Proof. We already know that both the Calabi and twisted de Rham complex associated to the Killing operator are formally exact, locally exact (Propositions 2 and 9) and both resolve the Killing sheaf, since the operators K and D_K are equivalent (Section 2.4). Thus, by Lemma 3, there exists a formal homotopy equivalence (realized by differential operators) between the two complexes. Noting that the exact same argument (with trivial changes) applies to the formal adjoint Calabi complex and the Killing–Yano sheaf concludes the proof. □

Corollary 11. *Provided the manifold M is countable at infinity (there is an exhaustion by a countable sequence of compact sets) or is of finite type (has a finite "good cover"), we have the following generalized Poincaré duality isomorphisms*

$$HC^i(M,g) \cong HC_i(M,g)^*, \qquad HC_c^i(M,g)^* \cong HC_i^{lf}(M,g), \tag{81}$$

$$HC^i(M,g)^* \cong HC_i(M,g), \qquad HC_c^i(M,g) \cong HC_i^{lf}(M,g)^*, \tag{82}$$

where isomorphisms are taken in the algebraic sense and duality is meant in the topological sense, as described in Proposition 8.

Note that in the case when all cohomology vector spaces are finite dimensional, the distinction between algebraic or topological isomorphisms and duals is irrelevant.

Proof. There are two ways to establish the desired duality isomorphisms, each relying on slightly different conditions on M, reflected in the hypotheses. We should note that both require an orientation on M. The existence off a non-degenerate metric on M implies that it is orientable. We then simply fix an orientation arbitrarily.

The Mayer–Vietoris argument (Proposition 6) establishes the duality isomorphisms

$$H^i(\Lambda^\bullet M \otimes V_K, D_K) \cong H_c^i(\Lambda^\bullet M \otimes V_K^*, D_K)^* \tag{83}$$

$$\text{and} \quad H^i(\Lambda^\bullet M \otimes V_{KY}, D_{KY}) \cong H_c^i(\Lambda^\bullet M \otimes V_{KY}^*, D_{KY})^*. \tag{84}$$

Under the finite type condition on M, an easy modification to the Mayer–Vietoris argument (Propositions 5.3.1 and 5.3.2 of [49]) also shows that each of these cohomology groups is finite dimensional, so the reverse duality isomorphisms hold as well. Finally, the formal homotopy equivalence of Corollary 10 translates these isomorphisms into the desired duality relations for Calabi cohomology and homology.

The Poincaré-Serre argument, applies by virtue of the ellipticity of the Calabi and its formal adjoint complexes (Proposition 9) and the hypothesis of countability at infinity. Combining the results of Propositions 7 and 8, easily establishes the desired duality isomorphisms directly. □

Corollary 12. *Assume the same hypotheses on M as in Corollary 11. The Calabi cohomology and homology, assuming they are finite dimensional, the following identities hold (with respect to algebraic duals):*

$$HC^i(M,g) \cong H^i(\mathcal{K}), \qquad HC_c^i(M,g) \cong H^{n-i}(\mathcal{KY})^*, \tag{85}$$

$$HC_i(M,g) \cong H^i(\mathcal{K})^* \qquad HC_i^{lf}(M,g) \cong H^{n-i}(\mathcal{KY}). \tag{86}$$

Note that we do expect the relevant cohomology and homology spaces to be finite dimensional in most applications. If the cohomology vector spaces happen to be infinite dimensional, then the correct (topological and algebraic) isomorphisms can be deduced from Corollary 11 and Proposition 8.

Proof. By Proposition 9 and Corollary 10 we already know that the Calabi and its formal adjoint complexes are locally exact differential complexes that respectively resolve the Killing and Killing–Yano sheaves, $\mathcal{K}$ and $\mathcal{KY}$. Then, Lemma 3 establishes the isomorphisms $HC^i(M,g) \cong H^i(\mathcal{K})$ and $HC_i^{lf}(M,g) \cong H^{n-i}(\mathcal{KY})$. Finally, the duality isomorphisms of Corollary 11 establish the rest of the desired identities. Note that we have added the finite dimensionality hypothesis only to avoid explicitly specifying a topology on the relevant cohomology vector spaces, so that the topological and algebraic duals coincide. □

4. The Killing sheaf and its cohomology

In this section we concentrate on possible effective ways of computing the Killing sheaf cohomology (or rather the cohomology of any locally constant sheaf) of a pseudo-Riemannian manifold (M,g) of constant curvature. For us, *effective* is used somewhat loosely and we take it to mean roughly to either consist of finitely many steps involving only finite-dimensional linear algebra or to reduce to calculation that has already been done in the literature. In particular, any such method would be more effective than the brute force approach of trying to solve the systems of differential equations appearing in the Calabi complex. Since the interest in the cohomology of the Killing sheaf may extend beyond the constant curvature context, we always discuss the more general situation, specializing to the constant curvature case when necessary.

There are two main possibilities, either the manifold M is simply connected or it is not. They are discussed respectively in Sections 4.1 and 4.2. In the simply connected case, the sheaf cohomology can be expressed completely in terms of the de Rham cohomology. The non-simply connected case is more complicated, where several complementary but potentially overlapping methods may be used. None of them, unfortunately, gives a complete solution.

Crucial to the discussion that follows (see Appendix D for relevant notation and concepts related to G-bundles) is the notion of the monodromy representation of the fundamental group $\pi = \pi_1(M)$ of a manifold with respect to a flat connection D on a vector bundle $V \to M$ (cf. Section 2.4). Let us identify $\pi_1(M) = \pi_1(M, x)$ for some $x \in M$. The connection D gives rise to a notion of parallel transport on V. Since the connection is flat, the parallel transport along a curve connecting $x, y \in M$ depends only on the homotopy class of the path with its endpoints fixed. Therefore, since parallel transport acts linearly, parallel transport along loops based at $x \in M$ induces a representation $\rho_V : \pi \to \mathrm{GL}(\bar{V})$, where $\bar{V} \cong V_x$ is the typical fiber of $V \to M$, called the *monodromy representation*. Another common term is the *holonomy representation*. However, we reserve the term *holonomy* for the same concept associated specifically to the g-compatible Levi-Civita connection on M. If $V \to M$ is a vector G-bundle, then there necessarily is an associated representation of the structure group on $\bar{V}$, $\sigma_V : G \to \mathrm{GL}(\bar{V})$. When the connection D preserves the G-bundle structure, parallel transport and hence monodromy factors through the associated representation. Hence $\rho_V = \sigma_V \circ \rho$, where $\rho : \pi \to G$ is the monodromy representation of π in the structure group.

Recall also that for any manifold M there exists a unique (up to diffeomorphism) connected, simply connected *universal cover* $\tilde{M} \to M$, where the projection map is a surjective local diffeomorphism. In fact, $\tilde{M} \to M$ is a π-principal bundle over M. The principal bundle action of π on $\tilde{M}$ by is called action by *deck transformations*. Note that $M \cong \tilde{M}/\pi$. Deck transformations, being diffeomorphisms, commute with the de Rham differential. Hence the action by deck transformations descends to de Rham cohomology. We call it the *deck representation* $\Delta^i : \pi \to \mathrm{GL}(H^i(M))$. The projection to M pulls the bundle $V \to M$ back to $\tilde{V} \to \tilde{M}$ and the connection D to $\tilde{D}$. Since the universal cover is simply connected, the pulled back bundle trivializes, $\tilde{V} \cong \bar{V} \times \tilde{M}$. Therefore, we have the isomorphism $H^i(\Lambda^\bullet \tilde{M} \otimes \tilde{V}, D) \cong H^i(\tilde{M}) \otimes \bar{V}$. It is not hard to see that the two sides are isomorphic not only as vector spaces but also as representations of the fundamental group π, with the right side transforming as the tensor product of the deck and monodromy representations $\Delta^i \otimes \rho_V$.

Let us fix the assumptions that (M, g) is connected and that its Killing sheaf $\mathcal{K}_g$ is locally constant, then concretize the above ideas to this case. Recall from Section 2.4 that $\mathcal{K}_g$ is then resolved by the twisted de Rham complex associated to the flat vector bundle (V_K, D_K). The typical fiber $\bar{V}_K$ of $V_K \to M$ consists of the germs of local Killing vector fields. Each local Killing vector field extends to a global one, and hence to an infinitesimal isometry, on the universal cover $(\tilde{M}, \tilde{g})$. Thus, we can identify $\bar{V}$ with the Lie algebra $\mathfrak{g}$ of the Lie group $G = \mathrm{Isom}(\tilde{M}, \tilde{g})$ of isometries of $(\tilde{M}, \tilde{g})$.

Infinitesimal isometries act on each other by the formula $\mathcal{L}_u v = [u, v]$, which corresponds to the infinitesimal adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. This representation integrates to the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, which is how finite isometries act on Killing vector fields. Also, it is clear by construction that deck transformations act on $(\tilde{M}, \tilde{g})$ by isometries. Let us denote this representation of the fundamental group $\pi = \pi_1(M)$ by isometries as $\rho\colon \pi \to G$. As described in Appendix D.3, this information is equivalent to specifying a flat principal G-bundle $P \to M$ with monodromy representation ρ of π in G. Further, it is clear that $V_K \cong \mathfrak{g}_P$ is the vector G-bundle over M associated to P with respect to the adjoint action Ad of G on $\mathfrak{g}$ and that D_K is the connection associated to the flat principal connection on P. The monodromy representation of π on $\bar{V}_K$ is then the *composite adjoint monodromy representation* $\rho_V = \mathrm{Ad}_\rho = \mathrm{Ad} \circ \rho$.

4.1. Simply connected case

The simplest case is when the manifold M is simply connected, that is, its fundamental group $\pi = \pi_1(M)$ is trivial. Let the locally constant sheaf $\mathcal{F}$ have stalk $\bar{F}$ so that it defines a flat vector bundle (F, D), with $\bar{F}$ the typical fiber of $F \to M$ (Sections 3.1 and 2.4). We know that the twisted de Rham differential complex $(\Lambda^\bullet M \otimes F, D)$ is an acyclic resolution of $\mathcal{F}$. Hence their cohomologies agree. On the other hand, since M is simply connected, we can choose a global D-flat basis frame for F and identify the twisted de Rham complex with $\operatorname{rk} F = \dim \bar{F}$ copies of the standard de Rham complex. This argument proves

Theorem 13. *Let (M, g) be a connected, simply connected pseudo-Riemannian manifold with locally constant Killing sheaf $\mathcal{K}_g$, resolved by the twisted de Rham complex $(\Lambda^\bullet M \otimes V_K, D_K)$. Let $\mathfrak{g} \cong \bar{V}_K$ be the Lie algebra of isometries of (M, g). Then the following isomorphisms hold:*

$$H^i(\mathcal{K}_g) \cong H^i(\Lambda^\bullet M \otimes V_K, D_K) \cong H^i(M) \otimes \mathfrak{g}. \tag{87}$$

In particular $H^0(\mathcal{K}_g) \cong \mathfrak{g}$ and $H^1(\mathcal{K}_g) = 0$.

4.2. Non-simply connected case

The non-simply connected case is of course more complicated and we can offer only partial results, which we summarize in this paragraph. The simplest sub-case is when the fundamental group $\pi = \pi_1(M)$ of the pseudo-Riemannian manifold

(M, g) is finite (Section 4.2.1). The Killing sheaf cohomology is then the π-invariant subspace of the de Rham cohomology of the universal covering space. If the fundamental group is not necessarily finite, we still have the following general result for the degree-1 cohomology of constant curvature spaces. We can equate $\dim H^1(\mathcal{K}_g)$ to the dimension of the space of possible infinitesimal deformations of the metric that preserve the constant curvature condition as well as the value of the scalar curvature itself. That observation was already made in the original work of Calabi [1] and in fact prompted his interest in a resolution of the Killing sheaf $\mathcal{K}_g$. This space of infinitesimal deformations can also be computed as the degree-1 group cohomology of π with coefficients in a certain representation on the Lie algebra of isometries of the universal cover of M (Section 4.2.2). Another result helps compute higher degree cohomology groups. The Killing sheaf, being locally constant, defines a *local system* or a system of *local coefficients* on M, a concept well known in algebraic topology. A general result from the theory of local systems is that the aforementioned group cohomology computes higher Killing sheaf cohomology groups up to the degree of asphericity of M (Section 4.2.3). Finally, there is a general method for completely computing the Killing sheaf cohomology based on a presentation of the manifold M as a finite simplicial set (Section 4.2.4).

4.2.1. *Finite fundamental group*

The basic idea here is to take advantage of the complete decomposability of representations of a finite group and then apply Schur's lemma. As will be clear from the proof, it is the complete decomposability that is important not the finiteness of π. So the same result actually holds under suitably weaker hypotheses.

Theorem 14. *Let (M, g) be a connected pseudo-Riemannian manifold with fundamental group $\pi = \pi_1(M)$ and Killing sheaf $\mathcal{K}_g$, resolved by the twisted de Rham complex $(\Lambda^\bullet M \otimes V_K, D_K)$. Let $\mathfrak{g} \cong \bar{V}_K$ be the Lie algebra of isometries of the universal cover $(\tilde{M}, \tilde{g})$. If π is finite, we have the following isomorphisms:*

$$H^i(\mathcal{K}_g) \cong (H^i(\tilde{M}) \otimes \mathfrak{g})^\pi, \tag{88}$$

where the superscript π denotes the π-invariant subspace with respect to the representation $\Delta^i \otimes \mathrm{Ad}_\rho$, the tensor product of the deck and composite adjoint monodromy representations. In particular $H^0(\mathcal{K}_g) \cong \mathfrak{g}^\pi$.

Proof. Consider the spaces of sections $\Omega_i = \Gamma(\Lambda^i \tilde{M} \otimes \tilde{V}_K)$, where $\tilde{V}_K \to \tilde{M}$ is the pullback of $V_K \to M$ along the universal covering projection $\tilde{M} \to M$. Let $\tilde{D}_K$ denote the pullback of the D_K. As we have already discussed at the top of Section 4, this pulled back bundle is trivial, $\tilde{V}_K \cong \mathfrak{g} \times M$. Moreover, by simple connectedness of $\tilde{M}$ and Theorem 13, we have the isomorphism $H^i = H^i(\Omega_i, \tilde{D}_K) \cong H^i(\tilde{M}) \otimes \mathfrak{g}$.

As also discussed at the top of Section 4, the spaces Ω_i carry representations of the fundamental group π, which also descends to the cohomologies H^i. Since π is finite, it is well known that any representation thereof is completely decomposable [60], that is, any subrepresentation has a direct sum complement subrepresentation. So, the subspace $\Omega_i^\pi \subset \Omega_i$ invariant under the action π (every element of π acts as the identity operator) has a direct sum complement $\Omega_i^{\hat{\pi}}$, so that $\Omega_i \cong \Omega_i^\pi \oplus \Omega_i^{\hat{\pi}}$. This direct sum induces the short exact sequence

$$0 \longrightarrow \Omega_i^\pi \longrightarrow \Omega_i \longrightarrow \Omega_i^{\hat{\pi}} \longrightarrow 0. \tag{89}$$

It is straightforward to note that, by construction of the universal cover $\tilde{M} \to M$, the π-invariant subcomplex $(\Omega_i^\pi, \tilde{D}_K)$ on $\tilde{M}$ is in fact cochain isomorphic to the complex $(\Gamma(\Lambda^\bullet M \otimes V_K), D_K)$ on M. Therefore the desired cohomology groups are $H^i(\Lambda^\bullet M \otimes V_K, D_K) \cong H^i(\Omega_\bullet^\pi, \tilde{D}_K)$.

The complement $\Omega_i^{\hat{\pi}}$ naturally does not contain any non-zero vectors invariant under the action of π. In representation theoretic terminology, these two complementary subspaces are *disjoint*. By Schur's lemma [61], the only equivariant map (*intertwiner*) between any two disjoint representations is the zero map. Note that the differentials $\tilde{D}_K$ and the maps in the short exact sequence (89) are in fact π-equivariant. By the general machinery of homological algebra (Appendix B) the short exact sequence (89) induces the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0_\pi \arrow[r] & H^0 \arrow[r] & H^0_{\hat{\pi}} \arrow[dll, rounded corners] \\ & H^1_\pi \arrow[r] & H^1 \arrow[r] & H^1_{\hat{\pi}} \arrow[r] & \cdots \end{tikzcd} \tag{90}$$

where $H^i_\pi = H^i(\Omega_\bullet^\pi, \tilde{D}_K)$, $H^i_{\hat{\pi}} = H^i(\Omega_\bullet^{\hat{\pi}}, \tilde{D}_K)$ and all the maps are also π-equivariant. It is clear that the representations carried by H^i_π and $H^i_{\hat{\pi}}$ are also disjoint. Therefore, the maps connecting the rows of diagram (90) are all zero. In other words, each of the rows becomes a short exact sequence on its own. Invoking again complete decomposability of representations of π, we can write $H^i \cong H^i_\pi \oplus H^i_{\hat{\pi}}$ and hence identify $H^i_\pi \cong (H^i)^\pi$ with the subspace of H^i on which π acts trivially.

Collecting the above arguments together, while recalling the sheaf cohomology identity $H^i(\mathcal{K}_g) \cong H^i(\Lambda^\bullet M \otimes V_K, D_K)$, we obtain the isomorphism $H^i(\mathcal{K}_g) \cong (H^i(\tilde{M}) \otimes \mathfrak{g})^\pi$. Noting the special cases $H^0(\tilde{M}) = \mathbb{R}$ and $H^1(\tilde{M}) = 0$, as in Theorem 13, concludes the proof. □

4.2.2. Degree-1 cohomology

Consider a 1-parameter family of n-dimensional pseudo-Riemannian manifolds $(M, g(t))$ where each $g(t)$, for t in some neighborhood of zero, has constant curvature, with scalar curvature independent of t: Riemann tensor equal to $\frac{k}{n(n-1)} g(t) \odot g(t)$. Let $g(0) = g$ and $\dot{g}(0) = h$. Then the linearization of the identity $R[g(t)] - \frac{k}{n(n-1)} g(t) \odot g(t) = 0$ at $t = 0$ will give (cf. Section 2.2)

$$\dot{R}[h] - k\frac{2}{n(n-1)}(g \odot h) = -\frac{1}{2}C_2[h] = 0. \tag{91}$$

In other words, h is a Calabi 1-cocycle. It is possible that not every Calabi 1-cocycle gives rise to an actual 1-parameter family of deformations, since there may be algebraic obstructions[3] to solving for higher order terms in the expansion parameter t. However, at the infinitesimal level, there are no other conditions and we can identify infinitesimal deformations with Calabi 1-cocycles. If the deformation family $g(t)$ is trivial, induced by a 1-parameter family of diffeomorphisms of the manifold M, then it is well known that $h = K[v]$ for some 1-form v (vector field generating the diffeomorphism family, with index lowered by the metric g), in other words a Calabi 1-coboundary. It is easy to see that Calabi 1-coboundaries can be identified with infinitesimal trivial deformations. Therefore, the Calabi cohomology vector space $HC^1(M, g)$, and hence the Killing cohomology vector space $H^1(\mathcal{K}_g)$ isomorphic to it, is in bijective correspondence with the space of infinitesimal deformations of the metric g within the space of constant curvature metrics of scalar curvature k, modulo infinitesimal diffeomorphisms.

There is another way to look at this infinitesimal deformation space. It is well known that the only geodesically complete, simply connected, constant curvature spaces are the pseudo-Euclidean ($k = 0$), pseudo-spherical ($k > 0$) and pseudo-hyperbolic ($k < 0$) spaces [63, Sec. 2.4]. In Riemannian signature, these are respectively the ordinary Euclidean, spherical and hyperbolic spaces. In Lorentzian signature, these are respectively the Minkowski, de Sitter and anti-de Sitter spaces. Thus, the elements of a family $(M, g(t))$ of geodesically complete, constant curvature, pseudo-Riemannian manifolds of fixed scalar curvature k all have isometric universal covers $(\tilde{M}, \tilde{g})$. Moreover, since the action of the fundamental group $\pi = \pi_1(M)$ on its universal cover via deck transformations is by isometries, there is an injective group homomorphism $\pi \to G = \mathrm{Isom}(\tilde{M}, \tilde{g})$, so that we have a subgroup $\rho(\pi) \subseteq G$ that acts on $\tilde{M}$ properly and discontinuously [63, Sec. 1.8]. Conversely, for any subgroup of $\pi' \subseteq G$ that acts on $\tilde{M}$ properly and discontinuously the quotient $(M', g') = (\tilde{M}, \tilde{g})/\pi'$ will be a manifold of the same constant curvature, but with fundamental group $\pi' = \pi_1(M')$. So, we have already noticed that all (M, g) with constant curvature arise in this way. Of course, any two subgroups $\pi', \pi'' \subseteq G$ that are conjugate, $\pi'' = a\pi' a^{-1}$ for some $a \in G$, give rise to isometric quotients. In fact, we have just argued that the infinitesimal deformations of the representation $\rho \colon \pi \to G$, up to conjugation by G, are in bijection with infinitesimal constant curvature deformations of the metric (M, g). It is well known that the deformations of the representation ρ are in bijection with certain degree-1 *group cohomology* of the fundamental group π. On the other hand, we have already seen that deformations of the constant curvature spaces are parametrized by the Killing sheaf cohomology $H^1(\mathcal{K}_g)$. Thus, computing the group cohomology may be an effective way of computing the Killing sheaf cohomology, at least in degree-1. The details of the definition of the representation ρ are described at the top of Section 4 and are also subsumed by the more general discussion below.

This connection between the degree-1 Killing sheaf cohomology, deformations of the geometry and group cohomology of the fundamental group π extends far beyond the case of manifolds of constant curvature. We base what follows on the remark at the top of Section 4 and the contents of Appendix D. If $(\tilde{M}, \tilde{g})$ is the universal cover of (M, g) and $G = \mathrm{Isom}(\tilde{M}, \tilde{g})$ with Lie algebra $\mathfrak{g}$, then there is a naturally defined flat principal G-bundle $P \to M$. Then, the infinitesimal deformations of this flat principal G-bundle are in bijections with $H^1(\mathcal{K}_g)$, the degree-1 Killing sheaf cohomology group. That is because the flat vector bundle (V_K, D_K), whose twisted de Rham complex resolves the Killing sheaf, is isomorphic to the associated bundle $\mathfrak{g}_P \to M$ with connection D induced by the flat principal connection on P. Recall that the fibers of $\mathfrak{g}_P$ transform under the adjoint representation $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ and that parallel transport with respect to the flat connection on P defines a representation $\rho \colon \pi \to G$ of the fundamental group $\pi = \pi_1(M)$. Their composition $\mathrm{Ad}_\rho = \mathrm{Ad} \circ \rho$, as already mentioned at the top of Section 4, is known as the *composite adjoint monodromy representation*. In the case of spaces of constant curvature, the infinitesimal deformations of the flat principal bundle $P \to M$ are the same thing as the infinitesimal deformations of the given constant curvature metric, fixing the value of the curvature.

Theorem 15. *Given the notations and hypotheses of the above paragraph, the following isomorphisms between the Killing sheaf cohomology and group cohomology of π with coefficients in Ad_ρ hold:*

$$H^0(\mathcal{K}_g) \cong H^0(\pi, \mathrm{Ad}_\rho) \cong \mathfrak{g}^\pi, \qquad H^1(\mathcal{K}_g) \cong H^1(\pi, \mathrm{Ad}_\rho). \tag{92}$$

[3] The study of these obstructions follows the general ideas outlined by Kodaira and Spencer [12,13]. See also the related phenomenon of linearization instabilities [62].

This result is a direct consequence of Proposition 17 of Appendix D. Unfortunately, we cannot use the same methods to establish isomorphisms between the group and sheaf cohomologies in higher degrees. See, however, Section 4.2.3. The connection between group cohomology of π and deformations of a flat principal bundle is well known, cf. [64]. The connection between, specifically, the cohomology of the Killing sheaf, infinitesimal deformations of the corresponding principal bundle, and group cohomology seems to be less well known, but is mentioned explicitly in [65, App. A.2].

4.2.3. Cohomology with local coefficients

We have just noted, in Section 4.2.2, a geometric relation between degree-1 locally constant sheaf cohomology and cohomology of the fundamental group. A more general connection between the cohomology of a locally constant sheaf, or equivalently cohomology with coefficients in a local system [66, Ch. VI], and group cohomology of the fundamental group has also been noticed in pure algebraic topology. In fact, that is how the notion of group cohomology first arose.

The original goal was to calculate the cohomology of a space (with or without coefficients in a non-trivial local system) in terms of data specifying its homotopy type. Following some early work by Hurewicz, Hopf and Eilenberg, Eilenberg and MacLane [67] introduced what are now known as $K(1,\pi)$ spaces (topological spaces with $\pi_1 = \pi$ and $\pi_i = 0$ for all $i > 0$) and computed all of their cohomology groups by introducing an algebraic construction based on the knowledge of the group π. We now call this construction *group cohomology* [68]. They further showed that the same construction works also for any topological space M, not just a $K(1,\pi)$, for the cohomologies in degree-i, with $0 < i \le p$, as long as the space M is p-aspherical, $\pi_i = 0$ for $0 < i \le p$. This result, applied to the Killing sheaf gives the following

Proposition 16. *Let (M,g) be a connected pseudo-Riemannian manifold with locally constant Killing sheaf $\mathcal{K}_g$ and universal cover $(\tilde{M},\tilde{g})$. Denote $G = \mathrm{Isom}(\tilde{M},\tilde{g})$ the group of isometries of the universal cover and let $\mathfrak{g}$ be its Lie algebra. The fundamental group $\pi = \pi_1(M)$ acts on $\mathfrak{g}$ via the composite adjoint monodromy representation $\mathrm{Ad}_\rho\colon \pi \to \mathrm{GL}(\mathfrak{g})$. If the manifold M is p-aspherical, meaning $\pi_i(M) = 0$ for $0 < i \le p$, then we have the following isomorphisms:*

$$H^i(\mathcal{K}_g) = H^i(\pi, \mathrm{Ad}_\rho) \quad \text{for } 0 \le i \le p. \tag{93}$$

For higher degree cohomology there are other contributions to the homology groups. There is still a homomorphism $H^i(\pi,\mathrm{Ad}_\rho) \to H^i(\mathcal{K}_g)$, but it need no longer be an isomorphism [46, Sec. 1.4.2].

Later, Postnikov [69,70] proposed a full solution for algebraically determining all the cohomology groups of a space based on its homotopy type. Postnikov's method encodes the full homotopy type of a space in terms of their homotopy groups and certain additional algebraic data known as a *Postnikov system* or *tower*. This construction is currently more commonly known in its topological form [66, Ch. IX]. For p-aspherical spaces and cohomology in degree i, with $0 < i \le p$, Postnikov's construction coincides with group cohomology. In general, the two constructions do differ in degrees higher than the degree of asphericity.

Unfortunately, both Postnikov's encoding of the homotopy type and his algebraic reconstruction of the cohomology are rather complicated and do not appear to have gained much popularity. They seem to be fully described only in his original monograph [70] or its translation [71], both being rather obscure references. At the moment, it is not clear to us what is the modern state of the art in terms of reconstructing the cohomology of a space with coefficients in a local system in terms of the space's homotopy type.

4.2.4. Simplicial set cohomology

The last mathematical tool, which we will discuss, that can aid in the computation of the cohomologies of a locally constant sheaf is simplicial cohomology with local coefficients. The idea is to substitute the underlying manifold M with a combinatorial structure like a *simplicial complex* or a *simplicial set*. Then, provided the combinatorial model is finite, the corresponding cohomology theory reduces to the computation of the cohomology of a finite dimensional cochain complex, and thus to finite dimensional linear algebra. We defer to the discussion in [72, Sec. I.4.7–10] for technical details.

A disadvantage of this method is that finite combinatorial models only cover the case of compact manifolds. Non-compact manifolds require either an infinite combinatorial model or a non-trivial extension of the formalism. Another inconvenience, besides the need for an explicit decomposition of M into simplices, is the need to define a discrete analog of parallel transport on the simplicial model to reproduce the composite adjoint monodromy representation Ad_ρ. That is usually done by associating a copy of $\mathfrak{g}$ to each vertex of the simplicial model for M and explicitly assigning a coherent set of linear isomorphisms between these copies to the edges connecting them, such that the composition of the isomorphisms of the edges along a closed loop is equal to the Ad_ρ action of the corresponding element of π. These choices may be simplified if all vertices could be collapsed into a single one, which is allowed for simplicial sets. Such a construction is always possible when M is compact and results in a so-called *reduced simplicial set* [73].

5. Application to linearized gravity

Recently, the symplectic and Poisson structure of linear classical field theories has been studied by the author within a very general framework [18,17] (see also [74–76] for related work), which admits in particular any linear field theory whose gauge fixed equations of motion can be formulated as a hyperbolic PDE system with possible constraints and residual gauge freedom. Certain sufficient geometric conditions need to be satisfied for a field theory to fit into that framework. The framework can then precisely characterize the degeneracies of the presymplectic and Poisson tensors on the solution space of the theory. These sufficient conditions require the gauge generator and the constraint operator to fit into differential complexes and the degeneracies of the presymplectic and Poisson tensors are then characterized using the cohomology of these complexes. Once known, these presymplectic and Poisson degeneracies are known to be of importance in classifying the charges, locality, superselection sectors and quantization of the corresponding classical theory.

The well known examples of Maxwell electromagnetism and Maxwell p-forms [20,21,24] fit into this framework [18, Sec. 4.2], invoking the well known de Rham complex. Linearized gravity on a constant curvature Lorentzian manifold also fits into this framework, with the role of the de Rham complex replaced by the Calabi complex or, as appropriate, the formal adjoint Calabi complex. For linearized gravity on an arbitrary background, we would need to make use of different differential complexes. The Calabi complex would be replaced by complexes defined by the property that they (at least formally) resolve the sheaf of Killing vectors on the given background (cf. Section 3.2). The corresponding formal adjoint complexes would play a role as well. This connection to the Killing sheaf, even without explicitly knowing the needed differential complexes, shows that the Killing sheaf cohomology plays a similar role both in the constant curvature context and more generally. Thus the ability to compute the Killing sheaf cohomology in as many circumstances as possible (as discussed in Section 4) should take us a large part of the way towards understanding the presymplectic and Poisson degeneracies of linearized gravity on general backgrounds. Unfortunately, about half the desired information would still be missing, since it is not clear which sheaf cohomology theory would control the cohomology of the formal adjoint differential complex. In the case of constant curvature, we were able to identify it as the cohomology of the sheaf of rank-$(n-2)$ Killing–Yano tensors, which is resolved by the formal adjoint Calabi complex (Section 2.3). It is currently not clear how to identify its analog in the case of a general background, without knowing the full differential complex that (formally) resolves the sheaf of Killing vectors.

Now, specialized to the case of linearized gravity on a constant curvature background (M,g), the analysis of [18,17] concludes that the presymplectic and Poisson tensors are actually non-degenerate (with spacelike compact support for solutions and compact support for smeared observables) if and only if the following two conditions are satisfied: *(global recognizability)* a certain bilinear pairing between degree-1 Calabi cohomology with spacelike compact supports and degree-1 Calabi homology, *(global parametrizability)* a certain bilinear pairing between on-shell degree-1 Calabi cohomology with spacelike compact supports and on-shell degree-1 timelike finite Calabi homology.

The descriptions of off-shell or on-shell Calabi cohomologies with *spacelike compact* supports, HC^i_{sc} or $HC^i_{\square,sc}$, and of off-shell or on-shell *timelike finite* Calabi homology, HC^{tf}_i or $HC^{\square,tf}_i$ go beyond the scope of the current work. However, they are defined and studied in detail in [77] (similar ideas appear also in [24]). In fact, the results of [77] show how to express these non-standard cohomologies in terms of the standard ones with unrestricted or compact supports, and similarly for homology. Recall also (Section 3.4) that the latter are isomorphic to appropriate cohomologies (or their linear duals) of the Killing or Killing–Yano sheaves, $\mathcal{K}_g$ or $\mathcal{KY}_g$. Using all of these results we are able to translate the non-degeneracy requirements as follows: *(global recognizability)* a certain bilinear pairing between

$$HC^1_{sc}(M,g) \cong H^{n-2}(M,\mathcal{KY}_g)^* \tag{94}$$

$$\text{and}\quad HC_1(M,g) \cong H^1(M,\mathcal{K}_g)^* \tag{95}$$

is non-degenerate, *(global parametrizability)* a certain bilinear pairing between

$$HC^1_{\square,sc}(M,g) \cong H^{n-1}(M,\mathcal{KY}_g)^* \oplus H^{n-2}(M,\mathcal{KY}_g)^*$$

$$\text{and}\quad HC^{\square,tf}_1(M,g) \cong H^1(M,\mathcal{K}_g)^* \oplus H^0(M,\mathcal{K}_g)^*$$

is non-degenerate. Notice that we have succeeded in expressing the vector spaces on which these pairings are defined purely in terms of Killing and Killing–Yano sheaf cohomologies.

Checking non-degeneracy of course requires an explicit expression for the required bilinear pairings. Such expressions can be obtained from the general framework of [18,17]. However, there are two cases were we do not need such detailed information, and these are the ones we shall content ourselves with here. For instance, if all the relevant cohomology vector spaces are trivial, then the only possible, trivial bilinear pairing is automatically non-degenerate. On the other hand, if the paired vector spaces have different dimensions, then every possible pairing between them must be degenerate.

We conclude this section by listing several well known Lorentzian backgrounds for which the methods of Section 4 allow us to determine all or a few of the cohomologies of the Killing sheaf. For the reasons discussed above, we make note of the Killing–Yano sheaf cohomologies only for constant curvature backgrounds.

The easiest case is that of simply connected spacetimes. Then, the Killing sheaf cohomology is just the de Rham cohomology tensored with the Lie algebra of global isometries (Section 4.1), with an analogous result for any other locally constant sheaf. Many of the well known exact solutions are in fact defined on simply connected underlying manifolds, including Minkowski space, black hole solutions and cosmological solutions. A few explicit examples are listed in Table 2. Note that only the Minkowski and de Sitter spaces are of constant curvature, so that the Calabi complex could be defined on them. For these backgrounds, it makes sense to also compute the Killing–Yano sheaf cohomologies $H^i(\mathcal{KY})$. However, since we know

Table 2
A list of some well known, simply connected solutions of (cosmological) vacuum Einstein equations, together with their topology and non-vanishing dimensions of Killing or Killing–Yano sheaf cohomologies. Note that b^0 always counts the number of independent global Killing vectors, and similarly for c^0. The Tangherlini solutions generalize the Schwarzschild one to higher dimensions and the Myers–Perry solutions do the same for Kerr [78]. For the latter, N counts the number of rotational symmetries, which varies depending on the variant of the solution. We only consider the exterior regions for black hole solutions.

Spacetime	Topology	$b^i = \dim H^i(\mathcal{K})$	$c^i = \dim H^i(\mathcal{KY})$
Minkowski	$\mathbb{R}^n$	$b^0 = \frac{n(n+1)}{2}$	$c^0 = \frac{n(n+1)}{2}$
Open FLRW	$\mathbb{R}^n$	$b^0 = \frac{(n-1)n}{2}$	
de Sitter	$\mathbb{R} \times S^{n-1}$	$b^0 = b^{n-1} = \frac{n(n+1)}{2}$	$c^0 = c^{n-1} = \frac{n(n+1)}{2}$
Closed FLRW	$\mathbb{R} \times S^{n-1}$	$b^0 = b^{n-1} = \frac{(n-1)n}{2}$	
Schwarzschild	$\mathbb{R}^2 \times S^2$	$b^0 = b^2 = 4$	
Tangherlini	$\mathbb{R}^2 \times S^{n-2}$	$b^0 = b^{n-2} = \frac{(n-2)(n-1)}{2}$	
Kerr	$\mathbb{R}^2 \times S^2$	$b^0 = b^2 = 2$	
Myers–Perry	$\mathbb{R}^2 \times S^{n-2}$	$b^0 = b^{n-2} = 1 + N$	

Table 3
Known values of $b^i = \dim H^i(\mathcal{K}_g)$ for a generic spatially homogeneous spacetime (M, g) with given topology and symmetry properties. See text for more details.

M	$\pi_1(M)$	Bianchi sym.	Additional sym.	b^0	b^1
$\mathbb{R} \times T^3$	$\mathbb{Z}^3$	$\mathbb{R}^3$	1	3	6
$\mathbb{R} \times T^3$	$\mathbb{Z}^3$	VII(0)	1	2	4
$\mathbb{R} \times T^3$	$\mathbb{Z}^3$	$\mathbb{R}^3$	$SO(2)$	3	6
$\mathbb{R} \times T^3$	$\mathbb{Z}^3$	$\mathbb{R}^3$	$SO(3)$	3	5

that the number of linearly independent rank-$(n-2)$ Killing–Yano tensors on these spaces is the same as the number of linearly independent Killing vectors (Section 2.3), the cohomology vector spaces are isomorphic, $H^i(\mathcal{KY}) \cong H^i(\mathcal{K})$.

In the non-simply connected case, we can rely on the results of Sections 4.2.1–4.2.3, according to which we can equate the Killing sheaf cohomologies with the group cohomology of the fundamental group with coefficients in the composite adjoint monodromy representation, at least up to the degree of asphericity of the underlying spacetime manifold. Unfortunately, there does not seem to exist a comprehensive list of exact solutions of Einstein's equations indexed by spacetime topology. So it takes some effort to find explicit examples of exact solutions on non-simply connected spacetimes. A rich source of examples comes from quotients of simply connected spacetimes (such as those mentioned in the preceding paragraph) by a discrete, freely acting subgroup π of the isometry group. The quotient is a manifold because the action of π is free and the metric descends to the quotient because the action of π on the original spacetime is by isometries. The group π then becomes the fundamental group of the quotient.

A nearly exhaustive study of possible quotients of 4-dimensional cosmological solutions (meaning spatially homogeneous ones) has been carried out in [79–81]. A complete presentation of the results is rather complicated and is relegated to the original references. A particular cosmological solution (M, g) is identified by (a) the topology of the spacetime M, (b) the topology and isometry group of the universal cover $(\tilde{M}, \tilde{g})$, (c) a number of continuous *metric parameters* specifying $\tilde{g}$, and (d) a number of continuous *moduli* (or *Teichmüller parameters*) specifying the quotient class. There may also be additional discrete parameters, but we ignore them here, since they do not affect the number of continuous parameters. According to the discussion of Section 4.2.2, the number of moduli (denoted by N_{m} in [81]) is in fact equal[4] to $b^1 = \dim H^1(\mathcal{K}_g)$. We shall not specify any metric parameters, since, as long as they take generic values they do not affect the number of moduli. For simplicity, we only consider the examples with toroidal spatial topology $M = \mathbb{R} \times T^3$, where $T^3 = S^1 \times S^1 \times S^1$. Hence, the fundamental group is $\pi_1(M) = \mathbb{Z}^3$ and the universal cover is $\mathbb{R}^4$. The (identity component) of the isometry group of $(\tilde{M}, \tilde{g})$ is then a semidirect product of a 3-dimensional transitive Bianchi group and an additional connected Lie group. Let us concentrate on the cases of either Bianchi type $I \cong \mathbb{R}^3$ or VII(0). Under these conditions, we can read off all the remaining possibilities and information form Table IV of [81]. They are summarized in Table 3. Note that $b^0 = \dim H^0(\mathcal{K}_g)$ counts the number of independent global Killing vectors on (M, g). The number of independent Killing vectors on $(\tilde{M}, \tilde{g})$ counts the dimension of the Bianchi group (always 3) and the dimension of the additional symmetry group. The number of independent Killing vectors not broken by compactification to T^3 can be deduced from the explicit

4 The space of moduli may not always be a smooth manifold, but may have algebraic singularities. Still, the number of moduli is the dimension of the generic subset of the moduli space, which is a smooth manifold. This dimension is also equal to $b^1 = \dim H^1(\mathcal{K})$. At singular points of the moduli space, b^1 may actually exceed the number of moduli, so at those points a more careful analysis is needed.

presentation of the isometry groups $\mathrm{Isom}(\tilde{M}, \tilde{g})$ and the discrete subgroups effecting the compactification, which for the examples given in Table 3 in [81, Sec. 3]. Many more examples can be read off from Tables IV, VII and Section 5.3 of [81].

It appears difficult to locate literature on explicit calculations that are equivalent to computing higher Killing sheaf cohomologies for other non-simply connected spacetimes.

6. Discussion and generalizations

We have reviewed in detail the algebraic, geometric and analytical properties of the Calabi differential complex [1].

In Section 2 we have defined the nodes of the complex in terms of Young symmetrized tensor bundles and given explicit formulas for the differential operators between them, verifying through explicit calculations that they in fact constitute a complex (Appendix A). Such explicit formulas are otherwise difficult to extract from the existing literature, especially in terms of tensor variables, as opposed to moving coframe variables used in Calabi's original work. Further, our formulas work for pseudo-Riemannian backgrounds of any signature, generalizing from the standard purely Riemannian context. We have also identified a differential operator cochain homotopy (Eqs. (2), (11)–(16)) that generates a cochain map from the complex to itself with a Laplacian-like principal symbol. This cochain homotopy map may be new. However, its lower order terms coincide with well known geometric operators known from the theory of linearized gravity (General Relativity). Another interesting and likely novel observation involved the formal adjoint complex (Section 2.3), whose initial differential operator turned out to be equivalent to the rank-$(n-2)$ Killing–Yano operator, in analogy with the Killing operator in the original complex.

In Sections 3 and 4 we showed that the Calabi complex is elliptic and locally exact. Hence, it resolves the sheaf of Killing vectors on the given constant curvature pseudo-Riemannian manifold. The same is true for the formal adjoint complex and the sheaf of rank-$(n-2)$ Killing–Yano tensors. Thus the cohomology of the Calabi complex could be expressed in terms of the Killing sheaf cohomology, while that of its formal adjoint in terms of the Killing–Yano sheaf cohomology. When a sheaf is locally constant (covering the relevant cases on constant curvature pseudo-Riemannian manifolds), its cohomology can be effectively computed in many circumstances using tools from algebraic topology, thus enabling effective computation of the Calabi cohomology. These methods were reviewed in Section 4, specialized to the Killing sheaf.

Finally, in Section 5, we discussed a physical application that motivated this work. Jointly, the results collected in this work, together with those of [77,18,17] imply that knowledge of Killing and Killing–Yano sheaf cohomologies allows some degree of control over the degeneracy subspaces of the presymplectic and Poisson structures within the classical field theory of linearized gravity on constant curvature backgrounds.

Unfortunately, the above results do not apply directly to linearized gravity on arbitrary Lorentzian manifolds, only those that have constant curvature and where the Calabi complex is defined. However, the Calabi complex serves as a case study for the more general situation and the same results partially generalize to general backgrounds. In particular, we can already make the following conclusions. In general, the Calabi complex will have to be replaced by a different differential complex, which will likely depend on some of the algebraic characteristics of the Lorentzian manifold (such as its isometries and the algebraic type of the curvature tensor and its derivatives). This complex would be identified, as was the Calabi complex [9,10], by the property of being a formally exact compatibility complex of the Killing operator. Such a complex is known to exist under general conditions and also have the property of being elliptic, since the Killing operator is itself elliptic [15,16]. Further, under a generic condition, it can be shown to be locally exact (Section 3.2). The local exactness property connects the cohomology of this complex to that of the Killing sheaf, which can be effectively computed, at least in many circumstances, when the sheaf is locally constant. Unfortunately, one piece of the puzzle remains incomplete. The connection between the cohomology of the formal adjoint complex and sheaf cohomology depends on the knowledge of the initial operator in that differential complex, which is the adjoint of the final operator of the differential complex resolving the Killing sheaf. In the Calabi case it is equivalent to the Killing–Yano operator. However, since the differential complex is expected to change depending on the Lorentzian manifold, so is this initial operator. Thus, it is not clear which sheaf cohomology will replace the Killing–Yano sheaf in the general case.

Hence, in future work, it would be very interesting to investigate these differential complex resolutions of the Killing sheaf, especially computing their differential operators explicitly. Besides the general existence results [15,16], such a complex has already been constructed for locally symmetric spaces ($\nabla_a R_{bcde} = 0$) [9,10]. Also, heuristic arguments suggest that they could be partially constructed by linearizing the so-called 'ideal' characterizations of certain exact families of solutions of Einstein's equations. These include Schwarzschild [82], Kerr [83] and some perfect fluid [84] solutions. An 'ideal' characterization consists of a number of tensor fields, locally and covariantly defined using the metric and its derivatives, which vanish iff the given metric is locally isometric to a particular geometry from the desired family. For instance, the vanishing of the Riemann tensor R is an ideal characterization of the flat geometry, while the vanishing of the corrected Riemann tensor $R-\bar{R}$ (Section 2.2) does the same for a constant curvature geometry. It should be clear from these examples, that the linearization of the tensors that constitute such an ideal characterization gives an operator whose composition with the Killing operator is formally exact. At the moment it is not completely clear what geometric interpretation can be given to subsequent differential operators in the desired formally exact differential complex.

Finally, one can easily imagine situations where the number of independent solutions to the Killing equations changes over the background pseudo-Riemannian manifold. The Killing sheaf is then no longer locally constant and many of the techniques described in this work are no longer applicable. In those cases, perhaps some insight can be gained from the theory of constructible sheaves [85, Ch. 4], [48, Ch. VIII], which are allowed to deviate from being locally constant in a controlled way.

Acknowledgments

The author would like to thank Mauro Carfora and Wouter von Limbek for helpful discussions on deformations of isometry and flat principal bundle structures. Thanks to Michał Wrochna for feedback on an earlier version of the manuscript. Thanks also to Benjamin Lang and Alexander Schenkel for useful references on the geometry of principal bundles. The kind hospitality of the Erwin Schrödinger Institute during the workshop "Algebraic Quantum Field Theory: Its Status and Its Future", where this work was first publicly presented, is also acknowledged.

Appendix A. Young tableaux and irreducible GL(n) representations

A.1. Basic background

A Young diagram of type $(r_1, r_2, \ldots)$ with k cells consists of a number of rows of cells of non-increasing lengths $r_i, r_{i+1} \le r_i$, such that $\sum_i r_i = k$. Example:

$$\text{type } (3, 3, 1) \text{ or } (3^2, 1), \quad \text{diagram } \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline \\ \cline{1-1} \end{array}.$$

Given a Young diagram with k cells, an instance of the corresponding GL(n) irrep can be realized as the image of the space of covariant k-tensors after two projections: assign an independent tensor index to each cell of the diagram, symmetrize over each row, anti-symmetrize over each column. The composition of these operations is called a *Young symmetrizer*, which we will denote by Y^d, where $d = (r_1, r_2, \ldots)$ is the type of the Young diagram. It will be convenient for us to group the indices of a symmetrized tensor by the columns of the corresponding diagram, separating them by a colon. For instance, we write $b_{abc:de}$ corresponding to the filling

$$\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1} \end{array}.$$

Here's an example of a simple Young symmetrizer:

$$\mathrm{Y}^{(2,1)}[t]_{ab:c} = \frac{1}{4}(t_{abc} + t_{cba} - t_{bac} - t_{abc}). \tag{96}$$

Different permutations of tensor indices filling a Young diagram create distinct Young symmetrizers, unless the permutation preserves the columns. The images of the Young symmetrizer for given diagram type with k-cells are all isomorphic as GL(n) representations, but are not necessarily all identical as subspaces of the space of covariant k-tensors. The reason for this observation is that the space of covariant k-tensors is a reducible GL(n) representation that decomposes into a sum of irreps corresponding to all possible diagram types with k-cells, but with, in general, non-trivial multiplicities. Both the dimension and the multiplicity of each occurring irrep can be computed with the so-called *hook formulas*. The *hook length* for a given cell is the number of cells constituting a hook with vertex at the given cell, extending to the right and down. *Multiplicity: k*! divided by the product of the hook lengths for each cell. *Dimension:* the product of shifted dimensions for each cell, divided by the product of hook lengths for each cell; the shifted dimensions of the cells are obtained by placing n in the top left cell, then always increasing by 1 to the right and decreasing by 1 down. Example:

$$\text{hook lengths } \begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}, \qquad \text{shifted dimensions } (n = 4) \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1} \end{array},$$

$$\text{multiplicity: } \frac{7!}{(5 \cdot 3 \cdot 2)(4 \cdot 2 \cdot 1)(1)} = 21, \qquad \text{dimension: } \frac{(4 \cdot 5 \cdot 6)(3 \cdot 4 \cdot 5)(2)}{(5 \cdot 3 \cdot 2)(4 \cdot 2 \cdot 1)(1)} = 60.$$

Note that when the number of rows exceeds n, the corresponding representation becomes zero-dimensional. This clearly follows from the dimension formula and from the more elementary observation that there do not exist non-trivial fully antisymmetric tensors of rank greater than n, the dimension of the fundamental representation of GL(n).

By construction, it is clear that every Young symmetrized subspace of covariant k-tensors is fully antisymmetric in the indices corresponding to each column of its Young diagram. However, this subspace will actually be even smaller and thus satisfy more identities. A complete set of identities selecting an irreducible GL(n) sub-representation of the space of covariant k-tensors identified by a diagram of type $(r_1, \ldots, r_l)$ filled with indices a_k^i (k being the row number and i the column number) consists of (i) *intracolumn* exchange identities and (ii) *intercolumn* exchange identities. The exchange of any two indices within a column changes the tensor by a sign. All such exchanges constitute the *intracolumn* identities. Let us define a two-column exchange as follows. Fix two columns $i < j$ and select the top k indices of column j. A two column-exchange consists of a swap between a set of k indices from column i and the top k indices of column j, without altering the internal order the substituted set of indices. For a fixed choice of such i, j, k an intercolumn identity consists of the equality of the tensor with unpermuted indices with the sum over all corresponding two-column exchanges. All such exchange identities with consistent choices of i, j, k constitute the *intercolumn* identities.

There already exists a special notation for antisymmetrization of a group of indices: inclusion in square brackets, $[a_1^i a_2^i \cdots]$. Let us introduce a special notation for the sum over all two column exchanges: fixing integers $i < j$ and k, we shall enclose the indices of column i in curly braces, $\{a_1^i a_2^i \cdots\}$, as well as the top k indices of column j, $\{a_1^j \cdots a_k^j\} a_{k+1}^j \cdots$. We give explicit examples of intracolumn and intercolumn identities for Young diagrams of type $(2, 2)$ and $(2, 2, 1)$:

$$r_{ab:cd} = r_{[ab]:cd} = \frac{1}{2}(r_{ab:cd} - r_{ba:cd}), \tag{97}$$

$$r_{ab:cd} = r_{ab:[cd]} = \frac{1}{2}(r_{ab:cd} - r_{ab:dc}), \tag{98}$$

$$r_{ab:cd} = r_{\{ab\}:\{c\}d} = r_{cb:ad} + r_{ac:bd}, \tag{99}$$

$$r_{ab:cd} = r_{\{ab\}:\{cd\}} = r_{cd:ab}, \tag{100}$$

$$b_{abc:de} = b_{[abc]:de} = \frac{1}{3}(b_{a[bc]:de} + b_{b[ca]:de} + b_{c[ab]:de}), \tag{101}$$

$$b_{abc:de} = b_{abc:[de]}, \tag{102}$$

$$b_{abc:de} = b_{\{abc\}:\{d\}e} = b_{dbc:ae} + b_{adc:be} + b_{abd:ce}, \tag{103}$$

$$b_{abc:de} = b_{\{abc\}:\{de\}} = b_{dec:ab} + b_{dbe:ac} + b_{ade:bc}. \tag{104}$$

It is remarkable, upon noticing the identity $r_{ab:cd} - r_{\{ab\}:\{c\}d} = 3r_{[ab:c]d}$, that according to Eqs. (97)–(100) a tensor $r_{ab:cd}$ with Young symmetry type $(2, 2)$ has the same algebraic symmetries as a Riemann curvature tensor (antisymmetry in ab and cd, interchange of ab with cd, and the algebraic Bianchi identity). This fact is well-known [86], but not often mentioned in textbooks on relativity.

A.2. Special algebraic and differential operators

Now, suppose that we are working on an n-dimensional pseudo-Riemannian manifold (M, g), with Levi-Civita connection ∇. As in Section 2.1, the Young symmetrizes introduced above define vector bundles of Young symmetrized covariant tensors $\mathrm{Y}^d T^*M \to M$, where d stands for a Young diagram. We define special linear algebraic and differential operators, already briefly discussed in Section 2.2, between these Young symmetrized tensor bundles occurring in the Calabi complex. Each of the corresponding Young diagrams has at most two columns, where the first column usually has at most n cells and the second column has at most two cells. The operator *trace* (tr) removes one row of cells, *metric exterior product* $(g \odot -)$ adds one row of cells, *left* or *right exterior derivative* (d_L and d_R) adds one cell to the left or right column respectively, and *left* or *right divergence* (δ_L and δ_R) removes one cell from the left or right column respectively. The name of each of these operators should be suggestive of their form, with the main complication being to maintain appropriate Young symmetry.

In principle, the Littlewood–Richardson decomposition rules uniquely fix the principal symbols of each of these operators up to a scalar multiple, with the Levi-Civita operator canonically converting a first order principal symbol into a first order operator. In practice, it takes a bit of work to find explicit formulas for them, given that a naive application a Young symmetrizer produces unmanageably large expressions. Moreover, the existence of the intracolumn and intercolumn symmetrization identities introduces non-uniqueness into possible explicit expressions. Below, we give explicit formulas for these operators. In case of ambiguity, the choice was dictated by practical convenience. Then, in Appendices A.3 and A.4 we show by explicit calculation that they satisfy the required symmetrization identities and thus carry the correct Young type.

$$\mathrm{tr}[b]_{a_1\cdots a_l:b} = b_{a_1\cdots a_l c:b}{}^{c}, \tag{105}$$

$$(g \odot t)_{a_1\cdots a_l:bc} = l(g_{b[a_1} t_{a_2\cdots a_l]:c} - g_{c[a_1} t_{a_2\cdots a_l]:b}), \tag{106}$$

$$\mathrm{d}_L[b]_{a_1\cdots a_l:bc} = l\nabla_{[a_1} b_{a_2\cdots a_l]:bc}, \tag{107}$$

$$\delta_L[b]_{a_1\cdots a_l:bc} = \nabla^a b_{aa_1\cdots a_l:bc} + 2l^{-1}\nabla^a b_{[b|a_1\cdots a_l:|c]a}, \tag{108}$$

$$\mathrm{d}_R[t]_{a_1\cdots a_l:bc} = 2\nabla_{[b} t_{|a_1\cdots a_l:|c]} + 2(l-1)^{-1}\nabla_{[\{b\}|} t_{\{a_1\cdots a_l\}:|c]}, \tag{109}$$

$$\delta_R[b]_{a_1\cdots a_l:b} = \nabla^c b_{a_1\cdots a_l:bc}. \tag{110}$$

Let us give an explicit example of (106) for $l = 2$, which appears in the formulas for the constant curvature Riemann tensor (124) and for the linearized Riemann curvature operator (8):

$$(g \odot h)_{a_1a_2:bc} = g_{a_1b}h_{a_2c} - g_{a_2b}h_{a_1c} - g_{a_1c}h_{a_2b} + g_{a_2c}g_{a_1b}, \tag{111}$$

$$(g \odot g)_{a_1a_2:bc} = 2(g_{a_1b}g_{a_2c} - g_{a_2b}g_{a_1c}), \tag{112}$$

$$\begin{aligned}(\nabla\nabla \odot h)_{a_1a_2:bc} &= \nabla\nabla_{a_1b}h_{a_2c} - \nabla\nabla_{a_2b}h_{a_1c} - \nabla\nabla_{a_1c}h_{a_2b} + \nabla\nabla_{a_2c}h_{a_1b}, \\ &\text{where } \nabla\nabla_{ab} = \nabla_{(a}\nabla_{b)} = \frac{1}{2}(\nabla_a\nabla_b + \nabla_b\nabla_a).\end{aligned} \tag{113}$$

In the last equation we used the $\odot$ operation to define another differential operator of definite Young type. This property follows directly from that of (106).

A.3. Preservation of Young type

Each of the operators (105)–(110) maps tensors of one Young type into another one, as is indicated by the index notation described in Appendix A.1. Below we explicitly demonstrate that, by showing that the result of applying one of these operators to a tensor of a given Young type always satisfies the required intracolumn and intercolumn identities.

First, we list some key identities satisfied by the idempotent antisymmetrization and column exchange operations. They follow from straightforward, though possibly lengthy, application of their definitions. Here, a tensor $t_{a_1\cdots a_l}$ is assumed to be fully antisymmetric. Also, to simplify the notation for nested operations, we use the notation $\{\cdots\}^k_l$, where the braces necessarily enclose the indices $a_k, a_{k+1}, \ldots, a_l$, though perhaps also others, to mean that we apply the appropriate column exchange operation to these a_i indices as if they appeared in the order $\{a_k \cdots a_l\}$.

$$(l+1)p_{[a}t_{a_1\cdots a_l]} = p_a t_{a_1\cdots a_l} - p_{\{a\}}t_{\{a_1\cdots a_l\}}, \tag{114}$$

$$p_{\{b\}}t_{\{a_1\cdots a_l\}} = p_{a_1}t_{ba_2\cdots a_l} + p_{\{b\}}t_{a_1\{a_2\cdots a_l\}}, \tag{115}$$

$$p_{\{a\}}t_{b\{a_1\cdots a_l\}} = -p_{\{b\}}t_{a\{a_1\cdots a_l\}}, \tag{116}$$

$$p_{\{b}q_{c\}}t_{\{a_1\cdots a_l\}} = p_{a_1}q_{\{c\}}t_{b\{a_2\cdots a_l\}} + p_{\{b}q_{c\}}t_{a_1\{a_2\cdots a_l\}}, \tag{117}$$

$$\left(p_{b'}t_{\{a_1\cdots a_l\}:c'} - p_{c'}t_{\{a_1\cdots a_l\}:b'}\right)\delta^{b'}_{\{b}\delta^{c'}_{c\}} = p_{\{b\}}t_{\{a_1\cdots a_l\}:c} - p_{\{c\}}t_{\{a_1\cdots a_l\}:b}, \tag{118}$$

$$\left(t_{b'\{a_1\cdots a_l\}:c'a} - t_{b'\{a_1\cdots a_l\}:c'a}\right)\delta^{b'}_{\{b}\delta^{c'}_{c\}} = -2(l-1)t_{[b|a_1\cdots a_l:|c]}, \tag{119}$$

$$p_{\{\{b\}\}}t_{\{\{a_1\cdots a_l\}\}} = l p_b t_{a_1\cdots a_l} - (l-1)p_{\{b\}}t_{\{a_1\cdots a_l\}}, \tag{120}$$

$$p_{\{b\}}q_{\{\{c\}}t_{\{a_1\cdots a_l\}\}^1_l} = p_{\{c\}}q_b t_{\{a_1\cdots a_l\}} + (p_{\{b}q_{c\}} - q_{\{b}p_{c\}})t_{\{a_1\cdots a_l\}}, \tag{121}$$

$$\left(p_{\{\{b'\}}t_{\{a_1\cdots a_l\}\}^1_l:c'} - p_{\{\{c'\}}t_{\{a_1\cdots a_l\}\}^1_l:b'}\right)\delta^{b'}_{\{b}\delta^{c'}_{c\}} = 2(l-1)p_{[b|}t_{a_1\cdots a_l:|c]} - 2(l-2)p_{[\{b\}|}t_{\{a_1\cdots a_l\}:|c]}, \tag{122}$$

$$p_{\{\{a\}}t_{\{a_1\cdots a_l\}\}^1_l:\{bc\}} = p_{\{a\}}t_{\{a_1\cdots a_l\}:bc} + 2p_{[\{b\}|}t_{\{a_1\cdots a_l\}:|c]a} - 2p_{[b|}t_{a_1\cdots a_l:|c]a}. \tag{123}$$

Next, we show how the above key identities can be used to explicitly demonstrate that the required symmetrization identities are satisfied. We try to indicate which of the key identities are used and where, while also silently making use of the symmetrization properties of the Young type tensors on which the operations are being performed.

For the *trace* (105), the intracolumn identities are obvious, so there is only one intercolumn identity to check:

$$\begin{aligned}\mathrm{tr}[b]_{\{a_1\cdots a_l\}:\{b\}} &= b_{\{a_1\cdots a_l\}c:\{b\}}{}^c \quad (115) \\ &= b_{\{a_1\cdots a_lc\}:\{b\}}{}^c - b_{\{a_1\cdots a_lb\}:c}{}^c = b_{a_1\cdots a_lc:b}{}^c \\ &= \mathrm{tr}[b]_{a_1\cdots a_l:b}.\end{aligned}$$

For the *metric exterior product* (106), the intracolumn identities are obvious, so there are two intercolumn identities to check:

$$\begin{aligned}(g\odot t)_{\{a_1\cdots a_l\}:\{b\}c} &= l(g_{\{b\}\{[a_1}t_{a_2\cdots a_l]\}:c} - g_{c\{[a_1}t_{a_2\cdots a_l]\}:\{b\}}) \quad (114) \\ &= -l(l+1)(g_{[b[a_1}t_{a_2\cdots a_l]]:c} - g_{c[[a_1}t_{a_2\cdots a_l]:b]}) + l(g_{b[a_1}t_{a_2\cdots a_l]:c} - g_{c[a_1}t_{a_2\cdots a_l]:b}) \\ &= (g\odot t)_{a_1\cdots a_l:bc}, \\ (g\odot t)_{\{a_1\cdots a_l\}:\{bc\}} &= l(g_{b'\{[a_1}t_{a_2\cdots a_l]\}:c'} - g_{c'\{[a_1}t_{a_2\cdots a_l]\}:b'})\delta^{b'}_{\{b}\delta^{c'}_{c\}} \quad (118) \\ &= l(g_{\{b\}\{[a_1}t_{a_2\cdots a_l]\}:c} - g_{\{c\}\{[a_1}t_{a_2\cdots a_l]\}:b}) \quad (114) \\ &= -l(l+1)(g_{[b[a_1}t_{a_2\cdots a_l]]:c} - g_{[c[a_1}t_{a_2\cdots a_l]]:b}) + l(g_{b[a_1}t_{a_2\cdots a_l]:c} - g_{c[a_1}t_{a_2\cdots a_l]:b}) \\ &= (g\odot t)_{a_1\cdots a_l:bc}.\end{aligned}$$

The double anti-symmetrizations vanished because of the identities $g_{[ab]} = 0$ and $t_{[a_2\cdots a_l:a_1]} = 0$, with the latter following from a combination of (114) and an intercolumn identity. Also, we have used the fact that $p_{[a_1}t_{a_2\cdots a_l]:b}$ is a tensor of the corresponding Young type, which follows from the identities in the paragraph below.

For the *left exterior derivative* (107), the intracolumn identities are obvious, so there are two intercolumn identities to check:

$$
\begin{aligned}
\mathrm{d}_L[b]_{\{a_1\cdots a_l\}:\{b\}c} &= l\nabla_{\{[a_1} b_{a_2\cdots a_l]\}:\{b\}c} \quad (114) \\
&= -l(l+1)\nabla_{[[a_1} b_{a_2\cdots a_l]:b]c} + l\nabla_{[a_1} b_{a_2\cdots a_l]:bc} = \mathrm{d}_L[b]_{a_1\cdots a_l:bc}, \\
\mathrm{d}_L[b]_{\{a_1\cdots a_l\}:\{bc\}} &= l\nabla_{\{[a_1} b_{a_2\cdots a_l]\}:\{bc\}} \quad (114) \\
&= \nabla_{\{a_1} b_{a_2\cdots a_l\}:\{bc\}} - \nabla_{\{\{a_1\}} b_{\{a_2\cdots a_l\}\}^1_l:\{bc\}} \quad (117) \\
&= \nabla_{a_1} b_{\{a_2\cdots a_l\}:\{bc\}} + \nabla_b b_{\{a_2\cdots a_l\}:a_1\{c\}} - \nabla_{\{\{a_1\}} b_{\{a_2\cdots a_l\}\}^2_l:\{bc\}} - \nabla_{\{\{b\}} b_{\{a_2\cdots a_l\}\}^2_l:a_1\{c\}} \quad (123),\ (121) \\
&= \nabla_{a_1} b_{a_2\cdots a_l:bc} + \nabla_b b_{a_2\cdots a_l:a_1c} - \nabla_c b_{\{a_2\cdots a_l\}:a_1\{b\}} - \nabla_{\{a_1\}} b_{\{a_2\cdots a_l\}:bc} \\
&\quad - 2\nabla_{[\{b\}|} b_{\{a_2\cdots a_l\}:|c]a_1} + 2\nabla_{[b|} b_{a_2\cdots a_l:|c]a_1} \\
&\quad - (\nabla_{b'} b_{\{a_2\cdots a_l\}:a_1c'} - \nabla_{c'} b_{\{a_2\cdots a_l\}:a_1b'})\delta^{c'}_{\{c}\delta^{b'}_{b\}} \quad (118) \\
&= l\nabla_{[a_1} b_{a_2\cdots a_l]:bc} - 2\nabla_{[\{b\}|} b_{\{a_2\cdots a_l\}:|c]a_1} + 2\nabla_{[\{c\}|} b_{\{a_2\cdots a_l\}:a_1|b]} \\
&= l\nabla_{[a_1} b_{a_2\cdots a_l]:bc} = \mathrm{d}_L[b]_{a_1\cdots a_l:bc}.
\end{aligned}
$$

For the *left divergence* (108), the intracolumn identities are obvious. It remains to check the two intercolumn identities:

$$
\begin{aligned}
\delta_L[b]_{\{a_1\cdots a_l\}:\{b\}c} &= \nabla^a (b_{a\{a_1\cdots a_l\}:\{b\}c} + l^{-1} b_{\{b\}\{a_1\cdots a_l\}:ca} - l^{-1} b_{c\{a_1\cdots a_l\}:\{b\}a}) \\
&= \nabla^a (b_{\{aa_1\cdots a_l\}:\{b\}c} + b_{ba_1\cdots a_l:ca} \quad (115) \\
&\quad - l^{-1}(l+1) b_{[ba_1\cdots a_l]:ca} + l^{-1} b_{ba_1\cdots a_l:ca} \quad (114) \\
&\quad - l^{-1} b_{\{ca_1\cdots a_l\}:\{b\}a} + l^{-1} b_{ba_1\cdots a_l:ca}) \quad (115) \\
&= \nabla^a (b_{aa_1\cdots a_l:bc} + 2l^{-1} b_{[b|a_1\cdots a_l:|c]a}) = \delta_L[b]_{a_1\cdots a_l:bc}, \\
\delta_L[b]_{\{a_1\cdots a_l\}:\{bc\}} &= \nabla^a (b_{a\{a_1\cdots a_l\}:\{bc\}} \quad (117) \\
&\quad + l^{-1}(b_{b'\{a_1\cdots a_l\}:c'a} - b_{c'\{a_1\cdots a_l\}:b'a})\delta^{b'}_{\{b}\delta^{c'}_{c\}}) \quad (119) \\
&= \nabla^a (b_{\{aa_1\cdots a_l\}:\{bc\}} + b_{b\{a_1\cdots a_l\}:\{c\}a} \quad (115) \\
&\quad - l^{-1}(l-1)(b_{ba_1\cdots a_l:ca} - b_{ca_1\cdots a_l:ba})) \\
&= \nabla^a (b_{aa_1\cdots a_l:bc} + b_{\{ba_1\cdots a_l\}:\{c\}a} - b_{ca_1\cdots a_l:ba} - (1-l^{-1})(b_{ba_1\cdots a_l:ca} - b_{ca_1\cdots a_l:ba})) \\
&= \nabla^a (b_{aa_1\cdots a_l:bc} + l^{-1} b_{ba_1\cdots a_l:ca} - l^{-1} b_{ca_1\cdots a_l:ba}) \\
&= \delta_L[b]_{a_1\cdots a_l:bc}.
\end{aligned}
$$

For the *right exterior derivative* (109), the following rewriting makes the intracolumn identities obvious:

$$
\begin{aligned}
\mathrm{d}_R[b]_{a_1\cdots a_l:bc} &= \nabla_b b_{a_1\cdots a_l:c} - \nabla_c b_{a_1\cdots a_l:b} + (l-1)^{-1}\left(\nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_{\{c\}} b_{\{a_1\cdots a_l\}:b}\right) \quad (114) \\
&= \nabla_b b_{a_1\cdots a_l:c} - \nabla_c b_{a_1\cdots a_l:b} - (l-1)^{-1}(l+1)(\nabla_{[b} b_{a_1\cdots a_l]:c} - \nabla_{[c} b_{a_1\cdots a_l]:b}) \\
&\quad + (l-1)^{-1}(\nabla_b b_{a_1\cdots a_l:c} - \nabla_c b_{a_1\cdots a_l:b}).
\end{aligned}
$$

There are also two intercolumn identities to check:

$$
\begin{aligned}
\mathrm{d}_R[b]_{\{a_1\cdots a_l\}:\{b\}c} &= \nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_c b_{\{a_1\cdots a_l\}:\{b\}} + (l-1)^{-1}(\nabla_{\{\{b\}\}} b_{\{\{a_1\cdots a_l\}\}:c} - \nabla_{\{\{c\}} b_{\{a_1\cdots a_l\}\}^1_l:\{b\}}) \\
&= \nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_c b_{a_1\cdots a_l:b} + (l-1)^{-1} l\nabla_b b_{a_1\cdots a_l:c} - \nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} \quad (120) \\
&\quad - (l-1)^{-1}\nabla_b b_{\{b_1\cdots a_l\}:\{c\}} \quad (121) \\
&\quad - (l-1)^{-1}(\nabla_{c'} b_{a_1\cdots a_l:b'} - \nabla_{b'} b_{a_1\cdots a_l:c'})\delta^{b'}_{\{b}\delta^{c'}_{c\}} \quad (118) \\
&= \nabla_b b_{a_1\cdots a_l:c} - \nabla_c b_{a_1\cdots a_l:b} + (l-1)^{-1}(\nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_{\{c\}} b_{\{a_1\cdots a_l\}:b}) \\
&= \mathrm{d}_R[b]_{a_1\cdots a_l:bc}, \\
\mathrm{d}_R[b]_{\{a_1\cdots a_l\}:\{bc\}} &= (\nabla_{b'} b_{\{a_1\cdots a_l\}:c'} - \nabla_{c'} b_{\{a_1\cdots a_l\}:b'})\delta^{b'}_{\{b}\delta^{c'}_{c\}} \quad (118),\ (122) \\
&\quad + (l-1)^{-1}(\nabla_{\{\{b'\}} b_{\{a_1\cdots a_l\}\}^a_l:c'} - \nabla_{\{\{c'\}} b_{\{a_1\cdots a_l\}\}^a_l:b'})\delta^{b'}_{\{b}\delta^{c'}_{c\}} \\
&= \nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_{\{c\}} b_{\{a_1\cdots a_l\}:b} + (\nabla_b b_{a_1\cdots a_l:c} - \nabla_c b_{a_1\cdots a_l:b}) \\
&\quad - (l-1)^{-1}(l-2)(\nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_{\{c\}} b_{\{a_1\cdots a_l\}:b}) \\
&= \nabla_b b_{a_1\cdots a_l:c} - \nabla_c b_{a_1\cdots a_l:b} + (l-1)^{-1}(\nabla_{\{b\}} b_{\{a_1\cdots a_l\}:c} - \nabla_{\{c\}} b_{\{a_1\cdots a_l\}:b}) \\
&= \mathrm{d}_R[b]_{a_1\cdots a_l:bc}.
\end{aligned}
$$

For the *right divergence* (110), the intracolumn identities are obvious, so there is only one intercolumn identity to check:

$$\delta_R[b]_{\{a_1\cdots a_l\}:\{b\}} = \nabla^c b_{\{a_1\cdots a_l\}:\{b\}c} = \nabla^c b_{a_1\cdots a_l:bc} = \delta_R[b]_{a_1\cdots a_l:b}.$$

A.4. Composition identities

Below, we list identities between some possible compositions of the operators (105)–(110). These will be instrumental in the following Appendix A.5, where they will be used to explicitly define the operators involved in the Calabi complex (2) and the necessary identities between them. We do not show the necessary explicit calculations, as they are lengthy but straightforward. It suffices to make use of the key identities (114)–(123), as explicitly illustrated in Appendix A.3.

Recall that ∇ denotes the Levi-Civita connection on a pseudo-Riemannian space of constant curvature with metric g and dimension n. The Riemann tensor on this space is defined by the convention $2\nabla_{[a}\nabla_{b]}\omega_c = \bar{R}_{ab:c}{}^d\omega_d$ and is explicitly equal to

$$\bar{R}_{ab:cd} = \frac{\lambda}{2}(g\odot g)_{ab:cd} = \lambda(g_{ac}g_{bd} - g_{ad}g_{bc}), \quad \text{with } \lambda = \frac{k}{n(n-1)}, \tag{124}$$

such that $k = g^{ac}\bar{R}_{ab:c}{}^b$ is the curvature constant.

The simplest composition identity is of two left exterior derivatives ($l \geq 4$):

$$\mathrm{d}_L \circ \mathrm{d}_L[b]_{a_1\cdots a_l:bc} = 0. \tag{125}$$

The principal symbols of these operators augment the left index column of the argument and antisymmetrize over it, thus composing to zero, as in the case of the de Rham differential. This means that, at worst, the result of the composition of the operators is of order zero and proportional to the background curvature $\bar{R}$ given in (124). Now, note that the background curvature is $\mathrm{GL}(n)$-equivariantly composed only out of the metric and the composition $\mathrm{d}_L \circ \mathrm{d}_L$ is also an equivariant operator (taking into account the transformation properties of the covariant derivative and the metric). Then the result of the composition (Young type $(2, 2, 1^{l-2})$) must be equivariantly composed only out of the metric g (Young type (2)) and the argument b (Young type $(2, 2, 1^{l-4})$). However, according the Littlewood–Richardson rules [29,42], there is no non-trivial combination of that kind. Therefore, the composition of these operator must vanish.

Next, we show the relation between the compositions $\delta_L \circ \mathrm{d}_L$ and $\mathrm{d}_L \circ \delta_L$, along with some auxiliary identities involving the curvature. These formulas hold when the length of the left index column of the output is $l > 2$.

$$2\nabla_{[a}\nabla_{b]}b_{a_1\cdots a_l:cd} = (\bar{R}\cdot b)_{ab\,a_1\cdots a_l:cd}, \tag{126}$$

$$\begin{aligned}(\bar{R}\cdot b)_{ab\,a_1\cdots a_l:cd} &= \bar{R}_{ab:\{e\}}{}^e b_{\{a_1\cdots a_l\}:cd} + \bar{R}_{ab:c}{}^e b_{a_1\cdots a_l:ed} + \bar{R}_{ab:d}{}^e b_{a_1\cdots a_l:ce} \\ &= \lambda(g_{a\{b\}} - g_{b\{a\}})b_{\{a_1\cdots a_l\}:cd} + \lambda(g_{ac}b_{a_1\cdots a_l:bd} - g_{bc}b_{a_1\cdots a_l:ad}) \\ &\quad - \lambda(g_{ad}b_{a_1\cdots a_l:bc} - g_{bd}b_{a_1\cdots a_l:ac});\end{aligned} \tag{127}$$

$$\begin{aligned}(l+1)(\bar{R}\cdot b)_{a[b\,a_1\cdots a_l]:cd} &= -l(l+1)\lambda g_{a[b}b_{a_1\cdots a_l]:cd} \\ &\quad - (l+1)\lambda(g_{c[b}b_{a_1\cdots a_l]:ad} - g_{d[b}b_{a_1\cdots a_l]:ac}), \\ (l+1)(\bar{R}\cdot b)^a{}_{[a\,a_1\cdots a_l]:bc} &= -(l(n-l)+2)\lambda b_{a_1\cdots a_l:bc} + (-)^l\lambda(g\odot \mathrm{tr}[b])_{a_1\cdots a_l:bc}, \\ (l+1)(\bar{R}\cdot b)^a{}_{[b\,a_1\cdots a_l]:ca} - (l+1)(\bar{R}\cdot b)^a{}_{[c\,a_1\cdots a_l]:ba} &= 0; \\ \delta_L\circ \mathrm{d}_L[b]_{a_1\cdots a_l:bc} &= \Box b_{a_1\cdots a_l:bc} - \mathrm{d}_L\circ\delta_L[b]_{a_1\cdots a_l:bc} + l^{-1}\mathrm{d}_R\circ\delta_R[b]_{a_1\cdots a_l:bc} \\ &\quad - (l(n-l)+2)\lambda b_{a_1\cdots a_l:bc} + (-)^l\lambda(g\odot\mathrm{tr}[b])_{a_1\cdots a_l:bc}.\end{aligned} \tag{128}$$

The main identity that will be useful in Appendix A.5 is the following:

$$\begin{aligned}(\delta_L\circ \mathrm{d}_L + \mathrm{d}_L\circ\delta_L)[b]_{a_1\cdots a_l:bc} &= \Box b_{a_1\cdots a_l:bc} + l^{-1}\mathrm{d}_R\circ\delta_R[b]_{a_1\cdots a_l:bc} \\ &\quad - (l(n-l)+2)\lambda b_{a_1\cdots a_l:bc} + (-)^l\lambda(g\odot\mathrm{tr}[b])_{a_1\cdots a_l:bc}.\end{aligned} \tag{129}$$

The composition $\delta_L \circ \mathrm{d}_L$ has a special form when the length of the left index column of the output is $l = 2$:

$$\nabla_{\{a\}}r_{\{a_1a_2\}:bc} = -(\nabla_{\{c\}}r_{\{a_1a_2\}:ba} + \nabla_{\{b\}}r_{\{a_1a_2\}:ca}); \tag{130}$$

$$\begin{aligned}(\bar{R}\cdot r)^a{}_{b\,a_1a_2:ca} &= -(n-1)\lambda r_{a_1a_2:bc} + \lambda(g_{ba_1}\,\mathrm{tr}[r]_{a_2:c} - g_{ba_2}\,\mathrm{tr}[r]_{a_1:c}), \\ (\bar{R}\cdot r)^a{}_{b\,a_1a_2:ca} - (\bar{R}\cdot r)^a{}_{c\,a_1a_2:ba} &= -2(n-1)\lambda r_{a_1a_2:bc} + \lambda(g\odot\mathrm{tr}[r])_{a_1a_2:bc}, \\ (\bar{R}\cdot r)^a{}_{[b\,a_1a_2]:ca} - (\bar{R}\cdot r)^a{}_{[c\,a_1a_2]:ba} &= 0; \\ \delta_L\circ\mathrm{d}_L[r]_{a_1a_2:bc} &= \Box r_{a_1a_2:bc} + \frac{1}{2}\mathrm{d}_R\circ\delta_R[r]_{a_1a_2:bc} - 2(n-1)\lambda r_{a_1a_2:bc} + \lambda(g\odot\mathrm{tr}[r])_{a_1a_2:bc}.\end{aligned} \tag{131}$$

Next, we show the relation between the compositions $\mathrm{tr}\circ\mathrm{d}_L$ and $\mathrm{d}_L\circ\mathrm{tr}$:

$$\mathrm{tr}\circ\mathrm{d}_L[b]_{a_1\cdots a_l:b} = \mathrm{d}_L\circ\mathrm{tr}[b]_{a_1\cdots a_l:b} + (-)^l\delta_R[b]_{a_1\cdots a_l:b}. \tag{132}$$

Next, we show the relations between the compositions $\mathrm{d}_L \circ \mathrm{d}_R$, $\mathrm{d}_R \circ \mathrm{d}_L$ and the operator $(\nabla\nabla \odot -)$, along with some auxiliary identities involving the curvature. Note that below we make use of the notation $[\cdots]^1_l$ which denotes idempotent antisymmetrization of the indices $a_1 \cdots a_l$, as if they were given in that position and ignoring any other indices appearing within the same brackets.

$$2\nabla_{[a}\nabla_{b]} b_{a_1\cdots a_l:c} = (\bar{R}\cdot b)_{ab\ a_1\cdots a_l:c}, \tag{133}$$

$$\begin{aligned}(\bar{R}\cdot b)_{ab\ a_1\cdots a_l:c} &= \bar{R}_{ab:\{d\}}{}^d b_{\{a_1\cdots a_l\}:c} + \bar{R}_{ab:c}{}^d b_{a_1\cdots a_l:d} \\ &= \lambda(g_{a\{b\}} - g_{b\{a\}}) b_{\{a_1\cdots a_l\}:c} + \lambda(g_{ac} b_{a_1\cdots a_l:b} - g_{bc} b_{a_1\cdots a_l:a});\end{aligned} \tag{134}$$

$$\begin{aligned}
l(\bar{R}\cdot b)_{b[a_1\ a_2\cdots a_l]:c} &= -l^2\lambda g_{b[a_1} b_{a_2\cdots a_l]:c} + \lambda(g\odot b)_{a_1\cdots a_l:bc}, \\
2l(\bar{R}\cdot b)_{[b|[a_1\ a_2\cdots a_l]:|c]} &= -(l-2)\lambda(g\odot b)_{a_1\cdots a_l:bc}, \\
l(\bar{R}\cdot b)_{\{b\}\{[a_1\ a_2\cdots a_l]\}:c} &= -l^2\lambda g_{b[a_1} b_{a_2\cdots a_l]:c} + \lambda(g\odot b)_{a_1\cdots a_l:bc}, \\
2l(\bar{R}\cdot b)_{[\{b\}|\{[a_1\ a_2\cdots a_l]\}:|c]} &= -(l-2)\lambda(g\odot b)_{a_1\cdots a_l:bc}, \\
Q_{ba_1\ a_2\cdots a_l:c} = (\bar{R}\cdot b)_{a_1\{b\}\ \{a_2\cdots a_l\}:c} &= -(\bar{R}\cdot b)_{ba_1\ a_2\cdots a_l:c} + l^2\lambda g_{a_1[b} b_{a_2\cdots a_l]:c} - \lambda(g\odot b)_{ba_2\cdots a_l:a_1c}, \\
l(\bar{R}\cdot b)_{[a_1\{b\}\ \{a_2\cdots a_l\}]^1_l:c} - l(\bar{R}\cdot b)_{[a_1\{c\}\ \{a_2\cdots a_l\}]^1_l:b} &= lQ_{b[a_1\ a_2\cdots a_l]:c} - lQ_{c[a_1\ a_2\cdots a_l]:b} \\
&= 2(l-2)\lambda(g\odot b)_{a_1\cdots a_l:bc};
\end{aligned}$$

$$\mathrm{d}_R \circ \mathrm{d}_L[b]_{a_1\cdots a_l:bc} = (l-1)^{-1} l(\nabla\nabla\odot b)_{a_1\cdots a_l:bc} - \frac{l(l-2)}{2(l-1)}\lambda(g\odot b)_{a_1\cdots a_l:bc}, \tag{135}$$

$$\mathrm{d}_L \circ \mathrm{d}_R[b]_{a_1\cdots a_l:bc} = (\nabla\nabla\odot b)_{a_1\cdots a_l:bc} + \frac{l}{2}\lambda(g\odot b)_{a_1\cdots a_l:bc}. \tag{136}$$

The main identity that will be useful in Appendix A.5 is the following:

$$l^{-1}\mathrm{d}_R \circ \mathrm{d}_L - (l-1)^{-1}\mathrm{d}_L \circ \mathrm{d}_R = -\lambda(g\odot b)_{a_1\cdots a_l:bc}. \tag{137}$$

A.5. Calabi complex and its homotopy formulas

Below, we use the special differential operators introduced earlier in Appendix A.2 to explicitly define the differential operators B_l, E_l and P_l that make up the Calabi complex and its homotopy formulas, as discussed in more detail in Section 2.2:

$$\begin{aligned}
B_1[v]_{a:b} &= \nabla_a v_b + \nabla_b v_a, \\
B_2[h]_{a_1a_2:bc} &= (\nabla\nabla\odot h)_{a_1a_2:bc} + \lambda(g\odot h)_{a_1a_2:bc}, \\
B_l[b]_{a_1\cdots a_l:bc} &= \mathrm{d}_L[b]_{a_1\cdots a_l:bc} \quad (l\geq 3), \\
E_1[h]_a &= \nabla^b h_{a:b} - \frac{1}{2}\nabla_a \operatorname{tr}[h], \\
E_2[b]_{a:b} &= \operatorname{tr}[b]_{a:b}, \\
E_{l+1}[b]_{a_1\cdots a_l:bc} &= \left(\delta_L - (-)^l l^{-1}\mathrm{d}_R\circ\operatorname{tr}\right)[b]_{a_1\cdots a_l:bc} \quad (l\geq 2).
\end{aligned}$$

Explicit formulas for B_l and E_l with low l have been given in Section 2.2.

Further, we make use of the identities given in Appendix A.4 to show that these operators satisfy the required identities, namely $B_{l+1}\circ B_l = 0$. The identities $B_2\circ B_1 = 0$ and $B_3\circ B_2 = 0$ have already been shown to follow in Section 2.2 from the usual transformation properties of the Riemann curvature tensor under diffeomorphisms and from its Bianchi identities. The identities $B_{l+1}\circ B_l = 0$ for $l > 2$ then follow directly from the composition identity $\mathrm{d}_L\circ\mathrm{d}_L = 0$ in Eq. (125).

Again, appealing to the identities of Appendix A.4, we give the homotopy formulas $P_l = E_{l+1}\circ B_{l+1} + B_l\circ E_l$ for $l\leq 2$:

$$\begin{aligned}
E_1\circ B_1[v]_a &= \nabla^b(\nabla_a v_b + \nabla_b v_a) - \frac{1}{2}\nabla_a(2\nabla^b v_b) \\
P_0 &= \Box v_a + \lambda(n-1)v_a, \\
(E_2\circ B_2 + B_1\circ E_1)[h]_{a:b} &= (\nabla\nabla\odot h)_{ac:b}{}^c + \lambda(g\odot h)_{ac:b}{}^c \\
&\quad + \nabla_a\left(\nabla^c h_{b:c} - \frac{1}{2}\nabla_b\operatorname{tr}[h]\right) + \nabla_b\left(\nabla^c h_{a:c} - \frac{1}{2}\nabla_a\operatorname{tr}[h]\right) \\
P_1 &= \Box h_{ab} - 2\lambda h_{ab} + 2\lambda g_{ab}\operatorname{tr}[h], \\
(E_3\circ B_3 + B_2\circ E_2)[r]_{a_1a_2:bc} &= \delta_L\circ\mathrm{d}_L[r]_{a_1a_2:bc} - \frac{1}{2}\mathrm{d}_R\circ\operatorname{tr}\circ\mathrm{d}_L[r]_{a_1a_2:bc} \\
&\quad + (\nabla\nabla\odot\operatorname{tr}[r])_{a_1a_2:bc} + \lambda(g\odot\operatorname{tr}[r])_{a_1a_2:bc} \\
P_2 &= \Box r_{a_1a_2:bc} - 2(n-1)\lambda r_{a_1a_2:bc} + 2\lambda(g\odot\operatorname{tr}[r])_{a_1a_2:bc}.
\end{aligned}$$

Finally, the same set of identities also implies the following formulas for P_l with $l > 2$:

$$\begin{aligned}(E_{l+1} \circ B_{l+1} + B_l \circ E_l)[b]_{a_1\cdots a_l:bc} &= (\delta_L \circ \mathrm{d}_L + \mathrm{d}_L \circ \delta_L)[b]_{a_1\cdots a_1:bc} \\ &\quad - (-)^l \left(l^{-1}\mathrm{d}_R \circ \mathrm{tr} \circ \mathrm{d}_L - (l-1)^{-1}\mathrm{d}_L \circ \mathrm{d}_R \circ \mathrm{tr}\right)[b]_{a_1\cdots a_l:bc} \\ P_l &= \Box b_{a_1\cdots a_l:bc} - (l(n-l)+2)\lambda b_{a_1\cdots a_l:bc} + (-)^l 2\lambda(g \odot \mathrm{tr}[b])_{a_1\cdots a_l:bc}.\end{aligned}$$

Explicit formulas for P_l with low l have also been given in Section 2.2. Recall that, as in Eq. (124), we have used the notation $\lambda = \frac{k}{n(n-1)}$.

A.6. An adjoint operator

Here we derive Eq. (31), which according to the general formula (23) implies that $-n^{-1}\delta_L$ is the formal adjoint of d_L when acting on tensors of Young type $c_{a_2\cdots a_n:bc}$.

$$\begin{aligned}\nabla_a(c^{aa_2\cdots a_n:bc} b_{a_2\cdots a_n:bc}) &= c^{aa_2\cdots a_n:bc}\nabla_a b_{a_2\cdots a_n:bc} + (\nabla_a c^{aa_2\cdots a_n:bc}) b_{a_2\cdots a_n:bc} \\ &= c^{aa_2\cdots a_n:bc}\nabla_{[a} b_{a_2\cdots a_n]:bc} + \delta_L[c]^{a_2\cdots a_n:bc} b_{a_2\cdots a_n:bc} \\ &\quad - \frac{1}{n-1}(\nabla_a c^{ba_2\cdots a_n:ca} - \nabla_a c^{ca_2\cdots a_n:ba}) b_{a_2\cdots a_n:bc} \\ &= \frac{1}{n} c^{aa_2\cdots a_n:bc}\mathrm{d}_L[b]_{aa_2\cdots a_n:bc} + \delta_L[c]^{a_2\cdots a_n:bc} b_{a_2\cdots a_n:bc} \\ &\quad - \frac{1}{n-1}(b_{[a_2\cdots a_n:b]c}\nabla_a c^{ba_2\cdots a_n:ca} + b_{[a_2\cdots a_n:c]b}\nabla_a c^{ca_2\cdots a_n:ba}) \quad (114) \\ &= \frac{1}{n} c^{aa_2\cdots a_n:bc}\mathrm{d}_L[b]_{aa_2\cdots a_n:bc} + \delta_L[c]^{a_2\cdots a_n:bc} b_{a_2\cdots a_n:bc}.\end{aligned} \tag{138}$$

We have simply used the definitions of the d_L and δ_L differential operators as well as the fact that the contraction of two tensors, one of which being totally anti-symmetric in a subset of indices, allows the insertion of an anti-symmetrization over the corresponding indices of the second tensor. Finally, some of the anti-symmetrizations annihilated the corresponding tensors, due to their intercolumn identities and the application of the identity (114).

Appendix B. Homological algebra

Below we introduce some basic notions from homological algebra. A standard text on the subject is [87], where more details can be found along with complete proofs.

Let A_i, also denoted $A_\bullet$, be a sequence of vector spaces (real vector spaces, for our purposes) with linear maps $A_i \to A_{i+1}$ between them. If each successive pair of maps $A_{i-1} \to A_i \to A_{i+1}$ composes to zero, this sequence is called a *complex (of vector spaces)* or a *cochain complex*, with an element $a \in A_i$ being referred to as a *cochain (of degree i)*, and the maps $A_i \to A_{i+1}$ referred to as *cochain differentials*. Any complex gives rise to cohomologies

$$H^i(A_\bullet) = \ker(A_i \to A_{i+1}) / \operatorname{im}(A_{i-1} \to A_i). \tag{139}$$

If all the cohomologies vanish, $H^i(A_\bullet) = 0$ or the image of each map is equal to the kernel of the subsequent map, the complex is called *exact* or an *exact sequence*. Given two complexes $A_\bullet$ and $B_\bullet$, the vertical maps in the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A_i \arrow[r] \arrow[d] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B_i \arrow[r] & B_{i+1} \arrow[r] & \cdots \end{tikzcd} \tag{140}$$

are called *cochain maps* provided they make the diagram commute. Furthermore, the diagonal maps in a diagram like

$$\begin{tikzcd} \cdots \arrow[r, "\mathrm{d}"] & A_i \arrow[r, "\mathrm{d}"] \arrow[d, "h"] \arrow[dl, dashed, "\delta"'] & A_{i+1} \arrow[r, "\mathrm{d}"] \arrow[d, "h"] \arrow[dl, dashed, "\delta"'] & \cdots \arrow[dl, dashed, "\delta"'] \\ \cdots \arrow[r, "\mathrm{d}"] & B_i \arrow[r, "\mathrm{d}"] & B_{i+1} \arrow[r, "\mathrm{d}"] & \cdots \end{tikzcd} \tag{141}$$

are called *cochain homotopies*. The homotopy maps induce vertical cochain maps by the formula $h = \mathrm{d}\delta + \delta\mathrm{d}$. It is a basic fact that cochain maps $A_\bullet \to B_\bullet$ naturally induce maps in cohomology $H^i(A_\bullet) \to H^i(B_\bullet)$. Of course, identity chain maps induce identity maps in cohomology and zero chain maps induce zero maps in cohomology. Also, two cochain maps induce the same map in cohomology when their difference is induced by a cochain homotopy.

A *short exact sequence*

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0 \tag{142}$$

between complexes $A_\bullet$, $B_\bullet$ and $C_\bullet$ consists of cochain maps between them such that each instance of

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0 \tag{143}$$

is an exact sequence of vector spaces. Another basic fact of homological algebra is that a short exact sequence of complexes induces the following *long exact sequence* in cohomology

$$\begin{tikzcd} \cdots \arrow[r] & H^i(A_\bullet) \arrow[r] & H^i(B_\bullet) \arrow[r] & H^i(C_\bullet) \arrow[dll, rounded corners] \\ & H^{i+1}(A_\bullet) \arrow[r] & H^{i+1}(B_\bullet) \arrow[r] & H^{i+1}(C_\bullet) \arrow[r] & \cdots \end{tikzcd}, \tag{144}$$

where the maps $H^i(A_\bullet) \to H^i(B_\bullet)$ and $H^i(B_\bullet) \to H^i(C_\bullet)$ are induced by the cochain maps from the short exact sequence and the *connecting* maps $H^i(C_\bullet) \to H^{i+1}(A_\bullet)$ are induced by the cochain differential.

Finally, another standard result is the so-called *5-lemma* (or a simple variant thereof). It states that the central vertical map in the commutative diagram

$$\begin{tikzcd} A_{-2} \arrow[r] \arrow[d, "\cong"] & A_{-1} \arrow[r] \arrow[d, "\cong"] & A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "\cong"] & A_2 \arrow[d, "\cong"] \\ B_{-2} \arrow[r] & B_{-1} \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & B_2 \end{tikzcd} \tag{145}$$

is an isomorphism, provided that the top and bottom rows are exact sequences and all the other vertical maps are isomorphisms themselves.

Appendix C. Jets and jet bundles

In this appendix, we briefly introduce jet bundles, fix the relevant notation and discuss differential operators in the context of jets. For simplicity, we restrict ourselves to fields taking values in vector bundles. However, the discussion could be straightforwardly generalized to general smooth bundles. More details, as well as a coordinate independent definition, can be found in the standard literature [88,89].

Given a vector bundle $F \to M$ over a connected n-dimensional smooth manifold M, the *k-jet bundle* $J^kF \to M$ is a vector bundle whose defining characteristic is that for any (possibly non-linear) differential operator $f : \Gamma(F) \to \Gamma(F')$ of order k, there exists a canonical factorization $f[u] = f \circ j^k u$ for any section $u : M \to F$, where the *k-jet prolongation* $j^k : \Gamma(F) \to \Gamma(J^kF)$ is composed with a smooth bundle map $f : J^kF \to F'$, which by a slight abuse of notation we denote using the same symbol as the original differential operator. Composing the differential operator f with an l-jet prolongation canonically defines a new differential operator $p_l f : J^{l+k}F \to J^lF'$ called its *l-prolongation*, $j^l f[u] = p_l f \circ j^k u$. Given a trivializable restriction $F_U \to U$ of F to a chart $U \subset M$ with local coordinates (x^i) and fiber-adapted local coordinates (x^i, u^a), there is a corresponding adapted chart $J^kF_U \subset J^kF$ with adapted local coordinates (x^i, u^a_I), where $I = i_1 \cdots i_l$ runs through multi-indices of orders $|I| = l = 0, \ldots, k$. In these coordinates, the k-jet prolongation is given by $j^k u(x) = (x^i, \partial_I u^a(x))$, while the l-prolongation is given by $p_l f[u](x) = (x^i, \partial_I f^b[u](x))$, where $f[u](x) = (x^i, f^b[u](x))$ in fiber-adapted local coordinates (x^i, v^b) on F'. For any $l > k$, discarding the information about all derivatives of order $> k$ defines a natural projection $J^lF \to J^kF$. The projective limit $J^\infty F := \varprojlim_{k\to\infty} J^kF$ defines the *∞-jet bundle*. The ∞-jet prolongation j^∞ and ∞-prolongation p_∞ are defined in the obvious way. By composing with the natural projection $J^\infty F \to J^kF$, the differential operator f also canonically defines the smooth bundle map $f : J^\infty F \to J^kF \xrightarrow{f} F'$, which is again denoted by the same symbol f. Conversely, due to the projective limit construction, any smooth bundle map $f : J^\infty F \to F'$ can only depend on finitely many coordinates of its domain, which means that there exists a $k \geq 0$ such that this bundle map canonically factors as $f : J^\infty F \to J^kF \xrightarrow{f} F'$, with the smallest such k being the *order* of f.

Given vector bundles $F \to M$, $E \to M$ and a differential operator $e : \Gamma(F) \to \Gamma(E)$, we write down the *partial differential equation* (PDE) $e[\psi] = 0$, with $\psi \in \Gamma(F)$. Sometimes it is convenient to refer to $F \to M$ as the *field bundle* and to $E \to M$ as the *equation bundle*. We will only consider linear PDEs below, where the differential operator e is linear. We denote the local spaces of solutions by $\mathcal{S}_e(U)$, where $U \subseteq M$ is open and $\psi \in \Gamma(F|_U)$ belongs to $\mathcal{S}_e(U)$ iff $e[\psi] = 0$ on U. The PDE is said to be of order k if it can be written as $e[\psi] = e(j^k\psi)$, where on the right-hand side we have a (linear) bundle map $e : J^kF \to E$.

In adapted coordinates (x^i, u^a) on F, the PDE $e[\psi] = 0$ has the form $e^I(x)\partial_I\psi(x) = 0$. When the PDE is of order k, the coefficients $e^I(x)$ vanish for multi-indices with $|I| > k$. The coefficients of the highest order derivatives, $e^I(x)$ with $|I| = k$, in fact transform as a tensor under coordinate changes and define a linear bundle map $\sigma e \colon F \otimes S^k T^*M \to E$ called the *principal symbol* of e. If we fix $(x, p) \in T^*M$, then the corresponding linear map $\sigma_{x,p} e = \sigma e(x) \cdot p^{\otimes k} \colon F_x M \to E_x M$ can also be referred to as the value of the principal symbol of e at (x, p).

The PDE $e[\psi] = 0$ is *equivalent* to the PDE $e'[\psi'] = 0$, with $e' \colon J^{k'} F' \to E'$, if they have isomorphic solution spaces. That is $e[\psi] = 0$ implies that $e'[f[\psi]] = 0$ and $e'[\psi'] = 0$ implies that $e[f'[\psi']] = 0$, for some differential operators f and f'. In fact, it can be shown that the two PDEs are equivalent precisely when they fit into the following diagram, where arrows are differential operators and the bundle labels stand in for the corresponding spaces of sections,

$$\begin{tikzcd} F \arrow[r, "e"'] \arrow[d, shift left, "f"] & E \arrow[l, dashed, bend right, "q"'] \arrow[d, shift left, "g"] \\ F' \arrow[r, "e'"] \arrow[u, shift left, "f'"] & E' \arrow[l, dashed, bend left, "q'"] \arrow[u, shift left, "g'"] \end{tikzcd}, \tag{146}$$

where differential operators satisfy the following identities:

$$e' \circ f = g \circ e, \qquad f' \circ f = \mathrm{id} + q \circ e, \tag{147}$$

$$e \circ f' = g' \circ e', \qquad f \circ f' = \mathrm{id} + q' \circ e'. \tag{148}$$

The reason we can express equivalence in this way, at least when all the differential operators are linear, follows from linear algebra on jets. If we replace the operators e, e', f and f' by the corresponding jet bundle maps, prolonged to the appropriate order, it follows from basic linear algebra that there exist jet bundle maps that ostensibly correspond to the operators g, g', q and q'. It then follows from a deeper analysis of the properties of linear PDEs [16,44,90] that once these differential operators are defined using prolongations of sufficiently high order, the appropriate identities hold at all higher orders. As a simple example, note that the equation $e[\psi] = 0$ and its prolongation $p_k e[\psi] = 0$ are equivalent, with $f = f' = \mathrm{id}$.

Consider vector bundles $E, F, G \to M$ and linear differential operators

$$f \colon \Gamma(G) \to \Gamma(F) \quad \text{and} \quad e \colon \Gamma(F) \to \Gamma(E), \tag{149}$$

of respective orders k and l, such that $e \circ f = 0$. We say that the composition of e and f is *formally exact* if the composition $p^{k+m} e \circ p^m f$ of jet bundle maps is exact in the usual linear algebra sense (the image of $p^m f$ is equal to the kernel of $p^{k+m} e$). Formal exactness is a powerful hypothesis. For instance, it implies that certain differential operators factorize through either e or f [16,11]. Namely, if g is any differential operator such that $g \circ f = 0$, then there must exist another differential operator g' such that $g = g' \circ e$. Similarly, if g is any differential operator such that $e \circ g = 0$, then there must exist another differential operator g' such that $g = f \circ g'$.

Appendix D. Deformations of flat principal bundles

The material below requires some familiarity with the theory of G-principal bundles [91,92,46,93]. Its main point is to show how one can reduce the computation of the degree-1 cohomology space of a certain locally constant sheaf on a manifold M to the computation of the degree-1 group cohomology of the fundamental group $\pi = \pi_1(M)$ with coefficients in a certain corresponding representation. This reformulation is a significant simplification because group cohomology calculations can often be reduced to finite dimensional linear algebra and many explicit calculations of that sort have already been performed and are available in the literature. The connection between these sheaf and group cohomologies is established by noticing that both of them describe equivalence classes of infinitesimal deformations of flat principal bundles. Unfortunately, this argument is not sufficient to establish an isomorphism between these sheaf and group cohomologies in higher degrees, but degree-1 is already interesting because it is the one relevant in the physical application we have in mind (Section 5).

We briefly recall some basic facts about principal G-bundles [91,92,46,93]. The total space of the *principal bundle* $P \to M$ has fibers that are right principal homogeneous spaces of the group G. A *right principal homogeneous space* is defined by the possession a free, transitive action of G. Thus, any principal homogeneous space is diffeomorphic to the manifold underlying the Lie group G and, if any particular point is identified with the unit element of G, the action of G coincides with right-multiplication. The fiber-wise right action of G on P allows us to construct so-called *associated bundles*. If F is a left G-space, with action $\rho \colon G \to \mathrm{Aut}(F)$, then we define the corresponding associated bundle, denoted sometimes F_ρ or F_P, as $P \times_\rho F \cong (P \times F)/G$, where the quotient identifies the points $(pg, f) = (p, gf)$, $p \in P, f \in F, g \in G$. In particular, we can define the associated bundles $G_P = P \times_{\mathrm{Ad}} G$ and $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$, where Ad denotes respectively the adjoint action of the Lie group on itself and its Lie algebra, $\mathrm{Ad}(b)a = bab^{-1}$ and $\mathrm{Ad}(b)\alpha = b\alpha b^{-1}$, with $a, b \in G$ and $\alpha \in \mathfrak{g}$. When convenient and for simplicity of notation, we shall implicitly treat Lie group and Lie algebra elements as if they were faithfully represented as matrices.

The principal G-bundle $P \to M$ is called *flat* when it is endowed with a flat connection or a notion of flat parallel transport, which are compatible with the structure group action. The details of these notions are discussed in the next subsections. The arguments presented therein roughly establish the following

Proposition 17. *Let $P \to M$ be a flat principal G-bundle and $\pi = \pi(M)$ be the fundamental group of M. We can define the following structures associated to it:* (a) *the sheaf $\mathcal{F}_\mathfrak{g}$ of locally flat sections of the associated bundle $\mathfrak{g}_P \to M$,* (b) *the twisted de Rham complex $(\Lambda^\bullet M \otimes \mathfrak{g}_P, D)$, and* (c) *the monodromy representation $\rho\colon \pi \to G$. Then the following cohomology groups (respectively the sheaf, twisted de Rham and group cohomologies) are all isomorphic, by reason of each being isomorphic to the space of equivalence classes of infinitesimal deformations of the flat principal G-bundle structure of $P \to M$:*

$$H^1(M, \mathcal{F}_\mathfrak{g}) \cong H^1(\Lambda^\bullet M \otimes \mathfrak{g}_P, D) \cong H^1(\pi, \mathrm{Ad}_\rho). \tag{150}$$

We defer to the standard references [92,46,93] for detailed proofs.

D.1. Flat principle bundle cocycle

There are multiple ways to construct a principal G-bundle over a manifold M. The one that will be important for us here defines also a bit more structure than principal bundle itself, it also defines a flat connection thereon. We shall refer to these structures as *flat principal G-bundles*. It is well known that this data can be specified as follows. Let $\mathcal{U} = (U_i)$ be an open cover of M and $(U, V) \mapsto t_{U,V} \in G$ an assignment of a structure group element to every ordered pair of opens $U, V \in \mathcal{U}$. Each $t_{U,V}$ is called a *transition map*. The transition maps define a principle G-bundle with a flat connection if they satisfy the following *cocycle identities*,

$$t_{U,V} t_{V,U} = \mathrm{id}, \tag{151}$$

$$t_{U,V} t_{V,W} t_{W,U} = \mathrm{id}. \tag{152}$$

A change of trivialization is an assignment $U \mapsto a_U \in G$ for every open $U \in \mathcal{U}$. The modified transition functions $t'_{U,V} = a_U t_{U,V} a_V^{-1}$ define an equivalent flat principal G-bundle.

Next, we describe infinitesimal deformations of a flat bundle cocycle $t_{U,V}$. Namely, suppose that $t_{U,V}(s)$ is a smooth 1-parameter family of flat bundle cocycles, with $t_{U,V}(0) = t_{U,V}$. Let us denote the derivative at $s = 0$ as $\dot{t}_{U,V} = \tau_{U,V} t_{U,V}$, with $\tau_{U,V} \in \mathfrak{g}$. Then, the defining relations (151) and (152) impose the following constraints on the infinitesimal deformation $\tau_{U,V}$:

$$\tau_{U,V} = -t_{V,U}^{-1} \tau_{V,U} t_{V,U}, \tag{153}$$

$$\tau_{U,V} + t_{U,V} \tau_{V,W} t_{U,V}^{-1} - \tau_{W,U} = 0. \tag{154}$$

On the other hand, suppose that $a_U(s)$ is a smooth 1-parameter family of trivialization changes, with $a_U(0) = \mathrm{id}$. Let us write the derivative at $s = 0$ as $\dot{a}_U = -\sigma_U$. The induced infinitesimal deformation in the transition functions $t_{U,V}(s) = a_U(s) t_{U,V} a_V^{-1}$ is

$$\tau_{U,V} = -\sigma_U + t_{U,V} \sigma_V t_{U,V}^{-1}. \tag{155}$$

The point of the above calculations is to show that infinitesimal deformations of the flat principal bundle cocycle, up to infinitesimal trivialization changes, correspond precisely to the cohomology classes of a certain sheaf. To complete the argument, we need only introduce the basic definitions of Čech cohomology, which is known to compute the cohomology vector spaces of a corresponding sheaf [47,48]. We will take the sheaf to be $\mathcal{F}_\mathfrak{g}$, where $\mathcal{F}_\mathfrak{g}(U)$ consists of the locally flat sections of the bundle $\mathfrak{g}_P \to M$, associated to the flat principal G-bundle $P \to M$. Let us now fix an open cover $\mathcal{U} = (U_i)$ of M such that each U_i is contractible and any multiple intersection of the U_i is also contractible. On a manifold, any open cover can be refined to such a *good cover* [49, Thm.5.1]. In particular, the flat principal bundle cocycle can always be refined to a good cover. The good cover hypothesis ensures that the Čech cohomology spaces are in fact isomorphic to the actual sheaf cohomologies.

We define a Čech q-cochain σ as an assignment $(U_{i_1}, \dots, U_{i_{q+1}}) \mapsto \sigma_{i_1 \cdots i_{q+1}} \in \mathcal{F}_\mathfrak{g}(U_i \cap \cdots \cap U_{i_{q+1}})$ to every ordered $(q + 1)$-tuple of opens from $\mathcal{U}$. By local flatness, for any $U \in \mathcal{U}$, $\mathcal{F}_\mathfrak{g}(U) \cong \bar{F}_\mathfrak{g} \cong \mathfrak{g}$ (cf. Section 3.1). It is convenient to think of a Čech cocycle $\sigma_{i_1 \cdots i_{q+1}}$ as taking values in $\mathfrak{g} \cong \mathcal{F}_\mathfrak{g}(U_i)$. This means that $\sigma_{i\dots}$ and $\sigma_{j\dots}$ restrict to the same element of $\mathcal{F}_\mathfrak{g}(U_i \cap U_j \cap \cdots)$ only if $\sigma_{i\dots} = \mathrm{Ad}(t_{U_i,V_i})\sigma_{j\dots} = (t_{U_i,V_j})\sigma_{j\dots}(t_{U_i,V_j}^{-1})$. We shall only need the Čech differential to be defined on 0- and 1-cochains:

$$\begin{aligned}(\delta\sigma)_{ij} &= \sigma_j|_{U_i \cap U_j} - \sigma_i|_{U_i \cap U_j} \\ &= \mathrm{Ad}(t_{U_i,U_j})\sigma_j - \sigma_i \\ &= t_{U_i,U_j} \sigma_j t_{U_i,U_j}^{-1} - \sigma_i,\end{aligned} \tag{156}$$

$$\begin{aligned}(\delta\tau)_{ijk} &= \tau_{jk}|_{U_i \cap U_j \cap U_k} - \tau_{ik}|_{U_i \cap U_j \cap U_k} + \tau_{ij}|_{U_i \cap U_j \cap U_k} \\ &= \mathrm{Ad}(t_{U_i,U_j})\tau_{jk} - \tau_{ik} + \tau_{ij} \\ &= t_{U_i,U_j} \tau_{jk} t_{U_i,U_j}^{-1} - \tau_{ik} + \tau_{ij}.\end{aligned} \tag{157}$$

The space of closed Čech q-cocycles modulo the Čech coboundaries then is isomorphic to the sheaf cohomology group in degree q, which in our case is $H^q(\mathcal{F}_{\mathfrak{g}})$.

It should now be clear, from Eqs. (154) and (155), that the infinitesimal deformation of the flat bundle cocycle defines a Čech 1-cocycle $\tau_{ij} = \tau_{U_i,U_j}$ and an infinitesimal change in trivialization defines a Čech coboundary $\tau_{ij} = (\delta\sigma)_{ij}$, with $\sigma_i = \sigma_{U_i}$.

D.2. Flat connection on a principal bundle

Another, ultimately equivalent, way to specify a principal bundle with a flat connection is as follows.

A *principal G-connection* on a principal G-bundle P is a $\mathfrak{g}$-valued 1-form ω on the total space P (an element of $\Omega^1(P)\otimes\mathfrak{g}$) such that (i) ω is Ad-equivariant ($R_a^*\omega = \mathrm{Ad}(a^{-1})\omega$, where $R_a\colon P\to P$ is the action of $a\in G$ on P by right multiplication) and (ii) $\omega(\beta)=\beta$ for any vertical vector $\beta\in TP$. Recall that vertical vectors are those annihilated by the tangent map of the projection $P\to M$ and that the vertical subspace of TP at any point of P may be naturally identified with $\mathfrak{g}$, which we have used in the preceding definition. The defining condition on the form ω is clearly linear inhomogeneous. Thus, the space of all principal G-connections forms an affine subspace of $\Omega^1(P)\otimes\mathfrak{g}$. So, the difference $A=\omega'-\omega$ between any two principal connections belongs to the subspace of $\Omega^1(P)\otimes\mathfrak{g}$ that is Ad-equivariant and horizontal (annihilates vertical vectors). This subspace is in fact isomorphic, by pullback along the projection $P\to P/G\cong M$, to the space of sections $\Gamma(\Lambda^1M\otimes\mathfrak{g}_P)$ of the associated bundle $\Lambda^1M\otimes\mathfrak{g}_P\to M$. In fact, we can identify the spaces of sections $\Gamma(\Lambda^pM\otimes\mathfrak{g}_P)$ with the Ad-equivariant, horizontal subspaces of $\Omega^p(P)\otimes\mathfrak{g}$. The first order differential operator $DA=\mathrm{d}A+[\omega\wedge A]$ (see below for notation) preserves these subspaces and hence can be projected down to a first order differential operator $D\colon \Gamma(\Lambda^pM\otimes\mathfrak{g}_P)\to\Gamma(\Lambda^{p+1}M\otimes\mathfrak{g}_P)$, which we shall refer to as the *twisted differential* (cf. Section 2.4).

The curvature Ω of a principal G-connection ω is defined to be the following $\mathfrak{g}$-valued 2-form on P:

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega\wedge\omega], \tag{158}$$

where the bracketed wedge product is definite to satisfy $[(\lambda\otimes\alpha)\wedge(\mu\otimes\beta)] = \lambda\wedge\mu\otimes[\alpha,\beta]$ for any $\lambda,\mu\in\Omega^1(P)$ and $\alpha,\beta\in\mathfrak{g}$. Since Ω is Ad-equivariant and horizontal, we can equally write $\Omega\in\Gamma(\Lambda^2M\otimes\mathfrak{g}_P)$. The twisted differential D is not nilpotent, $D^2\neq 0$. However, its square is $C^\infty(M)$-linear and so is a differential operator of order 0. In fact, we can compute it to be

$$D^2A = [\Omega\wedge A], \tag{159}$$

for any $A\in\Gamma(\Lambda^pM\otimes\mathfrak{g}_P)$. If $\Omega=0$, then the connection is said to be *flat*. This is a sufficient condition for the twisted differential to become nilpotent, $D^2=0$. A necessary and sufficient condition would simply be that the curvature Ω takes values in the center of $\mathfrak{g}$, upon local trivialization of $\mathfrak{g}_P$.

Given any two flat connections ω and ω', their difference can be represented by a section $\omega'-\omega = A\in\Gamma(\Lambda^1M\otimes\mathfrak{g}_P)$ (or rather its pullback to P) that necessarily satisfies the following equation:

$$0 = \mathrm{d}\omega' + \frac{1}{2}[\omega'\wedge\omega'] \tag{160}$$

$$= \mathrm{d}(\omega+A) + \frac{1}{2}[(\omega+A)\wedge(\omega+A)] \tag{161}$$

$$= \mathrm{d}A + [\omega\wedge A] + \frac{1}{2}[A\wedge A] \tag{162}$$

$$= DA + \frac{1}{2}[A\wedge A]. \tag{163}$$

Where the last expression can be interpreted as computed on M rather than on P. Equating this last expression to zero gives a differential equation on sections $A\in\Gamma(\Lambda^1M\otimes\mathfrak{g}_P)$ identifying those that parametrize the space of flat principal G-connections (relative to ω, which defines the twisted differential D).

An automorphism of a principal G-bundle $P\to M$ is a bundle map $f\colon P\to P$ that covers the identity on M and is equivariant with respect to the right action of G [94,93]. It is a standard fact that such maps can be expressed as functions $a_f\colon P\to G$ that are Ad-equivariant (where Ad is the left action of G on G by conjugation) with respect to the right action of G on P. In turn, the set of such maps is in bijection with the space of sections of the associated bundle $G_P = P\times_{\mathrm{Ad}} G$. Since the map $f\colon P\to P$ is an automorphism, the pullback connection $f^*\omega$ is considered equivalent to the original one. Given its equivariance, the map f corresponds to a section $a_f\in\Gamma(G_P)$. Equivalently, given a section $a\in\Gamma(G_P)$, we can define the corresponding automorphism map $f_a\colon P\to P$. It is not hard to compute that

$$f_a^*\omega = \mathrm{Ad}(a^{-1})\omega + a^{-1}\mathrm{d}a. \tag{164}$$

Naturally, the pullback $f^*\omega$ of a flat connection ω is also a flat connection.

Next, we describe infinitesimal deformations of a flat principle G-connection. Given a flat connection described by a $\mathfrak{g}$-valued 1-form ω on P, any smooth 1-parameter family of principal connection can be written as $\omega+A(s)$, with $A(0)=0$, where $A(s)$ can, for fixed s, be considered as a section of the associated bundle $\Lambda^1 M \otimes \mathfrak{g}_P \to M$. This family consists of flat connections if and only if the equation $DA(s) + \frac{1}{2}[A(s) \wedge A(s)] = 0$ is satisfied, where D is the twisted differential defined by ω. If $\dot{A}(0) = A$, then the preceding identity imposes the condition $DA = 0$ on this infinitesimal deformation. Also, if $a(s) \in \Gamma(G_P)$, with $a(0) = \mathrm{id}$ and $\dot{a}(0) = \alpha \in \Gamma(\mathfrak{g}_P)$, defines a smooth 1-parameter family of automorphisms $f_{a(s)} \colon P \to P$, then the corresponding infinitesimal deformation of the original flat connection is equal to $A = \mathrm{d}\alpha - [\alpha, \omega] = D\alpha$.

It should now be clear that infinitesimal deformations of a given flat principal G-connection, up to infinitesimal automorphisms of the underlying principal G-bundle, are in bijections with the cohomology vector space $H^1(\Lambda^\bullet \otimes \mathfrak{g}_P, D)$ of the twisted de Rham complex defined by the original flat connection.

D.3. Monodromy representation

A connection, in the sense of Ehresemann, can be defined as a splitting of the tangent space of P into $T_{x,a}P \cong T_x M \oplus \mathfrak{g}$, for $(x, a) \in P$, with the $\mathfrak{g}$ summand canonically identified with the subspace of vertical vectors, such that the splitting is smooth in x and equivariant in a. The $T_x M$ summand is called the *horizontal subspace* of $T_{x,a}P$. This formulation leads naturally to the idea of parallel transport. Given a point $(x, a) \in P$ and a smooth curve $\gamma \colon [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = 1$, there exists a unique lift $\tilde{\gamma}$ of γ to P such that $\tilde{\gamma}(0) = (x, a)$ and the tangent vector $\dot{\tilde{\gamma}}$ is always horizontal. With x and y fixed, the endpoint $(y, b) = \tilde{\gamma}(1)$ defines b as the image of a parallel transported along γ. Since the splitting of TP is equivariant with respect to the right action of G on P, so is parallel transport. It is easy to see that parallel transport does not depend on the parametrization of γ, is well defined also when γ is piecewise smooth, and respects concatenation, $a_{\gamma\eta} = a_\gamma a_\eta$ for $\gamma(1) = \eta(0)$ and $\gamma\eta$ being the concatenated curve. In particular, if γ is a closed curve based at $x \in M$ ($\gamma(0) = \gamma(1) = x$), then the effect of parallel transport is equivalent to the right action on the fiber P_x by some element $a_\gamma \in G$.

Let $\pi = \pi_1(M, x)$ be the fundamental group of M based at some point x. The connection splitting of TP is called *flat* when the parallel transport along any closed contractible curve γ is trivial, $a_\gamma = \mathrm{id}$, and thus the group element a_γ effecting parallel transport along a closed curve γ based at x depends only on its homotopy type $[\gamma] \in \pi$. In other words, parallel transport defines a homomorphism $\rho \colon \pi \to G$, $\rho([\gamma]) = a_\gamma$, which we call the *monodromy representation* of the fundamental group of M in the structure group of $P \to M$ (cf. the introduction to Section 4).

Thus, any flat principal G-bundle gives rise to a representation $\rho \colon \pi \to G$. Two isomorphic flat principal bundles give rise to equivalent monodromy representations, where two representations ρ' and ρ are equivalent if there exists an element $a \in G$ such that $\rho'([\gamma]) = a\rho([\gamma])a^{-1}$. Conversely, any homomorphism $\rho \colon \pi \to G$ allows us to construct a flat principal G-bundle with a monodromy representation equivalent to ρ. Namely, consider the universal cover $\tilde{M} \to M$ as a principal π-bundle and define the total space of the corresponding principal G-bundle as $P = \tilde{M} \times_\rho G$. A flat connection can be defined on the trivial principal G-bundle $\tilde{M} \times G$ using the construction of Appendix D.1 applied to a cover by contractible open sets and transition maps defined by ρ. This flat connection then projects down to P.

Next, we describe infinitesimal deformations of a fixed monodromy representation ρ. Let $\rho_s \colon \pi \to G$ be a smooth 1-parameter family of monodromy representations, with $\rho(s) = \rho$ and $\dot{\rho}_s(a) = \tau(s)\rho(a)$ for some $\tau \colon \pi \to \mathfrak{g}$. The representation property $\rho_s([\gamma][\eta]) = \rho_s([\gamma])\rho_s([\eta])$ imposes the following constraint on the infinitesimal deformation:

$$\tau([\gamma]) + \rho([\gamma])\tau([\eta])\rho([\gamma])^{-1} - \tau([\gamma][\eta]) = 0. \tag{165}$$

A family of trivial deformations is given by $\rho_s([\gamma]) = a_s\rho([\gamma])a_s^{-1}$ for a smooth 1-parameter family $a_s \in G$, with $a_0 = \mathrm{id}$ and $\dot{a}_0 = -\sigma \in \mathfrak{g}$. The corresponding infinitesimal deformation of the representation is given by

$$\tau([\gamma]) = -\sigma + \rho([\gamma])\sigma\rho([\gamma])^{-1}. \tag{166}$$

The point of the above calculations is to show that these infinitesimal deformations can be identified with certain *group cohomology* classes. To see that, we need to introduce some basic definitions [87, Ch. 6], [68]. Group cohomology is defined given a group and a representation thereof. We will give the definitions by directly taking the group to be π and the representation to be the composite adjoint representation of π on $\mathfrak{g}$, $\mathrm{Ad}_\rho = \mathrm{Ad} \circ \rho \colon \pi \to \mathrm{GL}(\mathfrak{g})$. The vector space $C^p(\pi, \mathrm{Ad}_\rho)$ of p-cochains consists of functions $\sigma \colon \pi^p \to \mathfrak{g}$, where $\pi^p = \pi \times \cdots \times \pi$ is the p-fold product. The cochain differentials $\delta \colon C^p(\pi, \mathrm{Ad}_\rho) \to C^{p+1}(\pi, \mathrm{Ad}_\rho)$ are defined by the formula

$$\begin{aligned}\delta\sigma([\gamma_1], \ldots, [\gamma_{p+1}]) = (-1)^{p+1}\sigma([\gamma_1], \ldots, [\gamma_p]) + \mathrm{Ad}_\rho([\gamma_1])\sigma([\gamma_2], \ldots, [\gamma_{p+1}]) \\ + \sum_{q=1}^{p} (-1)^q \sigma(\ldots, [\gamma_q][\gamma_{q+1}], \ldots).\end{aligned} \tag{167}$$

For 0- and 1-cochains, we have the following explicit formulas:

$$\begin{aligned}\delta\sigma([\gamma]) &= -\sigma + \mathrm{Ad}_\rho([\gamma])\sigma \\ &= -\sigma + \rho([\gamma])\sigma\rho([\gamma])^{-1},\end{aligned} \tag{168}$$

$$\begin{aligned}\delta\tau([\gamma], [\eta]) &= \tau([\gamma]) + \mathrm{Ad}_\rho([\gamma])\tau([\eta]) - \tau([\gamma][\eta]) \\ &= \tau([\gamma]) + \rho([\gamma])\tau([\eta])\rho([\gamma])^{-1} - \tau([\gamma][\eta]).\end{aligned} \tag{169}$$

It is worth noting that the degree-0 group cohomology is isomorphic to the subspace of the representation on which the group acts trivially, $H^0(\pi, \mathrm{Ad}_\rho) \cong \mathfrak{g}^\pi$.

It should now be clear from Eqs. (165) and (166) that infinitesimal deformations of a monodromy representations $\rho\colon \pi \to G$, up to deformations by conjugation, are in bijection with the group cohomology $H^1(\pi, \mathrm{Ad}_\rho)$ of the group π with coefficients in the composite adjoint representation of π on $\mathfrak{g}$.

References

[1] E. Calabi, On compact, Riemannian manifolds with constant curvature. I, in: C.B. Allendoerfer (Ed.), Differential Geometry, in: Proceedings of Symposia in Pure Mathematics, vol. 3, AMS, Providence, RI, 1961, pp. 155–180.
[2] J.-F. Pommaret, Lie Pseudogroups and Mechanics, in: Mathematics and its Applications, vol. 16, Gordon and Breach, New York, 1988.
[3] M. Eastwood, Variations on the de Rham complex, Notices Amer. Math. Soc. 46 (1999) 1368–1376. http://www.ams.org/notices/199911/fea-eastwood.pdf.
[4] C. Amrouche, P.G. Ciarlet, L. Gratie, S. Kesavan, On Saint Venant's compatibility conditions and Poincaré's lemma, C. R. Math. 342 (2006) 887–891.
[5] L.B. Bergery, J.P. Bourguignon, J. Lafontaine, Déformations localement triviales des variétés riemanniennes, in: Differential Geometry, Part 1, in: Proceedings of Symposia in Pure Mathematics, vol. 27, AMS, Providence, RI, 1975, pp. 3–32.
[6] A. Weil, On discrete subgroups of Lie groups (II), Ann. Mat. 75 (1962) 578–602.
[7] M. Berger, Encounter with a geometer: Eugenio Calabi, in: P. de Bartolomeis, F. Tricerri, and E. Vesentini (Eds.), in: Manifolds and Geometry, Symposia Mathematica, vol. 26, Cambridge, 1996, pp. 20–60.
[8] H. Goldschmidt, Duality theorems in deformation theory, Trans. Amer. Math. Soc. 292 (1985) 1.
[9] J. Gasqui, H. Goldschmidt, Déformations infinitésimales des espaces riemanniens localement symétriques. I, Adv. Math. 48 (1983) 205–285.
[10] J. Gasqui, H. Goldschmidt, Complexes of differential operators and symmetric spaces, in: M. Hazewinkel, M. Gerstenhaber (Eds.), Deformation Theory of Algebras and Structures and Applications, in: NATO ASI Series, vol. 247, Kluwer, Dordrecht, 1988, pp. 797–827.
[11] J.-F. Pommaret, Systems of partial differential equations and Lie pseudogroups, in: Mathematics and Its Applications, vol. 14, Gordon and Breach, New York, 1978.
[12] D.C. Spencer, Deformation of structures on manifolds defined by transitive, continuous pseudogroups Part I: Infinitesimal deformations of structure, Ann. Mat. 76 (1962) 306–398.
[13] D.C. Spencer, Deformation of structures on manifolds defined by transitive, continuous pseudogroups Part II: Deformations of structure, Ann. Mat. 76 (1962) 399–445.
[14] D.C. Spencer, Overdetermined systems of linear partial differential equations, Bull. Amer. Math. Soc. 75 (1969) 179–240.
[15] D.G. Quillen, Formal properties of over-determined systems of partial differential equations (Ph.D. thesis), Harvard University, 1964.
[16] H. Goldschmidt, Existence theorems for analytic linear partial differential equations, Ann. Mat. 86 (1967) 246–270.
[17] I. Khavkine, Characteristics, conal geometry and causality in locally covariant field theory. 2012. arXiv:1211.1914.
[18] I. Khavkine, Covariant phase space, constraints, gauge and the Peierls formula, Internat. J. Modern Phys. A 29 (2014) 1430009. arXiv:1402.1282.
[19] C. Dappiaggi, B. Lang, Quantization of Maxwell's equations on curved backgrounds and general local covariance, Lett. Math. Phys. 101 (2012) 265–287. arXiv:1104.1374.
[20] K. Sanders, C. Dappiaggi, T.-P. Hack, Electromagnetism, local covariance, the Aharonov-Bohm effect and Gauss' law, Comm. Math. Phys. 328 (2014) 625–667. arXiv:1211.6420.
[21] M. Benini, C. Dappiaggi, A. Schenkel, Quantized Abelian principal connections on Lorentzian manifolds, Comm. Math. Phys. 330 (2013) 123–152. arXiv:1303.2515.
[22] C.J. Fewster, D.S. Hunt, Quantization of linearized gravity in cosmological vacuum spacetimes, Rev. Math. Phys. 25 (2013) 1330003. arXiv:1203.0261.
[23] C.J. Fewster, B. Lang, Dynamical locality of the free Maxwell field, Ann. Henri Poincaré 17 (2015) 401–436. http://dx.doi.org/10.1007/s00023-015-0398-9, arXiv:1403.7083.
[24] M. Benini, Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies. 2014. arXiv:1401.7563.
[25] T.-P. Hack, Quantization of the linearized Einstein–Klein–Gordon system on arbitrary backgrounds and the special case of perturbations in inflation, Classical Quantum Gravity 31 (2014) 215004. http://dx.doi.org/10.1088/0264-9381/31/21/215004, arXiv:1403.3957.
[26] C. Becker, A. Schenkel, R.J. Szabo, "Differential cohomology and locally covariant quantum field theory". 2014. arXiv:1406.1514.
[27] N.N. Tarkhanov, Complexes of Differential Operators, in: Mathematics and Its Applications, vol. 340, Kluwer, Dordrecht, 1995.
[28] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Differential operators on the base affine space and a study of $\mathfrak{g}$-modules, in: I.M. Gelfand (Ed.), Lie Groups and Their Representations: Summer School of the Bolyai János Mathematical Society, Adam Hilger, Halsted, NY, 1975, pp. 21–64.
[29] W. Fulton, Young Tableaux: With Applications to Representation Theory and Geometry, in: London Mathematical Society Student Texts, vol. 35, Cambridge University Press, New York, 1997.
[30] R.M. Wald, General Relativity, University of Chicago Press, Chicago, 1984.
[31] G. Pérez-Nadal, A. Roura, E. Verdaguer, Stress tensor fluctuations in de Sitter spacetime, J. Cosmol. Astropart. Phys. 2010 (2010) 036. arXiv:0911.4870.
[32] J.M. Stewart, M. Walker, Perturbations of space-times in general relativity, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 341 (1974) 49–74.
[33] M. Benini, C. Dappiaggi, S. Murro, Radiative observables for linearized gravity on asymptotically flat spacetimes and their boundary induced states, J. Math. Phys. 55 (2014) 082301. http://dx.doi.org/10.1063/1.4890581, arXiv:1404.4551.
[34] M. Ryan, Teukolsky equation and Penrose wave equation, Phys. Rev. D 10 (1974) 1736–1740.
[35] A. Lichnerowicz, Propagateurs, commutateurs et anticommutateurs en relativité générale, in: C. DeWitt, B.S. DeWitt (Eds.), Relativity, Groups and Topology, Gordon and Breach, New York, 1964, pp. 821–861.
[36] D. Christodoulou, S. Klainerman, The global nonlinear stability of the Minkowski space, in: Princeton Mathematical Series, vol. 41, Princeton University Press, Princeton, NJ, 1993.
[37] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, in: Physics Series, W. H. Freeman, San Francisco, 1973.
[38] D. Bini, C. Cherubini, R.T. Jantzen, R. Ruffini, de Rham wave equation for tensor valued p-forms, Internat. J. Modern Phys. D 12 (2003) 1363–1384.
[39] I.M. Anderson, Introduction to the variational bicomplex, in: M.J. Gotay, J.E. Marsden, V. Moncrief (Eds.), Mathematical Aspects of Classical Field Theory, in: Contemporary Mathematics, vol. 132, American Mathematical Society, Providence, Rhode Island, 1992, pp. 51–73.
[40] I.M. Anderson, The variational bicomplex, Unpublished draft, 1989.
[41] Wikipedia, Levi-Civita symbol — Wikipedia, The Free Encyclopedia http://en.wikipedia.org/w/index.php?title=Levi-Civita_symbol&oldid=607146551, 2014 (Online; accessed 16.07.14).
[42] M.A.A. van Leeuwen, Littlewood-Richardson rule. http://www-math.univ-poitiers.fr/~maavl/LiE/form.html. Online calculator based on [95].
[43] S.E. Stepanov, The Killing-Yano tensor, Theoret. Math. Phys. 134 (2003) 333–338.
[44] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths, Exterior Differential Systems, in: Mathematical Sciences Research Institute Publications, vol. 18, Springer, 2011.
[45] S. Lang, Differential and Riemannian Manifolds, in: Graduate Texts in Mathematics, vol. 160, Springer, 1995.
[46] S. Morita, Geometry of Characteristic Classes, in: Translations of Mathematical Monographs, vol. 199, American Mathematical Society, 2001.
[47] G.E. Bredon, Sheaf theory, in: Graduate Texts in Mathematics, Springer, New York, 1997.

[48] M. Kashiwara, P. Schapira, Sheaves on Manifolds, in: Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 292, Springer, 1990.
[49] R. Bott, L.W. Tu, Differential Forms in Algebraic Topology, in: Graduate Texts in Mathematics, vol. 82, Springer, New York, 1982.
[50] L. Hörmander, The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis, in: Grundlehren Der Mathematischen Wissenschaften, vol. 256, Springer, 2012.
[51] R.T. Smith, Examples of elliptic complexes, Bull. Amer. Math. Soc. 82 (1976) 297–299.
[52] A.A. Shlapunov, N.N. Tarkhanov, A homotopy operator for Spencer's sequence in the C^∞-case, Sib. Adv. Math. 19 (2009) 91–127.
[53] L. Hörmander, The Analysis of Linear Partial Differential Operators III, in: Grundlehren Der Mathematischen Wissenschaften, vol. 274, Springer, 1985.
[54] M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. 1, third ed., Publish or Perish, 1999.
[55] W. Greub, S. Halperin, K. Vanstone, Connections, Curvature, and Cohomology. Vol. I: De Rham Cohomology of Manifolds and Vector Bundles, in: Pure and Applied Mathematics, vol. 47-I, Academic Press, 1972.
[56] J.-P. Serre, Un théorème de dualité, Comment. Math. Helv. 29 (1955) 9–26.
[57] L. Schwartz, Théorie des Distributions, Vols. 1–2, Hermann, Paris, 1951.
[58] F. Treves, Topological Vector Spaces, Distributions and Kernels, in: Dover Books on Mathematics, Dover Publications, 2013.
[59] M. Reed, B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1981.
[60] Wikipedia, Maschke's theorem — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=Maschke, 2014 (Online; accessed 29.07.14).
[61] Wikipedia, Schur's lemma — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=Schur, 2014 (Online; accessed 29.07.14).
[62] I. Khavkine, Topology, rigid cosymmetries and linearization instability in higher gauge theories, Ann. Henri Poincaré 16 (2015) 255–288. http://dx.doi.org/10.1007/s00023-014-0321-9, arXiv:1303.2406.
[63] J.A. Wolf, Spaces of Constant Curvature, sixth ed., in: AMS-Chelsea, vol. 372, American Mathematical Society, 2011.
[64] W.M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. Math. 54 (1984) 200–225.
[65] J. Ambjørn, M. Carfora, A. Marzuoli, The Geometry of Dynamical Triangulations, Springer, 1997.
[66] G.W. Whitehead, Elements of Homotopy Theory, in: Graduate Texts in Mathematics, vol. 61, Springer, New York, NY, 2012.
[67] S. Eilenberg, S. MacLane, Relations between homology and homotopy groups of spaces, Ann. Mat. 46 (1945) 480–509.
[68] Wikipedia, Group cohomology — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=Group_cohomology&oldid=611025879, 2014 (Online; accessed 27.07.14).
[69] M.M. Postnikov, Determination of the homology groups of a space by means of the homotopy invariants, Dokl. Akad. Nauk SSSR (N.S.) 76 (1951) 359–362.
[70] M.M. Postnikov, Issledovaniya po gomotopičeskoĭ teorii nepreryvnyh otobraženiĭ. I. Algebraičeskaya teoriya sistem. II. Natural'naya sistema i gomotopičeskiĭ tip, in: Trudy Matematicheskogo Instituta Steklova, vol. 46, Izdatelstvo Akademii Nauk SSSR, Moscow, 1955, http://mi.mathnet.ru/tm1182.
[71] M.M. Postnikov, Investigations in the homotopy theory of continuous mappings. I. The algebraic theory of systems. II. The natural system and homotopy type, in: American Mathematical Society Translations (2), vol. 7, American Mathematical Society, 1957.
[72] S.I. Gelfand, Y. Manin, Methods of Homological Algebra: Springer Monographs in Mathematics, second ed., in: Springer Monographs in Mathematics, Springer, New York, NY, 2003.
[73] *n*Lab, Reduced simplicial set. http://ncatlab.org/nlab/show/reduced+simplicial+set, 2014 (Online; accessed 02.07.14).
[74] M. Forger, S.V. Romero, Covariant Poisson brackets in geometric field theory, Comm. Math. Phys. 256 (2005) 375–410. arXiv:math-ph/0408008.
[75] R. Brunetti, K. Fredenhagen, P.L. Ribeiro, Algebraic structure of classical field theory I: Kinematics and linearized dynamics for real scalar fields. 2012. arXiv:1209.2148.
[76] T.-P. Hack, A. Schenkel, Linear bosonic and fermionic quantum gauge theories on curved spacetimes, Gen. Relativity Gravitation 45 (2013) 877–910. arXiv:1205.3484.
[77] I. Khavkine, Cohomology with causally restricted supports, Ann. Henri Poincaré (2016) http://dx.doi.org/10.1007/s00023-016-0481-x (online first) arXiv:1404.1932.
[78] R. Emparan, H.S. Reall, Black holes in higher dimensions, Living Rev. Relativ. 11 (6) (2008) arXiv:0801.3471.
[79] T. Koike, M. Tanimoto, A. Hosoya, Compact homogeneous universes, J. Math. Phys. 35 (1994) 4855–4888. arXiv:gr-qc/9405052.
[80] M. Tanimoto, T. Koike, A. Hosoya, Dynamics of compact homogeneous universes, J. Math. Phys. 38 (1996) 350–368. arXiv:gr-qc/9604056.
[81] H. Kodama, Canonical structure of locally homogeneous systems on compact closed 3-manifolds of types E^3, Nil and Sol, Progr. Theoret. Phys. 99 (1998) 173–236. arXiv:gr-qc/9705066.
[82] J.J. Ferrando, J.A. Sáez, An intrinsic characterization of the Schwarzschild metric, Classical Quantum Gravity 15 (1998) 1323–1330.
[83] J.J. Ferrando, J.A. Sáez, An intrinsic characterization of the Kerr metric, Classical Quantum Gravity 26 (2009) 075013. arXiv:0812.3310.
[84] B. Coll, J.J. Ferrando, Thermodynamic perfect fluid. its rainich theory, J. Math. Phys. 30 (1989) 2918–2922.
[85] A. Dimca, Sheaves in Topology. Universitext, Springer, Berlin, 2004.
[86] S.A. Fulling, R.C. King, B.G. Wybourne, C.J. Cummins, Normal forms for tensor polynomials. I. The Riemann tensor, Classical Quantum Gravity 9 (1992) 1151.
[87] C.A. Weibel, An introduction to homological algebra, in: Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
[88] P.J. Olver, Applications of Lie groups to differential equations, second ed., in: Graduate Texts in Mathematics, vol. 107, Springer-Verlag, New York, 1993.
[89] I. Kolař, P.W. Michor, J. Slovák, Natural Operations in Differential Geometry, Springer, 1993.
[90] W.M. Seiler, Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra, in: Algorithms and Computation in Mathematics, vol. 24, Springer, 2010.
[91] N.E. Steenrod, The Topology of Fibre Bundles, in: Princeton Mathematical Series, Princeton University Press, 1951.
[92] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Princeton University Press, 1987.
[93] H. Baum, Eichfeldtheorie: Eine Einführung in Die Differentialgeometrie Auf Faserbündeln, Springer, London, 2010.
[94] M.C. Abbati, R. Cirelli, A. Mania', P. Michor, The Lie group of automorphisms of a principle bundle, J. Geom. Phys. 6 (1989) 215–235.
[95] M.A.A. van Leeuwen, A.M. Cohen, B. Lisser, LiE, A Package for Lie Group Computations. Computer Algebra Nederland, Amsterdam, 1992. http://www-math.univ-poitiers.fr/~maavl/LiE/.

Ann. Henri Poincaré *Online First*

DOI 10.1007/s00023-016-0481-x

Annales Henri Poincaré

Cohomology with Causally Restricted Supports

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Abstract. De Rham cohomology with spacelike compact and timelike compact supports has recently been noticed to be of importance for understanding the structure of classical and quantum Maxwell theory on curved spacetimes. Similarly, causally restricted cohomologies of different differential complexes play a similar role in other gauge theories. We introduce a method for computing these causally restricted cohomologies in terms of cohomologies with either compact or unrestricted supports. The calculation exploits the fact that the de Rham–d'Alembert wave operator can be extended to a chain map that is homotopic to zero and that its causal Green function fits into a convenient exact sequence. As a first application, we use the method on the de Rham complex, then also on the Calabi (or Killing–Riemann–Bianchi) complex, which appears in linearized gravity on constant curvature backgrounds. We also discuss applications to other complexes, as well as generalized causal structures and functoriality.

1. Introduction

Recently, a number of works on the structure of classical and quantum field theory on curved spacetimes [5,6,15,19,31,34,36,47] have made use of de Rham cohomology with spacelike compact supports. It appears in the characterizations of the center of Poisson (or quantum) algebra of observables of the Maxwell field and also of the degeneracy of the bilinear pairing between spacelike compactly supported solutions and compactly supported smearing functions (see Proposition 1 for a specific statement). Similar considerations appear in more general field theories [34,36], though involving cohomologies of complexes that are different from the de Rham one. One example is the Calabi complex, which appears in linearized gravity on constant curvature backgrounds [36, Sect. 4.4] (see Proposition 10 for a specific statement). Note that cohomologies with timelike compact supports as well as on-shell cohomologies

Published online: 05 April 2016

(restricted to solution spaces of some particular hyperbolic differential operators) have also appeared in the same contexts. We shall loosely refer to all of these variations as causally restricted cohomologies or cohomologies with causally restricted supports.

It was noticed long ago [1] that non-trivial spacetime topology can influence in a non-trivial way the construction of the classical and quantum field theories. However, these effects had not been systematically investigated until recently. This may explain why neither the standard literature on differential geometry and topology, nor the literature on relativity seem to have considered[1] cohomologies with supports restricted by causal relations (like spacelike or timelike compactness). So, given their growing importance, they deserve independent investigation, which is the subject of this work. We introduce a method that allows us to compute the causally restricted cohomologies of a differential complex, provided that complex is equipped with extra structure similar to that found in Hodge theory [27,33]. The essentials of this method are illustrated on the case of the de Rham complex. Then, other applications and implications are discussed.

In Sect. 2, we briefly outline some well-known geometric properties of the de Rham complex on a Lorentzian spacetime, as well as some basic facts of homological algebra. These properties form the core of our method and are reminiscent of the structure found in Hodge theory. Our method of computing causally restricted cohomologies is then illustrated in Sect. 3 and is used to express the various causally restricted de Rham cohomologies in terms of the standard de Rham cohomologies with unrestricted and compact supports. Section 4 applies the same method to the Calabi differential complex. The Calabi complex plays a role in linearized gravity on a constant curvature background analogous to that of the de Rham complex for Maxwell theory. Its structure is briefly introduced and shown analogous to that highlighted in Sect. 2. Then, in Sect. 4.4, its causally restricted cohomologies are computed in analogy with Sect. 3. Section 5 discusses a few related questions that have appeared in the study of gauge theories in the framework of locally covariant classical and quantum field theory. In particular, Sects. 5.1 and 5.2 deal with the behavior of the causally restricted cohomology groups under changes of causal structure and under embeddings, and Sect. 5.3 briefly describes how the methods applied to the de Rham and Calabi examples could be generalized to other differential complexes that arise in the study of general field theories with constrains and gauge invariance [34,36]. Finally, Sect. 6 concludes with a discussion of our results.

It should be mentioned that results very similar to those in Sect. 3 have been obtained independently in a recent work [5], though by different methods.

[1] A notable exception is [39], which, as a byproduct of a different investigation, computed a few low degree cohomology groups with spacelike compact supports or restricted to solutions of the wave equation, but only on Minkowski space.

Those methods are very specific to the de Rham complex, including its invariance properties under topological homotopies. Such strong invariance properties certainly do not hold for other differential complexes. So it is noteworthy that the content of our Sects. 4 and 5 goes beyond [5] in several directions.

2. Preliminaries

Fix an n-dimensional smooth manifold M ($n \geq 2$) with a Lorentzian metric g such that (M, g) is an oriented, time-oriented space, globally hyperbolic spacetime [4,32,43,52]. Recall that, according to the Geroch splitting theorem, there exists a diffeomorphism $M \cong \mathbb{R} \times \Sigma$ (non-unique, of course) where the corresponding projection $t\colon M \to \mathbb{R}$ is a Cauchy temporal function [8,9,25]. Let $\Omega^p(M)$ denote the linear space of differential p-forms on M and let $\mathrm{d}\colon \Omega^p(M) \to \Omega^{p+1}(M)$ denote the de Rham differential, which together form the de Rham complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M) \longrightarrow 0, \tag{1}$$

This sequence of maps being a complex means that each pair of successive maps compose to zero, $\mathrm{d} \circ \mathrm{d} = 0$.

Its cohomology in degree p is defined and denoted by

$$H^p(M) := \frac{\ker(\mathrm{d}\colon \Omega^p(M) \to \Omega^{p+1}(M))}{\mathrm{im}(\mathrm{d}\colon \Omega^{p-1}(M) \to \Omega^p(M))}.$$

The cohomology of any other complex is defined in a similar way. It is well known that this de Rham cohomology is isomorphic, $H^p(M) \cong H^p(M, \mathbb{R})$, to the singular cohomology of M with coefficients in $\mathbb{R}$ [11, Theorem 15.8], to the Čech cohomology of M with coefficients in $\mathbb{R}$ [11, Theorem 8.9], and to the sheaf cohomology of M with coefficients in the sheaf of locally constant $\mathbb{R}$-valued functions [11, Proposition 10.6], all of which being isomorphic are denoted by $H^p(M, \mathbb{R})$. If we replace $\Omega^p(M)$ in (1) with $\Omega^p_c(M)$, the linear space of differential p-forms with compact support, the corresponding de Rham cohomology of M with compact supports, which satisfies the following isomorphism: $H^p_c(M)^* \cong H^p(M, \mathbb{R})$. That isomorphism is implemented by a non-degenerate bilinear pairing between $\Omega^p(M)$ and $\Omega^{n-p}_c(M)$,

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta, \tag{2}$$

which descends to a non-degenerate bilinear pairing between $H^p(M)$ and $H^p_c(M)$. This result is known as Poincaré duality [11, Remark 5.7].

Using the Hodge star operator $*\colon \Omega^p(M) \to \Omega^{n-p}(M)$ associated to the metric g, we can define the de Rham co-differential $\delta = *\mathrm{d}*\colon \Omega^p(M) \to \Omega^{p-1}(M)$. Next, we define the so-called de Rham–d'Alembertian or wave operator $\Box\colon \Omega^p(M) \to \Omega^p(M)$,

$$\Box = \mathrm{d}\delta + \delta\mathrm{d}. \tag{3}$$

This operator differs from the simple tensor d'Alembertian $\nabla_a \nabla^a$ by terms of lower differential order. From its very definition, we see that the d'Alembertian is a cochain map from the de Rham complex to itself, $\mathrm{d}\Box = \Box\mathrm{d}$, which is moreover cochain homotopic to zero, with the co-differential δ the corresponding cochain homotopy. That is, it induces the zero map from $H^p(M)$ to itself. The following diagram illustrates the discussion:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(M) \arrow[r, "\mathrm{d}"] \arrow[d, "\Box"] & \Omega^1(M) \arrow[r, "\mathrm{d}"] \arrow[d, "\Box"] \arrow[dl, dashed, "\delta"'] & \cdots \arrow[r, "\mathrm{d}"] \arrow[dl, dashed, "\delta"'] & \Omega^n(M) \arrow[r] \arrow[d, "\Box"] \arrow[dl, dashed, "\delta"'] & 0 \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "\mathrm{d}"] & \Omega^1(M) \arrow[r, "\mathrm{d}"] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^n(M) \arrow[r] & 0
\end{tikzcd} , \qquad (4)$$

where the rows constitute (de Rham) complexes, the solid arrows commute, and the dashed arrows illustrate the cochain homotopy. This is an important observation that will be used in an essential way in Sect. 3. Note that the formula (3) is analogous to the formula for the Hodge–de Rham Laplacian in Riemannian geometry. There, the observation that this Laplacian is homotopic to zero lies at the foundation of Hodge theory [27,33].

The causal structure on M defined by the Lorentzian metric g allows us to restrict the supports of differential forms in other ways as well. Recall that, for a subset $S \subseteq M$, by $J^\pm(S)$ we denote the subset of M that can be reached from S by piecewise smooth, future ($+$) or past ($-$) directed causal curves, while $J(S) = J^+(S) \cup J^-(S)$. A closed set $S \subseteq M$ is said to be retarded if $S \subseteq J^+(K)$ for some compact K, advanced if $S \subseteq J^-(K)$ for some compact K, spacelike compact if it $S \subseteq J(K)$ for some compact K, past compact if $S \cap J^-(K)$ is compact for every compact K, future compact if $S \cap J^+(K)$ is compact for every compact K, and timelike compact if S is both past and future compact [2,46]. Timelike compactness is also equivalent to the property of having compact intersection with every spacelike compact set. Let $\Omega^p_X(M)$, with $X = +, -, sc, pc, fc$ or tc, denote the linear space of differential p-forms with, respectively, retarded, advanced, spacelike compact, past compact, future compact or timelike compact supports. For brevity, we refer to these spaces as space of forms with causally restricted supports.

Of course, since differential operators preserve supports, $\Box$ also restricts to $\Box\colon \Omega^p_c(M) \to \Omega^p_c(M)$. By the same reasoning, the spaces of forms with causally restricted supports are also preserved by both d and $\Box$. We define de Rham cohomology with causally restricted supports in the obvious way and denote it by $H^p_X(M)$, with $X = +, -, sc, pc, fc$ or tc. Let $\Omega^p_\Box(M)$ and $\Omega^p_{\Box,X}(M)$ denote the kernel of the wave operator $\Box$, also known as its solution space, in the spaces of forms with corresponding supports. Finally, by the cochain map property, the de Rham differential restricts to the kernel of the wave operator, hence defining the de Rham cohomology groups $H^p_\Box(M)$ and $H^p_{\Box,X}(M)$ of solutions.

The specific way in which these causally restricted cohomologies are of importance in Maxwell gauge theory is summarized in the following proposition. For definiteness of notation let us fix a $\chi \in C^\infty(M)$ that is 1 in the future of a Cauchy surface Σ_+ and 0 in the past of another Cauchy surface Σ_-. The following is a special case of the general result [36, Theorem 3.2].

Proposition 1. *Maxwell gauge theory* [36, Sect. 4.2] *induces a symplectic form on* $\Omega^1_{\Box,sc}(M)$ [36, Definition 3.10] *that is non-degenerate when (a) the bilinear form on* $H^1_{sc}(M) \times H^{n-1}_c(M)$ *induced by* $\langle\alpha,\beta\rangle = \int_M \alpha\wedge\beta$ *is non-degenerate and (b) the bilinear form on* $H^1_{\Box,sc}(M)$ *induced by* $\langle\alpha,\beta\rangle_\Box = \int_M \alpha\wedge *\Box(\chi\beta)$ *is non-degenerate* (where $*$ denotes the Hodge dual).

From the proof of that proposition it also follows that degeneracies in (a) and (b) can imply degeneracies in the corresponding (pre-)symplectic structure.

The wave operator on a globally hyperbolic Lorentzian manifold is well known to be Green hyperbolic. That is, it has advanced and retarded Green functions denoted, respectively, G_+ and G_-, $\mathrm{G}_\pm\colon \Omega^p_c(M) \to \Omega^p_\pm(M)$. Since $\Box$ commutes with d, then so do G_+ and G_-. The form $\beta = \mathrm{G}_\pm[\alpha]$ is the unique solution of $\Box\beta = \alpha$ with, respectively, retarded or advanced support. The domain of definition of the Green functions can be extended, in a unique way, to $\Omega^p_X(M)$ for $X = +, -, pc$ or fc. Then, the maps

$$\Box\colon \Omega^p_Y(M) \to \Omega^p_Y(M), \quad \mathrm{G}_X\colon \Omega^p_Y(M) \to \Omega^p_Y(M) \tag{5}$$

are mutually inverse bijections, whenever $X = +$ and $Y = +$ or pc, or $X = -$ and $Y = -$ or fc. The combination $\mathrm{G} = \mathrm{G}_+ - \mathrm{G}_-$ is known as the causal Green function and fits into the following, in our terminology Green-hyperbolic, exact sequences [2,3,26,34,36]

$$0 \longrightarrow \Omega^p_c(M) \xrightarrow{\Box} \Omega^p_c(M) \xrightarrow{\mathrm{G}} \Omega^p_{sc}(M) \xrightarrow{\Box} \Omega^p_{sc}(M) \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow \Omega^p_{tc}(M) \xrightarrow{\Box} \Omega^p_{tc}(M) \xrightarrow{\mathrm{G}} \Omega^p(M) \xrightarrow{\Box} \Omega^p(M) \longrightarrow 0. \tag{7}$$

Note that, according to the above formulas, we can represent the space of solutions with spacelike compact or unrestricted support either as

$$\Omega^p_{\Box,X}(M) = \ker\Box \subset \Omega^p_X(M) \tag{8}$$

$$\text{or} \quad \Omega^p_{\Box,X}(M) = \mathrm{G}[\Omega^p_Y(M)] = \Omega^p_Y(M)/\Box\Omega^p_Y(M), \tag{9}$$

with $X = sc$ and $Y = c$, or X empty and $Y = tc$, respectively. On the other hand, we have trivial solution spaces $\Omega^p_{\Box,X}(M) = \{0\}$ when $X = +, -, pc$ or fc.

The existence of the Green-hyperbolic exact sequences will allow us to later make use of the following elementary result of homological algebra [11, p. 17]. Let $A^\bullet = (A^p, \mathrm{d})$ be a cochain complex, and similarly for $B^\bullet$ and $C^\bullet$. It is well known that a short exact sequence of cochain maps (maps commuting with the differentials d),

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0, \tag{10}$$

induces a long exact sequence in cohomology,

$$\begin{tikzcd} 0 \arrow[r] & H^0(A^\bullet,\mathrm{d}) \arrow[r, "{[f]}"] & H^0(B^\bullet,\mathrm{d}) \arrow[r, "{[g]}"] & H^0(C^\bullet,\mathrm{d}) \arrow[dll, rounded corners, "{[\mathrm{d}]}"'] \\ & H^1(A^\bullet,\mathrm{d}) \arrow[r, "{[f]}"] & H^1(B^\bullet,\mathrm{d}) \arrow[r, "{[g]}"] & H^1(C^\bullet,\mathrm{d}) \arrow[r, "{[\mathrm{d}]}"] & \cdots \end{tikzcd} \tag{11}$$

The maps $[f]$, $[g]$ are induced by the corresponding cochain maps, while the $[\mathrm{d}]$ maps are induced by the differentials of the complexes (hence our notation for them) and are known as connecting homomorphisms.

3. Computation of Cohomology Groups

In this section, we state and prove our main results on de Rham cohomology with causally restricted supports. We rely essentially on the properties of the wave operator and its Green functions, as summarized in Sect. 2. The important properties are that the wave operator $\Box$ is cochain homotopic to zero, and the way its range and kernel are characterized using the causal Green function G. In particular, we do not explicitly rely on the invariance properties of the de Rham complex under topological homotopies.

Theorem 2. *De Rham cohomology $H^p_X(M)$, with $X = +, -, pc$ or fc, is trivial.*

Proof. Let $X = +, -, pc$ or fc. Then, as was noted in Sect. 2, the wave operator is a cochain map of the corresponding de Rham complex into itself, is invertible [Eq. (5)] and cochain homotopic to zero [Eq. (3)]. Thus, it induces a map in cohomology that is both invertible and equal to zero, which can only mean that all the cohomologies are trivial. More concretely, given any closed $\alpha \in \Omega^p_X(M)$, the identity $\mathrm{d}(\delta \mathrm{G}_X[\alpha]) = \mathrm{G}_X[(\mathrm{d}\delta + \delta\mathrm{d})\alpha] = \alpha$ shows that it is also exact. $\square$

Theorem 3. *We have the isomorphisms*

$$H^p_{sc}(M) \cong H^{p+1}_c(M), \quad H^p_{\Box,sc} \cong H^p_c(M) \oplus H^{p+1}_c(M), \tag{12}$$

$$H^p_{tc}(M) \cong H^{p-1}(M), \quad \text{and} \quad H^p_{\Box}(M) \cong H^p(M) \oplus H^{p-1}(M), \tag{13}$$

with the convention that all cohomologies vanish in degree p for $p < 0$ or $p > n$.

Proof. Recall again from Sect. 2 that both the wave operator $\Box$ and its causal Green function G commute with d and hence constitute cochain maps between the de Rham complexes with appropriate supports, inducing maps in cohomology. Moreover, since $\Box$ is cochain homotopic to zero [Eq. (3)], it induces the zero map in cohomology.

Let us start with spacelike compact supports. We can break the exact sequence in (6) into two short exact sequences of complexes:

$$0 \longrightarrow \Omega^p_c(M) \xrightarrow{\ \Box\ } \Omega^p_c(M) \xrightarrow{\ \mathrm{G}\ } \Omega^p_{\Box,sc}(M) \longrightarrow 0, \tag{14}$$

$$0 \longrightarrow \Omega^p_{\Box,sc}(M) \xrightarrow{\ \subset\ } \Omega^p_{sc}(M) \xrightarrow{\ \Box\ } \Omega^p_{sc}(M) \longrightarrow 0. \tag{15}$$

Because $\Box$ always induces the zero map, $[\Box] = 0$, the corresponding long exact sequences in cohomology [cf. Eq. (11)] break up into the following short exact sequences:

$$0 \longrightarrow H^p_c(M) \xrightarrow{[G]} H^p_{\Box,sc}(M) \xrightarrow{[d]} H^{p+1}_c(M) \longrightarrow 0, \tag{16}$$

$$0 \longrightarrow H^{p-1}_{sc}(M) \xrightarrow{[d]} H^p_{\Box,sc}(M) \xrightarrow{[\subset]} H^p_{sc}(M) \longrightarrow 0, \tag{17}$$

again with the convention that any $H^p_X(M)$ vanishes for $p < 0$ or $p > n$. Since we are dealing with real vector spaces, any exact sequence splits, giving us the isomorphisms

$$H^p_c(M) \oplus H^{p+1}_c(M) \cong H^p_{\Box,sc}(M) \cong H^{p-1}_{sc}(M) \oplus H^p_{sc}(M).$$

Given that $H^0_c(M)$ and $H^{-1}_{sc}(M)$ both vanish (M is non-compact and there are no forms in degree $p = -1$), plugging $p = 0$ into the above isomorphism implies $H^0_{sc}(M) \cong H^1_c(M)$. Proceeding by induction on p, we can check that $H^p_{sc}(M) \cong H^{p+1}_c(M)$ for all p. Thus, we obtain the isomorophisms

$$H^p_{sc}(M) \cong H^{p+1}_c(M), \tag{18}$$

$$H^p_{\Box,sc}(M) \cong H^p_c(M) \oplus H^{p+1}_c(M). \tag{19}$$

Applying the same argument to the exact sequence (7), we obtain the isomorphisms

$$H^p_{tc}(M) \cong H^{p-1}(M), \tag{20}$$

$$H^p_{\Box}(M) \cong H^p(M) \oplus H^{p-1}(M). \tag{21}$$

This completes the proof. $\Box$

Let $\Sigma \subset M$ be a Cauchy surface. Recall that, by the smooth Geroch splitting theorem, we can always smoothly factor $M \cong \mathbb{R} \times \Sigma$. This observation results in

Corollary 4. *We have the isomorphisms*

$$H^p_{sc}(M) \cong H^p_c(\Sigma), \quad H^p_{\Box,sc}(M) \cong H^p_c(\Sigma) \oplus H^{p-1}_c(\Sigma) \tag{22}$$

$$H^p_{tc}(M) \cong H^{p-1}(\Sigma), \quad \textit{and} \quad H^p_{\Box}(M) \cong H^p(\Sigma) \oplus H^{p-1}(\Sigma), \tag{23}$$

with the convention that all cohomologies vanish in degree p for $p < 0$ or $p > n$.

Proof. The splitting $M \cong \mathbb{R} \times \Sigma$ shows that M is homotopic to Σ. Hence, by the homotopy invariance of de Rham cohomologies with unrestricted supports, we have the isomorphism $H^p(M) \cong H^p(\Sigma)$. On the other hand, Poincaré duality induces the isomorphism $H^p_c(M) \cong H^{p-1}_c(\Sigma)$. Therefore, the desired conclusion follows directly from these identities in combination with Theorem 3. $\Box$

Finally, knowing the respective de Rham cohomologies with spacelike and timelike compact supports, we have the following generalization of the Poincaré lemma.

Corollary 5. *The non-degenerate bilinear pairing between $\Omega^p_{sc}(M)$ and $\Omega^{n-p}_{tc}(M)$ descends to a non-degenerate bilinear pairing between $H^p_{sc}(M)$ and $H^{n-p}_{tc}(M)$. There exists also a non-degenerate bilinear pairing between $H^p_{\square,sc}(M)$ and $H^{n-p}_{\square}(M)$.*

Proof. A consequence of Theorem 3 is that $H^p_{sc}(M) \cong H^{p+1}_c(M)$ and $H^{n-p}_{tc}(M) = H^{n-p-1}(M)$. So, the usual Poincaré duality establishes that $H^p_{sc}(M)^* \cong H^{n-p}_{tc}(M)$. The isomorphism can be exhibited by bilinear pairing, which descends from the standard bilinear pairing between $\Omega^p_{sc}(M)$ and $\Omega^{n-p}_{tc}(M)$, tracing its effect throughout the proof of Theorem 3. Its non-degeneracy is also a consequence of the Poincaré lemma applied to $H^p_c(M)$ and $H^{n-p}(M)$.

It also follows from Theorem 3 that $H^p_{\square,sc}(M) \cong H^p_c(M) \oplus H^{p+1}_c(M)$ and $H^{n-p}_{\square}(M) \cong H^{n-p}(M) \oplus H^{n-p-1}(M)$. Again, the usual Poincaré duality establishes the isomorphism $H^p_{\square,sc}(M)^* \cong H^{n-p}_{\square}(M)$. The isomorphism can be exhibited by a bilinear pairing between $\Omega^p_{\square,sc}(M)$ and $\Omega^{n-p}_{\square}(M) \cong \Omega^{n-p}_{tc}(M)/\square\Omega^{n-p}_{tc}(M)$, defined by the latter identity and the self-adjointness of $\square$ with respect to our pairing between forms. Again, tracing this pairing through the proof of Theorem 3 and appealing to the standard Poincaré duality establishes its non-degeneracy. $\square$

As already discussed in Sect. 1, the importance of knowing the above cohomology groups is important for understanding the (pre)symplectic and Poisson structure of classical field theories, as emphasized in [5,6,34,36,47]. The same result as Corollary 4 was obtained independently in [5]. As a matter of fact, the method of [5] can be seen as a special case of our homological calculation, as discussed more explicitly at the end of Sect. 5.1.2.

4. Calabi or Killing–Riemann–Bianchi Complex

In [34,36], it was pointed out that the construction of the symplectic and Poisson structures on the phase space of field theories with constraints and/or gauge invariance can be done using a general framework, provided a given field theory satisfies certain geometric conditions. These conditions include the existence of certain differential complexes that extend the operators that constitute the constraints and that generate the gauge transformations. For Maxwell (and similar) theories, all of these complexes are invariably part of the de Rham complex [36, Sects. 4.2–.3]. On the other hand, for linearized gravity, one has to use something different. Unfortunately, the explicit form of these differential complexes is not currently known for linearized gravity on an arbitrary background [36, Sect. 4.4]. However, in the special case of constant curvature backgrounds, the answer is known and it is the so-called Calabi complex [13]. It is likely that, once an explicit understanding of the corresponding differential complexes for more general backgrounds is achieved, the general framework of [34,36] would supersede recent covariant treatments of the quantization of linearized gravity like [18,30].

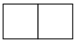
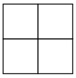
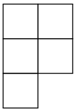
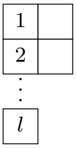
The Calabi complex provides a fine resolution [12, Sect. II.9] of the sheaf of Killing vectors, similarly to how the de Rham complex provides a fine resolution of the sheaf of locally constant functions. The cohomology of a sheaf (a rather abstract object) is isomorphic to the cohomology of the complex of global sections of a fine resolution of the same sheaf (a more concrete object), which is what makes fine resolutions significant [12, Theorem II.4.1]. As such, the Calabi complex has been studied in some literature on the deformation of constant curvature geometric structures [7,13,16,23,24,29,44]. Because its structure is substantially different from the de Rham complex, we summarize some of its relevant properties in Sects. 4.1 through 4.3 before concentrating on its causally restricted cohomologies in Sect. 4.4. Many of these properties are scattered throughout or are simply not available in the existing literature. We defer a fuller discussion of the Calabi complex, which collects these properties and their proofs, to [35]. However, all that we really need for the purposes of Sect. 4.4 is the existence of differential operators listed in Sect. 4.2 and the identities between them. Since these differential operators are explicitly given, the identities can in principle be verified by direct calculation.

4.1. Tensor Bundles

We will present later a differential complex whose nodes are sections of tensor bundles that are not so easy to express in conventional notation. So, let us introduce the following short-hands. We denote the cotangent bundle by $VM = T^*M$ and the bundle of metrics (symmetric, covariant 2-tensors) by $S^2M = S^2T^*M$. Let $RM \subset (T^*)^4M$ denote the sub-bundle of covariant 4-tensors that satisfy the algebraic symmetries of the Riemann tensor ($R_{(ab)cd} = R_{ab(cd)} = R_{abcd} - R_{cdab} = R_{[abc]d} = 0$). Next, we let $BM \subset (T^*)^5M$ denote the target bundle of the Bianchi operator $\nabla_{[a}R_{bc]de}$. At this point it is convenient to notice that the fiber of each of these bundles carries [20] an irreducible representation of $\mathrm{GL}(n)$, with $n = \dim M$. In fact, it is easiest to describe the remaining tensor bundles in terms of the irreducible $\mathrm{GL}(n)$ representation carried by their fibers. So let $C_lM \subset (T^*)^{l+2}M$ (with C standing for Calabi) denote the sub-bundles of covariant $(l+2)$-tensors with the corresponding irreducible representations listed in Table 1, which also lists their fiber ranks. It is consistent for us to assign $C_0M \cong VM$, $C_1M \cong S^2M$ and $C_2M \cong RM$ and $C_3M \cong BM$. Recall that, on an n-dimensional manifold, the largest rank of a fully antisymmetric tensor is n. So the bundles C_lM become trivial (zero fiber rank) for $l > n$.

The table below lists the tensor bundles of the Calabi complex, the corresponding irreducible $\mathrm{GL}(n)$ representations (labeled by Young diagrams), and their fiber ranks, for $\dim M = n$. The rank is given by the famous hook formula, which is the following fraction. The numerator is the product of the following numbers: place n in the top left cell, increase by 1 to the right and decrease by 1 down, until all cells are filled. The denominator is the product of the following numbers: fill a given cell with the number of cells constituting a hook with vertex at the given location, extending to the right and down [21].

Table 1. It is conventional to label irreducible GL(n) representations by Young diagrams [21]

Bundle	Young diagram	Fiber rank
$VM \cong C_0 M$		n
$S^2 M \cong C_1 M$		$\frac{n(n+1)}{2}$
$RM \cong C_2 M$		$\frac{n^2(n^2-1)}{12}$
$BM \cong C_3 M$		$\frac{n^2(n^2-1)(n-2)}{24}$
$C_l M$		$\frac{n^2(n^2-1)(n-2)\cdots(n-l+1)}{2(l+1)l(l-2)!}$

Recall that a Young diagram with k cells of type $(r_1, r_2, \ldots)$ consists of a number of rows of non-increasing lengths r_i, $r_{i+1} \leq r_i$, such that $\sum_i r_i = k$. Given a Young diagram with k cells, an instance of the corresponding irreducible GL(n) representation class can be realized as the image of the space of covariant k-tensors after two projections: assign an independent tensor index to each cell of the diagram, symmetrize over each row, antisymmetrize over each column

Given two $S^2 M$ tensors, we can construct an RM tensor out of them using the formula

$$(g \odot h)_{abcd} = g_{ac} h_{bd} - g_{bc} h_{ad} - g_{ad} h_{bc} + g_{bd} h_{ac}. \tag{24}$$

In fact, the above formula represents a GL(n)-equivariant map between $S^2 \otimes S^2$ and R (where we use the bundle prefixes to stand in for the corresponding irreducible representations). The decomposition of the $S^2 \otimes S^2$ tensor product has only one copy of R, so by Schur's lemma such a map is unique, up to an overall rescaling. The same argument can be repeated for the tensor product $S^2 \otimes Y$, where Y corresponds to any other Young diagram. This tensor product decomposes into irreducible subrepresentations without multiplicities. Then the projection onto any of the subrepresentations Y' is well defined up to a rescaling. If we fix sections g of $S^2 M$ and h of YM, these projections define a bilinear operation between g and h with the result a section of $Y'M$. We use the following explicit formulas:

$$\begin{aligned}(g \odot t)_{abc:de} = &+ g_{ad} t_{bc:e} + g_{bd} t_{ca:e} + g_{cd} t_{ab:e} \\ &- g_{ae} t_{bc:d} - g_{be} t_{ca:d} - g_{ce} t_{ab:d},\end{aligned} \tag{25}$$

$$\begin{aligned}(g \odot t)_{abcd:ef} = &+ g_{ae} t_{bcd:f} - g_{be} t_{cda:f} + g_{ce} t_{dab:f} - g_{de} t_{abc:f} \\ &- g_{af} t_{bcd:e} + g_{bf} t_{cda:e} - g_{cf} t_{dab:e} + g_{df} t_{abc:e}.\end{aligned} \tag{26}$$

Note that a tensor with indices written as in $t_{abc:de}$ has the symmetry type $(2, 2, 1)$, while $t_{abc:d}$ corresponds to the symmetry type $(2, 1, 1)$, and so on. The colon : is used purely as a visual aid to separate groups of indices belonging to different columns of a Young diagram.

The metric g_{ab} itself, an S^2M tensor, can now be used to produce an RM tensor,

$$(g \odot g)_{ab:cd} = 2(g_{ac}g_{bd} - g_{bc}g_{ad}), \tag{27}$$

which is obviously covariantly constant. In fact, a constant curvature spacetime must have (covariant) Riemann tensor, Ricci tensor and Ricci scalar of the following form

$$\bar{R}_{abcd} = \frac{k}{n(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad}), \quad \bar{R}_{ac} = \frac{k}{n}g_{ac}, \quad \bar{R} = k. \tag{28}$$

We have decorated these quantities with a bar to indicate the fact that we shall fix a constant curvature background metric g and consider perturbations on it. For our purposes, we also require that the Lorentzian manifold (M, g) is globally hyperbolic.

We should note that solutions of Einstein equations (including a possible cosmological constant term) with constant curvature include Minkowski space ($k = 0$), de Sitter space ($k > 0$) and anti-de Sitter space ($k < 0$). There is (up to isometry) a unique simply connected version of each of these cases [32, Sects. 5.1–2]. Other examples may be obtained by taking quotients thereof with respect to a discrete subgroup, thus changing the topology. The list of possibilities is thus exhausted by considering open subsets of such quotients. Some examples will not be globally hyperbolic (like anti-de Sitter space or quotients of Minkowski space with respect to timelike translations) and thus excluded from part of our discussion.

4.2. Differential Operators

Now, we introduce a number of differential operators between the tensor bundles that we have defined. For convenience of notation, we denote the space of sections of a bundle by the same symbol as the bundle itself. These operators fit into the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_0M \arrow[r, "B_1"] \arrow[d, "P_0"] & C_1M \arrow[r, "B_2"] \arrow[d, "P_1"] \arrow[dl, dashed, "E_1"'] & C_2M \cdots \arrow[r, "B_n"] \arrow[d, "P_2"] \arrow[dl, dashed, "E_2"'] & C_nM \arrow[r] \arrow[d, "P_n"] \arrow[dl, dashed, "E_n"'] & 0 \\
0 \arrow[r] & C_0M \arrow[r, "B_1"] & C_1M \arrow[r, "B_2"] & C_2M \cdots \arrow[r, "B_n"] & C_nM \arrow[r] & 0.
\end{tikzcd} \tag{29}$$

All the solid arrows commute and the rows constitute (cochain) complexes. The vertical maps are then necessarily cochain maps. They happen to satisfy the identities $P_l = E_{l+1} \circ B_{l+1} + B_l \circ E_l$, which means that they are null-homotopic, with the E_l supplying the corresponding cochain homotopy.

Below, we give explicit formulas for all these differential operators in dimension $n = 4$. More details can be found in [35], which draws from the earlier works [7,13,16,23,24,29,44]. As we shall see, for low indices they are well known in the relativity literature. However, the relations between them in terms of fitting into the above diagram do not seem to have been fully noted.

The Calabi differential complex is given by

$$B_1[v]_{ab} = \nabla_a v_b + \nabla_b v_a, \tag{30}$$

$$\begin{aligned} B_2[h]_{ab:cd} &= \big(\nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac}\big) \\ &\quad + k\frac{1}{n(n-1)}(g\odot h)_{ab:cd}, \end{aligned} \tag{31}$$

$$B_3[r]_{abc:de} = 3\nabla_{[a}r_{bc]:de} = \nabla_a r_{bc:de} + \nabla_b r_{ca:de} + \nabla_c r_{ab:de}, \tag{32}$$

$$B_4[b]_{abcd:ef} = 4\nabla_{[a}b_{bcd]:ef} \tag{33}$$

$$= \nabla_a b_{bcd:ef} - \nabla_b b_{cda:ef} - \nabla_c b_{dab:ef} - \nabla_d b_{abc:ef}, \tag{34}$$

$$B_l[b]_{a_1\cdots a_l:bc} = l\nabla_{[a_1}b_{a_2\cdots a_l]:bc} \quad (l\geq 3), \tag{35}$$

where $(a_1\cdots a_l)$ and $[a_1\cdots a_l]$ denote, respectively, complete idempotent symmetrization and antisymmetrization of a group of indices [52, Eqs. 2.4.3–4]. Recall also that the colon : is used purely as a visual aid to separate groups of indices belonging to different columns of the Young diagrams in Table 1. The details showing that these operators have the desired symmetry properties and indeed define a complex, $B_{l+1}\circ B_l = 0$, which is moreover elliptic,[2] can be found in [35].

It is interesting to note the following relations with well-known differential operators in relativity. The Killing operator is $K[h] = B_1[h]$. The linearized Riemann tensor is $\dot{F R}[h] = -\frac{1}{2}B_2[h] + k\frac{2}{n(n-1)}(g\odot h)$, where the all covariant non-linear Riemann tensor is expanded as $R[g+\lambda h]_{ab:cd} = \bar{R}_{ab:cd} + \lambda\dot{R}[h]_{ab:cd}$ (convention of [52]). The background Bianchi operator is $\bar{B}[r] = B_3[r]$, with $\bar{B}[\bar{R}] = 0$. Finally, though the name is not standard, it is meaningful to call $B_4[b]$ a higher Bianchi operator. Thus, it would also make sense to refer to the Calabi complex as the Killing–Riemann–Bianchi complex. This complex also happens to be locally exact[3] [13,35]. Thus, according to the general machinery of sheaf theory, the Calabi complex provides a fine resolution of the sheaf of Killing vectors (or Killing sheaf) $\mathcal{K}_g$ on the Lorentzian manifold (M,g) [35, Sect. 3]. This observation immediately gains us the following:

Proposition 6 (Calabi [13]). *The* (unrestricted) *cohomology* $HC^l(M,g) = \ker B_{l+1}/\operatorname{im} B_l$ *of the Calabi complex is isomorphic to the sheaf cohomology* $H^\bullet(M,\mathcal{K}_g)$ *of the sheaf of Killing vectors on any spacetime* (M,g) *of constant curvature.*

Calabi's proof was rather elementary and relied on the specific structure of this complex. Unfortunately, his method does not generalize easily to other

[2] A complex of differential operators is elliptic if the corresponding complex of symbol maps is exact for every non-zero covector.

[3] A differential complex on a manifold M is locally exact if every $x\in M$ has a neighborhood such that the complex restricted to it becomes exact. For example, this condition is fulfilled for the de Rham complex thanks to the Poincaré lemma.

differential complexes. So, we discuss below a different method to get local exactness, which relies mostly on the ellipticity of the Calabi complex, a property which is expected to be shared by other complexes of interest.

Next, we give explicitly the homotopy differential operators

$$E_1[h]_a = D[h]_a = \nabla^b h_{ab} - \frac{1}{2}\nabla_a h, \tag{36}$$

$$E_2[r]_{a:b} = \mathrm{tr}[r]_{a:b} = r_{ac:b}{}^c, \tag{37}$$

$$\begin{aligned} E_3[b]_{ab:cd} = \nabla^e b_{eab:cd} &+ \frac{1}{2}\nabla^e(b_{cab:de} - b_{dab:ce}) \\ &- \frac{1}{2}(\nabla_c b_{abe:d}{}^e - \nabla_d b_{abe:c}{}^e) \\ &- \frac{1}{2}(\nabla_a b_{cbe:d}{}^e - \nabla_a b_{dbe:c}{}^e \\ &+ \nabla_b b_{ace:d}{}^e - \nabla_b b_{ade:c}{}^e), \end{aligned} \tag{38}$$

$$\begin{aligned} E_4[b]_{abc:de} = \nabla^f b_{fabc:de} &+ \frac{1}{3}\nabla^f(b_{dabc:ef} - b_{eabc:df}) \\ &+ \frac{1}{3}(\nabla_d b_{abcf:e}{}^f - \nabla_e b_{abcf:d}{}^f) \\ &+ \frac{1}{6}(\nabla_a b_{dbcf:e}{}^f - \nabla_a b_{ebcf:d}{}^f \\ &+ \nabla_b b_{adcf:e}{}^f - \nabla_b b_{aecf:d}{}^f \\ &+ \nabla_c b_{abdf:e}{}^f - \nabla_c b_{abef:d}{}^f), \end{aligned} \tag{39}$$

$$\begin{aligned} E_{l+1}[b]_{a_1\cdots a_l:bc} = \nabla^a b_{aa_1\cdots a_l:bc} &+ l^{-1}\nabla^a(b_{ba_1\cdots a_l:ca} - b_{ca_1\cdots a_l:ba}) \\ &- \frac{(-1)^l}{l}(\nabla_b b_{a_1\cdots a_l a:c}{}^a - \nabla_c b_{a_1\cdots a_l a:b}{}^a) \\ &- \frac{(-1)^l}{l(l-1)}(\nabla_{\{b\}} b_{\{a_1\cdots a_l\}a:c}{}^a - \nabla_{\{c\}} b_{\{a_1\cdots a_l\}a:b}{}^a) \\ &\quad \text{for } (l \geq 2), \text{ where} \end{aligned} \tag{40}$$

$$p_{\{b\}} t_{\{a_1\cdots a_l\}} = \sum_{i=1}^{l} (-1)^{i+1} p_{a_i} t_{a a_1\cdots \hat{a}_i \cdots a_l} \quad (\hat{a}_i \text{ omitted}).$$

Their desired Young symmetry properties are demonstrated in [35]. Again, we find the following relations with classical differential operators from relativity. The de Donder operator is $D[h] = E_1[h]$. The trace from the Riemann to the Ricci tensors is given by $\bar{R}_{ab} = \bar{R}_{ac:b}{}^c = E_2[\bar{R}]_{ab}$. The higher homotopy operators E_l do not seem to be part of the classical literature. However, they are essentially modified divergence operators and are thus reminiscent of the de Rham co-differentials.

Finally, the cochain maps $P_l = E_{l+1} \circ B_{l+1} + B_l \circ E_l$ (with the edge cases $P_0 = E_1 \circ B_1$ and $P_n = B_n \circ E_n$) are given by

$$P_0[v]_a = \Box v_a + k\frac{1}{n}v_a, \tag{41}$$

$$P_1[h]_{ab} = \Box h_{ab} - k\frac{2}{n(n-1)}h_{ab} + 2k\frac{g_{ab}\,\mathrm{tr}[h]}{n(n-1)}, \tag{42}$$

$$P_2[r]_{ab:cd} = \Box r_{ab:cd} - k\frac{2}{n}r_{ab:cd} + 2k\frac{(g\odot \mathrm{tr}[r])_{ab:cd}}{n(n-1)}, \tag{43}$$

$$P_3[b]_{abc:de} = \Box b_{abc:de} - k\frac{(3n-7)}{n(n-1)}b_{abc:de} - 2k\frac{(g\odot \mathrm{tr}[b])_{abc:de}}{n(n-1)}, \tag{44}$$

$$P_4[b]_{abcd:ef} = \Box b_{abcd:ef} - k\frac{(4n-14)}{n(n-1)}b_{abcd:ef} + 2k\frac{(g\odot \mathrm{tr}[b])_{abcd:ef}}{n(n-1)}, \tag{45}$$

$$\begin{aligned} P_l[b]_{a_1\cdots a_l:bc} &= \Box b_{a_1\cdots a_l:bc} - k\frac{(ln-l^2+2)}{n(n-1)}b_{a_1\cdots a_l:bc} \\ &\quad + (-)^l 2k\frac{(g\odot \mathrm{tr}[b])_{a_1\cdots a_l:bc}}{n(n-1)} \quad (l\geq 3), \end{aligned} \tag{46}$$

where we have defined the traces as $\mathrm{tr}[h] = h_e{}^e$, $\mathrm{tr}[r]_{ab} = r_{ae:b}{}^e$, $\mathrm{tr}[b]_{ab:c} = b_{abe:c}{}^e$, $\mathrm{tr}[b]_{abc:d} = b_{abce:d}{}^e$, and $\mathrm{tr}[b]_{a_1\cdots a_l:b} = b_{a_1\cdots a_l a:b}{}^a$. The required null-homotopy identities $P_l = E_{l+1}\circ B_{l+1} + B_l\circ E_l$ (including the edge cases $P_0 = E_1\circ B_1$ and $P_n = B_n\circ E_n$) are demonstrated in [35]. These identities for $P_0[v]$ and $P_1[v]$ are well known and are tightly linked with the de Donder gauge fixing condition in linearized gravity [18,52]. The higher cochain maps and the corresponding identities appear to be new. Though, the identity for $P_2[r]$ is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the Lichnerowicz Laplacian [40, Sect. 1.3] (see also [14, Sect. 7.1], [41, Exr. 15.2], [10, Eq. 35]).

4.3. Cohomology with Unrestricted and Compact Supports

Let us denote the cohomology of the Calabi complex by $HC^l_X(M,g)$, where $X = c, +, -, fc, pc, sc, tc$ or empty, according to the conventions of Sect. 2. As in the case of the de Rham complex in Sect. 3, we will later relate the cohomology with causally restricted supports to that with unrestricted or compact supports. It remains still to find a means to calculate these cohomology groups. We will state some results in that direction below, referring to [35] for a fuller discussion.

An important observation is that each of the P_l operators is wave-like, that is, it has the same principal symbol as the wave operator $\Box_g$ with respect to the background Lorentzian metric g. This observation has a dual role. First, this means that each of the P_l operators is Green hyperbolic [2,3], while being cochain homotopic to zero, opening the door to using the methods of Sect. 3 to compute the cohomology with causally restricted supports.

The second role is more subtle:

Remark 1. Note that the principal symbols of the B_l maps in the Calabi complex are actually $\mathrm{GL}(n)$-equivariant and so do not actually involve the background metric g. On the other hand, the principal symbols of the cochain maps P_l do depend on g. This dependence comes purely from the cochain homotopy operators $E_l = E^g_l$ and the identity $P_l = P^g_l = E^g_{l+1}\circ B_{l+1} + B_l\circ E^g_l$, where we have used the subscript g to indicate that the background metric was used for covariant differentiation and index raising. On the other hand, we are completely free to define a different set of cochain maps $P_l^{g_R} = E_{l+1}^{g_R} \circ B_{l+1} + B_l \circ E_l^{g_R}$, which now depend on a different metric g_R with Riemannian signature. It is crucial to note that the principal symbol of $P_l^{g_R}$ depends only on the principal symbols of the $E_l^{g_R}$ and B_l. So, in fact, it is equal to the principal symbol of P_l^g, but with the Lorentzian metric g replaced by the Riemannian metric g_R. In other words, each of the $P_l^{g_R}$ operators is elliptic, since its principal symbol coincides with the Laplace operator Δ_{g_R}. Of course, $P_l^{g_R}$ would differ much more radically from the formulas we have given for P_l^g in the terms of subleading differential orders.

The ellipticity of the complex (together with a subtler property known as a δ-estimate, discussed in more detail in [35, 51]) results in the following

Proposition 7. *Let us denote by $\Gamma(C_l M)$ the space of smooth sections of the tensor bundle $C_l M \to M$. (a) The cohomology $HC^\bullet(M, g)$ of the Calabi complex $(\Gamma(C_l M), B_l)$ is isomorphic to the cohomology $H^\bullet(M, \mathcal{K}_g)$ of the sheaf $\mathcal{K}_g$ of Killing vectors on (M, g). (b) If (M, g) is a simply connected, constant curvature Lorentzian manifold, then $H^\bullet(M, \mathcal{K}_g) \cong H^\bullet(M) \otimes V_g$, where V_g is the vector space of all Killing vectors and $H^\bullet(M)$ is the de Rham cohomology group.*

Killing vectors (or rather covectors in our notation) are solutions $v \in \Gamma(T^*M)$ of the Killing equation $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a = 0$. On simply connected, constant curvature n-dimensional spacetimes, $\dim V_g = \binom{n+1}{2}$. Note also that the simple connectedness condition implies that $H^1(M) = 0$. The precise definition of a sheaf and its cohomology is not of particular importance for the moment. For present purposes, it suffices that the above result, at the very least, answers the question of what $HC^\bullet(M, g)$ is for the simply connected versions of Minkowski ($\mathbb{R}^n$), de Sitter ($\mathbb{R} \times S^{n-1}$ for $n \geq 3$, $\mathbb{R}^2$ for $n = 2$) and anti-de Sitter ($\mathbb{R}^n$) spacetimes. The proof, together with a partial discussion of the non-simply connected case, can be found in [35].

It remains to discuss Calabi cohomology with compact supports $HC_c^\bullet(M, g)$. First, we note that the chain complex $(\Gamma(C_l^* M), B_l^*)$ formally adjoint to the Calabi complex has the interesting property that equation $B_n^*[b] = 0$ is equivalent to the (rank-$(n-2)$) Killing–Yano equation $Y[w]_{abc_4\cdots c_n} = \nabla_{(a} w_{b)c_4\cdots c_n}$, where a solution with $w_{[bc_4\cdots c_n]} = w_{bc_4\cdots c_n}$ is called a (rank-$(n-2)$) Killing–Yano tensor on (M, g). We define Calabi homology $HC_l(M, g)$ as the cohomology of this adjoint complex $(\Gamma_c(C_l^* M), B_l^*)$ with compact supports and also locally finite Calabi homology as the cohomology of the adjoint complex $(\Gamma(C_l^* M), B_l^*)$ with unrestricted supports. Since taking formal adjoints preserves the homotopy identities and ellipticity, appealing to the same arguments as above (again, including a δ-estimate [35, 51]) we also have

Proposition 8. (a) *Locally finite Calabi homology $HC_l^{lf}(M, g)$ is isomorphic to the cohomology $H^\bullet(M, \mathcal{KY}_g)$ of the sheaf $\mathcal{KY}_g$ of Killing–Yano tensors on*

(M,g). (b) *If* (M,g) *is a simply connected, constant curvature Lorentzian manifold, then* $H^\bullet(M,\mathcal{KY}_g) \cong H^\bullet(M) \otimes W_g$*, where* W_g *is the vector space of all Killing–Yano tensors and* $H^\bullet(M)$ *is the de Rham cohomology group.*

On simply connected, constant curvature n-dimensional spacetimes, $\dim W_g = \binom{n+1}{2}$ [49]. Furthermore, using Remark 1 and some general results from the theory of elliptic differential complexes (see Example 5.1.11 of [51], which relies on the results of [48]), we have the following generalized Poincaré duality isomorphisms [35]:

Proposition 9. *When finite dimensional, Calabi homology is the linear dual of Calabi cohomology,* $HC_l(M,g) = HC^l(M,g)^*$*, while Calabi cohomology with compact supports is the linear dual of locally finite Calabi homology,* $HC^l_c(M,g) = HC^{lf}_l(M,g)^*$*. In both cases, the duality can be exhibited via the non-degeneracy of the pairing descended from the natural pairing between the chains and cochains of corresponding complexes.*

4.4. Cohomology with Causally Restricted Supports

Recall that, in Sect. 2, we defined de Rham cohomologies $H^p_X(M)$ with causally restricted supports $X = +, -, sc, tc, pc$ or fc by restricting the de Rham complex to forms with supports indicated by X, with the on-shell cohomologies $H^p_\Box(M)$ and $H^p_{\Box,sc}(M)$. Substituting the Calabi complex for the de Rham complex and the P_l operators for the d'Alembertians $\Box$, by direct analogy we can define the causally restricted Calabi cohomologies $HC^l_X(M,g)$, $HC^l_P(M,g)$ and $HC^l_{P,sc}(M,g)$. We can use the same definitions also in the case of the adjoint Calabi complex, with slightly altered notation. Let the causally restricted Calabi homology $HC^X_l(M,g)$ be the cohomology of the complex $(\Gamma_Y(C^*_l M), B^*_l)$ where the pair (X,Y) is one of retarded $(+, fc)$, advanced $(-, pc)$, spacelike locally finite (slf, tc), timelike locally finite (tlf, sc) , future locally finite $(flf, -)$ and past locally finite $(plf, +)$. Similarly, we define the on-shell Calabi homologies $HC^{P,lf}_l(M,g)$ and $HC^{P,tlf}(M,g)$ as the cohomologies of the complexes $(\ker P^*_l \cap \Gamma(C^*_l M), B^*_l)$ and $(\ker P^*_l \cap \Gamma_{sc}(C^*_l M), B^*_l)$, respectively. The above (X,Y) pairs are chosen specifically so that there is a bilinear pairing between Calabi homology $HC^X_l(M,g)$ and Calabi cohomology $HC^l_Y(M,g)$, which descends from the natural pairing between the corresponding spaces of sections of $C^*_l M$ and $C_l M$.

The specific way in which these causally restricted cohomologies are of importance in linearized gravity is summarized in the following proposition. For definiteness of notation let us fix a $\chi \in C^\infty(M)$ that is 1 in the future of a Cauchy surface Σ_+ and 0 in the past of another Cauchy surface Σ_-. The following is a special case of the general result [36, Theorem 3.2].

Proposition 10. *Linearized gravity on a constant curvature background* [36, Sect. 4.4] *induces a symplectic form on* $\Gamma_{P,sc}(C_1 M)$ [36, Definition 3.10] *that is non-degenerate when* (a) *the bilinear form on* $HC^1_{sc}(M,g) \times HC_1(M,g)$ *induced by* $\langle\alpha,\beta\rangle = \int_M \alpha\cdot\beta$ *is non-degenerate and* (b) *the bilinear form on* $HC^1_{P,sc}(M)$ *induced by* $\langle\alpha,\beta\rangle_P = \int_M \alpha\cdot P_1[\chi\beta]$ *is non-degenerate.*

From the proof of that proposition it also follows that degeneracies in (a) and (b) can imply degeneracies in the corresponding (pre-)symplectic structure.

With the above discussion in mind, we can see immediately that we are in a situation very similar to that of Sect. 3, with the de Rham complex replaced by the Calabi complex (or its adjoint complex) and the wave operators $\square$ replaced by the operators P_l (or P_l^*), which have wave-like principal symbols and are Green hyperbolic. So, repeating the arguments of Sect. 3, we immediately have the following

Theorem 11. *Consider a globally hyperbolic, constant curvature Lorentzian manifold (M, g). The Calabi cohomology $HC^l_X(M, g)$ with the causally restricted supports $X = +, -, pc$ or fc is trivial. Moreover, for the cases $X = sc, tc$, we have the isomorphisms*

$$HC^l_{sc}(M,g) \cong HC^{l+1}_c(M,g), \quad HC^l_{P,sc}(M,g) \cong HC^l_c(M,g) \oplus HC^{l+1}_c(M,g), \tag{47}$$

$$HC^l_{tc}(M,g) \cong HC^{l-1}(M,g), \quad HC^l_P(M,g) \cong HC^l(M,g) \oplus HC^{l-1}(M,g), \tag{48}$$

with the convention that all cohomologies vanish in degree l for $l < 0$ or $l > n$. Similarly, the Calabi homology $HC^X_l(M, g)$ with the causally restricted supports $X = +, -, plf$ or flf is trivial. Moreover, for the cases $X = tlf, slf$, we have the isomorphisms

$$HC^{tlf}_l(M,g) \cong HC_{l-1}(M,g), \;\; HC^l_{P,tlf}(M,g) \cong HC_l(M,g) \oplus HC_{l-1}(M,g), \tag{49}$$

$$HC^{slf}_l(M,g) \cong HC^{lf}_{l+1}(M,g), \quad HC^l_{P,lf}(M,g) \cong HC^{lf}_l(M,g) \oplus HC^{lf}_{l+1}(M,g), \tag{50}$$

again with the convention that all cohomologies vanish in degree l for $l < 0$ or $l > n$.

The Calabi cohomology with spacelike compact support in degree $l = 1$ is important in understanding the symplectic and Poisson structure of the classical field theory (and of course the quantization) of linearized gravitons on a background of constant curvature. This was pointed out explicitly in [36, Sect. 4.4] as a special case of a more general phenomenon (also discussed in [34]).

Remark 2. Using the above theorem and the results of Sect. 4.3, we can assert that for n-dimensional Minkowski space HC^l_{sc} vanishes in all degrees except $l = n - 1$, while $HC^l_{P,sc}$ vanishes in all degrees except $l = n, n - 1$. For n-dimensional de Sitter space HC^l_{sc} vanishes in all degrees except $l = n - 1$, while $HC^l_{P,sc}$ vanish in all degrees except $l = 0, n - 1, n$. Similar remarks apply to Calabi homologies.

5. Notes and Generalizations

5.1. Generalized Causal Structures

The notion of a causal structure on a manifold or even a topological space (in the sense of a partial order on events) can be generalized quite far beyond the context of Lorentzian geometry [22,37]. We will stick with the context of differential geometry, where a natural generalization consists of introducing at every point of a manifold an arbitrary convex cone in the tangent[4] bundle. Such a manifold could be called a conal manifold [34,38,42,50]. Various notions generated by the causal structure on Lorentzian manifolds survive almost without modification on conal manifolds, including spacelike and timelike compactness. The main question we will try to answer in this section is the following: is it possible to use the methods of Sect. 3 to compute causally restricted cohomologies on a conal manifold? We shall see that the answer is yes, even if the conal manifold is not Lorentzian.

5.1.1. Conal Manifolds. Before dealing with spacelike and timelike compactly supported forms, let us introduce the basics of conal manifolds and causal relations on them. Let M be a smooth manifold and $C \subset TM$ be an open subset, such that $C_x = C \cap T_x M$ is an open, convex cone in $T_x M$ that does not contain any affine line. It can be shown that the interior $C_x^{\circledast}$ of the polar dual (or convex dual) cone $T_x^* M \supset C_x^* = \{p \in T_x^* M \mid \forall v \in C_x \colon p \cdot v \geq 0\}$ satisfies the same conditions, with $C^{\circledast} = \sqcup_{x \in M} C_x^{\circledast}$. The pair (M, C) or $(M, C^{\circledast})$ is called a conal manifold, with C (or $C^{\circledast}$) called the tangent (or cotangent) cone distribution or cone bundle. For example, the subset of non-vanishing, future-pointing, timelike vectors on a Lorentzian manifold with a time orientation satisfies the above conditions. In general, the cones C_x need not even have elliptic cross sections, thus not be associated to any Lorentzian metric. The cones of future pointing timelike vectors of linear symmetric hyperbolic PDE systems also satisfy the same properties [34, Sect. 4.1]. Sometimes, it is also convenient to admit degenerate cases where the cones are not open or contain some affine lines, but some special care must be taken in those situations.

Given a conal manifold (M, C) we can define a chronological order relation on the points of M. Namely, $x \ll y$ if there exists a smooth curve $\gamma\colon [0,1] \to M$, such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot\gamma(t) \in C$ for all $t \in [0,1]$. It can be shown that the chronological order relation $I^+ \subset M \times M$ is open and transitive. We can also define the reverse chronological order, I^-, and chronological influence, $I = I^+ \cup I^-$, relations in the obvious way. We avoid defining the analog of the causal order relation usually denoted by J^+, simply because we have not made any hypotheses about the regularity of the set of causal vectors ($\overline{C}_x \subset T_x M$). Given any set $K \subseteq M$, we denote by $I^\pm(K)$ the set of all points of M that, respectively, chronologically precede ore are preceded by the points of K. In general, $I^\pm(K)$ is not closed, even if K is. So, for convenience we define $\overline{I}^\pm(K) = \overline{I^\pm(K)}$. We also use the notation $I(K) = I^+(K) \cup I^-(K)$

[4] One could equally do so in the cotangent bundle, and produce a tangent cone by convex (or polar) duality.

and $\bar{I}(K) = \bar{I}^+(K) \cup \bar{I}^-(K)$. Note that $\bar{I}^\pm \subseteq M \times M$ need not be transitive as relations.

The definition of a Cauchy surface $\Sigma \subset M$ is the usual one, every inextensible smooth curve with timelike tangents must intersect Σ exactly once. It has recently been shown that the smooth version of the Geroch splitting theorem [8,9,25] generalizes to conal manifolds [17]. So, globally hyperbolicity can be simply characterized by the existence of a Cauchy surface. Also, the results of [46] should also directly carry over to conal manifolds. Finally, we define the notions of advanced, retarded, spacelike compact, timelike compact, future compact and past compact exactly in the same way as in Sect. 2, with the exception that we use the relations $\bar{I}^\pm$ and $\bar{I}$ instead of the relations $J^\pm$ and J.[5]

5.1.2. Cohomology with Causally Restricted Supports. Let M be a globally hyperbolic conal manifold and g an auxiliary globally hyperbolic Lorentzian metric that induces another conal structure on M that is "slower" than the original one. That is, $\Omega^p_{\pm_g}(M) \subseteq \Omega^p_\pm(M)$, which also implies that $\Omega^p_{sc_g}(M) \subseteq \Omega^p_{sc}(M)$, while $\Omega^p_{fc_g,pc_g}(M) \supseteq \Omega^p_{fc,pc}(M)$, and hence $\Omega^p_{tc_g}(M) \supseteq \Omega^p_{tc}(M)$. Any conal manifold admits a nowhere vanishing vector field (contract each cone to a ray and select a vector from it), which is moreover everywhere future directed. So, the existence of such an auxiliary Lorentzian metric follows from the same known, general arguments that show the existence of Lorentzian metrics on manifolds of vanishing Euler characteristic (i.e., admitting a nowhere vanishing vector field) [4,43]. The "slowness" requirement is implemented by making sure that the Lorentzian timelike cones closely hug the directions singled out by the above everywhere timelike vector field.

Let $\mathrm{G}_\pm$ denote once again the advanced and retarded Green functions of the wave operator $\Box_g$ defined with respect to g. Then it is easy to see that the Green functions are still well defined and injective as maps $\mathrm{G}_\pm \colon \Omega^p_c(M) \to \Omega^p_\pm(M)$. Appealing to the same logic as in the standard proofs[6] [2,3,26,34,36], we can extend the Green functions to bijective maps $\mathrm{G}_\pm \colon \Omega^p_\pm(M) \to \Omega^p_\pm(M)$ and $\mathrm{G}_\pm \colon \Omega^p_{fc,pc}(M) \to \Omega^p_{fc,pc}(M)$, from which it is straightforward to establish exactness of the following sequences, with $\mathrm{G} = \mathrm{G}_+ - \mathrm{G}_-$:

$$0 \longrightarrow \Omega^p_0(M) \xrightarrow{\ \Box\ } \Omega^p_0(M) \xrightarrow{\ \mathrm{G}\ } \Omega^p_{sc}(M) \xrightarrow{\ \Box\ } \Omega^p_{sc}(M) \longrightarrow 0, \tag{51}$$

$$0 \longrightarrow \Omega^p_{tc}(M) \xrightarrow{\ \Box\ } \Omega^p_{tc}(M) \xrightarrow{\ \mathrm{G}\ } \Omega^p(M) \xrightarrow{\ \Box\ } \Omega^p(M) \longrightarrow 0, \tag{52}$$

[5] We are not concerned with possible minor inconsistencies this substitution introduces in the case of Lorentzian manifolds with ill-behaved causal structures. In any case, we shall only apply these notions for globally hyperbolic spacetimes, where these differences do not appear.

[6] Pick an exhaustion of M by compact sets and adapt a sequence of smooth "step functions" to this exhaustion. Precomposing $\mathrm{G}_\pm$ with multiplication by these step functions gives a sequence of operators which converges to an operator with the desired extended domain.

where the supports are restricted by the given conal structure on M and not by that induced by the auxiliary Lorentzian metric g. Note that the proofs would make use of the hypothesis that the given conal structure is globally hyperbolic, specifically in the construction of explicit splitting maps that demonstrate exactness [36, Lemma 2.1]. Thus, repeating the arguments Sect. 3, we establish the following generalization of Theorems 2 and 3.

Theorem 12. *Consider a globally hyperbolic conal manifold M. Its de Rham cohomology $H^p_X(M)$ with causally restricted supports $X = +, -, pc$ or fc is trivial. Moreover, we have the isomorphisms*

$$H^p_{sc}(M) \cong H^{p+1}_c(M), \quad H^p_{\Box,sc} \cong H^p_c(M) \oplus H^{p+1}_c(M), \tag{53}$$

$$H^p_{tc}(M) \cong H^{p-1}(M), \quad \textit{and} \quad H^p_{\Box}(M) \cong H^p(M) \oplus H^{p-1}(M), \tag{54}$$

with the convention that all cohomologies vanish in degree p for $p < 0$ or $p > n$.

It should be clear from the preceding discussion that there is nothing inherently special in our use of the d'Alembertian $\Box_g$, when it comes to the calculation of de Rham cohomologies with causally restricted supports on a globally hyperbolic conal manifold M. It is merely one of multiple possible auxiliary hyperbolic differential operators that can serve the same purpose. Here are the key required properties for such an operator h: (a) h must be a cochain map that is homotopic to zero with respect to the de Rham complex, (b) it must possess retarded and advanced Green functions, (c) these Green functions must be causal with respect to the given conal structure on M. In fact, the conclusion of our Theorem 3 was reached independently in the recent paper [5] by following an argument structurally similar to ours, with the d'Alembertian replaced by the Lie derivative $\mathcal{L}_v$ with respect to a complete timelike vector field v. It is clearly (Green) hyperbolic [2,3,34,36] with Green functions simply given by integration (into the future or past) along the flow lines of v. Moreover, it is cochain homotopic to zero because of the well-known magic formula of Cartan: $\mathcal{L}_v = \iota_v \mathrm{d} + \mathrm{d}\iota_v$.

5.2. Functoriality

Recall that ordinary de Rham cohomology is defined on any finite dimensional manifold and the pullback of differential forms along a smooth map between manifolds induces a map between their cohomologies (in the direction opposite the original smooth map). This observation has the following well-known formalization: de Rham cohomology in degree p, $H^p(-)$, is a contravariant functor[7] from the category of smooth manifolds to the category of real vector spaces. The same cannot be said for de Rham cohomology with compact supports, $H^p_c(-)$, because the pullback of a compactly supported form need not be compactly supported itself. This pullback problem is

[7] We shall not delve here into the details of category theory. It suffices to say that any statement that we shall make involving functors and categories will be simply a very terse transcription of some other property that will be spelled out in more elementary terms. More details about the functorial properties of de Rham cohomology can be found in [11].

fixed by considering only proper[8] smooth maps between manifolds. So, given a proper smooth map $f\colon M \to N$, pullback along it induces a contravariant map between de Rham cohomologies in degree p with compact support, $f^*\colon H^p_c(N) \to H^p_c(M)$. If the map f satisfies a different restrictive condition, namely that it is an open embedding, it is possible to define a covariant pushforward map $f_*\colon H^p_c(M) \to H^p_c(N)$: we can identify M with its image $f(M)$, an open subset of N, and extend by zero any compactly supported form defined M to all of N. In short, de Rham cohomology with compact supports, $H^p_c(-)$, defines a contravariant functor on the category of smooth manifolds with proper maps as morphisms, when paired with the pullback, while it defines a covariant functor on the category of smooth manifolds with open embeddings as morphisms, when paired with the pushforward.

A natural question is the following: do similar properties hold, and under what precise conditions, for de Rham cohomologies with causally restricted supports? For instance, this question was briefly raised, but without any definite answer, in [5]. In fact, it is straight forward to present causally restricted cohomologies as functors, provided we modify the domain category by adding generalized causal structures to manifolds (as in Sect. 5.1) and by modifying the notion of a proper map with respect to the causal structure.

Consider two conal manifolds M and N, with a smooth map $f\colon M \to N$ between them. We call the map f reflectively spacelike-proper if the preimage of any spacelike compact set is also spacelike compact, while we call it reflectively timelike-proper if the preimage of any timelike compact set is also timelike compact. When the map f is an open embedding, we also introduce the terminology monotonically spacelike-proper for the case when the image of any spacelike compact set is itself spacelike-compact and monotonically timelike-proper for the case when the image of any timelike compact set is timelike compact. We should note that the above terminology is partly inspired by some general notions from the theory of partially ordered sets. A map $f\colon M \to N$ between two partially ordered sets $(M, \leq)$ and $(N, \leq)$ is said to be monotonic if $x \leq y$ implies $f(x) \leq f(y)$ and, on the other hand, it is said to be order-reflecting if $f(x) \leq f(y)$ implies $x \leq y$. The following theorem is a straight forward generalization of the previous arguments for the simpler case of compact supports.

Theorem 13. *Let $\mathfrak{CMan}_{sc}$ and $\mathfrak{CMan}_{tc}$ be the categories of conal manifolds with, respectively, reflectively spacelike-proper and reflectively timelike-proper, smooth maps as morphisms, while the $\mathfrak{CMan}^e_{sc}$ and $\mathfrak{CMan}^e_{tc}$ categories have, respectively, monotonically spacelike-proper and monotonically timelike-proper open embeddings as morphisms. Then, de Rham cohomologies with spacelike and timelike supports, $H^p_{sc}(-)$ and $H^p_{tc}(-)$, are contravariant functors on $\mathfrak{CMan}_{sc}$ and $\mathfrak{CMan}_{tc}$, respectively. Similarly, $H^p_{tc}(-)$ and $H^p_{sc}(-)$ are covariant functors on $\mathfrak{CMan}^e_{tc}$ and $\mathfrak{CMan}^e_{sc}$, respectively.*

---

[8] A continuous map is proper if the preimage of any compact set is compact.

Proof. The proof is a direct parallel of the above arguments for the case with compact supports, since the definitions have been specifically adapted to that argument. □

To show that the definitions of spacelike- and timelike-proper maps are in some sense natural, we give a couple of examples.

Lemma 14. *Let M be a manifold and two conal structures on it, $C \subseteq C' \subseteq TM$ (C is "slower" than C')* (Sect. 5.1). *The identity map is a reflectively spacelike-proper from (M, C') to (M, C) and reflectively timelike-proper from (M, C) to (M, C').*

Proof. Let $K \subseteq M$ be any compact subset. Then, by hypothesis, the C-influence set is smaller than the C'-influence set, $\overline{I}_C(K) \subseteq \overline{I}_{C'}(K)$. Therefore, any C-spacelike compact set is also C'-spacelike and hence the identity from (M, C') to (M, C) is reflectively spacelike-proper. On the other hand, if $U \subseteq M$ is C'-timelike compact, then we have the inclusion $\overline{I}_C(K) \cap U \subseteq \overline{I}_{C'}(K) \cap U$, the latter being compact. Therefore, U is also C-timelike compact and the identity from (M, C) to (M, C') is reflectively timelike-proper. □

Lemma 15. *Let (M, g) and (N, h) be two globally hyperbolic Lorentzian manifolds and $f : M \to N$ an open isometric embedding, such that the image of a Cauchy surface of M is a Cauchy surface of N. Then, f is monotonically timelike-proper.*

Proof. Let $U \subseteq M$ be timelike compact. According to [46], this is equivalent to U being contained between two Cauchy surfaces in (M, g), say $\Sigma_1, \Sigma_2 \subset M$. This means that the image, $f(U)$ is contained between $f(\Sigma_1)$ and $f(\Sigma_2)$, with the latter, by hypothesis, being Cauchy surfaces in (N, h). Thus, $f(U)$ is also timelike compact and the map f is monotonically timelike-proper. □

5.3. Other Differential Complexes

Our interest in computing the de Rham and Calabi cohomologies with causally restricted supports has was motivated by their importance in understanding the geometric structure of classical and quantum field theories [5, 6, 15, 19, 31, 34, 36, 47]. Namely, for a general class of linear field theories, one can formulate sufficient conditions for the non-degeneracy of the theory's Poisson structure and the completeness of compactly supported smeared fields as physical observables in terms of the cohomologies of corresponding differential complexes. Non-linear field theories can be studied in terms of their linearizations about arbitrary background solutions. To Maxwell electrodynamics corresponds the de Rham complex [36, Sect. 4.2]. To linearized gravity on constant curvature backgrounds, corresponds the Calabi complex [36, Sect. 4.4]. Similarly, to Yang–Mills linearized about a flat connection corresponds a twisted de Rham complex.

Each of these examples can be treated using the methods presented in this paper. Few other explicit examples of differential complexes corresponding to other field theories of physical interest seem to be known. In particular, they do not seem to be known for linearized gravity on non-constant curvature backgrounds and, perhaps, not even for Yang–Mills linearized about non-flat connections. On the other hand, there are strong abstract reasons to believe that such differential complexes do indeed exist [28, 44, 45].

If such a differential complex also shares the apparently crucial property of admitting cochain homotopies that generate hyperbolic and elliptic cochain maps (cf. the E_l^g, P_l^g, $E_l^{g_R}$ and $P_l^{g_R}$ maps of Sects. 4.2 and 4.3), then its causally restricted cohomologies can be related to those with unrestricted and compactly supported ones, as in Theorems 3 and 11.

If, in addition, such a differential complex could also be seen as resolving a locally constant sheaf, its unrestricted cohomologies could be computed by algebraic means, without actually solving complicated systems of differential equations, as in Sect. 4.3. The latter requirement is closely related to the initial differential operator in the complex having only a finite dimensional space of solutions (being of finite type), as is the case for the locally constant (de Rham) and Killing (Calabi) conditions.

The compactly supported cohomologies could also be obtained if the corresponding formally adjoint complex satisfied similar requirements, as illustrated in Sect. 4.3 by the appearance of the locally constant sheaf of Killing–Yano tensors.

6. Discussion

We have shown how to compute the de Rham cohomology with causally restricted supports (retarded, advanced, past compact, future compact, spacelike compact and timelike compact) on a globally hyperbolic Lorentzian spacetime, using special properties of the d'Alembert wave operator and its Green functions. The result (Theorems 2, 3; Corollary 5) expresses these causally restricted cohomologies in terms of the standard de Rham cohomologies of the spacetime manifold, with either unrestricted or compact supports. These results, confirm the independent similar results of the recent work [5]. However, since our method does not rely on the strong invariance properties of the de Rham complex under topological homotopies, we have also obtained further results. In particular, our method is also applicable to the Calabi complex (Theorem 11). The Calabi complex appears in linearized gravity on constant curvature backgrounds in a way similar to the de Rham complex in Maxwell theory. These results answer some questions that have naturally arisen in recent investigations of classical and quantum gauge theories on curved spacetimes.

Finally, we have also made comments about other questions that have naturally appeared in these investigations. Namely, we discussed the covariance of causally restricted cohomologies under specific types of morphisms between spacetimes, adapted to their causal structure, and under changes of the causal structure itself.

We have presented almost the bare minimum of information about the Calabi complex that is needed to obtain our results. A fuller discussion of this interesting complex, including relevant geometric properties that are difficult to locate in or are absent from the current literature, is deferred to future work [35]. In the future, it will also be interesting to find the analogs of the Calabi complex on more general Lorentzian backgrounds, which would consist of differential complexes resolving the sheaf of Killing vectors on a given background. However, we conjecture that the Hodge-like structure that we have used to compute causally restricted cohomologies will be shared by all of them.

Acknowledgements

The author would like to thank Marco Benini for useful discussions and also Romeo Brunetti, Claudio Dappiaggi, and Valter Moretti for their support in the course of this work.

References

[1] Ashtekar, A., Sen, A.: On the role of spacetime topology in quantum phenomena: Superselection of charge and emergence of nontrivial vacua. J. Math. Phys. **21**, 526–533 (2008)

[2] Baer, C.: Green-hyperbolic operators on globally hyperbolic spacetimes (2013). arXiv:1310.0738

[3] Baer, C., Ginoux, N., Pfaeffle, F.: Wave Equations on Lorentzian Manifolds and Quantization. Vol. 2 of ESI Lectures in Mathematics and Physics. European Mathematical Society (2007). arXiv:0806.1036

[4] Beem, J.K., Ehrlich, P., Easley, K.: Global Lorentzian Geometry. Vol. 202 of Pure and Applied Mathematics. Marcel Dekker, New York (1996)

[5] Benini, M.: Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies (2014). arXiv:1401.7563

[6] Benini, M., Dappiaggi, C., Schenkel, A.: Quantized Abelian principal connections on Lorentzian manifolds. Commun. Math. Phys. **330**, 123–152 (2013). arXiv:1303.2515

[7] Bergery, L.B., Bourguignon, J.-P., Lafontaine, J.: Déformations localement triviales des variétés riemanniennes. In: Differential Geometry, Part 1, Vol. 27 of Proceedings of Symposia in Pure Mathematics, pp. 3–32. AMS, Providence, RI (1975)

[8] Bernal, A., Sánchez, M.: Further results on the smoothability of cauchy hypersurfaces and cauchy time functions. Lett. Math. Phys. **77**, 183–197 (2006). arXiv:gr-qc/0512095

[9] Bernal, A.N., Sánchez, M.: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. Commun. Math. Phys. **257**, 43–50 (2005). arXiv:gr-qc/0401112

[10] Bini, D., Cherubini, C., Jantzen, R.T., Ruffini, R.: de Rham wave equation for tensor valued p-forms. Int. J. Modern Phys. D **12**, 1363–1384 (2003)

[11] Bott, R., Tu, L.W.: Differential Forms in Algebraic Topology. Vol. 82 of Graduate Texts in Mathematics. Springer, New York (1982)

[12] Bredon, G.E.: Sheaf Theory. Graduate Texts in Mathematics. Springer, New York (1997)

[13] Calabi, E.: On compact, Riemannian manifolds with constant curvature. I. In: Allendoerfer, C.B. (ed.) Differential Geometry, Vol. 3 of Proceedings of Symposia in Pure Mathematics, pp. 155–180. AMS, Providence (1961)

[14] Christodoulou, D., Klainerman, S.: The global nonlinear stability of the Minkowski space. Vol. 41 of Princeton Mathematical Series. Princeton University Press, Princeton (1993)

[15] Dappiaggi, C., Lang, B.: Quantization of Maxwell's equations on curved backgrounds and general local covariance. Lett. Math. Phys. **101**, 265–287 (2012). arXiv:1104.1374

[16] Eastwood, M.: Variations on the de Rham complex. Notices Am. Math. Soc. **46**, 1368–1376 (1999). http://www.ams.org/notices/199911/fea-eastwood

[17] Fathi, A., Siconolfi, A.: On smooth time functions. Math. Proc. Camb. Philos. Soc. **152**, 303–339 (2011)

[18] Fewster, C.J., Hunt, D.S.: Quantization of linearized gravity in cosmological vacuum spacetimes. Rev. Math. Phys. **25**, 1330003 (2013). arXiv:1203.0261

[19] Fewster, C.J., Lang, B.: Dynamical locality of the free Maxwell field (2014). arXiv:1403.7083

[20] Fulling, S.A., King, R.C., Wybourne, B.G., Cummins, C.J.: Normal forms for tensor polynomials. I. The Riemann tensor. Class. Quantum Gravity **9**, 1151 (1992)

[21] Fulton, W.: Young Tableaux: With Applications to Representation Theory and Geometry. Vol. 35 of London Mathematical Society Student Texts. Cambridge University Press, New York (1997)

[22] García-Parrado, A., Senovilla, J.M.M.: Causal structures and causal boundaries. Class. Quantum Gravity **22**, R1–R84 (2005). arXiv:gr-qc/0501069

[23] Gasqui, J., Goldschmidt, H.: Déformations infinitésimales des espaces riemanniens localement symétriques. I. Adv. Math. **48**, 205–285 (1983)

[24] Gasqui, J., Goldschmidt, H.: Complexes of differential operators and symmetric spaces. In: Hazewinkel, M., Gerstenhaber, M. (eds.) Deformation Theory of Algebras and Structures and Applications, Vol. 247 of NATO ASI Series, pp. 797–827. Kluwer, Dordrecht (1988)

[25] Geroch, R.: Domain of dependence. J. Math. Phys. **11**, 437–449 (1970)

[26] Ginoux, N.: Linear wave equations. In: Baer, C., Fredenhagen, K. (eds.) Quantum Field Theory on Curved Spacetimes: Concepts and Methods, Vol. 786 of Lecture Notes in Physics. Springer, Berlin (2009)

[27] Goldberg, S.I.: Curvature and Homology. Dover, Mineola (1998)

[28] Goldschmidt, H.: Existence theorems for analytic linear partial differential equations. Ann. Math. **86**, 246–270 (1967)

[29] Goldschmidt, H.: Duality theorems in deformation theory. Trans. Am. Math. Soc. **292**, 1 (1985)

[30] Hack, T.-P.: Quantization of the linearised Einstein–Klein–Gordon system on arbitrary backgrounds and the special case of perturbations in inflation (2014). arXiv:1403.3957.

[31] Hack, T.-P., Schenkel, A.: Linear bosonic and fermionic quantum gauge theories on curved spacetimes. Gen. Relativ. Gravit. **45**, 877–910 (2013). arXiv:1205.3484

[32] Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-Time. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (1973)

[33] Jost, J.: Riemannian Geometry and Geometric Analysis. Springer, Berlin (2011)

[34] Khavkine, I.: Characteristics, conal geometry and causality in locally covariant field theory (2012). arXiv:1211.1914

[35] Khavkine, I.: The Calabi complex and Killing sheaf cohomology (2014). arXiv:1409.7212

[36] Khavkine, I.: Covariant phase space, constraints, gauge and the Peierls formula. Int. J. Mod. Phys. A **29**, 1430009 (2014). arXiv:1402.1282

[37] Kronheimer, E.H., Penrose, R.: On the structure of causal spaces. Math. Proc. Camb. Philos. Soc. **63**, 481–501 (1967)

[38] Lawson, J.D.: Ordered manifolds, invariant cone fields, and semigroups. Forum Math. **1**, 273–308 (1989)

[39] Leyland, P., Roberts, J.E.: The cohomology of nets over Minkowski space. Commun. Math. Phys. **62**, 173–189 (1978)

[40] Lichnerowicz, A.: Propagateurs, commutateurs et anticommutateurs en relativité générale. In: DeWitt, C., DeWitt, B.S. (eds.) Relativity, Groups and Topology, pp. 821–861. Gordon and Breach, New York (1964)

[41] Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Physics Series. W. H. Freeman, San Francisco (1973)

[42] Neeb, K.H.: Conal orders on homogeneous spaces. Invent. Math. **104**, 467–496 (1991)

[43] O'Neill, B.: Semi-Riemannian Geometry With Applications to Relativity. Vol. 103 of Pure and Applied Mathematics. Academic Press, San Diego (1983)

[44] Pommaret, J.-F.: Systems of Partial Differential Equations and Lie Pseudogroups. Vol. 14 of Mathematics and its Applications. Gordon and Breach, New York (1978)

[45] Quillen, D.G.: Formal properties of over-determined systems of partial differential equations. PhD thesis, Harvard University (1964)

[46] Sanders, K.: A note on spacelike and timelike compactness. Class. Quantum Gravity **30**, 115014 (2012). arXiv:1211.2469

[47] Sanders, K., Dappiaggi, C., Hack, T.-P.: Electromagnetism, local covariance, the Aharonov–Bohm effect and Gauss' law. Commun. Math. Phys. **328**, 625–667 (2014), arXiv:1211.6420

[48] Serre, J.-P.: Un théorème de dualité. Commentarii Mathematici Helvetici **29**, 9–26 (1955)

[49] Stepanov, S.E.: The Killing–Yano tensor. Theor. Math. Phys. **134**, 333–338 (2003)

[50] Sullivan, D.: Cycles for the dynamical study of foliated manifolds and complex manifolds. Inventiones Mathematicae **36**, 225–255 (1976)

[51] Tarkhanov, N.N.: Complexes of Differential Operators. Vol. 340 of Mathematics and its Applications. Kluwer, Dordrecht (1995)

[52] Wald, R.M.: General Relativity. University of Chicago Press, Chicago (1984)

Cohomology with Causally Restricted Supports

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Communicated by Karl-Henning Rehren.
Received: November 6, 2015.
Accepted: February 21, 2016.

IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. 35 (2018) 035013 (37pp) https://doi.org/10.1088/1361-6382/aa9f61

IDEAL characterization of isometry classes of FLRW and inflationary spacetimes

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Received 22 May 2017, revised 23 October 2017
Accepted for publication 5 December 2017
Published 8 January 2018

Abstract

In general relativity, an IDEAL (Intrinsic, Deductive, Explicit, ALgorithmic) characterization of a reference spacetime metric g_0 consists of a set of tensorial equations $T[g] = 0$, constructed covariantly out of the metric g, its Riemann curvature and their derivatives, that are satisfied if and only if g is locally isometric to the reference spacetime metric g_0. The same notion can be extended to also include scalar or tensor fields, where the equations $T[g, \phi] = 0$ are allowed to also depend on the extra fields ϕ. We give the first IDEAL characterization of cosmological FLRW spacetimes, with and without a dynamical scalar (inflaton) field. We restrict our attention to what we call regular geometries, which uniformly satisfy certain identities or inequalities. They roughly split into the following natural special cases: constant curvature spacetime, Einstein static universe, and flat or curved spatial slices. We also briefly comment on how the solution of this problem has implications, in general relativity and inflation theory, for the construction of local gauge invariant observables for linear cosmological perturbations and for stability analysis.

Keywords: differential invariants, cosmology, equivalence up to isometry

(Some figures may appear in colour only in the online journal)

1. Introduction

In this work, we are interested in an intrinsic characterization of homogeneous and isotropic cosmological spacetimes (also known as Friedmann–Lemaître–Robertson–Walker or *FLRW spacetimes*), either with or without the presence of a scalar field (aka *inflationary spacetimes*). By a spacetime (M, g), we mean a smooth manifold M with a Lorentzian metric g. While 'intrinsic' generally does preclude direct reference to the form of the spacetime metric in a special coordinate system, it is a vague enough term to have multiple interpretations. To be specific, we refer to an IDEAL[5] or *Rainich-type* characterization that has been used, for instance, in the works [6, 12, 14, 15, 20, 27, 33]. It consists of a list of tensorial equations ($T_a[g] = 0$, $a = 1, 2, \ldots, N$), constructed covariantly out of the metric (g) and its derivatives (concomitants of the Riemann tensor) that are satisfied if and only if the given spacetime locally belongs to the desired class, possibly narrow enough to be the isometry class of a reference spacetime geometry. This notion has a natural generalization ($T_a[g, \phi] = 0$) to spacetimes equipped with scalar or tensor fields (ϕ), with equivalence still given by isometric diffeomorphisms that also transform the additional scalars or tensors into each other. A nice historical survey of this and other local characterization results can be found in [22].

An IDEAL characterization neither requires the existence of any extra geometric structures, nor the translation of the metric and of the curvature into a frame formalism. Thus, it is an alternative to the Cartan–Karlhede characterization [31, chapter 9], which is based on Cartan's moving frame formalism. Intrinsic characterizations, of various types, have been of long standing and independent interest in geometry and general relativity. But, in addition, they can be helpful in deciding when a metric, given for instance by some complicated coordinate formulas, corresponds to one that is already known. In this regard, an IDEAL characterization is especially helpful if one would like to find an algorithmic solution to this recognition problem. In numerical relativity, the near-satisfaction of the tensor equations $T_a[g] \approx 0$ may signal the local proximity of a numerical spacetime to a desired reference geometry. In addition, the approach to zero of $T_a[g] \to 0$ could be used to study either linear or nonlinear stability of reference geometries, in an unambiguous and gauge independent way.

The following particular application should be noted. By the Stewart–Walker lemma [32, lemma 2.2], the vanishing of a tensor concomitant $T_a[g] = 0$ for a metric g implies that its linearization $\dot{T}_a[h]$ ($T_a[g + \varepsilon h] = T_a[g] + \varepsilon \dot{T}_a[g] + O(\varepsilon^2)$) is invariant under linearized diffeomorphisms. Thus, any quantity of the form $\dot{T}_a[h]$ defines a gauge invariant observables in linearized gravity, when Einstein or Einstein-matter equations are linearly perturbed about a background solution g. A straight forward argument shows that an IDEAL characterization provides a list $\dot{T}_a[h]$, $a = 1, \ldots, N$, of gauge invariant observables that is also complete (it suffices to check that $T_a[g + h]$ do not approach zero at $O(h^2)$ or higher order). That is, the joint kernel of $\dot{T}_a[h] = 0$ locally consists only of pure gauge modes ($h = \mathcal{L}_v g$ for some vector field v). The use of such local observables (given by differential operators) can be advantageous both in theoretical and practical investigations of classical and quantum field theoretical models because they cleanly separate the local (or ultraviolet) and global (or infrared) aspects of the theory. This is of particular and current relevance to some controversies in inflationary models of early universe cosmology [23, 34]. Despite their importance, complete lists of (linearized) local gauge invariant observables have been explicitly produced only in very few cases, by ad hoc methods. For instance, in the case of Einstein equations coupled to a single inflaton field, a complete list has been produced only recently [17]. On the other hand,

[5] The acronym, explained in [14] (footnote, p.2), stands for intrinsic, deductive, explicit and algorithmic.

linearising the equations of an IDEAL characterization provides a systematic method of construction. The results of this method can be compared to those of [17] and are equivalent [18]. Since these two sets of results naturally appear in rather different forms, a detailed comparison is beyond the scope of this work and will be presented elsewhere.

A similar geometric approach to the construction of gauge invariant linearized observables was taken in [11], using what we would call a partial IDEAL characterization of cosmological spacetimes. No proof of their completeness was ever given. In a sense, we complete the earlier literature in this regard.

In this work, we add the cases of FLRW and inflationary spacetimes to the (unfortunately still small) literature concerning IDEAL characterizations of isometry classes of individual reference geometries. Other IDEAL characterizations for geometries of interest in general relativity include Schwarzschild [12], Reissner–Nordström [13], Kerr [15], Lemaître–Tolman–Bondi [14], Stephani universes [16] (see references for complete lists and details) and of course the classic cases of constant curvature spaces, which are known to be fully characterized by the structure of the Riemann tensor (by theorems of Riemann and Killing–Hopf).

The synopsis of the paper is the following: in section 1.1 we fix our notation and we outline our main results on the IDEAL characterization of FLRW spacetimes (theorem 1.4) and inflationary spacetimes (theorem 1.5). Our main goal there is to discuss our findings without dwelling on the technical proofs, which are left to the next sections. Hence a reader who wishes to focus more on the physical aspects of this paper should refer mainly to this part of the paper. In addition, still in section 1.1, we provide flowcharts for classifying spacetimes into FLRW and inflationary isometry classes, visually summarizing the contents of theorems 1.4 and 1.5. In section 2 we collect relevant information on the geometry of FLRW and inflationary spacetimes. In section 3, we distinguish the possible local isometry classes of FLRW or inflationary geometries and prove our main theorems.

1.1. Main results

In this subsection, our goal is to introduce our conventions and to outline our main results. Therefore we will not dwell on the mathematical proofs, but we will focus on the basic technical tools, necessary to formulate and to understand the physical significance of our findings.

In this work, a *spacetime* or *Lorentzian manifold* (M, g) will be a smooth finite dimensional manifold M (also Hausdorff, second countable, connected and orientable) of $\dim M = m + 1 \geqslant 2$, with a Lorentzian metric g (with signature $-+\cdots+$). A *spacetime with scalar* will consist of a triple (M, g, ϕ), where (M, g) is a Lorentzian manifold and $\phi\colon M \to \mathbb{R}$ is a smooth scalar field. Obviously, we could always consider the spacetime (M, g) as the special spacetime with zero scalar, $(M, g, 0)$. In addition, with inflationary spacetimes, we will be assuming that the metric and the scalar field satisfy the coupled Einstein–Klein–Gordon equations, possibly with a nonlinear potential.

These observations should be kept in mind while reading the following

Definition 1.1 (Locally isometric). A spacetime with scalar (M_1, g_1, ϕ_1) is *locally isometric at* $x_1 \in M_1$ *to* a spacetime with scalar (M_2, g_2, ϕ_2) *at* $x_2 \in M$ if there exist open neighbourhoods $U_1 \ni x_1$, $U_2 \ni x_2$ and a diffeomorphism $\chi\colon U_1 \to U_2$ such that $\chi(x_1) = x_2$, $\chi^* g_2 = g_1$ and $\chi^* \phi_2 = \phi_1$. If we can choose $U_1 = M_1$ and $U_2 = M_2$ then they are *(globally) isometric*. If for every $x_1 \in M$ there is $x_2 \in M_2$ such that (M_1, g_1, ϕ_1) at x_1 is locally isometric to (M_2, g_2, ϕ_2) at x_2, we simply say that (M_1, g_1, ϕ_1) is *locally isometric to* (M_2, g_2, ϕ_2) (note the asymmetry in the definition). If (M_1, g_1, ϕ_1) is locally isometric to (M_2, g_2, ϕ_2), as well as vice versa, we say that they are *locally isometric to each other* (which constitutes an equivalence relation). All spacetimes with scalar that are locally isometric to a reference (M, g, ϕ) constitute its *local isometry class*.

Our main results give an IDEAL characterization of local isometry classes of *regular* FLRW and inflationary spacetimes. In the following we give their precise definition, which is motivated in more detail in sections 2.3 and 2.4. Starting from the first case:

Definition 1.2 (Regular FLRW spacetime). Let us fix a constant $\kappa \neq 0$. Denote by the triple (m, α, f), of a dimension $m \geqslant 1$, a constant $\alpha \in \mathbb{R}$ and a smooth positive function $f\colon I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$, the corresponding FLRW spacetime $(M, g) = (I \times F, -\mathrm{d}t^2 + f^2 g^F)$ (definition 2.2), with α the sectional curvature of (F, g^F) and $F \cong S^m$ (when $\alpha > 0$) or $F \cong \mathbb{R}^m$ (when $\alpha \leqslant 0$).

We call (M, g) a *regular FLRW spacetime* if it belongs to one of the parametrized families identified below.

(a) Constant curvature spacetime, with spacetime sectional curvature K:

$$\mathrm{CC}^m_K = \begin{cases} \{(m, K, \cosh(\sqrt{K}t)) & \mid K > 0,\ I = \mathbb{R}\}, \\ \{(m, 0, 1) & \mid K = 0,\ I = \mathbb{R}\}, \\ \{(m, K, \cos(\sqrt{-K}t)) & \mid K < 0,\ I = \mathbb{R}\}. \end{cases} \tag{1}$$

(b) Einstein static universe, with spatial sectional curvature $K \neq 0$:

$$\mathrm{ESU}^m_K = \{(m, K, 1) \mid m > 1,\ I = \mathbb{R}\}. \tag{2}$$

(c) Spatially flat constant scalar curvature spacetime, with spacetime scalar curvature $m(m+1)K$ and such that $\frac{f'^2}{f^2}(I) = J$:

$$\begin{aligned} \mathrm{CSC}^{m,0}_{K,J} = \{(m, \alpha, f) \mid m > 1,\ & f' \neq 0, \\ \left(\frac{f''}{f} - \frac{f'^2}{f^2}\right) + \frac{(m+1)}{2}\left(\frac{f'^2}{f^2} - K\right) &= 0\Big\}. \end{aligned} \tag{3}$$

(d) Generic constant scalar curvature spacetime, with spacetime scalar curvature $m(m+1)K$, normalized radiation density constant Ω and such that $\frac{\alpha}{f^2}(I) = J$:

$$\begin{aligned} \mathrm{CSC}^m_{K,\Omega,J} = \{(m, \alpha, f) \mid m > 1,\ & \alpha \neq 0,\ f' \neq 0, \\ \frac{f'^2}{f^2} + \frac{\alpha}{f^2} &= K + \Omega\frac{|\alpha|^{(m+1)/2}}{f^{m+1}}\Big\}. \end{aligned} \tag{4}$$

(e) Spatially flat FLRW spacetime with normalized pressure function P defined on an open interval J, with $0 < \frac{f'^2}{f^2}(I) = J$ and

$$P(u)\left[\partial_u P(u) - \frac{1}{2\kappa}\right] \neq 0 \tag{5}$$

everywhere on J:

$$\mathrm{FLRW}^{m,0}_{P,J} = \left\{(m, 0, f) \mid \left(\frac{f''}{f} - \frac{f'^2}{f^2}\right) + \frac{m}{2}\frac{f'^2}{f^2} = -\kappa P\left((f'/f)^2\right)\right\}. \tag{6}$$

(f) Generic FLRW spacetime with normalized energy function E defined on an open interval J, with $0 \notin \frac{\alpha}{f^2}(I) = J$ and

$$\partial_u\left[u\partial_u E(u)-\frac{(m+1)}{2}\right]\neq 0 \tag{7}$$

everywhere on J:

$$\mathrm{FLRW}^m_{E,J}=\left\{(m,\alpha,f)\mid m>1,\ \alpha\neq 0,\ \frac{f'^2}{f^2}+\frac{\alpha}{f^2}=\kappa E(\alpha/f^2)\right\}. \tag{8}$$

Next, we focus our attention to the inflationary spacetimes, following the more detailed motivation from sections 2.5 and 2.6:

Definition 1.3 (Regular inflationary spacetime). Let us fix a constant $\kappa\neq 0$. Denote by the quadruple (m,α,f,ϕ), of dimension $m>1$, constant $\alpha\in\mathbb{R}$, and smooth functions $f,\phi\colon I\to\mathbb{R}$ defined on an interval $I\subseteq\mathbb{R}$, with f positive, the corresponding inflationary spacetime $(M,g,\phi)=(I\times F,-\mathrm{d}t^2+f^2g^F,\bar\phi)$ (definition 2.10), with $\bar\phi$ being the composition of standard projection $I\times F\to I$ with ϕ, with α the sectional curvature of (F,g^F) and $F\cong S^m$ (when $\alpha>0$) or $F\cong\mathbb{R}^m$ (when $\alpha\leqslant 0$).

We call $(M,g,\bar\phi)$ a *regular inflationary spacetime* if it belongs to one of the parametrized families identified below.

(a) Constant scalar, with scalar value Φ, on a constant curvature spacetime with scalar curvature K:

$$\mathrm{CC}^m_K\mathrm{CS}_\Phi=\{(m,\alpha,f,\Phi)\mid(m,\alpha,f)\in\mathrm{CC}^m_K\}. \tag{9}$$

(b) Constant energy scalar, with energy density $\rho>0$ and $J=\phi(I)$, on an Einstein static universe with spatial sectional curvature $K=\frac{2}{m(m-1)}\kappa\rho$, or equivalently with cosmological constant $\Lambda=\frac{(m-1)}{m}\kappa\rho$:

$$\mathrm{ESU}^m_K\mathrm{CES}_{\rho,J}=\{(m,K,1,\sqrt{2\rho/m}t)\mid I=J/\sqrt{2\rho/m}\}. \tag{10}$$

(c) Spatially flat massless minimally-coupled scalar spacetime, with cosmological constant Λ, $J=\phi(I)$ and $J'=\frac{f'}{f}(I)\not\ni 0$ and $\frac{2\Lambda/\kappa}{m(m-1)}<\frac{1}{\kappa}(J')^2$:

$$\mathrm{MMS}^{m,0}_{\Lambda,J,J'}=\left\{(m,0,f,\phi)\mid\phi'<0,\ \frac{f'}{f}\neq 0,\right.$$
$$\left.\frac{f'^2}{f^2}=\frac{\kappa\phi'^2+2\Lambda}{m(m-1)},\ \left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)+m\frac{f'^2}{f^2}=\frac{2\Lambda}{(m-1)}\right\}. \tag{11}$$

(d) Generic massless minimally-coupled scalar spacetime, with cosmological constant Λ, normalized scalar energy constant $\Omega>0$, $J=\phi(I)$ and $J'=\frac{f'}{f}(I)\not\ni 0$:

$$\mathrm{MMS}^m_{\Lambda,\Omega,J,J'}=\left\{(m,\alpha,f,\phi)\mid\alpha\neq 0,\ \frac{f'}{f}\neq 0,\right.$$
$$\left.\phi'=-\sqrt{\Omega}\frac{|\alpha|^{\frac{m}{2}}}{f^m},\ \frac{f'^2}{f^2}+\frac{\alpha}{f^2}=\frac{2\Lambda+\kappa\Omega|\alpha|^m/f^{2m}}{m(m-1)}\right\}. \tag{12}$$

(e) Spatially flat nonlinear Klein–Gordon spacetime, with non-constant scalar self-coupling potential $V\colon J\to\mathbb{R}$, with $J=\phi(I)$, and expansion profile $\Xi\colon J\to\mathbb{R}$, satisfying $\Xi(u)\neq 0$, $\frac{1}{\kappa}\partial_u\Xi(u)>0$ and $\mathfrak{H}_V(\Xi)=0$ in the notation of (18):

$$\mathrm{NKG}^{m,0}_{V,\Xi,J}=\left\{(m,0,f,\phi)\mid\frac{f'}{f}=\Xi(\phi),\phi'=-\frac{(m-1)}{\kappa}\partial_\phi\Xi(\phi)\right\}. \tag{13}$$

(f) Generic nonlinear Klein–Gordon spacetime, with non-constant scalar potential $V\colon J \to \mathbb{R}$, with $J = \phi(I)$, and expansion profile $(\Pi, \Xi)\colon J \to \mathbb{R}^2$, satisfying $\Pi < 0$, $\Xi \neq 0$, $\kappa \frac{\Pi^2+V}{m(m-1)} \neq \Xi^2$ and $\mathfrak{G}_V(\Pi, \Xi) = 0$ in the notation of (20):

$$\mathrm{NKG}^m_{V,\Pi,\Xi,J} = \Big\{ (m, \alpha, f, \phi) \mid \alpha \neq 0,\ \frac{f'}{f} \neq 0, \\ \phi' = \Pi(\phi),\ \tfrac{f'}{f} = \Xi(\phi),\ \tfrac{f'^2}{f^2} + \tfrac{\alpha}{f^2} = \kappa \tfrac{\phi'^2+V(\phi)}{m(m-1)} \Big\}. \tag{14}$$

Below we directly give the list of tensor equations, covariantly constructed from the metric, the Riemann curvature, and its derivatives, that characterize the corresponding local isometry classes. Observe that an IDEAL characterization is not unique. Given one, many others can be produced by covariant and invertible transformations. Our choices are based on various conventions used in relativity and cosmology.

To be specific, our conventions for the relations between the metric g_{ij}, covariant derivative ∇_i, Riemann curvature, as well as Ricci tensor and scalar are the following:

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)v_k = R_{ijk}{}^l v_l, \quad R_{ijkh} = R_{ijk}{}^l g_{lh},$$
$$R_{ik} = R_{ikj}{}^k, \quad \mathcal{R} = R_{ij}g^{ij}, \quad \mathcal{B} = R_{ij}R_{kl}g^{ik}g^{jl}.$$

It is convenient to define the following product (sometimes also known as the *Kulkarni–Nomizu product*) that builds an object with the symmetries of the Riemann tensor out of symmetric 2-tensors A_{ij}, B_{ij}:

$$(A \odot B)_{ijkh} = A_{ik}B_{jh} - A_{jk}B_{ih} - A_{ih}B_{jk} + A_{jh}B_{ik}. \tag{15}$$

Note that $A \odot B = B \odot A$ and $UU \odot UU = 0$, with $(UU)_{ij} = U_iU_j$ and U_i any vector field. For $\dim M = m + 1 > 2$, our formula for the Weyl tensor is

$$W_{ijkh} = R_{ijkh} - \frac{1}{(m-1)}(g \odot R)_{ijkh} + \frac{1}{2m(m-1)}\mathcal{R}(g \odot g)_{ijkh}. \tag{16}$$

Note that W_{ijkh} vanishes precisely when $R_{ijkh} = (g \odot A)_{ijkh}$ for some symmetric A_{ij}. As usual, we denote idempotent symmetrization and anti-symmetrization by $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$, $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$.

The first theorem classifies just the Lorentzian spacetime, without reference to a scalar field. The definitions for the various scalar and tensor fields introduced below may seem ad hoc, but they have straightforward geometrical meanings. The vector field U^i plays the role of a future-pointing unit timelike vector field, orthogonal to the cosmological spatial slices. It is defined as a normalized gradient of a curvature scalar, with the choice of curvature scalar depending on the precise case being considered. The tensors $\mathfrak{P}_{ij}$ and $\mathfrak{D}_{ij}$ encode in them the shear, twist and geodesic character of U^i and are non-zero when the spacetime deviates from a generalized Robertson–Walker (GRW) spacetime (a possibly non-homogeneous geometry undergoing cosmological expansion or contraction). The expansion ξ of the vector field U^i also plays the role of the Hubble rate, while $\boldsymbol{\eta}$ that of the Hubble acceleration. The tensor $\mathfrak{E}_{ijkh}$ measures the deviation from the spatial slices from homogeneity and isotropy, while the scalar ζ, together with $\mathfrak{E}_{ijkh}$, measures the deviation of spatial slices from flatness.

Theorem 1.4. *Consider a Lorentzian manifold (M, g) of $\dim M = m + 1 \geqslant 2$, $\kappa \neq 0$ a fixed constant. With U a unit timelike vector field, consider the following notations, which are defined when possible:*

$$
\begin{aligned}
&\xi := \frac{\nabla^i U_i}{m}, \quad \boldsymbol{\eta} := U^i \nabla_i \xi, \quad \zeta := \frac{\mathcal{R} - 2m\boldsymbol{\eta} - \frac{1}{2}m(m+1)\xi^2}{m(m-1)}, \\
&\mathfrak{P}_{ij} := U_{[i}\nabla_{j]}\xi, \quad \mathfrak{D}_{ij} := \nabla_i U_j - \frac{\nabla^k U_k}{m}(g_{ij} + U_i U_j), \\
&\mathfrak{Z}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{\xi^2}{2} g - \boldsymbol{\eta} UU \right] \right)_{ijkh}, \\
&\mathfrak{C}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{(\xi^2 + \zeta)}{2} g - (\boldsymbol{\eta} - \zeta) UU \right] \right)_{ijkh}, \\
&U_{\mathcal{R}} := \frac{-\nabla \mathcal{R}}{\sqrt{-(\nabla \mathcal{R})^2}}, \quad U_{\mathcal{B}} := \frac{-\nabla \mathcal{B}}{\sqrt{-(\nabla \mathcal{B})^2}}.
\end{aligned}
\tag{17}
$$

Given $x \in M$, table 1 gives the list of inequalities and equations (right column, written using the above notation, with a specific choice of U) that are satisfied on a neighborhood of x if and only if the Lorentzian manifold belongs to the corresponding local isometry class at x (left column) of a regular FLRW spacetime (definition 1.2). Each local isometry class belongs to a family, parametrized by real constants, intervals or functions (middle column). By continuity, each inequality need only be checked at x.

In addition, since both theorem 1.4 and table 1 are densely packed with information, we include a graphical flowchart summaries of the same information in figure 1. The notation is the same as in the original theorems.

Finally, we state the theorem classifying inflationary spacetimes, those endowed with scalar and satisfying the coupled Einstein–Klein–Gordon equations, where the equation for the scalar ϕ may be nonlinear due to a potential $V(\phi)$. The reader is referred to the paragraph preceding theorem 1.4 for an explanation of the notation. The new scalar $\mathfrak{H}_V$ roughly corresponds to the Hamilton–Jacobi equation of spatially flat single field inflation, while $\mathfrak{G}_V$ is its generalization to the non-flat case. See the end of section 2.6 for a more detailed motivation.

Theorem 1.5. *Consider an inflationary spacetime (M, g, ϕ) of $\dim M = m + 1 > 2$, $\kappa \neq 0$ a fixed constant. With U a unit timelike vector field, recall the notation of theorem 1.4, supplemented with*

$$
(-)' := U^i \nabla_i (-), \quad U_\phi := \frac{\nabla \phi}{\sqrt{-(\nabla \phi)^2}}, \tag{18}
$$

$$
\mathfrak{H}_V(\Xi) := (\partial_u \Xi)^2 - \kappa \frac{m\Xi^2}{(m-1)} + \kappa^2 \frac{V}{(m-1)^2}, \tag{19}
$$

$$
\mathfrak{G}_V(\Pi, \Xi) := \begin{pmatrix} \Pi \left(\partial_u \Xi + \kappa \frac{\Pi}{(m-1)} \right) - \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) \\ \partial_u \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) + 2 \frac{\Xi}{\Pi} \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) \end{pmatrix}, \tag{20}
$$

where $\Xi = \Xi(u)$ and $\Pi = \Pi(u)$. Let g and ϕ satisfy the coupled Einstein–Klein–Gordon equations with scalar potential $V(\phi)$,

$$
\nabla^i \nabla_i \phi - \frac{1}{2} \partial_\phi V(\phi) = 0, \tag{21}
$$

Table 1. IDEAL characterization of local isometry classes of regular FLRW spacetimes (theorem 1.4).

Class	Parameters/U	Inequalities/equations
(a) Constant curvature		
CC^m_K		$R_{ijkh} - \frac{K}{2}(g \odot g)_{ijkh} = 0$
(b) Einstein static universe		
ESU^m_K	$m > 1$, $K \neq 0$	$\exists V^i\colon \left(g_{ij} - \frac{R_{ij}}{(m-1)K}\right) V^i V^j < 0$
		$W_{ijkh} = 0, \quad \nabla_i R_{jk} = 0,$ $R_i^j\left(R_j^k - (m-1)K\delta_j^k\right) = 0,$ $\mathcal{R} - m(m-1)K = 0$
(c) Spatially flat constant scalar curvature		
$\mathrm{CSC}^{m,0}_{K,J}$	$m > 1$, $0 < J \subset \mathbb{R}$, $(U = U_{\mathcal{B}})$	$(\nabla\mathcal{B})^2 < 0, \quad \xi^2(x) \in J$
		$\mathfrak{P}_{ij} = 0, \quad \mathfrak{D}_{ij} = 0,$ $\mathfrak{Z}_{ijkh} = 0,$ $\eta + \frac{(m+1)}{2}(\xi^2 - K) = 0$
(d) Generic constant scalar curvature		
$\mathrm{CSC}^m_{K,\Omega,J}$	$m > 1, \quad \Omega \neq 0,$ $0 \notin J \subset \mathbb{R}$, $(U = U_{\mathcal{B}})$	$(\nabla\mathcal{B})^2 < 0, \quad \zeta(x) \in J$
		$\nabla_i U_j - \frac{\nabla_i\zeta}{2\zeta} U_j - \xi g_{ij} = 0,$ $\mathfrak{C}_{ijkh} = 0,$ $\xi^2 + \zeta - K - \Omega\lvert\zeta\rvert^{\frac{m+1}{2}} = 0$
(e) Spatially flat FLRW		
$\mathrm{FLRW}^{m,0}_{P,J}$	$0 < J \subset \mathbb{R}$, $P\colon J \to \mathbb{R}$, $P[\partial_u P - \frac{1}{2\kappa}] \neq 0$ $(U = U_{\mathcal{R}})$	$(\nabla\mathcal{R})^2 < 0, \quad \eta \neq 0, \quad \xi^2(x) \in J$
		$\mathfrak{P}_{ij} = 0, \quad \mathfrak{D}_{ij} = 0,$ $\mathfrak{Z}_{ijkh} = 0,$ $\eta + \frac{m}{2}\xi^2 + \kappa P(\xi^2) = 0$
(f) Generic FLRW		
$\mathrm{FLRW}^m_{E,J}$	$m > 1, \quad 0 \notin J \subset \mathbb{R}$, $E\colon J \to \mathbb{R}, \ \kappa E(u) > u$, $\partial_u[u\partial_u E - \frac{(m+1)}{2}E] \neq 0$ $(U = U_{\mathcal{R}})$	$(\nabla\mathcal{R})^2 < 0, \quad \xi \neq 0, \quad \zeta(x) \in J$
		$\nabla_i U_j - \frac{\nabla_i\zeta}{2\zeta} U_j - \xi g_{ij} = 0,$ $\mathfrak{C}_{ijkh} = 0,$ $\xi^2 + \zeta - \kappa E(\zeta) = 0$

$$R_{ij} - \frac{1}{2}\mathcal{R}g_{ij} = \kappa\left((\nabla_i\phi)(\nabla_j\phi) - \frac{1}{2}g_{ij}[(\nabla\phi)^2 + V(\phi)]\right). \tag{22}$$

Given $x \in M$, table 2 gives the list of inequalities and equations (right column) that are satisfied on some neighborhood of x if and only if the inflationary spacetime belongs to the corresponding local isometry class at x (left column) of a regular inflationary spacetime. Each

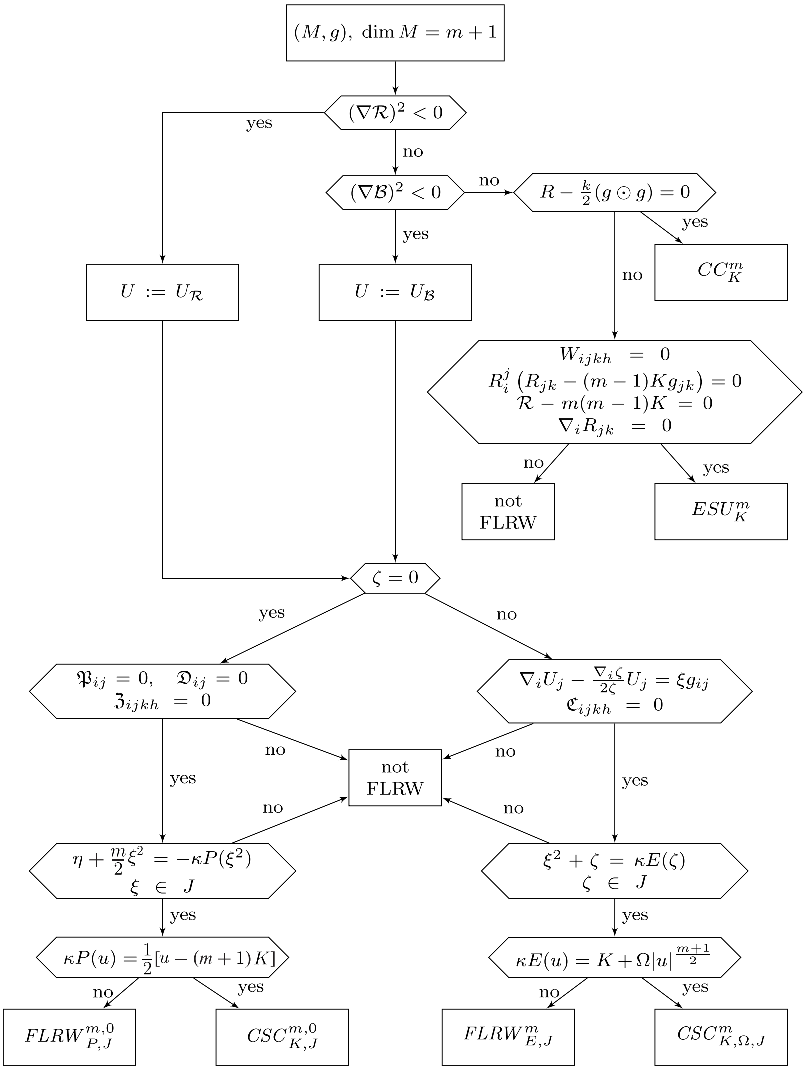

Figure 1. IDEAL characterization of local isometry classes of regular FLRW spacetimes (theorem 1.4, table 1).

local isometry class belongs to a family, parametrized by real constants, intervals or functions (middle column). By continuity, each inequality needs only to be checked at x.

In addition, since both theorem 1.5 and table 2 are densely packed with information, we include a graphical flowchart summaries of the same information in figure 2. The notation is the same as in the original theorems.

Note that when we make the choice $U = U_\phi$, it automatically follows that $\phi' = -\sqrt{-(\nabla\phi)^2} < 0$. This convention is common in the study of inflation, where $\phi(t)$ starts off at a high value and then 'rolls down hill' as t increases. This is reflected in the inequalities in table 2.

Table 2. IDEAL characterization of local isometry classes of regular inflationary spacetimes (theorem 1.5).

Class	Parameters	Inequalities/equalities
(a) Constant scalar		
$\mathrm{CC}^m_K\mathrm{CS}_\Phi$	$V(u)=\frac{2}{\kappa}\Lambda$, $K=\frac{2}{m(m-1)}\Lambda$	$R_{ijkh}-\frac{\Lambda}{m(m-1)}(g\odot g)_{ijkh}=0$, $\phi=\Phi$
(b) Constant energy scalar		
$\mathrm{ESU}^m_K\mathrm{CES}_{\rho,J}$	$V(u)=\frac{2(m-1)}{m}\rho$, $\rho>0,\ K=\frac{2\kappa}{m(m-1)}\rho$ $(U=U_\phi)$	$(\nabla\phi)^2<0,\quad \frac{\zeta}{\kappa}>0,\quad \phi(x)\in J$ $\nabla_i U_j=0,\ \mathfrak{C}_{ijkh}=0$, $(\nabla\phi)^2=-\frac{2}{m}\rho,\ \zeta=\frac{2\kappa\rho}{m(m-1)}$
(c) Spatially flat massless minimally-coupled scalar		
$\mathrm{MMS}^{m,0}_{\Lambda,J,J'}$	$0\notin J'$, $\frac{2\Lambda/\kappa}{m(m-1)}<\frac{1}{\kappa}(J')^2$ $V(u)=\frac{2}{\kappa}\Lambda$ $(U=U_\phi)$	$(\nabla\phi)^2<0,\frac{1}{\kappa}(\xi^2-\frac{2\Lambda}{m(m-1)})>0$, $\phi(x)\in J,\ \xi(x)\in J'$ $\nabla_iU_j-\frac{\nabla_i\phi'}{m\phi'}U_j-\xi g_{ij}=0$, $\mathfrak{Z}_{ijkh}=0,\ \boldsymbol{\eta}+m\xi^2=\frac{2\Lambda}{(m-1)}$, $\xi^2=\frac{\kappa\phi'^2+2\Lambda}{m(m-1)}$
(d) Generic massless minimally-coupled scalar		
$\mathrm{MMS}^m_{\Lambda,\Omega,J,J'}$	$V(u)=\frac{2}{\kappa}\Lambda$, $0\notin J',\ \Omega>0$ $(U=U_\phi)$	$(\nabla\phi)^2<0$, $\phi(x)\in J,\ \xi(x)\in J'$ $\nabla_iU_j-\frac{\nabla_i\phi'}{m\phi'}U_j-\xi g_{ij}=0$, $\mathfrak{C}_{ijkh}=0,\ \phi'=-\sqrt{\Omega}\lvert\zeta\rvert^{\frac{m}{2}}$, $\xi^2+\zeta=\frac{2\Lambda+\kappa\Omega\lvert\zeta\rvert^m}{m(m-1)}$
(e) Spatially flat nonlinear Klein–Gordon		
$\mathrm{NKG}^{m,0}_{V,\Xi,J}$	$V,\Xi\colon J\to\mathbb{R}$, $\Xi(u)\neq0,\frac{1}{\kappa}\Xi'(u)>0$, $V'(u)\neq0,\mathfrak{H}_V(\Xi)=0$ $(U=U_\phi)$	$(\nabla\phi)^2<0,\ \xi\neq0,\ \frac{1}{\kappa}\boldsymbol{\eta}<0$ $\mathfrak{P}_{ij}=0,\ \mathfrak{D}_{ij}=0,\ \mathfrak{Z}_{ijkh}=0$, $\phi'=-\frac{(m-1)}{\kappa}\partial_\phi\Xi(\phi)$, $\xi=\Xi(\phi)$
(f) Generic nonlinear Klein–Gordon		
$\mathrm{NKG}^m_{V,\Pi,\Xi,J}$	$V,\Xi,\Pi\colon J\to\mathbb{R}$, $\Pi<0,\ \Xi\neq0$, $\kappa\frac{\Pi^2+V}{m(m-1)}\neq\Xi^2$, $V'(u)\neq0,\mathfrak{G}_V(\Pi,\Xi)=0$ $(U=U_\phi)$	$(\nabla\phi)^2<0,\ \xi\neq0$, $\zeta\neq0,\ \frac{\boldsymbol{\eta}-\zeta}{\kappa}<0$ $\mathfrak{P}_{ij}=0,\ \mathfrak{D}_{ij}=0,\ \mathfrak{C}_{ijkh}=0$, $\zeta=\kappa\frac{\phi'^2+V(\phi)}{m(m-1)}-\xi^2$, $\phi'=\Pi(\phi),\ \xi=\Xi(\phi)$

Our characterization theorems cover what we have called *regular* FLRW or inflationary spacetimes (definitions 1.2 and 1.3), which are required to satisfy the inequalities listed in tables 1 and 2 everywhere.

2. Geometry of FLRW and inflationary spacetimes

Definition 2.1. Let (F,g^F) be a m-dimensional Riemannian manifold, $m \geqslant 1$, $I \subseteq \mathbb{R}$ an open interval with standard coordinate t and endowed with the usual reversed metric $-\mathrm{d}t^2$ and $f \in C^\infty(I)$, with $f > 0$. A *generalized Robertson Walker (GRW) spacetime* is a product manifold $M = I \times F$ endowed with the metric g defined as

$$g = -\pi_I^* \mathrm{d}t^2 + (f \circ \pi_I)^2 \pi_F^* g^F \tag{23}$$

where π_I and π_F are respectively the projections on I and F. Furthermore I is called the *base*, F the *fiber* and f the *warping function* (also *scale factor*, in the literature on cosmology).

To simplify notation in the sequel, let us introduce the notation $\tilde{T} = \pi_F^* T$ for any completely covariant tensor T defined on F.

The definition implies that around every point of $M = I \times F$, there exists a coordinate system (x^0, x^i) adapted to the product structure, such that, denoting $t = x^0$,

$$g_{ij} = -(\mathrm{d}t)_i(\mathrm{d}t)_j + f^2(t) g^F_{ij}, \tag{24}$$

where g^F_{ij} depends only on the x^i coordinates with $i > 0$ and $g^F_{ij}(\partial_t)^i = 0$. The only obstacle to making the last statement global on M is that the F factor may not admit a global coordinate system.

Definition 2.2. A *Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime* is a Lorentzian manifold (M, g) that is a GRW spacetime (definition 2.1) where the fiber (F,g^F) is simply connected, complete and has *constant curvature* with sectional curvature α (some constant), that is, the Riemann curvature tensor R^F_{ijkh} of (F,g^F) has the form

$$R^F_{ijkh} = \alpha(g^F_{ik} g^F_{hj} - g^F_{jk} g^F_{hi}). \tag{25}$$

When $\dim M = 2$, only $\alpha = 0$ is possible, since any 2-dimensional (F,g^F) is flat.

It is well known that any simply connected, complete Riemannian manifold of constant curvature, meaning that its Riemann curvature tensor is of the form (25), is isometric to either a round sphere ($\alpha > 0$), flat Euclidean space ($\alpha = 0$), or a hyperbolic space ($\alpha < 0$) [36, section 2.4]. If the *complete* and *simply connected* hypotheses are dropped, then a constant curvature Riemannian manifold is still locally isometric to one of these model spaces.

Similarly, in the sequel, we will be interested in Lorentzian spacetimes that are locally isometric (definition 1.1) to GRW or FLRW models.

2.1. Riemann curvature in GRW spacetimes

Below, we describe the Riemann curvature R_{ijkh} in a GRW spacetime, in terms of the curvature of (F,g^F), the warping function f and the vector field $U_i = -(\mathrm{d}t)_i$. For reference, let us denote the Riemann tensor on the (F,g^F) factor by R^F_{ijkh}, with $R^F_{ij} = (g^F)^{kh} R^F_{ikjh}$ and $\mathcal{R}^F = (g^F)^{ij} R^F_{ij}$ denoting respectively the corresponding Ricci tensor and scalar. Recall also the notation $\tilde{R}^F_{ijkh} = \pi_F^* R^F_{ijkh}$, $\tilde{R}^F_{ij} = \pi_F^* R^F_{ij}$ and $\tilde{\mathcal{R}}^F = \pi_F^* \mathcal{R}^F$.

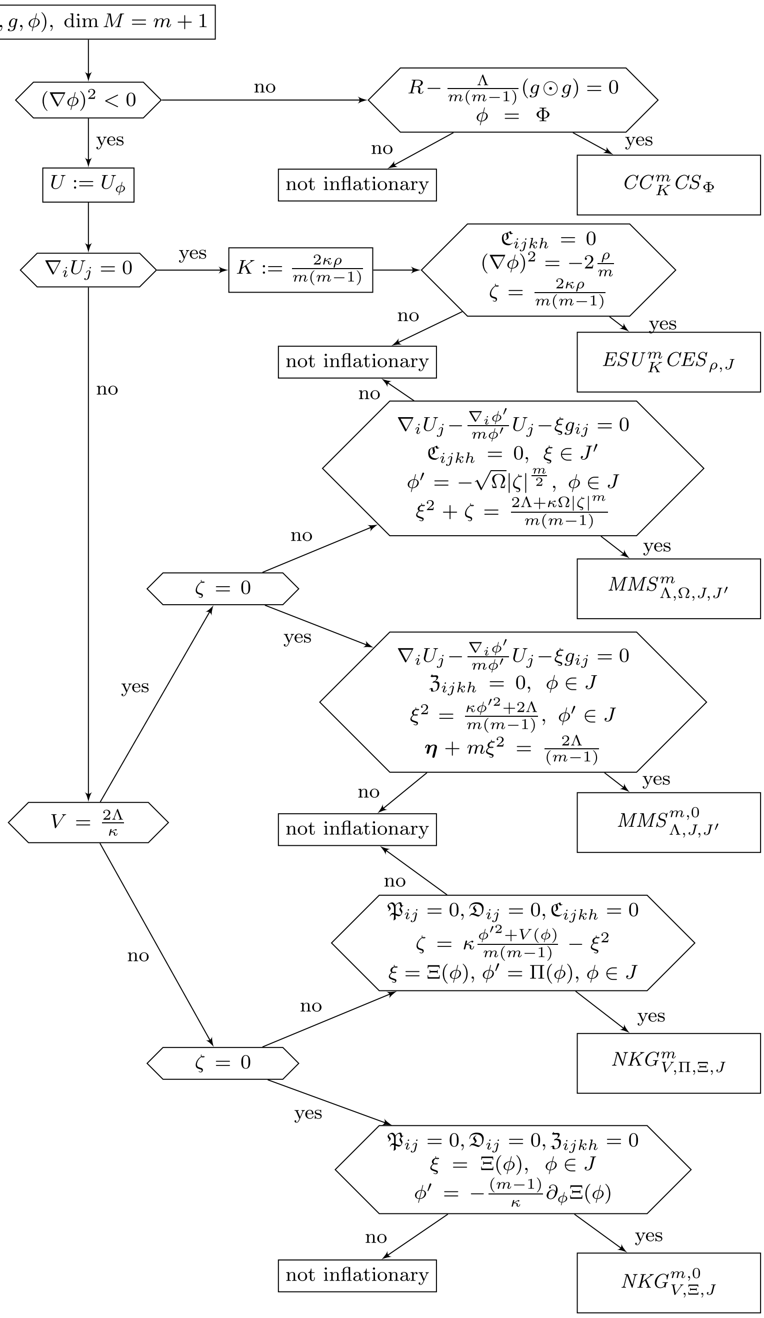

Figure 2. IDEAL characterization of local isometry classes of regular inflationary spacetimes (theorem 1.5, table 2).

Adapting the more general results on the covariant derivative on warped products [26, proposition 7.35], the action of the spacetime covariant derivative is determined by

$$\nabla_i(fU_j) = f' g_{ij}, \quad \nabla_i \tilde{X}_j = \widetilde{\nabla_i X_j} - 2\frac{f'}{f}U_{(i}\tilde{X}_{j)}, \tag{26}$$

for any X_j defined on F. Recalling the notation already used in the introduction, for any U_i we can define the temporal derivative $(-)' := U^i\nabla_i(-)$ and also

$$\xi := \frac{\nabla^i U_i}{m}, \quad \boldsymbol{\eta} := \xi' = U^j\nabla_j\xi. \tag{27}$$

With our choice of U on a GRW spacetime, we will be making repeated use of the identifies

$$\xi = \frac{f'}{f}, \quad \boldsymbol{\eta} = \frac{f''}{f} - \frac{f'^2}{f^2}. \tag{28}$$

Geometrically ξ is called the *expansion* of the vector field U, while its GRW value f'/f is known as the *Hubble rate* in the literature on cosmology.

Next, adapting the more general result [26, proposition 7.42] of how to write the Riemann tensor of a warped product manifold in terms of the curvatures of its factors and the warping function, it is possible to give the following general expression for the Riemann tensor of GRW spacetimes:

$$\begin{aligned} R_{ijkh} &= f^2\tilde{R}^F_{ijkh} + \left(g \odot \left[\frac{1}{2}\frac{f'^2}{f^2}g - \left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)\mathrm{d}t^2\right]\right)_{ijkh} \\ &= f^2\tilde{R}^F_{ijkh} + \left(g \odot \left[\frac{\xi^2}{2}g - \boldsymbol{\eta}UU\right]\right)_{ijkh}, \end{aligned} \tag{29}$$

where $(\mathrm{d}t^2)_{ij} = (\mathrm{d}t)_i(\mathrm{d}t)_j$ and where we have used the product notation (15). When $m = 1$, the tensors $g \odot UU$ and $g \odot g$ are no longer linearly independent, in fact $g \odot UU = -\frac{1}{2}g \odot g$. Moreover, the Riemann curvature for a 1-dimensional (F, g^F) is always zero. Hence, in the special $m = 1$ case we have the simplification

$$R_{ijkh} = \frac{(\boldsymbol{\eta} + \xi^2)}{2}(g \odot g)_{ijkh} = \frac{f''}{f}\frac{1}{2}(g \odot g)_{ijkh}. \tag{30}$$

As a consequence, using the identities

$$g^{kh}\tilde{R}^F_{ikjh} = \frac{1}{f^2}\pi_F^*((g^F)^{kh}R^F_{ikjh}) = \frac{1}{f^2}\tilde{R}^F_{ij}, \tag{31}$$

$$g^{ij}\tilde{R}^F_{ij} = \frac{1}{f^2}\pi_F^*((g^F)^{ij}R^F_{ij}) = \frac{1}{f^2}\tilde{\mathcal{R}}^F, \tag{32}$$

we get the following formulas for the Ricci tensor $R_{ij} = g^{kh}R_{ikjh}$ and scalar $\mathcal{R} = g^{ij}R_{ij}$:

$$\begin{aligned} R_{ij} &= \tilde{R}^F_{ij} - (m-1)\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)U_iU_j + \left(\frac{f''}{f} + (m-1)\frac{f'^2}{f^2}\right)g_{ij} \\ &= \tilde{R}^F_{ij} - (m-1)\boldsymbol{\eta}U_iU_j + (\boldsymbol{\eta} + m\xi^2)g_{ij}, \end{aligned} \tag{33}$$

$$\begin{aligned} \mathcal{R} &= \frac{1}{f^2}\tilde{\mathcal{R}}^F + 2m\frac{f''}{f} + m(m-1)\frac{f'^2}{f^2} \\ &= \frac{1}{f^2}\tilde{\mathcal{R}}^F + 2m\boldsymbol{\eta} + m(m+1)\xi^2. \end{aligned} \tag{34}$$

For completeness, we also compute the value of the scalar square of the Ricci tensor:

$$\begin{aligned} \mathcal{B} &= \frac{\tilde{\mathcal{B}}^F}{f^4} + 2\left(\frac{f''}{f} + (m-1)\frac{f'^2}{f^2}\right)\frac{\tilde{\mathcal{R}}^F}{f^2} + m\left(\frac{f''}{f} + (m-1)\frac{f'^2}{f^2}\right)^2 + m^2\frac{f''^2}{f^2} \\ &= \frac{\tilde{\mathcal{B}}^F}{f^4} + 2(\boldsymbol{\eta} + m\xi^2)\frac{\tilde{\mathcal{R}}^F}{f^2} + m(\boldsymbol{\eta} + m\xi^2)^2 + m^2(\boldsymbol{\eta} + \xi^2)^2, \end{aligned} \tag{35}$$

where $\mathcal{B}^F = (g^F)^{ik}(g^F)^{jh}R^F_{ij}R^F_{kh}$.

The above formulas motivate the following definitions, which can be used to isolate the spatial curvature R^F_{ijkh} from the knowledge of the spacetime curvature R_{ijkh} and of U_i.

Definition 2.3. Consider a Lorentzian manifold (M, g) with a unit timelike vector field U. Recall also the scalars ξ and $\boldsymbol{\eta}$ scalars from (27).

(a) We define the *zero (spatial) curvature deviation (ZCD) tensor* as

$$\mathfrak{Z}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{\xi^2}{2}g - \boldsymbol{\eta}UU\right]\right)_{ijkh}. \tag{36}$$

(b) Provided $m > 1$, we define the *spatial curvature scalar* as

$$\zeta := \frac{\mathfrak{Z}_{ij}{}^{ij}}{m(m-1)} = \frac{\mathcal{R} - 2m\boldsymbol{\eta} - m(m+1)\xi^2}{m(m-1)} \tag{37}$$

and if $m = 1$, we set $\zeta = 0$.

(c) We define the *constant (spatial) curvature deviation (CCD) tensor* as

$$\mathfrak{C}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{(\xi^2 + \zeta)}{2}g - (\boldsymbol{\eta} - \zeta)UU\right]\right)_{ijkh}. \tag{38}$$

On GRW spacetimes, the usefulness of these definitions lies in the identities

$$\mathfrak{Z}_{ijkh} = f^2\tilde{R}^F_{ijkh}, \quad \zeta = \frac{1}{m(m-1)}\frac{\tilde{\mathcal{R}}^F}{f^2}, \tag{39}$$

$$\mathfrak{C}_{ijkh} = f^2\left(\tilde{R}^F_{ijkh} - \frac{1}{m(m-1)}\frac{\tilde{\mathcal{R}}^F}{2}(\tilde{g}^F \odot \tilde{g}^F)_{ijkh}\right). \tag{40}$$

2.2. Riemann curvature in FLRW spacetimes

Next, we specialize the main formulas obtained in the preceding section from GRW to FLRW spacetimes (definition 2.2), by making use of their spatial curvature structure

$$R^F_{ijkh} = \frac{\alpha}{2}(g^F \odot g^F)_{ijkh}, \quad R^F_{ij} = (m-1)\alpha g^F_{ij}, \quad \mathcal{R}^F = m(m-1)\alpha, \tag{41}$$

and of the identity

$$\begin{aligned} f^2 \frac{1}{2}(\tilde{g}^F \odot \tilde{g}^F)_{ijkh} &= \frac{1}{f^2}\frac{1}{2}\left((g+UU)\odot(g+UU)\right)_{ijkh} \\ &= \frac{1}{f^2}\left((g\odot UU)_{ijkh} + \frac{1}{2}(g\odot g)_{ijkh}\right) = \frac{1}{f^2}\left(g \odot \left[\frac{1}{2}g + UU\right]\right)_{ijkh}, \end{aligned} \tag{42}$$

where we have recalled that $f^2\tilde{g}^F = g + UU$. Recall also the definitions of $U_i = (\mathrm{d}t)_i$, the scalars ξ and $\boldsymbol{\eta}$ from (27), and note the identity

$$\zeta = \frac{\alpha}{f^2} \tag{43}$$

for the spatial curvature scalar (definition 2.3) when $m > 1$. When $m = 1$, we always have $R^F_{ijkh} = 0$, so it is consistent to take $\zeta = 0$, as we do.

Thus, for FLRW spacetimes of spatial sectional curvature α, we have

$$R_{ijkh} = \left(g \odot \left[\frac{(\xi^2+\zeta)}{2}g - (\boldsymbol{\eta}-\zeta)UU\right]\right)_{ijkh}, \tag{44}$$

$$R_{ij} = -(m-1)(\boldsymbol{\eta}-\zeta)U_iU_j + [(\boldsymbol{\eta}-\zeta) + m(\xi^2+\zeta)]g_{ij}, \tag{45}$$

$$\mathcal{R} = m\left[2(\boldsymbol{\eta}-\zeta) + (m+1)(\xi^2+\zeta)\right], \tag{46}$$

$$\mathcal{B} = m[(\boldsymbol{\eta}-\zeta) + m(\xi^2+\zeta)]^2 + m^2(\boldsymbol{\eta}+\xi^2)^2, \tag{47}$$

where we have also used $\mathcal{B}^F = m(m-1)^2\alpha^2$. In the special $m = 1$ case, the above formulas simplify to

$$R_{ijkh} = \frac{(\boldsymbol{\eta}+\xi^2)}{2}(g\odot g)_{ijkh}, \tag{48}$$

$$R_{ij} = (\boldsymbol{\eta}+\xi^2)g_{ij}, \tag{49}$$

$$\mathcal{R} = 2(\boldsymbol{\eta}+\xi^2), \tag{50}$$

$$\mathcal{B} = 2(\boldsymbol{\eta}+\xi^2)^2. \tag{51}$$

Because of the frequent appearance of the combinations $\boldsymbol{\eta} - \zeta = \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}$ and $\xi^2 + \zeta = \frac{f'^2}{f^2} + \frac{\alpha}{f^2}$, in the sequel we will need the identity

$$(\xi^2+\zeta)' = 2\xi(\boldsymbol{\eta}-\zeta) \quad \text{or} \quad \left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right)' = 2\frac{f'}{f}\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right). \tag{52}$$

2.3. Perfect fluid interpretation

An arbitrary FLRW spacetime will in general not satisfy the vacuum Einstein equations. But it could be interpreted, when $m > 1$, as a solution of Einstein equations with a perfect fluid stress energy tensor

$$R_{ij} - \frac{1}{2}\mathcal{R}g_{ij} + \Lambda g_{ij} = \kappa T_{ij} = \kappa(\rho + p)U_i U_j + \kappa p g_{ij}, \tag{53}$$

where Λ is the cosmological constant, ρ is the *energy density* and p is the pressure. When $m = 3$, the coupling constant usually has the value $\kappa = 8\pi G/c^4$, where G is Newton's constant and c the speed of light. In other dimensions, there are at least two conventions: either keeping the value of κ the same, or setting it to $\kappa = 2\sigma_m G/c^4$, where $\sigma_m = 2\pi^{\frac{m-1}{2}}/\Gamma(\frac{m-1}{2})$ is the area of the unit $(m-1)$-sphere. We will simply keep it as an unspecified but fixed constant $\kappa \neq 0$. The cosmological constant could of course be shifted to $\Lambda \mapsto 0$ by the redefinitions $p \mapsto p - \Lambda/\kappa$, $\rho \mapsto \rho + \Lambda/\kappa$. When $m = 1$, the fluid interpretation is no longer possible, simply because the Einstein tensor $R_{ij} - \frac{1}{2}\mathcal{R}g_{ij}$ is identically zero in two spacetime dimensions.

Defining $\mathcal{T} = g^{ij}T_{ij}$, an equivalent form of Einstein's equations is

$$R_{ij} = \kappa T_{ij} - \frac{\kappa}{m-1}\mathcal{T}g_{ij} = \kappa(\rho + p)U_i U_j + \kappa\frac{\rho - p}{m-1}g_{ij}. \tag{54}$$

Hence, for FLRW spacetimes, these equations translate to

$$\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \frac{2}{m(m-1)}\kappa\rho, \qquad \kappa\rho = \frac{m(m-1)}{2}\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right), \tag{55}$$

$$\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} = -\frac{1}{m-1}\kappa(\rho + p), \qquad \kappa p = -(m-1)\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) - \frac{m(m-1)}{2}\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right), \tag{56}$$

On the top-left we have the *Friedmann equation*, while on the bottom-left we have the *acceleration equation*. These equations agree with the formulas previously obtained in [4], which was one of the first to consider perfect fluid cosmologies in higher spacetime dimensions. The Bianchi identity $\nabla^i(R_{ij} - \frac{1}{2}\mathcal{R}g_{ij}) = 0$ implies the stress–energy conservation $\nabla^i T_{ij} = 0$ condition, which translates to the *energy conservation* or *continuity equation*

$$\rho' + m\frac{f'}{f}(\rho + p) = 0. \tag{57}$$

2.4. Special FLRW classes

Below, we list the forms of FLRW spacetimes (definition 2.2) satisfying some special geometric conditions. Throughout this section, consider an FLRW spacetime (M, g), $\dim M = m + 1 \geqslant 2$, with warping function $f\colon I \to \mathbb{R}$ and spatial sectional curvature α. Whenever parameters are present, they must be chosen to respect $f(t) > 0$ for all $t \in I$, even if not explicitly indicated, as well as $\alpha = 0$ when $m = 1$.

Lemma 2.4. *The complete list of possible triples $(m, \alpha, f(t))$ satisfying the flat (or Minkowski space) condition, $R_{ijkh} = 0$, consists of*

$$\begin{cases} \frac{f'}{f} = 0\colon & (m, 0, A) \quad (A > 0); \\ \frac{f'}{f} \neq 0\colon & \begin{cases} m = 1\colon \frac{f''}{f} = 0 & (1, 0, A(t - t_0)) \quad (A \neq 0); \\ m > 1\colon \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = 0 & (m, \alpha, \pm\sqrt{-\alpha}(t - t_0)) \quad (\alpha < 0). \end{cases} \end{cases}$$

Proof. From equation (44), the necessary and sufficient conditions are $\frac{f''}{f} = 0$ and $(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}) = 0$, when $m > 1$, or only $f''/f = 0$, when $m = 1$. It is easy to see that the desired conclusion exhausts the solutions of these equations under the constraint that $f(t) \neq 0$ everywhere. □

Lemma 2.5. *The complete list of possible triples $(m, \alpha, f(t))$ satisfying the constant curvature (or (anti-)de Sitter space) condition, $R_{ijkh} = \frac{K}{2}(g \odot g)_{ijkh}$, with sectional curvature $K \neq 0$, consists of (A constant)*

$$\begin{cases} m = 1\colon \frac{f''}{f} = K & \begin{cases} K > 0\colon & \begin{cases} \left(1, 0, A\cosh(\sqrt{K}(t - t_0))\right), \\ \left(1, 0, A\mathrm{e}^{\pm\sqrt{K}(t-t_0)}\right) \\ \quad (A > 0); \end{cases} \\ K < 0\colon & \begin{cases} \left(1, 0, A\cos(\sqrt{-K}(t - t_0))\right) \\ \quad (A > 0); \end{cases} \end{cases} \\ m > 1\colon \begin{array}{c} \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = K, \\ \frac{f'}{f} \neq 0 \end{array} & \begin{cases} K > 0\colon & \begin{cases} \left(m, \alpha, \sqrt{\alpha/K}\cosh(\sqrt{K}(t - t_0))\right), \\ \left(m, 0, A\mathrm{e}^{\pm\sqrt{K}(t-t_0)}\right) \\ \quad (\alpha > 0, A > 0); \end{cases} \\ K < 0\colon & \begin{cases} \left(m, \alpha, \sqrt{\alpha/K}\cos(\sqrt{-K}(t - t_0))\right) \\ \quad (\alpha < 0). \end{cases} \end{cases} \end{cases}$$

Proof. Again, referring to equation (44), the necessary and sufficient conditions are $\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} = 0$ and $\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = K$, when $m > 1$, or only $f''/f = K$, when $m = 1$. If $m > 1$ and $f'/f = 0$, we must have $K = \alpha/f^2 = 0$, which contradicts the $K \neq 0$ hypothesis. Otherwise, it is easy to see that the desired conclusion exhausts the solutions of these equations under the constraint that $f(t) \neq 0$ everywhere. □

Lemma 2.6. *The complete list of possible triples $(m, \alpha, f(t))$ satisfying both conditions $\mathcal{R}' = 0$ and $\mathcal{B}' = 0$, but not of constant curvature, consists of (A, K constant)*

$$(m, KA^2, A) \quad (m > 1, K \neq 0, A > 0).$$

This is is the *Einstein static universe* [35, section 16.2] with spatial sectional curvature K, which solves the Einstein equation, $R_{ij} - \frac{1}{2}\mathcal{R}g_{ij} + \Lambda g_{ij} = 0$, with the cosmological constant $\Lambda = \frac{(m-1)(m-2)}{2}K$.

Proof. From equations (46) and (47), both $\mathcal{R}' = 0$ and $\mathcal{B}' = 0$ are third order equations in f. Eliminating f''' from both of them, we obtain the integrability condition

$$m^2(m-1)^2\frac{f'}{f}\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) = 0. \tag{58}$$

Obviously, it is trivial when $m = 1$. This is not surprising, because then $\mathcal{R}$ is the only independent curvature component and $\mathcal{R}' = 0$ already implies that the spacetime is of constant curvature, which is excluded by the hypotheses.

Further, this integrability condition splits into the cases $f'/f = 0$ and $f'/f \neq 0$. In the latter, it implies $\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} = 0$ and $(\frac{f'^2}{f^2} + \frac{\alpha}{f^2})' = 0$ (see equation (52)). But these are precisely the necessary and sufficient conditions for the spacetime to be of constant curvature (lemma 2.5), which is excluded by our hypotheses. Thus, we are left with the only possibility $f'/f = 0$ and the desired conclusion clearly exhausts the solutions of this equation. □

Lemma 2.7. *The complete list of possible triples* $(m, \alpha, f(t))$ *satisfying the constant scalar curvature condition* $\mathcal{R}' = 0$*, but with* $\mathcal{B}' \neq 0$*, consists of*

$$\begin{cases} \alpha = 0\colon & (m, 0, f) \ \left(m > 1,\ \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{(m+1)}{2}(\frac{f'^2}{f^2} - K) = 0,\ \frac{f'^2}{f^2} - K \neq 0\right); \\ \alpha \neq 0\colon & (m, \alpha, f) \ \left(m > 1,\ \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = K + \kappa\Omega\frac{|\alpha|^{\frac{m+1}{2}}}{f^{m+1}},\ \Omega \neq 0\right). \end{cases}$$

These are FLRW spacetimes with cosmological constant $\Lambda = \frac{m(m-1)}{2}K$ and *radiation perfect fluid* of energy density $\Omega_r = \frac{1}{\kappa}\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2} - K\right)$, where $\Omega_r = C/f^{m+1}$ for some constant C. We refer to Ω_r as the radiation energy density because the term with the power law $1/f^{m+1}$ in the Friedmann equation

$$\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = K + \kappa\frac{C}{f^{m+1}}, \tag{59}$$

when considered by itself gives rise to the constitutive relation $p_r(\rho) = \rho/m$, which is characteristic of radiation in thermal equilibrium [25]. If $\Omega_\alpha = \alpha/f^2$ is the energy density due to spatial curvature, when it is nonzero, the ratio $\Omega = \Omega_r/\Omega_\alpha$ defines our *normalized radiation density* constant Ω.

Proof. If $f'/f = 0$, then $\mathcal{R} = m(m-1)\alpha/f^2$ and $\mathcal{B} = m(m-1)^2\alpha^2/f^4$. Hence $\mathcal{R}' = 0$ implies $\mathcal{B}' = 0$. The same implication holds if $m = 1$ (see proof of lemma 2.5). Therefore, by the $\mathcal{B}' \neq 0$ hypothesis, we can assume that $m > 1$ and $f'/f \neq 0$.

From equation (46), the constant scalar curvature condition $\mathcal{R} = m(m+1)K$ (with some constant K)

$$2\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) + (m+1)\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right) = (m+1)K, \tag{60}$$

after multiplying both sides by the integrating factor $(f'/f)f^{m+1}$ and using identity (52), it is equivalent to

$$f^{m+1}\left(\frac{f'^2}{f^2}+\frac{\alpha}{f^2}-K\right)=\kappa C, \tag{61}$$

for some constant C. If $C=0$, we are back to the case of constant curvature (lemma 2.5), which is excluded by the $\mathcal{B}'\neq 0$ hypothesis. When $\alpha\neq 0$, we can normalize this constant as $C=\Omega|\alpha|^{\frac{m+1}{2}}$, with some $\Omega\neq 0$. Thus, the desired conclusion clearly consists of the necessary and sufficient conditions for $\mathcal{R}'=0$ and $\mathcal{B}'\neq 0$ to hold. □

Lemma 2.8. *For any triple $(m,\alpha,f(t))$ for which $\alpha=0$, $(f'^2/f^2)'\neq 0$ and $(\nabla\mathcal{R})^2<0$, there is a unique smooth function $P\colon J\to\mathbb{R}$, where $J=\frac{f'^2}{f^2}(I)$, $I\subseteq\mathbb{R}$ and*

$$\frac{f''}{f}-\frac{f'^2}{f^2}+\frac{m}{2}\frac{f'^2}{f^2}=-\kappa P\left((f'/f)^2\right). \tag{62}$$

The function $P(u)$ will also satisfy the following condition for each $u\in J$:

$$P(u)\left[\kappa\partial_u P(u)-\frac{1}{2}\right]\neq 0. \tag{63}$$

We will call P the *normalized pressure function* because, when $m>1$, the spacetime admits a perfect fluid interpretation (section 2.3) with energy density $\kappa\rho(t)=\frac{m(m-1)}{2}(f'/f)^2$, pressure $p(t)=(m-1)P\left((f'/f)^2\right)$, which admits the constitutive relation $p=p(\rho)$, where

$$p(\rho)=(m-1)P\left(\frac{2}{m(m-1)}\kappa\rho\right). \tag{64}$$

When $m=1$, the triviality of the Einstein equations does not allow such an interpretation, so without loss of generality the function P simply determines the differential equation satisfied by f.

Proof. Under our hypotheses, the existence of a unique function $P(u)$ is an elementary consequence of the implicit function theorem. If $P(u)=0$, then we are back to the case of flat or constant curvature spacetime (lemmas 2.4 and 2.5), while $P(u)=\frac{1}{2\kappa}[u-(m+1)K]$ brings us back to the $\mathcal{R}'=0$ case (lemma 2.7), both of which contradict the $(\nabla\mathcal{R})^2<0$ hypothesis. For any other value of $P(u)$, we have $\nabla\mathcal{R}\neq 0$, which can then only be timelike. □

Lemma 2.9. *For any triple $(m,\alpha,f(t))$ for which we have $\alpha\neq 0$, $f'/f\neq 0$ and $(\nabla\mathcal{R})^2<0$, there is a unique smooth function $E\colon J\to\mathbb{R}$, where $J=\frac{\alpha}{f^2}(I)$, $I\subseteq\mathbb{R}$ and*

$$\frac{f'^2}{f^2}+\frac{\alpha}{f^2}=\kappa E(\alpha/f^2). \tag{65}$$

The function $E(u)$ will also satisfy the following conditions for each $u\in J$:

$$\kappa E(u)-u>0,\quad \partial_u\left[u\partial_u E(u)-\frac{(m+1)}{2}E(u)\right]\neq 0. \tag{66}$$

We will call E the *normalized energy function* because, when $m>1$, the spacetime admits a perfect fluid interpretation (section 2.3) with energy density $\rho(t)=\frac{m(m-1)}{2}E(\alpha/f^2)$ and pressure $p(t) = -(f\rho')/(mf') - \rho$ given by the continuity equation (57). When $m = 1$, the triviality of the Einstein equations does not allow such an interpretation, so without loss of generality the function E simply determines the differential equation satisfied by f.

Proof. Under our hypotheses, the existence of a unique function $E(u)$ is an elementary consequence of the implicit function theorem. Since $f'^2/f^2 > 0$, we must also have $\kappa E(u) - u > 0$. Finally, we want to make sure that $\kappa E(u) \neq K + \kappa\Omega u^{\frac{m+1}{2}}$, which would imply $\mathcal{R}' = 0$ (lemma 2.7), contrary to our hypothesis that $(\nabla\mathcal{R})^2 < 0$. With K and Ω arbitrary, these right-hand-sides precisely exhaust the solutions of the equation $u\partial_u(u\partial_u - \frac{m+1}{2})E(u) = 0$. Thus, the second inequality in (66) is sufficient to ensure that $\nabla\mathcal{R} \neq 0$, which can then only be timelike. □

2.5. Scalar field

In this section, we will be interested in the geometry of Lorentzian spacetimes that are endowed with a scalar field and satisfying the coupled Einstein equations. To make non-trivial use of Einstein equations, throughout this section, we will assume that the spacetime dimension is $m + 1 > 2$. This information will later be used in section 3.4 to classify the local isometry classes (definition 1.1) of such spacetimes.

Definition 2.10. We call a spacetime with scalar (M, g, ϕ), with $\dim M = m + 1 > 2$, an *inflationary spacetime* when (M, g) can be put in FLRW form (23), $(M, g) \cong (I \times F, -\mathrm{d}t^2 + f^2 g^F)$ such that $\phi = \phi(t)$ is only a function of the t-coordinate, and for some constant Λ and smooth function $V(\phi)$ the coupled Einstein–Klein–Gordon equations are satisfied

$$\nabla^i\nabla_i\phi - \frac{1}{2}\partial_\phi V(\phi) = 0, \tag{67}$$

$$R_{ij} - \frac{1}{2}g_{ij}\mathcal{R} + \Lambda g_{ij} = \kappa T_{ij},$$
$$\text{where} \quad T_{ij} = (\nabla_i\phi)(\nabla_j\phi) - \frac{1}{2}g_{ij}[(\nabla\phi)^2 + V(\phi)]. \tag{68}$$

Equation (67) is in general the *nonlinear Klein–Gordon equation* with $V(\phi)$ the self-coupling potential, though in the special case that the potential is a quadratic polynomial it becomes linear. It is easy to see that we can set $\Lambda \mapsto 0$ by the redefinition $V(\phi) \mapsto V(\phi) + \frac{2}{\kappa}\Lambda$. We will adopt this convention from now on.

On an FLRW background, when $\phi = \phi(t)$, the stress energy tensor and the wave operator are given by

$$T_{ij} = \phi'^2 U_i U_j + \frac{1}{2}[\phi'^2 - V(\phi)]g_{ij}, \tag{69}$$

$$\nabla^i\nabla_i\phi = -\phi'' - m\frac{f'}{f}\phi'. \tag{70}$$

Hence, the coupled Einstein–Klein–Gordon equations reduce to the system of ODEs

$$\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \kappa\frac{\phi'^2 + V(\phi)}{m(m-1)}, \tag{71}$$

$$\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} = -\kappa \frac{\phi'^2}{(m-1)}, \tag{72}$$

$$\phi'' + \frac{1}{2}\partial_\phi V(\phi) = -m\frac{f'}{f}\phi', \tag{73}$$

which we will refer to as the *Friedmann equation* (71), the *(Einstein) acceleration equation* (72), and the *nonlinear Klein–Gordon equation*. When $\phi' \neq 0$, the nonlinear Klein–Gordon equation is not independent from the other two and follows from the continuity equation (57) applied to this situation. Note that the potential $V(\phi)$ can be isolated from the following combination of the Friedmann and acceleration equations:

$$\begin{aligned}\kappa\frac{V(\phi)}{(m-1)} &= \left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) + m\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right) \\ &= \frac{f''}{f} + (m-1)\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right).\end{aligned} \tag{74}$$

While we will eventually give a characterization of local isometry classes of inflationary spacetimes with a specific scalar potential $V(\phi)$, it is an interesting question how to recognize when an FLRW spacetime can be interpreted as part of a solution to an Einstein–Klein–Gordon system with *some* potential $V(\phi)$. This is a coarser version of the question that asks for a Rainich-type characterization with a *specific* potential $V(\phi)$. The latter finer question was answered in theorem 4 of [20], on which we base the following considerations.

Our starting point are the equations

$$-\kappa\frac{\phi'^2}{(m-1)} = \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}, \tag{75}$$

$$\kappa\frac{V(\phi)}{(m-1)} = \left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) + m\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right). \tag{76}$$

To answer our question, we will be happy with some reasonable conditions on a given (α, f) for the existence of $\phi(t)$ and $V(\phi)$ such that the above equations are satisfied. Supposing that the potential $V(\phi)$ has a smooth inverse, $V(\phi) = u \iff \phi = W(u)$, we have the relation $(V(\phi))'/\phi' = 1/W'(V(\phi))$, which is of course consistent only if both expressions remain both finite and non-zero. On the other hand, knowing $W'(u)$, we can recover W up to the ambiguity $W(u) \mapsto W(u) + \phi_0$, which determines V up to the ambiguity $V(\phi) \mapsto V(\phi - \phi_0)$. Thus, under the hypotheses

$$-\frac{1}{\kappa}\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) > 0, \tag{77}$$

$$\left[\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) + m\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right)\right]' \neq 0, \tag{78}$$

using the last left-hand-side as the independent variable in an application of the implicit function theorem, we define functions W' by the formula

$$\frac{\frac{(m-1)}{\kappa}\left[\left(\frac{f''}{f}-\frac{f'^2}{f^2}-\frac{\alpha}{f^2}\right)+m\left(\frac{f'^2}{f^2}+\frac{\alpha}{f^2}\right)\right]'}{\pm\sqrt{-\frac{(m-1)}{\kappa}\left(\frac{f''}{f}-\frac{f'^2}{f^2}-\frac{\alpha}{f^2}\right)}} = \frac{1}{W'\left(\frac{(m-1)}{\kappa}\left[\left(\frac{f''}{f}-\frac{f'^2}{f^2}-\frac{\alpha}{f^2}\right)+m\left(\frac{f'^2}{f^2}+\frac{\alpha}{f^2}\right)\right]\right)}, \tag{79}$$

which fixes W uniquely up to the ambiguity, $W(u)\mapsto \pm W(u)+\phi_0$. Hence, we can let $V(\phi)=W^{-1}(\phi)$ and

$$\phi(t)=W\left(\frac{(m-1)}{\kappa}\left[\left(\frac{f''}{f}-\frac{f'^2}{f^2}-\frac{\alpha}{f^2}\right)+m\left(\frac{f'^2}{f^2}+\frac{\alpha}{f^2}\right)\right]\right), \tag{80}$$

which are unique up to the ambiguity $V(\phi)\mapsto V(\pm[\phi-\phi_0])$ and $\phi(t)\mapsto\pm[\phi(t)-\phi_0]$. With these definitions for $\phi(t)$ and $V(t)$, (α,f) will satisfy the desired coupled Einstein–Klein–Gordon equations. Thus, any FLRW spacetime satisfying the inequalities (77) and (78) can be thought of as part of a solution of the Einstein–Klein–Gordon equations with some non-constant potential. On the other hand, the conditions on α and f to be part of a solution of Einstein–Klein–Gordon equations with a constant potential are considered in lemma 2.13.

2.6. Special inflationary classes

Below, we list the forms of inflationary spacetimes (definition 2.10) satisfying some special geometric conditions. Throughout this section, consider an inflationary spacetime (M,g,ϕ), $\dim M=m+1>2$, with scalar field $\phi\colon I\to\mathbb{R}$, warping function $f\colon I\to\mathbb{R}$ and spatial sectional curvature α. Whenever parameters are present, they must be chosen to respect $f(t)>0$ for all $t\in I$, even if not explicitly indicated.

Lemma 2.11. *The complete list of possible quadruples $(m,\alpha,f(t),\phi(t))$ satisfying the constant scalar condition, $\phi(t)=\Phi$, as well as $f'/f\neq 0$, consists of (m,α,f,Φ) with (m,α,f) satisfying the constant curvature condition, $R_{ijkh}=\frac{K}{2}(g\odot g)_{ijkh}$, with some spacetime sectional curvature constant K. The Einstein–Klein–Gordon equations are satisfied with the choice $V(\phi)=\frac{2}{\kappa}\Lambda$, where the cosmological constant $\Lambda=\frac{m(m-1)}{2}K$.*

Proof. Since $\phi'=0$, the Einstein–Klein–Gordon equations reduce to $R_{ij}-\frac{1}{2}\mathcal{R}g_{ij}=-\Lambda g_{ij}$, or $R_{ij}=mKg_{ij}$, with $K=\frac{2}{m(m-1)}\Lambda$, which together with the FLRW property is precisely the necessary and sufficient to be of constant curvature. □

Further on, in several cases, we will require the condition $f'/f\neq 0$. So first, we explore the special case $f'/f=0$, of static backgrounds. We know from lemma 2.6 that the only static FLRW backgrounds are flat or Einstein static universes, with the flat case already covered by lemma 2.11. What is special about this case is that the energy $\frac{1}{2}(\phi'^2+V(\phi))$ of the scalar field is conserved. It turns out that the converse is also true and it is only consistent with $V(\phi)$ being constant.

Lemma 2.12. *The complete list of possible quadruples $(m,\alpha,f(t),\phi(t))$ satisfying the constant energy condition $\frac{1}{2}(\phi'^2+V(\phi))=\rho$, with some constant ρ, but with $(m,\alpha,f(t))$ not of constant curvature, consists of*

$$(m,KA^2,A,\pm\sqrt{2\rho/m}(t-t_0))\quad (A>0,\ \rho>0). \tag{81}$$

Proof. We can presume that $\phi' \neq 0$, since otherwise the spacetime is of constant curvature (lemma 2.11). The Friedmann equation (71) reduces to $f'^2/f^2 + \alpha/f^2 = \kappa\rho$. Using the identity (52) and the acceleration equation (72), we conclude that $f'/f = 0$. Plugging this conclusion back into the Friedmann and acceleration equation, we find that each of $K = \alpha/f^2$, $\frac{2}{\kappa}\Lambda = V(\phi)$ and ϕ'^2 must be individually constant, with K interpreted as the spatial sectional curvature and Λ the cosmological constant. If we take ρ as an independent constant, the rest are given by $K = \frac{2}{m(m-1)}\kappa\rho$, $\phi'^2 = \frac{(m-1)}{\kappa}K = \frac{2}{m}\rho$ and $\Lambda = \frac{(m-1)}{m}\kappa\rho$. □

Whenever the scalar potential $V(\phi)$ is a constant, the Klein–Gordon equation is just the wave equation $\nabla^i\nabla_i\phi = 0$, which we also call the *massless minimally-coupled Klein–Gordon equation*.

Lemma 2.13. *The complete list of possible quadruples* $(m, \alpha, f(t), \phi(t))$ *with* $V(\phi) = \frac{2}{\kappa}\Lambda$ *a constant, where the scalar field is not constant nor of constant energy, consists of*

$$\begin{cases} \alpha = 0: & (m, 0, f, \phi) \begin{pmatrix} \left(\frac{f''}{f} - \frac{f'^2}{f^2}\right) + m\frac{f'^2}{f^2} = \frac{2\Lambda}{(m-1)}, \\ \frac{f'^2}{f^2} = \frac{\kappa\phi'^2 + 2\Lambda}{m(m-1)} \end{pmatrix} \\ \alpha \neq 0: & (m, \alpha, f, \phi) \begin{pmatrix} \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \frac{2\Lambda}{(m-1)} + \frac{\kappa}{m(m-1)}\Omega\frac{|\alpha|^m}{f^{2m}}, \\ \phi' = \pm\sqrt{\Omega}\frac{|\alpha|^{\frac{m}{2}}}{f^m}, \ \Omega > 0 \end{pmatrix} \end{cases}$$

Proof. Recall from (76) that a constant potential $V(\phi) = \frac{2}{\kappa}\Lambda$ implies the equation

$$\left(\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2}\right) + m\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2}\right) = 2\frac{\Lambda}{(m-1)}, \tag{82}$$

which is also supplemented by the Friedmann equation (71)

$$\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \frac{\kappa\phi'^2 + 2\Lambda}{m(m-1)} \tag{83}$$

is clearly equivalent to the Einstein equations with a massless minimally-coupled scalar field stress energy tensor and, because of our hypothesis that $\phi' \neq 0$ and the comments below equation (73), which are equivalent to the full coupled Einstein–Klein–Gordon system. Setting $\alpha = 0$ completes the proof of the first part of the lemma.

The hypothesis of non-constant energy and lemma 2.12 imply that $f'/f \neq 0$. Thus, we obtain the following equivalent form of (82) after multiplying it by the integrating factor $2(f'/f)f^{2m}$:

$$f^{2m}\left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2} - \frac{2}{m(m-1)}\Lambda\right) = \frac{\kappa}{m(m-1)}C, \tag{84}$$

for some constant C. When $\alpha \neq 0$, we can normalize C by a power of $|\alpha|$ to get

$$\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \frac{2\Lambda + \kappa\Omega\frac{|\alpha|^m}{f^{2m}}}{m(m-1)}, \tag{85}$$

with another constant Ω. Provided that $\Omega > 0$, we can determine $\phi(t)$ by the equation $\phi' = \pm\sqrt{\Omega}\frac{|\alpha|^{\frac{m}{2}}}{f^m}$, which is equivalent to the massless minimally-coupled Klein–Gordon equation

$$\frac{1}{f^m}\left(f^m\phi'\right)' = \phi'' + m\frac{f'}{f}\phi' = 0. \tag{86}$$

With the above expression for ϕ', plugging it into the Friedmann equation gives exactly equation (85). This observation completes the proof of the second part of the lemma. □

Next, we will transform the Einstein–Klein–Gordon equations (71)–(73) under the hypothesis that $\phi' \neq 0$ everywhere. If we use the Friedmann equation to eliminate α/f^2 from the acceleration equation, while also multiplying the Klein–Gordon equation by ϕ' and adding to it a multiple of the acceleration equation, they can be equivalently expressed as

$$\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2} = \frac{\alpha}{f^2}, \tag{87}$$

$$\left(\frac{f'}{f}\right)' + \kappa\frac{\phi'^2}{(m-1)} = \left(\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right), \tag{88}$$

$$\left(\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)' = -2\frac{f'}{f}\left(\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right). \tag{89}$$

The equations (88) and (89) are second order, while (87) is first order. To see that there are no integrability conditions, note that differentiating the first order equation gives the identity

$$\begin{aligned}&\left[f^2\left(\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)\right]' \\ &\quad = f^2\left[\left(\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)' + 2\frac{f'}{f}\left(\kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)\right],\end{aligned} \tag{90}$$

where the right-hand-side is clearly proportional to (89).

Since we are assuming that $\phi' \neq 0$, we can use ϕ as the independent variable and convert all t-derivatives as $(-)' = \phi'\partial_\phi(-)$. Denoting $\pi = \phi'$ and $\xi = f'/f$, we get the equations

$$f^2\left(\kappa\frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right) = \alpha, \tag{91}$$

$$\pi\left[\partial_\phi\xi + \kappa\frac{\pi}{(m-1)}\right] = \left(\kappa\frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right), \tag{92}$$

$$\partial_\phi\left(\kappa\frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right) = -2\frac{\xi}{\pi}\left(\kappa\frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right), \tag{93}$$

where ξ, π and f are now all considered as functions of ϕ. With fixed $V(\phi)$, the system (92) and (93) closes in the (π, ξ) variables, with the symmetry $(\pi, \xi) \mapsto (-\pi, -\xi)$ corresponding to the coordinate transformation $t \mapsto -t$, and can be solved for the highest derivatives $\partial_\phi\xi$ and $\partial_\phi\pi$ (always assuming that $\pi \neq 0$). In the notation of (20), we can use the short-hand $\mathfrak{G}_V(\pi, \xi) = 0$

for this system. Hence the space of solutions $\xi = \Xi(\phi)$, $\pi = \Pi(\phi)$, will be two-dimensional. We will always leave these parameters implicit in the choice of the solution $(\Pi(\phi), \Xi(\phi))$. With (Π, Ξ) fixed, the equations $\phi' = \Pi(\phi)$, $f'/f = \Xi(\phi)$ and $\alpha = f^2[\kappa \frac{\Pi^2(\phi)+V(\phi)}{m(m-1)} - \Xi^2(\phi)]$ have a two-dimensional family of solutions, parametrized essentially by the transformations

$$(\alpha, f(t), \phi(t)) \mapsto (A^2\alpha, Af(t-t_0), \phi(t_0 - t_0)), \tag{94}$$

which are the isometries preserving FLRW form (proposition 3.6). So the parameters determining (α, f, ϕ) that are invariant under these transformations are essentially exhausted by the choice of $V(\phi)$ and the solution (Π, Ξ). We summarize as follows.

Lemma 2.14. *For any quadruple* $(m, \alpha, f(t), \phi(t))$ *for which* $\alpha \neq 0$, $f'/f \neq 0$ *and* $(\nabla\phi)^2 < 0$, *there is a unique smooth function* $(\Pi, \Xi)\colon J \to \mathbb{R}^2$, *where* $J = \phi(I)$ *and*

$$\phi' = \Pi(\phi), \quad \frac{f'}{f} = \Xi(\phi), \quad \frac{\alpha}{f^2} = \kappa\frac{\Pi^2(\phi) + V(\phi)}{m(m-1)} - \Xi^2(\phi). \tag{95}$$

For each $u \in J$, *these functions will also satisfy* $\Pi(u) \neq 0$, $\Xi(u) \neq 0$ *and* $\kappa\frac{\Pi^2(u)+V(u)}{m(m-1)} \neq \Xi^2(u)$, *they will satisfy* $\mathfrak{G}_V(\Pi, \Xi) = 0$, *in the notation of (20).*

When $\alpha = 0$, the above discussion can be greatly simplified. The Einstein–Klein–Gordon system reduces to the following equivalent forms, using the same notation as above and always supposing that $\pi \neq 0$ everywhere:

$$\begin{cases} \kappa\frac{\pi+V(\phi)}{m(m-1)} - \xi^2 & = 0 \\ \pi\left[\partial_\phi\xi + \kappa\frac{\pi}{(m-1)}\right] & = 0 \end{cases} \iff \begin{cases} (\partial_\phi\xi)^2 = \kappa\frac{m(m-1)\xi^2 - \kappa V(\phi)}{(m-1)^2} \\ \pi = -\frac{(m-1)}{\kappa}\partial_\phi\xi \end{cases} \tag{96}$$

In a way, this simplification comes from eliminating $\pi = \phi'$ from the equations. In the notation of (18), we use the short-hand $\mathfrak{H}_V(\xi) = 0$ for the equation satisfied by $\xi(\phi)$, which retains the symmetry $\xi \mapsto -\xi$. With $V(\phi)$ fixed, under the hypothesis $\partial_\phi\Xi \neq 0$, this equation will have a one-dimensional family of solutions $\xi = \Xi(\phi)$. We will always leave the corresponding parameter implicit in the choice of the solution $\Xi(\phi)$. With Ξ fixed, the equations $\phi' = -\frac{(m-1)}{\kappa}\partial_\phi\Xi(\phi)$, $f'/f = \Xi(\phi)$ have a two-dimensional family of solutions, again parametrized by the transformations (94). So the parameters determining (f, ϕ) that are invariant under these transformations are essentially exhausted by the choice of $V(\phi)$ and the solution Ξ. We summarize as follows.

Lemma 2.15. *For any quadruple* $(m, \alpha, f(t), \phi(t))$ *for which* $\alpha = 0$, $f'/f \neq 0$ *and* $(\nabla\phi)^2 < 0$, *there is a unique smooth function* $\Xi\colon J \to \mathbb{R}$, *where* $J = \phi(I)$ *and*

$$\phi' = -\frac{(m-1)}{\kappa}\partial_\phi\Xi(\phi), \quad \frac{f'}{f} = \Xi(\phi). \tag{97}$$

For each $u \in J$, *these functions will also satisfy* $\Xi(u) \neq 0$, $\partial_\phi\Xi^2(u) \neq 0$ *and it will satisfy* $\mathfrak{H}_V(\Xi) = 0$, *in the notation of (18).*

In the spatially flat ($\alpha = 0$) case, the equation $\mathfrak{H}_V(\Xi) = 0$ is sometimes known as the *Hamilton–Jacobi equation* of single field inflation [24, 28]. The more general system $\mathfrak{G}_V(\Pi, \Xi) = 0$ needed in the generic case ($\alpha \neq 0$) does not seem to have been considered before. In the cosmology literature, in the case of non-zero α, an alternative system of equations has been used [30], though one less convenient for our purposes. There, a complex scalar field $Z(\phi)$ is introduced, and plays the role of a 'super-potential' (in the sense of super-symmetry) for a 'pseudo-Killing' spinor. The isometry class of (α, f, ϕ) determines the integrability conditions for $Z(\phi)$, an algebraic relation between ϕ, $Z(\phi)$ and $Z'(\phi)$.

3. Geometric characterization

In this section, we leverage the information from section 2 to give necessary and sufficient conditions to belong to the local isometry class of a regular FLRW or inflationary spacetime, eventually proving our main theorems 1.4 and 1.5.

The resulting systems of conditions will be of the IDEAL type, as discussed in the Introduction, consisting of a list $\{T_a[g,\phi]=0\}$, $a=1,\ldots,N$, of tensor equations built covariantly out of a metric g, a scalar field ϕ, and their derivatives. Each set of equations will consist of roughly three parts: for the GRW structure, for the FLRW structure, and for the specific isometry subclass.

3.1. Special cases

The two cases of FLRW spacetimes whose local isometry classes need to be characterized separately from the general pattern given in the sequel are the constant curvature spacetimes (lemmas 2.4 and 2.5) and Einstein static universes (lemma 2.6).

Proposition 3.1. *Consider a Lorentzian manifold (M,g),* $\dim M = m+1 \geqslant 2$.

(a) *Given a fixed constant K, if (M,g) everywhere satisfies*

$$R_{ijkh} - K\frac{1}{2}(g\odot g)_{ijkh} = 0, \tag{98}$$

then it is locally isometric to any other spacetime satisfying the same condition.

(b) *Given a fixed constant K, if $m>1$ and (M,g) everywhere satisfies*

$$W_{ijkh}=0, \qquad R_i^{\,j}\,(R_{jk}-(m-1)Kg_{jk})=0, \tag{99}$$

$$\nabla_i R_{jk}=0, \qquad \mathcal{R}-m(m-1)K=0, \tag{100}$$

while the 1-dimensional kernel of $R_i^{\,j}$ is timelike, it is locally isometric to an Einstein static universe with spatial sectional curvature K. The value $K=0$ coincides with the flat case, $R_{ijkh}=0$.

Proof.

(a) This is standard; see for instance theorem 2.4.11 in [36].

(b) When $m=1$, spatial slices are always flat, hence it is impossible to have $K\neq 0$ spatial sectional curvature. When $K=0$, we are back in the flat case, characterized by $R_{ijkh}=0$, a special case of part (a). This is why we take $m>1$. Direct calculation (see 2.2) shows that the above equations hold when (M,g) is an Einstein static universe with spatial sectional curvature $K\neq 0$.

Conversely, assume that we only know about (M,g) that the above equations hold, with $K\neq 0$. The algebraic equations on the $R_i^{\,j}$ tangent space endomorphism guarantee that it is diagonalizable with precisely two distinct eigenvalues, 0 and $(m-1)K$, with the kernel being

1-dimensional. Since R_{ij} is symmetric, the kernel can only be either timelike or spacelike (not null)[6], with the hypotheses constraining it to be timelike. Since R_{ij} is also covariantly constant, so is any unit vector U^i in its kernel. That is, $R_{ij}\nabla_X U^j = \nabla_X(R_{ij}U^j) = 0$ for any X^i, which implies that $\nabla_X U^j = A_X U^j$ and $A_X = -U_j\nabla_X U^j = -\frac{1}{2}\nabla_X(U_jU^j) = 0$. This gives us the desired $\nabla_i U^j = 0$ conclusion.

The existence of a covariantly constant unit vector U^i implies that for any $x \in M$ and contractible open neighborhood $O \ni x$, the holonomy action of $(O, g|_O)$ at x leaves invariant the subspace spanned by U^i at x as well as its orthogonal complement (simply note that contraction with U^i commutes with parallel transport). Under these conditions (proposition IV.5.2 in [19]), it is possible to locally factor $(O, g|_O)$ into a direct product of a 1-dimensional and an m-dimensional pseudo-Riemannian manifold, $(I, -\mathrm{d}t^2) \times (F, g^F)$, with g^F of Riemannian signature. Furthermore, the algebraic conditions on W_{ijkh} and R_{ij} imply that $W^F_{ijkh} = 0$ and $R^F_{ij} = (m-1)Kg^F_{ij}$, which means that the spatial factor (F, g^F) is locally of constant curvature with sectional curvature K. In other words, we can locally describe (M, g) as an FLRW spacetime with $\alpha = K$ and $f(t) = 1$, which belongs precisely to the desired Einstein static universe class. □

3.2. FLRW spacetimes

An FLRW spacetime (definition 2.2) is a GRW spacetime (definition 2.1) whose spatial slices have constant curvature (equation (25)). GRW spacetimes have been geometrically characterized in two different but related ways by the existence of a spatially conformal vector field U by Sánchez [29] and of a concircular vector field v by Chen [5]. Given Chen's vector field v, the vector field $U = v/\sqrt{-v^2}$ satisfies the conditions of Sánchez. A recent survey of these and related geometric characterization results of GRW spacetimes can be found in [21].

Chen's condition is somewhat simpler, but we will only be able to make use of it to characterize spatially curved, but not spatially flat FLRW spacetimes. In one case it will be possible to produce Chen's vector field v directly from the spacetime curvature, in the other not. Sánchez's conditions work equally well also in the spatially flat case. So, motivated by providing the simplest set of equations when possible, we present both characterizations.

Proposition 3.2 (Sánchez's conditions). *Let (M, g) be a Lorentzian manifold, $\dim M = m + 1 \geqslant 2$. It is locally GRW at $x \in M$ if and only if there exists, on a neighborhood of x, a unit timelike vector field U that satisfies the conditions*

$$\mathfrak{P}_{jk} := U_{[j}\nabla_{k]}\frac{\nabla^i U_i}{m} = 0, \tag{101}$$

$$\mathfrak{D}_{ij} := \nabla_i U_j - \frac{\nabla_k U^k}{m}(g_{ij} + U_iU_j) = 0. \tag{102}$$

Proof. In one direction, given an FLRW metric in the form (23), direct calculation shows that the above conditions are satisfied with $U^i = (\partial_t)^i$.

[6] Suppose the 1-dimensional kernel N of R_i^j is null. From its invariant factors and the symmetry of R_{ij}, we have the following splittings of invariant subspaces: $N^\perp = N \oplus S$ and $S^\perp = N \oplus N'$, where S is necessarily spacelike, meaning that N' is 1-dimensional and has a non-zero eigenvalue. But, by the well-known Segre classification [31, section 5.1], on $S^\perp$, R_i^j can either have only a single degenerate eigenvalue or no null eigenvectors.

In the other direction, Sánchez's theorem 2.1 from [29] shows that locally (M, g) can be put into the form (23), with $U^i = (\partial_t)^i$. Sánchez's original conditions look more complicated, but they follow from ours by easy algebraic manipulations. Sánchez's hypotheses also include connectedness and simple connectedness. But, from the proof, these can all be dropped for the local result that we want. □

We have based the above result on the characterization of GRW spacetimes that Sánchez obtained independently [29, theorem 2.1] in the process of a detailed investigation of the geometry of GRW spacetimes. However, this characterization (existence of a *shear*-free, $\mathfrak{D}_{(ij)} = 0$, and *twist*-free, $\mathfrak{D}_{[ij]} = 0$, vector field U, with *expansion* ξ constant in directions orthogonal to U, $\mathfrak{P}_{ij} = 0$), at least when applied to FLRW spacetimes, has been known already as far back as [8, theorem 2.5.1, 9], and has been referenced for instance in [11, section III.B], [10, section 5.1]. Another independent source for these conditions seems to be the unpublished thesis [7], which has been referenced in at least [2, p.124].

Proposition 3.3 (Chen's conditions). *Consider a Lorentzian manifold (M, g), $\dim M = m + 1 \geqslant 2$. It is locally GRW at $x \in M$ if and only if there exists, on a neighborhood of x, a timelike vector field v and a scalar μ that satisfy the condition*

$$\nabla_i v_j = \mu g_{ij}. \tag{103}$$

A vector field satisfying (103) is called *concircular*.

Proof. In one direction, given GRW metric in the form (23), direct calculation shows that we can take $v^i = f(\partial_t)^i$ and $\mu = f'$.

Chen's theorem 1 from [5] shows that locally (M, g) can be put into the form (23), with $v^i = f(\partial_t)^i$. Chen stated this result for $m + 1 \geqslant 3$. However, the same proof also works when $m + 1 = 2$. It is easiest to see by showing that the concircular condition (103) implies that $U^i = v^i/\sqrt{-v^2}$ satisfies Sánchez's conditions, independently of the dimension. Let $\phi = \sqrt{-v^2}$, so that $v^i = \phi U^i$. From the $U^j \nabla_i U_j = 0$ identity, the concircular condition decomposes into

$$\begin{aligned} U_j \nabla_i \phi + \phi \nabla_i U_j &= -\mu U_i U_j + \mu(g_{ij} + U_i U_j) \\ \iff \nabla_i \phi = -\mu U_i, &\quad \nabla_i U_j = \frac{\mu}{\phi}(g_{ij} + U_i U_j). \end{aligned} \tag{104}$$

Then $U_{[i} U_{j]} = 0$ implies $U_{[i} \nabla_{j]} \phi = 0$, and $\nabla_{[i} \nabla_{j]} \phi = 0$ implies $U_{[i} \nabla_{j]} \mu = 0$. Finally, noting that $\mu = U^i \nabla_i \phi = \frac{\phi}{m} \nabla^i U_i$ and eliminating both ϕ and μ gives us Sánchez's conditions $\mathfrak{P}_{ij} = 0$ and $\mathfrak{D}_{ij} = 0$. □

The concircular condition can be rewritten slightly for our convenience.

Lemma 3.4. *Let U be a vector field, ν and ϕ smooth functions, with $\phi > 0$, and k a constant. Then the condition*

$$\nabla_i U_j + k \frac{\nabla_i \phi}{\phi} U_j = \nu g_{ij} \tag{105}$$

implies that $v = \phi^k U$ is a concircular vector field. In particular, $U_{[i} \nabla_{j]} \phi = 0$.

Proof. The concircular condition with $v = \phi^k U$ and $\mu = \phi^k \nu$ is equivalent to $\phi^{-k} \nabla_i (\phi^k U_j) = \phi^{-k} \mu g_{ij}$, which when expanded gives precisely equation (105). In GRW form (23), $\phi^k U^i = f(\partial_t)^i$ and $\nabla_j f = -f' U_j$, from which follows the desired condition on $\nabla_j \phi$. □

Proposition 3.5. *Consider a GRW spacetime* $(M,g) \cong (I \times F, -\mathrm{d}t^2 + f^2 g^F)$, $\dim M = m+1 \geqslant 2$. *Set* $U^i = (\partial_t)^i$ *and recall the notation of definition 2.3.*

The (F,g^F) factor is locally of constant curvature if and only if the CCD tensor (see definition 2.3) vanishes and the spatial scalar curvature is constant,

$$\mathfrak{C}_{ijkh} = 0 \quad \text{and} \quad U_{[i}\nabla_{j]}\zeta = 0. \tag{106}$$

If in addition the spatial scalar curvature or equivalently the ZCD tensor (see definition 2.3) also vanishes, $\zeta = 0$ *or*

$$\mathfrak{Z}_{ijkh} = 0, \tag{107}$$

then (F,g^F) is actually flat.

Proof. From equation (40), $\mathfrak{C}_{ijkh} = 0$ is equivalent to

$$R^F_{ijkh} = \frac{1}{m(m-1)} \frac{\mathcal{R}^F}{2} (g^F \odot g^F)_{ijkh}. \tag{108}$$

while $U_{[i}\nabla_{j]}\zeta = 0$ and (39) imply that $\mathcal{R}^F$ is a constant. Hence, (F,g^F) is of constant curvature. Furthermore, either of the conditions $\zeta = 0$ or $\mathfrak{Z}_{ijkh} = 0$ implies that $R^F_{ijkh} = 0$ and hence that (F,g^F) is flat. □

3.3. FLRW local isometry classes

Within the class of FLRW spacetimes, two metrics in the form (23) with different (α,f) parameters may or may not be isometric. Below, we give the results that allow us to classify FLRW metrics into isometry classes.

The obvious form-preserving transformations, time translation, reflection and rescaling, relate any FLRW metric to a 2-parameter family of (locally) isometric metrics. We state this result directly for FLRW spacetimes with scalar, which will come in useful later in section 3.4. As mentioned in the introduction, we can reduce to the case of no scalar field by setting the scalar field to zero.

Proposition 3.6. *Consider two inflationary spacetimes* (M_i, g_i, ϕ_i), $i = 1,2$, *with corresponding spatial sectional curvature, warping function and scalar field triples* (α_i, f_i, ϕ_i), $i = 1,2$. *If for every* $x \in M_1$ *with* $t_1 = t(x)$ *in the domain of* (f_1, ϕ_1) *there exists an open interval* $(t_1 - \delta, t_1 + \delta)$ *still in the domain of* (f_1, ϕ_1), *with* $\delta > 0$, *and an interval* $(t_2 - \delta, t_2 + \delta)$ *in the domain of* (f_2, ϕ_2) *such that*

$$\begin{cases} \alpha_1 &= A^2 \alpha_2, \\ f_1(t) &= A f_2(st + t_0), \\ \phi_1(t) &= \phi_2(st + t_0), \end{cases} \tag{109}$$

for some constants $s \in \{+1, -1\}$, $A \neq 0$ *and every* $t \in (t_1 - \delta, t_1 + \delta)$, *then* (M_1, g_1, ϕ_1) *is locally isometric to* (M_2, g_2, ϕ_2) *at* $x \in M_1$.

Proof. The result follows from noting that an FLRW metric in standard form $-\mathrm{d}t^2 + f(t)^2 \tilde{g}^F$ is locally isometric to each of $-\mathrm{d}t^2 + f(-t)^2 \tilde{g}^F$, $-\mathrm{d}t^2 + f(t+t_0)^2 \tilde{g}^F$ and to $-\mathrm{d}t^2 + (Af(t))^2 (\tilde{g}^F / A^2)$. □

We will now show that, under certain conditions, two FLRW metrics with parameters (α_1,f_1) and (α_2,f_2) are locally isometric *if and only if* they belong to the same 2-parameter family as in proposition 3.6. To describe such a 2-parameter family of (α,f) intrinsically, we will look for a differential equation satisfied by every element of that family and only elements of that family. Heuristically, we should look for either a second order equation for f or a first order equation for f depending also on the parameter α, either of which will generically have a 2-parameter general solution.

The following helpful lemma follows easily from standard ODE existence and uniqueness theory [1].

Lemma 3.7. *Consider a smooth real function G defined on an open interval J, two nonzero real constants α_1 and α_2, and two nowhere vanishing smooth real functions $f_1(t)$ and $f_2(t)$ defined respectively on the open intervals I_1 and I_2.*

(a) Suppose $G>0$ and that the pairs (α_1,f_1) and (α_2,f_2) both satisfy the differential equation

$$(f'/f)^2 = G(\alpha/f^2) \tag{110}$$

and that there exist $t_1 \in I_1$ and $t_2 \in I_2$ such that $\frac{\alpha_1}{f_1(t_1)^2} = \frac{\alpha_2}{f_2(t_2)^2} \in J$. Then there exist constants $s \in \{+1,-1\}$, t_0, $A \neq 0$ and $\delta > 0$ such that $t_2 = st_1 + t_0$, as well as

$$\alpha_1 = A^2\alpha_2 \quad \text{and} \quad f_1(t) = Af_2(st+t_0) \tag{111}$$

for every $t \in (t_1-\delta, t_1+\delta)$.

(b) Suppose that the functions f_1 and f_2 both satisfy the differential equation

$$f''/f = G\left((f'/f)^2\right) \tag{112}$$

and that there exist $t_1 \in I_1$ and $t_2 \in I_2$ such that $\frac{f_1'(t_1)}{f_1(t_1)} = \frac{f_2'(t_2)}{f_2(t_2)} \in J$. Then there exist constants $s \in \{+1,-1\}$, t_0, $A \neq 0$ and $\delta > 0$ such that $t_2 = st_1 + t_0$, as well as

$$f_1(t) = Af_2(st+t_0) \tag{113}$$

for every $t \in (t_1-\delta, t_1+\delta)$.

We are finally in a position to define and classify all regular FLRW spacetimes into families and to describe the parameters needed identify an isometry class within each family.

Lemma 3.8. *Two regular FLRW spacetimes (those belonging to one of the families identified in definition 1.2) are isometric to each other (definition 1.1) if and only if they belong to the same parametrized family and the corresponding parameters are identical.*

Proof. Let us fix m, noting that two isometric spacetimes must have the same dimension. To show that two spacetimes cannot be isometric, it is sufficient to point out an identity or inequality that is satisfied by curvature scalars or tensors on one spacetime but not on the other. With that in mind, recall (in the notation of theorem 1.4) that for FLRW spacetimes, $\xi = f'/f$, $\eta = f''/f - f'^2/f^2$ and $\zeta = \alpha/f^2$, which are all curvature scalars as long as they are defined with respect to a vector field U that is also defined from pure, such as the choices $U = U_{\mathcal{R}}$ or $U_{\mathcal{B}}$. To show that all the representatives of a family with identical parameters are all isometric to each other, there will be two possibilities to consider. Either the representative is unique, which is the trivial case. Or, all representatives are selected by satisfying a differential equation. By invoking lemma 3.7, we can be sure that two solutions to such an equation (with all parameters fixed), if they can be matched up at at least one point, are in fact locally isometric around that point. If the domains of these solutions can also be matched up, then it is clear that they are also globally isometric.

(a) For each K, there is a unique representative in CC^m_K. The scalar curvature $\mathcal{R} = m(m+1)K$ distinguishes the different values of K.
(b) Again, for each $K \neq 0$, there is a unique representative in ESU^m_K. The scalar curvature $\mathcal{R} = m(m-1)K$ distinguishes the different values of K. Comparing the formulas from section 2.2 and proposition 3.1(b), the structure of the Ricci tensor R_{ij} distinguish ESU^m_K from any spacetime of constant curvature.
(c) The representatives of $\mathrm{CSC}^{m,0}_{K,J}$ satisfy an equation like in lemma 3.7(b). The scalar curvature $\mathcal{R} = m(m+1)K$ distinguishes the different values of K, and setting $U = U_{\mathcal{B}}$ the range $J = \xi^2(I)$ distinguishes the different intervals J. Also, from lemma 2.7, $(\nabla\mathcal{B})^2 < 0$ distinguishes these spacetimes from those of parts (a) and (b), where $\mathcal{B}' = 0$.
(d) The representatives of the class $\mathrm{CSC}^m_{K,\Omega,J}$ satisfy an equation like in lemma 3.7(a). The scalar curvature $\mathcal{R} = m(m+1)K$ distinguishes the different values of K, and setting $U = U_{\mathcal{B}}$, the constant $\kappa\Omega = (\xi^2 + \zeta - K)/|\zeta|^{\frac{m+1}{2}}$ (lemma 2.7) and range $J = \zeta(I)$ distinguishes the different values of Ω and J. Again, $(\nabla\mathcal{B})^2 < 0$ distinguishes these spacetimes from those of parts (a) and (b), while $\zeta \neq 0$ distinguishes them from those of part (c) where $\zeta = 0$.
(e) The representatives of the class $\mathrm{FLRW}^{m,0}_{P,J}$ satisfy an equation like in lemma 3.7(b). Setting $U = U_{\mathcal{R}}$, the identity $\boldsymbol{\eta} + \frac{m}{2}\xi^2 = -\kappa P(\xi^2)$ and the range $J = \xi^2(I)$ distinguish different values of the P and J parameters. Also, combining the constraints on P and lemma 2.8, $(\nabla\mathcal{R})^2 < 0$ distinguishes these spacetimes from those of parts (a), (b), (c) and (d), where $\mathcal{R}' = 0$.
(f) The representatives of the class $\mathrm{FLRW}^m_{E,J}$ satisfy an equation like in lemma 3.7(b). Setting $U = U_{\mathcal{R}}$, the identity $\xi^2 + \zeta = \kappa E(\zeta)$ and the range $J = \zeta(I)$ distinguish different values of the E and J parameters. Again, combining the constraints on E and lemma 2.9, $(\nabla\mathcal{R})^2 < 0$ distinguishes these spacetimes from those of parts (a), (b), (c) and (d), while $\zeta \neq 0$ distinguishes them from those of part (e), where $\zeta = 0$. □

We are now finally in a position to prove our main result about IDEAL characterizations of regular FLRW spacetimes.

Proof of theorem 1.4. The goal is to prove that, for each of the cases listed in table 1, a spacetime satisfies the listed equations (and inequalities) if and only if it is locally isometric (definition 1.1) to one of the regular FLRW spacetimes listed in definition 1.2. In one direction (a regular FLRW spacetime satisfies the corresponding conditions), this is essentially the content of lemma 3.8. It remains to show the converse.

(a) The constant curvature case is standard (proposition 3.1(a)).
(b) We have already proven the desired conclusion in the Einstein static universe case in proposition 3.1(b).

(c) and (e) With the appropriate definition of the unit timelike vector field U, according to proposition 3.2, the equations $\mathfrak{P}_{ij} = 0$ and $\mathfrak{D}_{ij} = 0$ are sufficient to locally put the spacetime in GRW form (23), while according to proposition 3.5 the equation $\mathfrak{Z}_{ijkh} = 0$ implies that the spatial slices are flat and hence the spacetime is locally FLRW. The remaining conditions place the spacetime in the unique corresponding local regular FLRW isometry class, as per lemmas 3.8(c) and (e).

(d) and (f) With the appropriate definition of the unit timelike vector field U, according to proposition 3.3 and lemma 3.4, the equation $\nabla_i U_j - \frac{\nabla_i \zeta}{2\zeta} U_j - \xi g_{ij} = 0$ is sufficient to locally put the spacetime in GRW form (23) and show that ζ is constant along the spatial slices, while according to proposition 3.5 the additional equation $\mathfrak{C}_{ijkh} = 0$ implies that the spatial slices are of constant curvature and hence the spacetime is locally FLRW. The remaining conditions place the spacetime in the unique corresponding local regular FLRW isometry class, as per lemma 3.8(c) and (e).

3.4. Inflationary local isometry classes

Within the class of inflationary spacetimes (M, g, ϕ), two spacetimes in the form (23) and with $\phi = \phi(t)$, with different (α, f, ϕ) parameters may or may not be isometric. Below, we give the results that allow us to classify inflationary spacetimes into isometry classes (definition 1.1).

Recall that proposition 3.6 gives a sufficient condition for local isometry. We will now show that, under certain conditions, two inflationary spacetimes with parameters (α_i, f_i, ϕ_i), $i = 1,2$, are locally isometric *if and only if* they belong to the same 2-parameter family as in proposition 3.6. As in section 3.3, we will look for an ODE system, jointly satisfied by any locally isometric (α, f, ϕ) triples, with a 2-parameter general solution. The following helpful lemma, the analog of lemma 3.7, again follows easily from standard ODE existence and uniqueness theory [1].

Lemma 3.9. *Consider a smooth real function $V\colon J \to \mathbb{R}$ defined on an open interval, two non-zero real constants α_i, $i = 1,2$, and two pairs of smooth real functions (f_i, ϕ_i) defined on intervals I_i, $i = 1,2$, with either f_i nowhere vanishing.*

(a) Suppose that $\Pi, \Xi\colon J \to \mathbb{R}$ are smooth real functions that satisfy the $\mathfrak{G}_V(\Pi, \Xi) = 0$, in the notation of (20). Suppose also that the triples (α_i, f_i, ϕ_i), $i = 1,2$, both satisfy the system of differential equations

$$\begin{aligned} \phi' &= \Pi(\phi), \\ \frac{f'}{f} &= \Xi(\phi), \\ \frac{\alpha}{f^2} &= \kappa \frac{\Pi^2(\phi) + V(\phi)}{m(m-1)} - \Xi^2(\phi), \end{aligned} \tag{114}$$

and that there exist $t_i \in I_i$, $i = 1,2$, such that $\phi_1(t_1) = \phi_2(t_2) \in J$ and $\frac{\alpha_1}{f_1(t_1)^2} = \frac{\alpha_2}{f_2(t_2)^2}$. Then there exist constants t_0, $A \neq 0$ and $\delta > 0$ such that

$$\alpha_1 = A^2 \alpha_2, \quad f_1(t) = A f_2(t + t_0) \quad \text{and} \quad \phi_1(t) = \phi_2(t + t_0) \tag{115}$$

for every $t \in (t_1 - \delta, t_1 + \delta)$.

(b) Suppose that $\Xi\colon J \to \mathbb{R}$ is a smooth real function. Suppose also that the pairs (f_i, ϕ_i), $i = 1,2$, both satisfy the system of differential equations

$$\begin{aligned} \phi' &= -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi), \\ \frac{f'}{f} &= \Xi(\phi), \end{aligned} \tag{116}$$

and that there exist $t_i \in I_i$, $i = 1,2$, such that $\phi_1(t_1) = \phi_2(t_2) \in J$. Then there exist constants t_0, $A \neq 0$ and $\delta > 0$ such that

$$f_1(t) = Af_2(t + t_0) \quad \text{and} \quad \phi_1(t) = \phi_2(t + t_0) \tag{117}$$

for every $t \in (t_1 - \delta, t_1 + \delta)$.

We are finally in a position to define and classify all regular inflationary spacetimes into families and to describe the parameters needed to identify an isometry class within each family.

Lemma 3.10. *Two regular inflationary spacetimes (those belonging to one of the families identified in definition 1.3) are isometric to each other (definition 1.1) if and only if they belong to the same parametrized family and the corresponding parameters are identical.*

The following proofs are very much analogous to the proofs of lemma 3.8 and theorem 1.4, but we will write them in a mostly self-contained way.

Proof of lemma 3.10. Let us fix m, noting that two isometric spacetimes must have the same dimension. To show that two spacetimes with scalar cannot be isometric, it is sufficient to point out an identity or inequality that is satisfied by curvature scalars or tensors, possibly together also with scalars or tensors covariantly obtained from the scalar field, on one spacetime but not on the other. With that in mind, recall (in the notation of theorems 1.4 and 1.5), that for inflationary spacetimes $\xi = f'/f$, $\boldsymbol{\eta} = f''/f - f'^2/f^2$ and $\zeta = \alpha/f^2$, which are all curvature scalars, as long as they are defined with respect to a vector field U that is also defined from either pure curvature or from the scalar field, such as the choices $U = U_{\mathcal{R}}$, $U_{\mathcal{B}}$ or U_ϕ. To show that all the representatives of a family with identical parameters are all isometric to each other, there will be two possibilities to consider. Either the representative is unique, which is the trivial case. Or, all representatives are selected by satisfying a differential equation. By invoking lemmas 3.9 or 3.7, we can be sure that two solutions to such an equation (with all parameters fixed), if they can be matched up at at least one point, are in fact locally isometric around that point. If the domains of these solutions can also be matched up, then it is clear that they are also globally isometric.

(a) For each Λ (hence $K = \frac{2}{m(m-1)}\Lambda$) and Φ, there is a unique representative in $\mathrm{CC}^m_K\mathrm{CS}_\Phi$. The scalar curvature $\mathcal{R} = m(m+1)K$ and the scalar field $\phi = \Phi$ distinguish the different values of these parameters.

(b) For each $\rho > 0$ (hence $K = \frac{2}{m(m-1)}\kappa\rho$) and interval $J \subseteq \mathbb{R}$, there is a unique representative in $\mathrm{ESU}^m_K\mathrm{CES}_{\rho,J}$. The scalar curvature $\mathcal{R} = m(m-1)K$ and the range $J = \phi(I)$ distinguish different values ρ and J. The condition $(\nabla\phi)^2 < 0$ distinguishes these spacetimes from those in part (a), where $\phi' = 0$.

(c) The representatives of $\mathrm{MMS}^{m,0}_{\Lambda,J,J'}$ satisfy the equations $f''/f + (m-1)f'^2/f^2 = \frac{2\Lambda}{(m-1)}$ which is like in lemma 3.7(a), and

$$\phi' = -\sqrt{\frac{1}{\kappa}\left(\frac{f'^2}{f^2} - \frac{2\Lambda}{m(m-1)}\right)}, \tag{118}$$

since by hypothesis $\phi' < 0$. Thus, the first equation shows that the underlying Lorentzian spacetimes are isometric for identical Λ and J'. The second equation shows, by applying once again standard ODE existence and uniqueness theory, that the inflationary spacetimes are also isometric (as spacetimes with scalar) for identical J. With the choice $U = U_\phi$, the curvature scalars $\boldsymbol{\eta} + m\xi^2 = \frac{2\Lambda}{(m-1)}$ and the ranges of $J = \phi(I)$, $J' = \xi(I)$ distinguishes different Λ, J and J'. The implication that $\xi = f'/f \neq 0$ and $\phi' \neq 0$ distinguish these spacetimes from those of parts (a) and (b).

(d) The representatives of the class $\mathrm{MMS}^m_{\Lambda,\Omega,J}$ satisfy an equation like in lemma 3.9(a), namely

$$\phi' = -\sqrt{\Omega}\frac{|\alpha|^{\frac{m}{2}}}{f^m}, \quad \frac{f'}{f} = \pm\sqrt{\frac{2\Lambda + \kappa\Omega|\alpha|^m/f^{2m}}{m(m-1)} - \frac{\alpha}{f^2}}, \tag{119}$$

where the $\pm$ sign is determined by whether $0 < J'$ or $J' < 0$. With the choice $U = U_\phi$, the curvature scalars $\boldsymbol{\eta} + m\xi^2 = \frac{2\Lambda}{(m-1)}$, $|\zeta|^{-m}(\xi^2 + \zeta) = \frac{\kappa\Omega}{m(m-1)}$ and the range $J = \phi(I)$ distinguish different Λ, Ω and J. The implication that $\xi = f'/f \neq 0$ and $\phi' \neq 0$ distinguish these spacetimes from those of parts (a) and (b), while $\zeta = \kappa\frac{\Pi^2(\phi)+V(\phi)}{m(m-1)} - \Xi^2(\phi) \neq 0$ distinguishes them from those of part (c), where $\zeta = 0$.

(e) The representatives of the class $\mathrm{NKG}^{m,0}_{V,\Xi,J}$ satisfy an equation like in lemma 3.9(b). With the choice $U = U_\phi$, the identities $\phi' = -\frac{(m-1)}{\kappa}\partial_\phi\Xi(\phi)$, $\xi = \Xi(\phi)$ and the range $J = \phi(I)$ distinguish different Ξ, and J. It is important to note that for any solution of $\mathfrak{H}_V(\Xi)$, $-\Xi$ is also a solution that defines another spacetime isometric to a given one via $t \mapsto -(t - t_0)$ for some t_0. We have broken this degeneracy by the $\frac{1}{\kappa}\partial_u\Xi(u) > 0$ requirement (due to using U_ϕ and not $-U_\phi$), so distinct Ξ imply non-isometric spacetimes. The identity $\boldsymbol{\eta} + m\xi^2 = \kappa\frac{V(\phi)}{(m-1)}$, with non-constant $V(\phi)$, distinguishes these spacetimes from those in parts (a), (b), (c) and (d), where the left-hand-side would have been constant.

(f) The representatives of the class $\mathrm{NKG}^m_{V,\Pi,\Xi,J}$ satisfy an equation like in lemma 3.9(a). With the choice $U = U_\phi$, the identities $\phi' = \Pi(\phi)$, $\xi = \Xi(\phi)$ and range $J = \phi(I)$ distinguish different Π, Ξ and J. It is important to note that for any solution of $\mathfrak{G}_V(\Pi, \Xi)$, $(-\Pi, -\Xi)$ is also a solution that defines another spacetime isometric to a given one via $t \mapsto -(t - t_0)$ for some t_0. We have broken this degeneracy by the $\Pi < 0$ requirement (due to using U_ϕ and not $-U_\phi$), so distinct Ξ imply non-isometric spacetimes. The identity $\boldsymbol{\eta} + m\xi^2 = \kappa\frac{V(\phi)}{(m-1)}$, with non-constant $V(\phi)$, distinguishes these spacetimes from those in parts (a), (b), (c), and (d), where the left-hand-side would have been constant, while $\zeta \neq 0$ distinguishes them from those of part (e), where $\zeta = 0$. □

We are now finally in a position to prove our main result about IDEAL characterizations of regular inflationary spacetimes.

Proof of theorem 1.5. The goal is to prove that, for each of the cases listed in table 2, a spacetime satisfies the listed equations (and inequalities) if and only if it is locally isometric (definition 1.1) to one of the regular inflationary spacetimes listed in definition 1.3. In one direction (a regular inflationary spacetime satisfies the corresponding condition), this is essentially the content of lemma 3.10. It remains to show the converse.

(a) When $\phi = \Phi$ is a constant, so is $V(\phi) = \frac{2}{\kappa}\Lambda$, which we have parametrized for our convenience with Λ. Then the Einstein–Klein–Gordon equations become the cosmological vacuum equations $R_{ij} - \frac{1}{2}\mathcal{R}g_{ij} + \Lambda g_{ij} = 0$, which under the FLRW hypotheses have only the constant curvature solution.

(b) The existence of a timelike covariantly constant vector $U = U_\phi$, $\nabla_i U_j = 0$, implies that the spacetime decomposes into a direct sum, with one of the factors being of constant curvature, since the CCD tensor $\mathfrak{C}_{ijkh} = 0$ (see definition 2.3) vanishes and the spatial scalar curvature $\zeta = \frac{2\kappa}{m(m-1)}\rho$ is constant (proposition 3.5); see the proof of proposition

3.1(b) for details. The conclusion, as desired, is that the spacetime is an Einstein static universe and the equation $\phi' = -\sqrt{2\rho/m}$ means that we can choose the time coordinate to put $\phi(t)$ precisely into the form in lemma 2.12.

(c) and (d) With the vector field $U = U_\phi$, according to proposition 3.3 and lemma 3.4, the equation $\nabla_i U_j - \frac{\nabla_i \phi'}{m\phi'} U_j - \xi g_{ij} = 0$ is sufficient to locally put the spacetime into GRW form (23) and show that ϕ' is constant along spatial slices. In case (c), the vanishing of the ZCD tensor $\mathfrak{Z}_{ijkh} = 0$ (see definition 2.3) implies that the spatial slices are flat. In case (d), the equation $\phi' = -\sqrt{\Omega}|\zeta|^{\frac{m}{2}}$ shows that ζ is also constant on spatial slices, and together with the vanishing of the CCD tensor $\mathfrak{C}_{ijkh} = 0$ this implies that the spatial slices are of constant curvature. In both cases we have referred to proposition 3.5, and in both case we have established that the spacetime is locally FLRW. Now, recalling the identities $\xi = f'/f$, $\boldsymbol{\eta} = f''/f - f'^2/f^2$ and $\zeta = \alpha/f^2$, the remaining conditions in each case clearly show that the spacetime is locally isometric to the corresponding reference class in definitions 1.3(c) or (d).

(e) and (f) With the vector field $U = U_\phi$, according to proposition 3.2, the equations $\mathfrak{P}_{ij} = 0$ and $\mathfrak{D}_{ij} = 0$ are sufficient to locally put the spacetime into GRW form (23). In case (e), the vanishing of the ZCD tensor $\mathfrak{Z}_{ijkh} = 0$ implies that the spatial slices are flat. In case (f), the equations $\phi' = \Pi(\phi)$, $\xi = \Xi(\phi)$ show that $\zeta = \kappa\frac{\phi'^2 + V(\phi)}{m(m-1)} - \xi^2$ is then constant along spatial slices (slices of constant ϕ), and together with the vanishing of the CCD tensor $\mathfrak{C}_{ijkh} = 0$ this implies that the spatial slices are of constant curvature. In both cases we have referred to proposition 3.5, and in both cases we have established that the spacetime is locally FLRW. Now, recalling the identities $\xi = f'/f$, $\boldsymbol{\eta} = f''/f - f'^2/f^2$ and $\zeta = \alpha/f^2$, the remaining conditions in each case clearly show that the spacetime is locally isometric to the corresponding reference class in definitions 1.3(e) or (f). □

Acknowledgments

The authors thank M Sánchez for some helpful discussions and Y Urakawa for pointing out reference [30]. Some of the results presented here are based on the MSc thesis (Dept. of Physics, Pavia, 2016) of GC [3]. The work of CD was supported by the University of Pavia. GC acknowledges partial support of SNF Grant No. 200020_172498/1. GC was also (partly) supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation, and by the COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology). IK was partially supported by the ERC Advanced Grant 669240 QUEST 'Quantum Algebraic Structures and Models' at the University of Rome 2 (Tor Vergata). IK also thanks the University of Zürich for its hospitality during the completion of part of this work.

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References

[1] Arnol'd V I 1992 *Ordinary Differential Equations* (*Universitext*) 3rd edn (New York: Springer)

[2] Beem J K, Ehrlich P and Easley K 1996 *Global Lorentzian Geometry* (*Pure and Applied Mathematics* vol 202) (New York: Marcel Dekker)
[3] Canepa G 2016 An ideal characterization of Friedmann–Lemaître–Robertson–Walker space-times *Master's Thesis* University of Pavia
[4] Chatterjee S and Bhui B 1990 Homogeneous cosmological model in higher dimension *Mon. Not. R. Astron. Soc.* **247** 57–61 (http://adsabs.harvard.edu/abs/1990MNRAS.247...57C)
[5] Chen B Y 2014 A simple characterization of generalized Robertson–Walker spacetimes *Gen. Relativ. Gravit.* **46** 1833
[6] Coll B and Ferrando J J 1989 Thermodynamic perfect fluid its Rainich theory *J. Math. Phys.* **30** 2918–22
[7] Easley K L 1991 Local existence of warped product metrics *PhD Thesis* University of Missouri-Columbia
[8] Ehlers J 1961 Beiträge zur relativistischen Mechanik kontinuierlicher Medien *Mainz Akad. Wiss. Math.-Nat. Kl.* **11** 792–837
[9] Ehlers J 1993 Contributions to the relativistic mechanics of continuous media *Gen. Relat. Gravit.* **25** 1225–66 (Engl. Transl.)
[10] Ellis G F E 1971 Relativistic cosmology *General Relativity and Cosmology* ed R K Sachs (New York: Academic) pp 104–82
[11] Ellis G F R and Bruni M 1989 Covariant and gauge-invariant approach to cosmological density fluctuations *Phys. Rev.* D **40** 1804–18
[12] Ferrando J J and Sánchez J A 1998 An intrinsic characterization of the Schwarzschild metric *Class. Quantum Grav.* **15** 1323
[13] Ferrando J J and Sáez J A 2002 On the classification of type D spacetimes *J. Math. Phys.* **45** 652–67
[14] Ferrando J J and Sáez J A 2009 An intrinsic characterization of the Kerr metric *Class. Quantum Grav.* **26** 075013
[15] Ferrando J J and Sáez J A 2010 An intrinsic characterization of spherically symmetric spacetimes *Class. Quantum Grav.* **27** 205024
[16] Ferrando J J and Sáez J A 2017 Labeling spherically symmetric spacetimes with the Ricci tensor *Class. Quantum Grav.* **34** 045002
[17] Fröb M B, Hack T P and Higuchi A 2017 Compactly supported linearised observables in single-field inflation *J. Cosmol. Astropart. Phys.* **2017** 043
[18] Fröb M B, Hack T P and Khavkine I 2017 private communication
[19] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol I (New York: Wiley)
[20] Krongos D S and Torre C G 2015 Geometrization conditions for perfect fluids, scalar fields, and electromagnetic fields *J. Math. Phys.* **56** 072503
[21] Mantica C A and Molinari L G 2017 Generalized Robertson–Walker spacetimes—a survey *Int. J. Geom. Methods Mod. Phys.* **14** 1730001
[22] Mars M 2017 On local characterization results in geometry and gravitation *From Riemann to Differential Geometry and Relativity* ed L Ji *et al* (Berlin: Springer) ch 18, pp 541–70
[23] Miao S P and Woodard R P 2012 Issues concerning loop corrections to the primordial power spectra *J. Cosmol. Astropart. Phys.* JCAP12(2012)008
[24] Muslimov A G 1990 On the scalar field dynamics in a spatially flat Friedman universe *Class. Quantum Grav.* **7** 231–7
[25] Nemiroff R J and Patla B 2007 Adventures in Friedmann cosmology: an educationally detailed expansion of the cosmological Friedmann equations *Am. J. Phys.* **76** 265–76
[26] O'Neill B 1983 *Semi-Riemannian Geometry With Applications to Relativity* (*Pure and Applied Mathematics* vol 103) (New York: Elsevier)
[27] Rainich G Y 1925 Electrodynamics in the general relativity theory *Trans. Am. Math. Soc.* **27** 106–36
[28] Salopek D S and Bond J R 1990 Nonlinear evolution of long-wavelength metric fluctuations in inflationary models *Phys. Rev.* D **42** 3936–62
[29] Sánchez M 1998 On the geometry of generalized Robertson–Walker spacetimes: geodesics *Gen. Relativ. Gravit.* **30** 915–32
[30] Skenderis K and Townsend P K 2006 Hidden supersymmetry of domain walls and cosmologies *Phys. Rev. Lett.* **96** 191301
[31] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's* Field Equations (Cambridge: Cambridge University Press)
[32] Stewart J M and Walker M 1974 Perturbations of space-times in general relativity *Proc. R. Soc. Lond.* **341** 49–74

[33] Takeno H 1952 On the spherically symmetric space-times in general relativity *Prog. Theor. Phys.* **8** 317–26
[34] Urakawa Y and Tanaka T 2009 Influence on observation from IR divergence during inflation. I: single field inflation *Prog. Theor. Phys.* **122** 779–803
[35] Weinberg S 1972 *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (New York: Wiley)
[36] Wolf J 2011 *Spaces of Constant Curvature* (Providence, RI AMS Chelsea Publishing)

IDEAL characterization of higher dimensional spherically symmetric black holes

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October 6, 2018

Abstract

In general relativity, an IDEAL (Intrinsic, Deductive, Explicit, ALgorithmic) characterization of a reference spacetime metric g_0 consists of a set of tensorial equations $T[g] = 0$, constructed covariantly out of the metric g, its Riemann curvature and their derivatives, that are satisfied if and only if g is locally isometric to the reference spacetime metric g_0. We give the first IDEAL characterization of generalized Schwarzschild-Tangherlini spacetimes, which consist of Λ-vacuum extensions of higher dimensional spherically symmetric black holes, as well as their versions where spheres are replaced by flat or hyperbolic spaces. The standard Schwarzschild black hole has been previously characterized in the work of Ferrando and Sáez, but using methods highly specific to 4 dimensions. Specialized to 4 dimensions, our result provides an independent, alternative characterization. We also give a proof of a version of Birkhoff's theorem that is applicable also on neighborhoods of horizon and horizon bifurcation points, which is necessary for our arguments.

1 Introduction

In this work, we are interested in an intrinsic characterization of higher dimensional generalizations of the Schwarzschild black hole. These spherically symmetric, asymptotically flat vacuum spacetimes were first studied by Tangherlini [33]. By a spacetime $(\mathcal{M}, g)$, we mean a smooth manifold $\mathcal{M}$ with a Lorentzian metric g, though a similar discussion can be carried out for any pseudo-Riemannian geometry. While "intrinsic" generally does preclude direct reference to the form of the spacetime metric in a special coordinate system, it is a vague enough term to have multiple interpretations. To be specific, we refer to an *IDEAL*[1] or *Rainich-type* characterization that has been used, for instance, in the works [27, 32, 3, 6, 7, 9, 15, 14, 20, 2]. It consists of a list of tensorial equations ($T_k[g] = 0$, $a = 1, 2, \ldots, N$), constructed covariantly out of the metric (g) and its derivatives (concomitants of the Riemann tensor) that are satisfied if and only if the given spacetime locally belongs to the desired class, possibly narrow enough to be the isometry class of a single reference spacetime geometry. This notion has a natural generalization ($T_k[g, \phi] = 0$) to spacetimes equipped with scalar or tensor fields (ϕ), with equivalence still given by isometric diffeomorphisms that also transform the additional scalars or tensors into each other, though we will not make use of this generalization here. A nice historical survey of this and other local characterization results can be found in [23].

An IDEAL characterization requires neither the existence of any extra geometric structures, nor the translation of the metric and of the curvature into a frame formalism. Thus, it is an alternative to the Cartan-Karlhede characterization [30, Ch.9], which is based on Cartan's moving frame formalism. Intrinsic characterizations, of various types, have been of long standing and independent interest in geometry and General Relativity. But, in addition, they can be helpful in deciding when a metric, given for instance by some complicated coordinate formulas, corresponds

[1]The acronym, explained in [9] (footnote, p.2), stands for Intrinsic, Deductive, Explicit and ALgorithmic.

to one that is already known. In this regard, an IDEAL characterization is especially helpful if one would like to find an algorithmic solution to this recognition problem. In numerical relativity, the near-satisfaction of the tensor equations $T_k[g] \approx 0$ may signal the local proximity of a numerical spacetime to a desired reference geometry. In addition, the approach to zero $T_k[g] \to 0$ could be used to study either linear or nonlinear stability of reference geometries, in an unambiguous and gauge independent way.

The following additional application should be noted. By the Stewart-Walker lemma [31, Lem.2.2], the vanishing of a tensor concomitant $T_k[g] = 0$ for a metric g implies that its linearization $\dot{T}_k[h]$ ($T_k[g+\varepsilon h] = T_k[g] + \varepsilon \dot{T}_k[g] + O(\varepsilon^2)$) is invariant under linearized diffeomorphisms. Thus, any quantity of the form $\dot{T}_k[h]$ defines a gauge invariant observable in linearized gravity, when Einstein or Einstein-matter equations are linearly perturbed about a background solution g. A straight forward (though heuristic) argument shows that an IDEAL characterization of a local isometry class provides a list $\dot{T}_k[h]$, $k = 1, \ldots, N$, of gauge invariant observables that should also complete: the joint kernel of $\dot{T}_k$ should coincide with the tangent space to the isometry orbit (to make this argument completely rigorous, it suffices to check that $T_k[g+h]$ do not approach zero at $O(h^2)$ or higher order). That is, the joint kernel of $\dot{T}_k[h] = 0$ locally consists only of pure gauge modes ($h = \mathcal{L}_v g$ for some vector field v). The use of such local observables (given by differential operators) can be advantageous both in theoretical and practical investigations of classical and quantum field theoretical models because they cleanly separate the local (or ultraviolet) and global (or infrared) aspects of the theory. This may be of interest in the problem of reconstructing the metric of a linear gravitational wave from its complete set of gauge invariant observables [25], or in the problem of determining the decay properties of linear gravitational waves in a gauge-independent way [4].

In this work, we add the family of *generalized Schwarzschild-Tangherlini geometries* to the (unfortunately still small, but slowly growing) literature concerning IDEAL characterizations of isometry classes of individual reference geometries. That family consists of all Λ-vacuum $2+m$-warped products, where the warped m-dimensional factor is maximally symmetric. When the latter factor is a round sphere, we recover the asymptotically flat or (anti-)de Sitter generalization of the Schwarzschild-Tangherlini black holes [33, 19]. Replacing the sphere by flat Euclidean space, we get the higher dimensional generalizations of *Taub's plane symmetric* spacetimes [34]. Replacing the sphere by hyperbolic space, we obtain so-called *pseudo-Schwarzschild wormhole* spacetimes [22]. Other IDEAL characterizations for geometries of interest in General Relativity include (4-dimensional) Schwarzschild [6, 15], Reissner-Nordström [5], Kerr [7, 14], Lemaître-Tolman-Bondi [9], Stephani universes [10] (see references for complete lists and details), and most recently FLRW and inflationary spacetimes (in any dimension) [2]. Of course, for completeness, we have to mention the classic cases of constant curvature spaces (cf. (14)), which are known to be fully characterized by the structure of the Riemann tensor (by theorems of Riemann and Killing-Hopf [36]).

For definiteness, let us state what we mean by a *local isometry* and *local isometry class.*

Definition 1 (locally isometric)**.** *A pseudo-Riemannian geometry* $(\mathcal{M}_1, g_1)$ *is* locally isometric at $x_1 \in \mathcal{M}_1$ to *a pseudo-Riemannian geometry* $(\mathcal{M}_2, g_2)$ at $x_2 \in \mathcal{M}_2$ *if there exist open neighbourhoods* $U_1 \ni x_1$*,* $U_2 \ni x_2$ *and a diffeomorphism* $\chi\colon U_1 \to U_2$ *such that* $\chi(x_1) = x_2$ *and* $\chi^* g_2 = g_1$*. If we can choose* $U_1 = \mathcal{M}_1$ *and* $U_2 = \mathcal{M}_2$ *then they are* (globally) isometric*. If for every* $x_1 \in \mathcal{M}$ *there is* $x_2 \in \mathcal{M}_2$ *such that* $(\mathcal{M}_1, g_1)$ *at* x_1 *is locally isometric to* $(\mathcal{M}_2, g_2)$ *at* x_2*, we simply say that* $(\mathcal{M}_1, g_1)$ *is* locally isometric to $(\mathcal{M}_2, g_2)$ *(note the asymmetry in the definition). If* $(\mathcal{M}_1, g_1)$ *is locally isometric to* $(\mathcal{M}_2, g_2)$*, as well as vice versa, we say that they are* locally isometric to each other *(which constitutes an equivalence relation). All pseudo-Riemannian geometries that are locally isometric to a reference* $(\mathcal{M}, g)$ *constitute its* local isometry class*.*

The synopsis of the paper is the following: In Section 2 we define and exhibit the main geometric features of $2+m$-warped product geometries. Proposition 3 gives a geometric characterization of $2+m$-warped products in terms of a symmetric projector whose covariant derivative satisfies a special constraint. In Section 2.1 we introduce the family of *generalized Schwarzschild-Tangherlini* (gST) geometries, with special attention to the structure of their Riemann curvature. Section 2.2 states and proves a version of Birkhoff's theorem, according to which a locally maximally symmetric $2+m$-warped product that is also a Λ-vacuum must locally coincide with one of the gST geometries. The main reason to include a proof is to pay special attention to the applicability of this result to points lying on a (Killing) horizon. Finally, Theorem 7 in Section 3 puts all the pieces together to give an IDEAL characterization of the local isometry classes of the gST geometries. Due to a quirk of the structure of the gST Riemann curvature in $n = 4$ dimensions, the final result looks slightly different in $n = 4$ and $n \geq 5$ dimensions. This difference is accounted for by Theorem 8. In the case of spherical symmetry in $n = 4$ dimensions with $\Lambda = 0$, our results provide an independent alternative characterization of the standard Schwarzschild spacetime, which was first characterized in [6]. All other instances of the results from Section 3 are new. Finally, in Section 4, we conclude with a discussion of our results and of directions for future work.

Throughout the paper we follow the conventions of [35]: $(-+\cdots+)$ for Lorentzian signature, and $2\nabla_{[\mu}\nabla_{\nu]}\omega_\lambda = R_{\mu\nu\lambda}{}^{\kappa}\omega_\kappa$ for curvature. Unless otherwise specified, all functions will be considered C^∞ smooth.

2 $2 + m$-warped products

Below, we consider $2 + m$-warped product geometries. That is, pseudo-Riemannian geometries on an n-dimensional manifold, which can be represented as a warped product of a 2-dimensional and an m-dimensional geometry. Our main family of examples consists of the *generalized Schwarzschild-Tangherlini* spacetimes (Section 2.1), which includes the spherically symmetric black holes in four and higher dimensions. We will discuss the structure of Riemann curvature tensor of the warped product and the consequences for the 2-dimensional base factor when the product satisfies the Einstein equation (*Birkhoff's theorem*, Section 2.2).

Definition 2 (warped product)**.** *A pseudo-Riemannian geometry $(\bar{\mathcal{M}}, \bar{g}) \cong (\mathcal{M}, g) \times_r (\mathcal{S}, \Omega)$ is a* warped product *with* warping function r *when $\bar{\mathcal{M}} \cong \mathcal{M} \times \mathcal{S}$ and the metric can be written as*

$$\bar{g} = g + r^2\Omega, \tag{1}$$

where the metric tensors g and Ω are lifted to the product space by pulling back along the projections $\bar{\mathcal{M}} \to \mathcal{M}$ and $\bar{\mathcal{M}} \to \mathcal{S}$, while r is the pullback of a nowhere vanishing function on $\mathcal{M}$. We call $(\mathcal{S}, \Omega)$ the warped factor *and $(\mathcal{M}, g)$ the* base factor*.*

Let us now introduce some notational conventions that will simplify subsequent discussions. Denote by $\bar{\nabla}_\mu$, ∇_a and D_A the canonical Levi-Civita connections on $(\bar{\mathcal{M}}, \bar{g}) \cong (\mathcal{M}, g) \times_r (\mathcal{S}, \Omega)$, $(\mathcal{M}, g)$ and $(\mathcal{S}, \Omega)$, respectively. We will use Greek indices $(\alpha\beta\cdots)$ for tensors on $\bar{\mathcal{M}}$, lower case Latin indices $(ab\cdots)$ for tensors on $\mathcal{M}$, and upper case Latin indices $(AB\cdots)$ on $\mathcal{S}$. Using the product structure $\bar{\mathcal{M}} \cong \mathcal{M} \times \mathcal{S}$, any tensor or differential operator on $\mathcal{M}$ or $\mathcal{S}$ can be canonically transferred to $\bar{\mathcal{M}}$. We will do so for the usual Riemann and Ricci curvature tensors and the derivative of the warping function:

$$\begin{gathered} R_{abcd}[g], R_{ab}[g], R[g] \to R_{\mu\nu\lambda\kappa}, R_{\mu\nu}, R, \quad R_{ABCD}[\Omega], R_{AB}[\Omega], R[\Omega] \to S_{\mu\nu\lambda\kappa}, S_{\mu\nu}, S, \\ r_a = \nabla_a r \to r_\mu = \nabla_\mu r. \end{gathered} \tag{2}$$

We will also use the obvious notation $\bar{R}_{\mu\nu\lambda\kappa} = R_{\mu\nu\lambda\kappa}[\bar{g}]$, $\bar{R}_{\mu\nu} = R_{\mu\nu}[\bar{g}]$, $\bar{R} = R[\bar{g}]$. We will use the following self-explanatory convention when raising and lowering indices on tensors transferred to $\bar{\mathcal{M}}$ from one of the factors: $\bar{g}^{\mu\nu}g_{\nu\lambda} = g^{\mu}{}_{\lambda}$, but $\bar{g}^{\mu\nu}r^2\Omega_{\nu\lambda}\bar{g}^{\lambda\kappa} = r^{-2}\Omega^{\mu\kappa}$.

The covariant derivative on the warped product geometry acts as

$$\bar{\nabla}_\mu X_\nu = \nabla_\mu X_\nu + D_\mu X_\nu - 2X^\lambda(r^2\Omega)_{\lambda(\nu}\bar{\nabla}_{\mu)}\log r + X^\lambda(\bar{\nabla}_\lambda \log r)(r^2\Omega)_{\mu\nu}, \tag{3}$$

where the action of $\bar{\nabla}$ on scalars is just through the exterior derivative. In particular, we get

$$\bar{\nabla}_\mu g_{\nu\lambda} = 2\frac{r^2\Omega_{\mu(\nu}r_{\lambda)}}{r}, \quad \text{and} \quad \bar{\nabla}_\mu(r^2\Omega)_{\nu\lambda} = -2\frac{r^2\Omega_{\mu(\nu}r_{\lambda)}}{r}. \tag{4}$$

The curvature tensors are given by

$$\bar{R}_{\mu\nu\lambda\kappa} = r^2 S_{\mu\nu\lambda\kappa} + R_{\mu\nu\lambda\kappa} - \left(\frac{\nabla\nabla r}{r} \odot r^2\Omega\right)_{\mu\nu\lambda\kappa} - \frac{r_\sigma r^\sigma}{2r^2}(r^2\Omega \odot r^2\Omega)_{\mu\nu\lambda\kappa}, \tag{5}$$

$$\bar{R}_{\mu\nu} = S_{\mu\nu} + R_{\mu\nu} - m\frac{\nabla_\mu\nabla_\nu r}{r} - \frac{\Box r}{r} r^2\Omega_{\mu\nu} - (m-1)\frac{r_\sigma r^\sigma}{r^2} r^2\Omega_{\mu\nu}, \tag{6}$$

$$\bar{R} = \frac{S}{r^2} + R - 2m\frac{\Box r}{r} - m(m-1)\frac{r_\sigma r^\sigma}{r^2}, \tag{7}$$

where for convenience we have introduced the *Kulkarni-Nomizu product* of symmetric tensors:

$$(A \odot B)_{\mu\nu\lambda\kappa} = A_{\mu\lambda}B_{\nu\kappa} - A_{\nu\lambda}B_{\mu\kappa} - A_{\mu\kappa}B_{\nu\lambda} + A_{\nu\kappa}B_{\mu\lambda}. \tag{8}$$

Note that we could rewrite the ∇-derivatives using $\bar{\nabla}$-derivatives in the above formulas with the help of the identity

$$\frac{\nabla_\mu\nabla_\nu r}{r} = \frac{\bar{\nabla}_\mu\bar{\nabla}_\nu r}{r} - \frac{r_\lambda r^\lambda}{r^2}(r^2\Omega)_{\mu\nu} = \bar{\nabla}_\mu\frac{\bar{\nabla}_\nu r}{r} + \frac{r_\mu r_\nu}{r^2} - \frac{r_\lambda r^\lambda}{r^2}(r^2\Omega)_{\mu\nu}. \tag{9}$$

These formulas can be extracted from [26, Prp.7.42]. But the quickest way check them is to notice that $\bar{g}_{\mu\nu} = r^2(r^{-2}g_{\mu\nu} + \Omega_{\mu\nu})$, in which a pair of nested conformal transformation relates $\bar{g}$ and a product metric, and use the standard formulas for the conformal transformations of covariant derivatives and curvatures [35, App.D].

Proposition 3 ([8], [13, Thm.16(2)])**.** *A pseudo-Riemannian geometry $(\bar{\mathcal{M}}, \bar{g})$ can be locally put into the form of a $2+m$-warped product*

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + r^2\Omega_{\mu\nu} \tag{10}$$

iff there exist a 1-form ℓ_μ and a symmetric tensor $\bar{\Omega}_{\mu\nu} = \bar{\Omega}_{(\mu\nu)}$ that together satisfy the following conditions

$$\begin{gathered} \bar{\nabla}_{[\mu}\ell_{\nu]} = 0, \quad \ell^\mu\bar{\Omega}_{\mu\nu} = 0, \\ \bar{\Omega}_\mu{}^\nu\bar{\Omega}_{\nu\lambda} = \bar{\Omega}_{\mu\lambda}, \quad \bar{\Omega}_\mu{}^\mu = m, \quad \bar{\nabla}_\mu\bar{\Omega}_{\nu\lambda} = -2\bar{\Omega}_{\mu(\nu}\ell_{\lambda)}. \end{gathered} \tag{11}$$

Then we can choose $g_{\mu\nu} = \bar{g}_{\mu\nu} - \bar{\Omega}_{\mu\nu}$ and $\Omega_{\mu\nu} = r^{-2}\bar{\Omega}$, with r satisfying $\bar{\nabla}_\mu \log|r| = \ell_\mu$.

Proof. In one direction, starting with the definitions of $\bar{\Omega}_{\mu\nu}$ and ℓ_μ in terms of r and $\Omega_{\mu\nu}$, verifying the above identities is a matter of direct calculation (cf. Equation (4)).

In the other direction, the key observation is that a warped product metric is conformal to a direct product metric, namely $\bar{g}_{\mu\nu} = r^2\hat{g}_{\mu\nu}$, with $\hat{g}_{\mu\nu} = r^{-2}g_{\mu\nu} + \Omega_{\mu\nu}$, where the warping function r^2 plays the role of the conformal factor. Given our first condition on ℓ_μ, and since we are working locally, we can always find a smooth function r satisfying $\bar{\nabla}_\mu \log|r| = \bar{\nabla}_\mu r/r = \ell_\mu$. Again locally, we can choose r to be nowhere vanishing. The choice is unique, up to a multiplicative constant.

Define $\hat{g}_{\mu\nu} = r^{-2}\bar{g}_{\mu\nu}$, $\hat{g}^{\mu\nu}$ its inverse, $\Omega_{\mu\nu} = r^{-2}\bar{\Omega}_{\mu\nu}$ and let $\hat{\nabla}$ be the $\hat{g}$-compatible Levi-Civita connection. A straight forward calculation shows that our conditions on $\bar{\Omega}$ translate to

$$\Omega_{\mu\nu}\hat{g}^{\nu\lambda}\Omega_{\lambda\kappa} = \Omega_{\mu\kappa}, \quad \hat{g}^{\mu\nu}\Omega_{\mu\nu} = m, \quad \text{and} \quad \hat{\nabla}_\mu\Omega_{\nu\lambda} = 0. \tag{12}$$

It is well known [8] that the existence of such a rank-m $\hat{\nabla}$-covariantly constant symmetric projector $\Omega_{\mu\nu}$ implies that $\hat{g}_{\mu\nu}$ can be locally put into $2+m$-product form $\hat{g}_{\mu\nu} = (\hat{g}_{\mu\nu} - \Omega_{\mu\nu}) + \Omega_{\mu\nu}$. Our second condition on ℓ_μ implies that r does not depend on the Ω-factor. Thus, without loss of generality, we can define $g_{\mu\nu} = r^2(\hat{g}_{\mu\nu} - \Omega_{\mu\nu})$ and write the product form as $\hat{g}_{\mu\nu} = r^{-2}g_{\mu\nu} + \Omega_{\mu\nu}$.

Undoing the conformal transformation, we end up with the desired local $2+m$-warped product form $g_{\mu\nu} = g_{\mu\nu} + r^2\Omega_{\mu\nu}$. □

2.1 Generalized Schwarzschild-Tangherlini geometries

Consider an integer $n \geq 4$ and a triple of real numbers (α, M, Λ), where $M \neq 0$. The 2-dimensional metric

$$g_{ab} = -f\,dt_a dt_b + \frac{1}{f}dr_a dr_b, \quad f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)}r^2, \tag{13}$$

is well-defined and Lorentzian in the interiors of the r-intervals separated by $r = 0$ and the roots of $f(r) = 0$. It is well-known that each one of these regions has a unique maximal analytic, connected and simply-connected extension [21, 28]. Each region with $r > 0$ generates the same extension (topologically $\mathbb{R}^2$), and similarly for each region with $r < 0$. When n is even, the extensions with $r > 0$ and $r < 0$ are distinct. However, the $r < 0$ extension is isometric to the $r > 0$ extension with M replaced by $-M$, by sending $r \mapsto -r$. When n is odd, the extensions with $r > 0$ and $r < 0$ are isometric, again by sending $r \mapsto -r$, but the geometry obtained by replacing M by $-M$ is different. Thus, for book keeping convenience, let us denote by $(\mathcal{M}, g)_{n,\alpha,M,\Lambda}$ the disjoint union of the $r > 0$ and $r < 0$ extensions with the same M parameter when n is even, and the disjoint union of the $r > 0$ extension with parameter M and the $r < 0$ extension with parameter $-M$ when n is odd. In either case, $\mathcal{M} \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$. Naturally, by our construction, each of these maximally extended geometries is accompanied by the distinguished scalar function r, taking on all non-zero real values, which was analytically extended along with the metric.

The precise way in which the (t, r) charts are glued together along *horizons* and *horizon bifurcation* points to form the analytic extension can be glimpsed from Proposition 6, where (a) corresponds to a *generic* points covered by an (t, r) chart, (b) corresponds to a *horizon* points, and (c) corresponds to a *horizon bifurcation* point of the extension. The gluing is done with the help of the *tortoise coordinate* r_* from (49). Penrose conformal diagrams for the extensions can be found in [21, 28].

Recall that $(\mathcal{S}, \Omega)$ is of *constant curvature* [36], with sectional curvature α, if its Riemann curvature tensor is

$$R_{ABCD}[\Omega] = \frac{\alpha}{2}(\Omega \odot \Omega)_{ABCD}. \tag{14}$$

When an m-dimensional Riemannian geometry $(\mathcal{S}, \Omega)$ is simply connected, geodesically complete and of constant curvature it can only be one of the following [36, Sec.2.4]: Euclidean m-space, round m-sphere, hyperbolic m-space. These are called *maximally symmetric* spaces. Let us denote the corresponding maximally symmetric space with sectional curvature α by $(\mathcal{S}, \Omega)_{m,\alpha}$.

Definition 4 (generalized Schwarzschild-Tangherlini spacetime)**.** *Fix a dimension $m \geq 2$ and a triple of real numbers (α, M, Λ), with $M \neq 0$. Set $n = 2 + m$ and denote the warped product $(\bar{\mathcal{M}}, \bar{g})_{\alpha,M,\Lambda} \cong (\mathcal{M}, g)_{(n,\alpha,M,\Lambda)} \times_r (\mathcal{S}, \Omega)_{m,\alpha}$, where the base factor and the warping function r are defined in the discussion following (13), and the warped factor is the m-dimensional maximally symmetric space of sectional curvature α. We call $(\bar{\mathcal{M}}, \bar{g})_{\alpha,M,\Lambda}$ a n-dimensional* generalized Schwarzschild-Tangherlini (gST) spacetime.

If we had included the $M = 0$ cases, then each such geometry would correspond to a particular representation of a subset of a maximally symmetric geometry (de Sitter or anti-de Sitter spacetime), as we will see shortly (Equation (23)). Since this case has already been extensively studied (e.g., see our previous works [17, 2]), we exclude it from consideration.

For tensors with two or four indices, we define contractions

$$(A \cdot B)_{\mu\nu} = A_\mu{}^\lambda B_{\lambda\nu}, \quad \text{and} \quad (R \cdot S)_{\mu\nu\lambda\kappa} = R_{\mu\nu}{}^{\sigma\tau} S_{\sigma\tau\lambda\kappa}. \tag{15}$$

Recalling also the definition of the Kulkarni-Nomizu product (8), when A, B, C and D are symmetric, we have the useful identities

$$[(A \odot B) \cdot (C \odot D)]_{\mu\nu\lambda\kappa} = 2[(A \cdot C) \odot (B \cdot D) + (A \cdot D) \odot (B \cdot C)]_{\mu\nu\lambda\kappa}, \tag{16}$$

$$(A \odot B)_{\mu\nu}{}^\nu{}_\kappa = [A \cdot B - (\operatorname{tr} A)B - A(\operatorname{tr} B) + B \cdot A]_{\mu\kappa}. \tag{17}$$

Now, we compute the curvature of the gST geometries that we have defined above. Let us start with the 2-dimensional $(\mathcal{M}, g)$ factor. We basically follow the presentation from [24]. Working in the (t, r) chart, clearly $t^a = (\partial_t)^a$ is a timelike Killing vector. For convenience, we also introduce the notation $t_a = g_{ab}t^b = -f dt_a$ and $r_a = dr_a$. They are related as $t^a = -\varepsilon^{ab} r_b$, where $\varepsilon_{ab} = (dt \wedge dr)_{ab}$. Then, of course, $r_a r^a = f$ and $t_a t^a = -1/f$. The action of the covariant derivative is summarized by

$$\nabla_a t_b = \frac{f_1}{2r}\varepsilon_{ab} \quad \text{and} \quad \nabla_a r_b = \frac{f_1}{2r} g_{ab}. \tag{18}$$

For the record, let us write out in full the following identities for $(\mathcal{M}, g)$:

$$\frac{r_a r^a}{r^2} = \frac{f}{r^2} = \frac{\alpha}{r^2} - 2\frac{M}{r^{n-1}} - \frac{2\Lambda}{(n-1)(n-2)}, \tag{19}$$

$$\frac{\nabla_a \nabla_b r}{r} = \left((n-3)\frac{M}{r^{n-1}} - \frac{2\Lambda}{(n-1)(n-2)}\right) g_{ab}, \tag{20}$$

$$R_{abcd} = \frac{R}{4}(g \odot g)_{abcd}, \quad R_{ab} = \frac{R}{2} g_{ab}, \tag{21}$$

$$R = \frac{4\Lambda}{(n-1)(n-2)} + 2(n-2)(n-3)\frac{M}{r^{n-1}}. \tag{22}$$

They can be directly plugged into (5), the formula for the Riemann tensor of a $2+m$-warped product, to get the explicit expression for the Riemann tensor $\bar{R}_{\mu\nu\lambda\kappa}$ of a gST geometry $(\bar{\mathcal{M}}, \bar{g})$.

$$\begin{aligned}\bar{R}_{\mu\nu\lambda\kappa} &= \frac{\Lambda}{(n-1)(n-2)}(\bar{g} \odot \bar{g})_{\mu\nu\lambda\kappa} \\ &+ \frac{M}{r^{n-1}}\left[\frac{(n-2)(n-3)}{2}(g \odot g)_{\mu\nu\lambda\kappa} + (r^2\Omega \odot r^2\Omega)_{\mu\nu\lambda\kappa} - (n-3)(g \odot r^2\Omega)_{\mu\nu\lambda\kappa}\right].\end{aligned} \tag{23}$$

Next, let us define several tensors and scalars built out of the Riemann tensor and its derivatives:

$$\bar{T}_{\mu\nu\lambda\kappa}[\bar{g}] := \bar{R}_{\mu\nu\lambda\kappa} - \frac{\Lambda}{(n-1)(n-2)}(\bar{g} \odot \bar{g})_{\mu\nu\lambda\kappa}, \tag{24}$$

$$\rho[\bar{g}] := \left[\frac{(\bar{T} \cdot \bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu}}{8(n-1)(n-2)(n-3)[(n-2)(n-3)(n-4)+2]}\right]^{\frac{1}{3}}, \tag{25}$$

$$\ell_\mu[\bar{g}] := -\frac{1}{(n-1)}\frac{\bar{\nabla}_\mu \rho}{\rho}, \tag{26}$$

$$A[\bar{g}] := \ell_\mu \ell^\mu + 2\rho + \frac{2\Lambda}{(n-1)(n-2)}. \tag{27}$$

For future reference, we also compute some algebraic combinations among these tensors (see [18, Sec.3.3] for more intermediate steps of the calculations):

$$\bar{T} = \frac{M}{r^{n-1}}\left[\frac{(n-2)(n-3)}{2}(g \odot g) + (r^2\Omega \odot r^2\Omega) - (n-3)(g \odot r^2\Omega)\right], \tag{28}$$

$$\bar{T} \cdot \bar{T} = \left(\frac{M}{r^{n-1}}\right)^2 \left[(n-2)^2(n-3)^2(g \odot g) + 4(r^2\Omega \odot r^2\Omega) + 2(n-3)^2(g \odot r^2\Omega)\right] \tag{29}$$

$$\bar{T} \cdot \bar{T} \cdot \bar{T} = \left(\frac{M}{r^{n-1}}\right)^3 \left[2(n-2)^3(n-3)^3(g \odot g) + 16(r^2\Omega \odot r^2\Omega) - 4(n-3)^3(g \odot r^2\Omega)\right], \tag{30}$$

$$(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\nu}{}_{\kappa} = -\left(\frac{M}{r^{n-1}}\right)^2 \left[2(n-1)(n-2)(n-3)^2 g_{\mu\kappa} + 4(n-1)(n-3)r^2\Omega_{\mu\kappa}\right], \tag{31}$$

$$(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu} = 4(n-1)(n-2)^2(n-3)\left(\frac{M}{r^{n-1}}\right)^2, \tag{32}$$

$$(\bar{T} \cdot \bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu} = 8(n-1)(n-2)(n-3)[(n-2)(n-3)(n-4)+2]\left(\frac{M}{r^{n-1}}\right)^3, \tag{33}$$

$$\rho = \frac{M}{r^{n-1}}, \quad \ell_\mu = \frac{r_\mu}{r}, \quad A = \frac{\alpha}{r^2}. \tag{34}$$

Given the last row of identities, it is clear that

$$\operatorname{sgn} A = \operatorname{sgn} \alpha, \quad \text{and} \quad A^{n-1}\rho^{-2} = \alpha^{n-1}M^{-2}. \tag{35}$$

Next, we find a way to express the projector $r^2\Omega_{\mu\nu}$ onto the warped factor in terms of the curvature. Here we find a slight dimension dependence (as already noted in [18, Sec.3.3]). In dimension $n \geq 5$, one can find a formula that involves only products and contractions $\bar{g}$ of $\bar{T}$:

$$(r^2\Omega)_{\lambda\kappa} = \frac{2(n-2)^2}{(n-1)(n-4)} \frac{(\bar{T}\cdot\bar{T})_{\lambda\nu}{}^{\nu}{}_{\kappa}}{(\bar{T}\cdot\bar{T})_{\mu\nu}{}^{\mu\nu}} + \frac{(n-2)(n-3)}{(n-1)(n-4)}\bar{g}_{\lambda\kappa}. \tag{36}$$

Obviously, the above formula has poles and hence fails when $n = 4$. On the other hand, the following slightly more complex formula works both in $n = 4$ as well as higher dimensions:

$$(r^2\Omega)_{\lambda\kappa} = -\frac{1}{(n-1)(n-3)\,\rho\,\ell^2}\left(\bar{T}_{\mu\lambda\nu\kappa} - \frac{(n-2)(n-3)}{2}\rho\,(\bar{g}\odot\bar{g})_{\mu\lambda\nu\kappa}\right)\ell^{\lambda}\ell^{\kappa}. \tag{37}$$

The complexity of the second formula is due to the presence of the ℓ_μ vector, which is itself defined as the gradient of a scalar of a gradient constructed from $\bar{T}$. Thus formula (36) may be preferable in $n \geq 5$ to (37), even if the latter also works in higher dimensions.

The following result simply identifies the invariant parameters that can be used to exhaustively label the distinct isometry classes of gST reference geometries as we have defined them earlier. Note that below we adopt the convention that the sign function satisfies $\operatorname{sgn} 0 = 0$.

Proposition 5. *A gST geometry $(\bar{\mathcal{M}}, \bar{g})_{\alpha,M,\Lambda}$ is locally isometric at $x \in \bar{\mathcal{M}}$ to another gST geometry $(\bar{\mathcal{M}}', \bar{g}')_{\alpha',M',\Lambda'}$ at $x' \in \bar{\mathcal{M}}'$ iff*

$$((\operatorname{sgn}\alpha)\,|\alpha|^{n-1}\,M^{-2}, \Lambda) = ((\operatorname{sgn}\alpha')\,|\alpha'|^{n-1}\,M'^{-2}, \Lambda'), \quad \rho[\bar{g}](x) = \rho[\bar{g}'](x') \tag{38}$$

and one of the following holds

(a) x is a generic point, $\ell^2[\bar{g}](x) \neq 0 \neq \ell^2[\bar{g}'](x')$, or

(b) x is a horizon point, $\ell_\mu[\bar{g}](x) \neq 0 \neq \ell_\mu[\bar{g}'](x')$ and $\ell^2[\bar{g}](x) = 0 = \ell^2[\bar{g}'](x')$, or

(c) x is a horizon bifurcation point, $\ell_\mu[\bar{g}](x) = 0 = \ell_\mu[\bar{g}'](x')$.

Hence, $(\bar{\mathcal{M}}, \bar{g})_{\alpha,M,\Lambda}$ and $(\bar{\mathcal{M}}', \bar{g}')_{\alpha',M',\Lambda'}$ are isometric iff

$$((\operatorname{sgn}\alpha)\,|\alpha|^{n-1}\,M^{-2}, \Lambda) = ((\operatorname{sgn}\alpha')\,|\alpha'|^{n-1}\,M'^{-2}, \Lambda'^2). \tag{39}$$

In other words, the pair or real numbers

$$\big((\operatorname{sgn}\alpha)|\alpha|^{n-1}M^{-2}, \Lambda\big) \in \mathbb{R}^2 \tag{40}$$

uniquely and exhaustively identifies all isometry classes among the gST geometries (Definition 4). Moreover, non-isometric gST geometries are not even locally isometric.

Proof. Many of the arguments below are based on the fact that the existence of a local isometry linking the points x and x' forces the pairwise equality of all curvature scalars respectively evaluated at these points. A slight generalization of this idea to covariant identities involving curvature tensors immediately establishes our claims (a), (b) and (c). Also, recall that, from our definition of a reference gST geometry, the coordinate transformation $r \mapsto -r$ always corresponds to the parameter flip $M \mapsto -M$, independent of the parity of the dimension n. Finally, we will assume that the points $x \in \bar{\mathcal{M}}$ and $x' = \bar{\mathcal{M}}'$ belong to the regions where we can introduce the (t, r) and (t', r') coordinates, as in (13), on the base factors. Then, simple coordinate transformations on (t, r) extend to globally defined diffeomorphisms of $\bar{\mathcal{M}}$ or $\bar{\mathcal{M}}'$ by analyticity. When such coordinates are ill-defined on neighborhoods of x or x', the same logic applies, but where we need to directly apply the diffeomorphisms defined by analytic extension.

First, note that $\Lambda = \Lambda'$ is necessary for local isometry. Relying on (23), we can obtain this constant from the Ricci scalar of the geometry, $\bar{R}[\bar{g}] = \frac{2n\Lambda}{(n-2)}$ and $\bar{R}[\bar{g}'] = \frac{2n\Lambda'}{(n-2)}$. Let us assume the equality $\Lambda = \Lambda'$ from now on.

Next, relying on (35), note that

$$\operatorname{sgn} A[\bar{g}] = \operatorname{sgn}\alpha, \quad A[\bar{g}]^{n-1}\rho[\bar{g}]^{-2} = \alpha^{n-1}M^{-2} \tag{41}$$

$$\text{and} \quad \operatorname{sgn} A[\bar{g}'] = \operatorname{sgn}\alpha', \quad A[\bar{g}']^{n-1}\rho[\bar{g}']^{-2} = \alpha'^{n-1}M'^{-2}. \tag{42}$$

Hence, since knowledge of $(\operatorname{sgn}\alpha)|\alpha|^{n-1}M^{-2}$ is equivalent to the knowledge of both $\operatorname{sgn}\alpha$ and $\alpha^{n-1}M^{-2}$, the equality $(\operatorname{sgn}\alpha)|\alpha|^{n-1}M^{-2} = (\operatorname{sgn}\alpha')|\alpha'|^{n-1}M^{-2}$ is also necessary for local isometry. Let us assume this equality from now on. It remains only to check that both equalities guarantee the existence of an isometry.

When $\operatorname{sgn}\alpha = 0 = \operatorname{sgn}\alpha'$, apply the coordinate transformations $(1/t, r) \mapsto |M|^{\frac{1}{n-1}}(1/t, r)$ and $(1/t', r') \mapsto |M'|^{\frac{1}{n-1}}(1/t', r')$, together with a possible $r \mapsto -r$ and/or $r' \to -r'$ flip, depending on the signs of M and M', to bring the base factors to isometric form with $M = 1 = M'$. To keep r and r' as the warping functions, these transformations must be accompanied by the conformal rescalings $\Omega \mapsto |M|^{\frac{2}{n-1}}\Omega$ and $\Omega' \mapsto |M|^{\frac{2}{n-1}}\Omega'$. However, since $\alpha = 0 = \alpha'$ and the two warped factors are flat, these rescalings do not affect their isometry class. Hence, the two gST geometries must be isometric since they have the same warped product structure.

Now, assume that $\alpha \neq 0 \neq \alpha'$, while necessarily $\operatorname{sgn}\alpha = (-1)^k = \operatorname{sgn}\alpha'$. Then the coordinate redefinitions $(1/t, r) \mapsto |\alpha|^{\frac{1}{2}}(1/t, r)$ and $(1/t', r') \mapsto |\alpha|^{\frac{1}{2}}(1/t', r')$ bring them both to $\alpha = (-1)^k = \alpha'$. Let us assume this equality from now on.

The only possible remaining difference between the parameters is that $\operatorname{sgn} M \neq \operatorname{sgn} M'$, while $|M| = |M'|$. But then, applying $r \mapsto -r$ or $r' \mapsto -r'$ brings about the equality $M = M'$ and hence the desired isometry.

Clearly, the above arguments apply both to local isometries as well as to global isometries. This concludes the proof. □

2.2 Birkhoff's theorem

It is well-known that being a $2+m$-warped product solution of cosmological Einstein's equations is highly restrictive. In particular, the geometry of the base factor is restricted to locally coincide with one of the base factors of a gST geometry, whether the warped factor is spherically symmetric or not. This rigidity result (though usually stated with the spherical symmetry assumption) is known as Birkhoff's theorem [21, 29, 28, 1]. Below, we state and prove a version that is convenient for our purposes. The main reason to include a proof is to make sure that we can cover the corner cases (when ∇r becomes null or even vanishes) that are often skipped in the literature.

Recall that a metric $\bar{g}_{\mu\nu}$ is called a Λ-*vacuum* when it satisfies the source-free Einstein equations with cosmological constant Λ:

$$\bar{R}_{\mu\nu}[\bar{g}] - \frac{1}{2}\bar{R}[\bar{g}]\bar{g}_{\mu\nu} + \Lambda\bar{g}_{\mu\nu} = 0 \iff \bar{R}_{\mu\nu}[\bar{g}] - \frac{2\Lambda}{(n-2)}\bar{g}_{\mu\nu} = 0. \tag{43}$$

Proposition 6 ([21, 29, 28, 1])**.** *Consider a pseudo-Riemannian geometry $(\bar{\mathcal{M}}, \bar{g})$ of dimension $n = 2+m$ that locally, say at $\bar{x} \in \bar{\mathcal{M}}$, has the form $(\mathcal{M}, g) \times_r (\mathcal{S}, \Omega)$ of a $2+m$-warped product, with*

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + r^2\Omega_{\mu\nu}, \tag{44}$$

where $(\mathcal{M}, g)$ is Lorentzian and r is not locally constant at $x \in \mathcal{M}$, the projection of $\bar{x}$ to $\mathcal{M}$. When $\bar{g}_{\mu\nu}$ is a Λ-vacuum that is not locally of constant curvature at $\bar{x}$, the metric of the base factor can be locally put into one of the following forms at x:

(a) when $\nabla_a r \neq 0$, $(\nabla r)^2 \neq 0$ at x, in local (t, r) coordinates,

$$g_{ab} = -f\, dt_a dt_b + \frac{1}{f}\, dr_a dr_b, \tag{45}$$

(b) when $\nabla_a r \neq 0$, $(\nabla r)^2 = 0$, $r = r_H$ at x, in local (v, r) coordinates,

$$g_{ab} = -f\, dv_a dv_b + 2\, dv_{(a} dr_{b)}, \tag{46}$$

(c) when $\nabla_a r = 0$, $r = r_H$ at x, in local (U, V) coordinates,

$$r = r(UV) = r_H + r_H UV + O^3(U, V), \quad \textit{with} \quad z r'(z) = \frac{f(r)}{f'(r_H)}, \quad \textit{and}$$

$$g_{ab} = \frac{-4fe^{-h}}{f'(r_H)^2\,(1 - r/r_H)}\, dU_{(a} dV_{b)}, \quad \textit{with} \quad h(r) = \int_{r_H}^{r} ds \left(\frac{f'(r_H)}{f(s)} - \frac{1}{s - r_H} \right), \tag{47}$$

where in each case

$$f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)} r^2, \tag{48}$$

for some constants α and $M \neq 0$. In cases (b) and (c), $r = r_H$ is a root of $f(r) = 0$; in case (c) the root is always simple.

Thus, $(\mathcal{M}, g)$ is locally isometric at x to either (a) a generic point, (b) a horizon point, or (c) a horizon bifurcation point of a gST, as classified in Proposition 5.

Proof. We address the last statement first. The transitions between the different charts in (a), (b) and (c) are effected with the help of the *tortoise coordinate*

$$r_* = \frac{1}{f'(r_H)} \log\left(\frac{r}{r_H} - 1 \right) + \frac{h(r)}{f'(r_H)}, \quad \text{which satisfies} \quad dr_* = \frac{dr}{f(r)}, \tag{49}$$

implicitly defining $h(r)$ as in (47). The null coordinate from (b) has the form $v = t + r_*$, while the double null coordinates from (c) have the form $U = -e^{-f'(r_H)(t-r_*)/2}$, $V = e^{f'(r_H)(t+r_*)/2}$. Direct calculation shows that the metrics g_{ab} expressed in these charts agree on overlaps. Hence, charts (b) and (c) constitute analytic extensions of the charts from (a), which when glued in a simply connected way, form the maximal analytic extension, of the gST geometry from Definition 4. The correspondence between points (a), (b) and (c) from the current Proposition with those from Proposition 5 is obvious.

Before entering further specific arguments, we use formulas (6) and (9) to project the Einstein equations (43) onto the base factor of the warped product:

$$R_{ab} - (n-2)\frac{\nabla_a \nabla_b r}{r} - \frac{2\Lambda}{(n-2)} g_{ab} = 0. \tag{50}$$

Recalling that in 2 dimensions $R_{ab} = \frac{1}{2} R g_{ab}$, the equation decomposes into its trace and traceless parts:

$$R - (n-2)\frac{\Box r}{r} - \frac{4\Lambda}{(n-2)} = 0, \quad (n-2)\frac{\nabla_a \nabla_b r}{r} - \frac{\Box r}{2r} g_{ab} = 0. \tag{51}$$

Contracting the traceless part with ε_{ab} shows that $t_a = -\varepsilon_{ab} r^b$ is Killing, $\nabla_{(a} t_{b)} = 0$.

Suppose that x is critical a point of r, that is, $\nabla_a r(x) = 0$, and hence also $t_a(x) = 0$. We will now argue that this critical point must be non-degenerate and hence isolated (cf. Remark 2.9 in [1]). From the projected Einstein equations above, $\nabla_a \nabla_b r \propto g_{ab}$, hence either $\nabla_a \nabla_b r(x) = 0$ or the critical point is non-degenerate. If indeed $\nabla_a \nabla_b r(x) = 0$, then the formula $t_a = -\varepsilon_{ab} \nabla^b r$ tells us that $t_a(x) = 0$ and $\nabla_a t_b(x) = 0$ as well. In turn, this implies that locally $t_a \equiv 0$, and hence also $\nabla_a r \equiv 0$, which contradicts our hypothesis that r is not locally constant. The reason is that t_a solves the Killing equation, which is an equation of *finite type*. In short, knowing $t_a(x)$ and $\nabla_a t_b(x)$ determines t_a uniquely in a neighborhood of x [16, App.B], which in this case gives $t_a \equiv 0$.

Next, we address each of the possibilities stated in the theorem. The function $f(r)$ from (48) always appears as the general solution, parametrized by constants α and M, of the differential equation

$$r\,(rf' + (n-3)f)' = -\frac{4\Lambda}{(n-2)} r^2. \tag{52}$$

(a) When $\nabla_a r \neq 0$ is non-null, we are free to pick orthogonal coordinates (t, r), with $(\partial_t)^a \propto t^a$ Killing. Then, the most general ansatz for the metric is $g_{ab} = -f(r) dt_a dt_b + 1/h(r) dr_a dr_b$. The projected Einstein equations reduce to

$$\frac{h'}{h} = \frac{f'}{f}, \quad r\,(rf' + (n-3)f)' = -\frac{4\Lambda}{(n-2)} r^2. \tag{53}$$

Up to rescaling t by a constant, $h(r) = f(r)$ and $f(r)$ is as stipulated. The metric g_{ab} is singular only when $\alpha = M = \Lambda = 0$, meaning that all other values of the parameters are allowed.

(b) When $\nabla r(x) \neq 0$ is null, we are free to pick coordinates (v, r), with $\nabla_a v$ null everywhere, with $(\partial_v)^a \propto t^a$ Killing. Then, the most general ansatz for the metric is $g_{ab} = -f(r)dv_a dv_b + 2h(r)dv_{(a}dr_{b)}$. The projected Einstein equations reduce to

$$h' = 0, \quad r(rf' + (n-3)f)' = -\frac{4\Lambda}{(n-2)}r^2. \tag{54}$$

Up to rescaling v by a constant, $h(r) = 1$ and $f(r)$ is as stipulated. For $\nabla_a r$ to be null at x as well, we must have $f(r_H) = 0$.

(c) When x is a non-degenerate critical point of r, we are free to pick double null coordinates (U, V) centered at x. Let $r_H = r(x)$, which must be a non-zero constant. In 2 dimensions, double null coordinates are unique up to permutation and individual reparametrization of each coordinate. Then, the most general ansatz for the metric is

$$g_{ab} = 2F(U, V)\, dU_{(a}dV_{b)}. \tag{55}$$

Our hypotheses on r force its Taylor expansion to start, up to a constant rescaling, as

$$r(U, V) = r_H + \frac{1}{2L}g_{ab}(U\partial_U)^a(V\partial_V)^b + O^3(U, V) = r_H + \frac{F(0,0)}{L}UV + O^3(U, V), \tag{56}$$

for some constant $L \neq 0$, which will be constrained later on. The precise form of $r = r(U, V)$ is to be determined from the equations. The traceless part of the projected Einstein equations reduces to

$$\partial_U \frac{\partial_U r}{F} = 0, \quad \partial_V \frac{\partial_V r}{F} = 0 \iff \frac{\partial_U r}{\xi(U)} = V\frac{F(U, V)}{L\xi(U)\eta(V)}, \quad \frac{\partial_V r}{\eta(V)} = U\frac{F(U, V)}{L\xi(U)\eta(V)}, \tag{57}$$

for some arbitrary $\xi(U)$ and $\eta(V)$, though with $\xi(0) = 1 = \eta(0)$ as needed to maintain our hypotheses on the Taylor expansion of r. We are free to change our ansatz by $F(U, V) \mapsto F(U, V)\xi(U)\eta(V)$ and reparametrize the coordinates subject to $\xi(U)dU \mapsto dU$, $\eta(V)dV \mapsto dV$, effectively setting $\xi = 1 = \eta$. Then, two immediate integrability conditions follow:

$$U(\partial_U r - VF) - V(\partial_V r - UF) = U\partial_U r - V\partial_V r = 0, \tag{58}$$

$$U\partial_U(\partial_V r - UF) - V\partial_V(\partial_U r - VF) = UV(U\partial_U F - V\partial_V F) = 0. \tag{59}$$

Therefore, both F and r are constant along the flow lines of the vector field $U\partial_U - V\partial_V$, which are the connected components of the level sets of UV. Without loss of generality, we can restrict to a neighborhood where each level set of UV consists of exactly two connected components, exchanged by the flip $(U, V) \mapsto (-U, -V)$.

At this point, we would like to conclude that $r = r(UV)$ and $F = F(UV)$ for some smooth functions $r(z)$ and $F(z)$, but this conclusion must be postponed due to the technical complication (not shared by *polynomial* or *analytic functions*) that a *smooth function* invariant under the flow of $U\partial_U - V\partial_V$ takes such a form only on those open regions where the product UV may play the role of a simple coordinate (no critical points, connected level sets). For instance, $F = F_{\{U>0\}}(UV)$ on $U > 0$ and $F = F_{\{V>0\}}(UV)$ on $V > 0$, but $F_{\{U>0\}}(z)$ and $F_{\{V>0\}}(z)$ may be different smooth functions. Of course, these functions have to agree on overlapping regions, namely for $z > 0$, in this case. Below, we will presume that we are restricting to one of the open regions $U > 0$, $U < 0$, $V > 0$, or $V < 0$.

It turns out that it is more convenient to write everything in terms of r, $UV = UV(r)$ and $F = F(r)$, which is always possible to do locally, away from $(U, V) = (0, 0)$ and subject to the above caveats. Taking advantage of the usual identity $(UV)' = 1/r'$, our previous integrating step (57) simply gives $F = L/(UV)'$. Thus, the remaining trace part of the projected Einstein equations reduces to

$$r\partial_r(r\partial_r + (n-3))\frac{2}{L}\frac{UV}{(UV)'} = -\frac{4\Lambda}{(n-2)}r^2 \iff \frac{UV}{(UV)'} = \frac{L}{2}f(r), \tag{60}$$

with $f(r)$ as stipulated. At $(U,V) = 0$, the above left-hand side evaluates to 0. Hence, $r = r_H$ must solve $f(r) = 0$. If it is a multiple root, that is $f(r) = C(r-r_H)^k + O(r-r_H)^{k+1}$ for some constants C and $k > 1$, then asymptotically $UV \sim e^{D/(r-r_H)^{k-1}}$ for some constant D as $r \to r_H$, which is not compatible with our requirement that UV be a smooth function of r at $r = r_H$ and vice versa. Hence, $r = r_H$ must also be a *simple root* of $f(r) = 0$, meaning that $f'(r_H) \neq 0$.

Thus, given that $f(r) = 0$ has a simple root at r_H, we can rewrite the equation for UV as

$$\frac{(UV)'}{UV} = \frac{2}{Lf'(r_H)(r-r_H)} + h'(r) \iff UV = C(r/r_H - 1)^{\frac{2}{Lf'(r_H)}} e^{h(r)}, \tag{61}$$

for some constant C, with smooth

$$h(r) = \frac{2}{Lf'(r_H)} \int_{r_H}^{r} ds \left(\frac{f'(r_H)}{f(s)} - \frac{1}{s - r_H} \right). \tag{62}$$

For this formula to be consistent with the Taylor expansion $r(U,V) = r_H + \frac{F(0,0)}{L} UV + O^3(U,V)$, we must have $L = 2/f'(r_H)$ and $C = r_H L/F(0,0)$. The final form of the solution is then

$$UV = \frac{2r_H}{f'(r_H)F(0,0)} \left(\frac{r}{r_H} - 1 \right) e^h, \quad F = \frac{L}{(UV)'} = \frac{f'(r_H)F(0,0)}{2r_H} \frac{-2fe^{-h}}{f'(r_H)^2(1 - r/r_H)}. \tag{63}$$

To bring the metric into the desired form, it remains only to choose the value of $F(0,0) = \frac{2r_H}{f'(r_H)}$, which could be done by appropriately rescaling U or V by a constant. This also finally fixes the initial coefficients in Taylor expansion $r = r_H + r_H UV + O^3(U,V)$.

Finally, recall that the above discussion, determining the precise form of $UV = UV(r)$ and $F = F(r)$, applies separately in each of the open regions $U > 0$, $U < 0$, $V > 0$ or $V < 0$, though that precise form of the functions $UV(r)$ and $F(r)$ may differ from region to region. It is obvious that the only differences may be in the values of the constants α and M, which a priori may be different in these different regions. However, $UV(r)$ and $F(r)$ must agree on the intersection whenever two of these regions overlap (e.g., $U > 0$ and $V > 0$), and this is only possible if the values of α and M agree between the overlapping regions. Considering all possible overlaps, the values of α and M must then agree in all these regions. In other words, the formulas in (63) actually hold on a whole neighborhood of $(U,V) = (0,0)$ without any more reservations (the extension to the origin is by continuity). This concludes the proof. □

3 Characterization

In this Section, we state and prove our main result on the IDEAL characterization of local isometry classes (Definition 1) of generalized Schwarzschild-Tangherlini (gST) geometries (Definition 4). The result comes in two versions, one valid for any dimension $n \geq 5$ (Theorem 7), and the other valid for $n = 4$ as well as higher dimensions (Theorem 8). The only difference between them is the covariant formula for extracting the idempotent projector $\bar{\Omega}_{\mu\nu}$ from the curvature. In higher dimensions, $n \geq 5$, it can be obtained by a simpler formula than in $n = 4$. However, the more complicated formula that works in $n = 4$ also generalizes to higher dimensions. When restricted to the standard spherically symmetric, $\Lambda = 0$, $n = 4$ Schwarzschild solution, our Theorem 8 provides an independent alternative IDEAL characterization compared to the one previously obtained in [6].[2] For other values of the dimension n, and the parameters α, M and Λ, the results of this Section are new.

Theorem 7. *Consider a Lorentzian geometry $(\bar{\mathcal{M}}, \bar{g})$, with $\dim \bar{\mathcal{M}} = n \geq 5$. Given a constant Λ,*

[2]The notation in [6] is somewhat hard to follow. A transcription of the key formulas into more standard tensor notation can be found in [15].

define the following tensors and scalars from the metric and the curvature:

$$\bar{T}_{\mu\nu\lambda\kappa} := \bar{R}_{\mu\nu\lambda\kappa} - \frac{\Lambda}{(n-1)(n-2)}(\bar{g}\odot\bar{g})_{\mu\nu\lambda\kappa}, \tag{64a}$$

$$\rho := \left[\frac{(\bar{T}\cdot\bar{T}\cdot\bar{T})_{\mu\nu}{}^{\mu\nu}}{8(n-1)(n-2)(n-3)[(n-2)(n-3)(n-4)+2]}\right]^{\frac{1}{3}}, \tag{64b}$$

$$\ell_\mu := -\frac{1}{(n-1)}\frac{\bar{\nabla}_\mu\rho}{\rho}, \tag{64c}$$

$$A := \ell_\mu\ell^\mu + 2\rho + \frac{2\Lambda}{(n-1)(n-2)}, \tag{64d}$$

$$\bar{\Omega}_{\mu\nu} := \frac{2(n-2)^2}{(n-1)(n-4)}\frac{(\bar{T}\cdot\bar{T})_{\mu\lambda}{}^{\lambda}{}_{\nu}}{(\bar{T}\cdot\bar{T})_{\lambda\kappa}{}^{\lambda\kappa}} + \frac{(n-2)(n-3)}{(n-1)(n-4)}\bar{g}_{\mu\nu}, \tag{64e}$$

$$g_{\mu\nu} := \bar{g}_{\mu\nu} - \bar{\Omega}_{\mu\nu}, \tag{64f}$$

$$\begin{aligned} Z_{\mu\nu\lambda\kappa} &:= \bar{T}_{\mu\nu\lambda\kappa} \\ &\quad - \rho\left[\frac{(n-2)(n-3)}{2}(g\odot g)_{\mu\nu\lambda\kappa} + (\bar{\Omega}\odot\bar{\Omega})_{\mu\nu\lambda\kappa} - (n-3)(g\odot\bar{\Omega})_{\mu\nu\lambda\kappa}\right]. \end{aligned} \tag{64g}$$

Then the geometry $(\bar{\mathcal{M}},\bar{g})$ is locally isometric at $\bar{x}\in\bar{\mathcal{M}}$ to a gST geometry $(\bar{\mathcal{M}}',\bar{g}')_{\alpha',M',\Lambda'}$ (Definition 4) iff $\Lambda=\Lambda'$ and the following conditions hold on some neighborhood of $\bar{x}$:

$$\rho\not\equiv 0, \quad \ell_\mu\not\equiv 0, \quad \bar{\Omega}_{\mu\nu}\geq 0, \tag{65a}$$

$$\begin{gathered} \bar{\nabla}_{[\mu}\ell_{\nu]} = 0, \quad \ell^\mu\bar{\Omega}_{\mu\nu} = 0, \\ \bar{\Omega}_\mu{}^\mu = (n-2), \quad \bar{\Omega}_\mu{}^\nu\bar{\Omega}_{\nu\lambda} = \bar{\Omega}_{\mu\lambda}, \quad \bar{\nabla}_\mu\bar{\Omega}_{\nu\lambda} = -2\bar{\Omega}_{\mu(\nu}\ell_{\lambda)}, \end{gathered} \tag{65b}$$

$$Z_{\mu\nu\lambda\kappa} = 0, \quad (\operatorname{sgn} A)\,|A|^{n-1}\,\rho^{-2} = (\operatorname{sgn}\alpha')\,|\alpha'|^{n-1}\,M'^{-2}. \tag{65c}$$

By the inequality $\bar{\Omega}_{\mu\nu}\geq 0$ we mean that the quadratic form $\bar{\Omega}(v,v)=\bar{\Omega}_{\mu\nu}v^\mu v^\nu$ is positive-semidefinite.

Note that, in choosing the precise form of the conditions (65), we have aimed for clarity rather than any particular kind of minimality. So, for instance, the condition $\bar{\Omega}_\mu{}^\mu=(n-2)$ is automatically satisfied by virtue of our definition of $\bar{\Omega}_{\mu\nu}$, the same being true for the condition $\bar{\nabla}_{[\mu}\ell_{\nu]}=0$.

Proof. In the easy direction, the direct calculations from Section 2.1 show that all of the identities (65) hold for any gST geometry.

In the other direction, we first note that the conditions (65b) involving ℓ_μ and $\bar{\Omega}_{\mu\nu}$ are precisely needed by Proposition 3 to locally put the geometry into $2+m$-warped product form $(\bar{\mathcal{M}},\bar{g}) = (\mathcal{M},g)\times_r(\mathcal{S},\Omega)$, with $m=n-2$ and $\bar{g}_{\mu\nu}=g_{\mu\nu}+r^2\Omega_{\mu\nu}$, where $\Omega_{\mu\nu}=r^{-2}\bar{\Omega}_{\mu\nu}$ and $\ell_\mu=\bar{\nabla}_\mu\log|r|$, r not locally constant by $\ell_\mu\not\equiv 0$. Since, $\bar{\Omega}_{\nu\nu}\geq 0$ and $\bar{g}_{\mu\nu}$ is Lorentzian, $g_{\mu\nu}$ must be a Lorentzian metric when restricted to the base factor of the warped product.

Next, taking the trace of the $Z_{\mu\nu\lambda\kappa}=0$ identity, we obtain precisely the Λ-vacuum Einstein equations, forcing the equality $\Lambda=\Lambda'$. Appealing to Birkhoff's theorem (Proposition 6), we can conclude that the metric g_{ab} on the *base* of the warped product has the gST form $(\mathcal{M},g)_{n,\alpha,M,\Lambda}$ (Section 2.1), for some values of the parameters α and M (we have not yet drawn any conclusion about the *warped factor*). Note that our version of Birkhoff's theorem is applicable as long as the warping function r is not locally constant at $\bar{x}\in\bar{\mathcal{M}}$, without other restrictions on $\bar{\nabla}_\mu r(\bar{x})$, with the different possibilities listed as parts (a), (b) and (c) in Proposition 6.

Two immediate consequences, again following the direct calculations in Section 2.1, are the identities

$$\rho = \frac{M}{r^{n-1}} \quad \text{and} \quad A = \frac{\alpha}{r^2}.$$

Now, knowing that our geometry is locally of $2+m$-warped product form, implies that its Riemann tensor takes the form (5), where we can replace $r_\sigma r^\sigma/r^2=\ell^2$. Hence, projecting the identity $Z_{\mu\nu\lambda\kappa}=0$ by $\bar{\Omega}_{\mu\nu}$ on each index, we obtain

$$\bar{\Omega}^{\mu}_{\mu'}\bar{\Omega}^{\nu}_{\nu'}\bar{\Omega}^{\lambda}_{\lambda'}\bar{\Omega}^{\kappa}_{\kappa'}\left(r^2 S_{\mu\nu\lambda\kappa} - \frac{A}{2}(\bar{\Omega}\odot\bar{\Omega})_{\mu\nu\lambda\kappa}\right) = 0. \tag{66}$$

Substituting in what we already know about $\bar{\Omega}_{\mu\nu}$ and A into this formula, it reduces to the equality $S_{ABCD} = \frac{\alpha}{2}(\Omega \odot \Omega)_{ABCD}$ on the warped factor $(\mathcal{S}, \Omega)$. In other words, the *warped* factor is locally of constant curvature, with sectional curvature α. Hence, our geometry $(\bar{\mathcal{M}}, \bar{g})$ is indeed locally isometric at $\bar{x} \in \bar{M}$ to a gST geometry $(\bar{\mathcal{M}}', \bar{g}')_{\alpha,M,\Lambda}$ with parameters α, M and Λ (Definition 4).

Finally, referring again to the direct calculations from Section 2.1, and in particular the identity (35), the last identity from (65) implies the equality

$$(\operatorname{sgn}\alpha)\,|\alpha|^{n-1}\,M^{-2} = (\operatorname{sgn}\alpha')\,|\alpha'|^{n-1}\,M'^{-2}. \tag{67}$$

So, invoking Proposition 5, we can at last conclude that the gST geometry that we have identified locally about $\bar{x} \in \bar{\mathcal{M}}$ is indeed isometric to the desired reference gST geometry, $(\bar{\mathcal{M}}', \bar{g}')_{\alpha,M,\Lambda} \cong (\bar{\mathcal{M}}', \bar{g}')_{\alpha',M',\Lambda'}$. □

For a version of the above result that holds also when $n = 4$ we need only replace formula (64e) for $\bar{\Omega}_{\mu\nu}$ by formula (68) below (recall the discussion around Equations (36) and (37) in Section 2.1). Hence, the proof of the following result proceeds in an exactly analogous way.

Theorem 8. *Consider a Lorentzian geometry $(\bar{\mathcal{M}}, \bar{g})$, with $\dim \bar{\mathcal{M}} = n \geq 4$. Then the same statement as in Theorem 7 holds, with the exception that we must change the definition*

$$\bar{\Omega}_{\mu\nu} := -\frac{1}{(n-1)(n-3)\,\rho\,\ell^2}\left(\bar{T}_{\mu\lambda\nu\kappa} - \frac{(n-2)(n-3)}{2}\rho\,(\bar{g}\odot\bar{g})_{\mu\lambda\nu\kappa}\right)\ell^\lambda \ell^\kappa. \tag{68}$$

4 Discussion

We have given an IDEAL characterization (Theorems 7, 8) of each spacetime from the family of local isometry classes *generalized Schwarzschild-Tangherlini (gST)* spacetimes (Definition 4), which consists of maximally symmetric Λ-vacuum $2+m$-warped products. In particular, this family includes the higher dimensional spherically symmetric black holes, which generalize the 4-dimensional Schwarzschild solution and which were first investigated by Tangherlini [33].

Our strategy, inspired by the related recent work on the characterization of cosmological FLRW spacetimes [2], was to first identify a geometric characterization of the $2+m$-warped product structure in terms of a rank-m symmetric projector $\bar{\Omega}$ (Proposition 3) [8, 13] and then to identify a covariant formula for $\bar{\Omega}$ in terms of the curvature of a given gST geometry. The previously existing IDEAL characterization of the 4-dimensional Schwarzschild geometry [6, 15] relied much more on an intricate algebraic classification of the Riemann tensor, special to 4 dimensions. Unfortunately, we could not generalize the latter approach to higher dimensions directly. On the other hand, our general strategy succeeds also in 4 dimensions (Theorem 8) and thus provides an alternative characterization of the Schwarzschild geometry, which should be compared to that of [6]. We leave such a comparison to future work.

As discussed in [11], the linearization of the tensors of an IDEAL characterization of a given reference geometry provides a set of gauge invariants with respect to linearized gauge transformations (diffeomorphisms) of linearized gravity on that geometry. Heuristically, this set of invariants is a good candidate for being complete, but to be rigorous its completeness should be proven separately. In the recent work [18], we have explicitly exhibited (by a different method) complete sets of linear invariants for each geometry in the gST family. Relating these invariants to the linearization of the IDEAL characterization tensors, as well as vice versa, can accomplish two goals: give a geometric interpretation to the invariants of [18] and to prove the completeness of the linearized invariants that can be obtained from the present work.

A natural direction for related future work is to extend it to an IDEAL characterization of other black hole spacetimes. For instance, the generalization to charged spherical symmetric black holes, the Reissner-Nordström geometry and its higher dimensional generalizations, should be straight forward. A bigger challenge would be to generalize it to higher dimensional rotating black holes, the Myers-Perry generalizations of the Kerr geometry, perhaps building on the existing characterization of the 4-dimensional Kerr spacetime [7]. Eventually, it would be interesting to extend the characterization to the full Kerr-Taub-NUT-(A)dS family [12] and higher dimensional versions.

An important future application of the above results could be an intrinsic and invariant characterization of asymptotic flatness. Usually, asymptotic flatness is defined by an asymptotic condition on the metric in a special coordinate system. On the other hand, this definition is supposed to capture the asymptotic approach to flatness or the asymptotic end of a black hole spacetime. Thus, having an on hand an IDEAL characterization of these reference geometries may give us a chance to intrinsically and invariantly define asymptotic approach to them, providing an alternative definition of asymptotic flatness.

Acknowledgments. Thanks to Willie Wong and Markus Fröb for helpful discussions, as well as to Alfonso García-Parrado Gómez-Lobo for bibliographic suggestions. The author was partially supported by the GAČR project 18-07776S and RVO: 67985840.

References

[1] X. An and W. W. Y. Wong, "Warped product space-times," *Classical and Quantum Gravity* **35** (2018) 025011, arXiv:1707.01483.

[2] G. Canepa, C. Dappiaggi, and I. Khavkine, "IDEAL characterization of isometry classes of FLRW and inflationary spacetimes," *Classical and Quantum Gravity* **35** (2018) 035013, arXiv:1704.05542.

[3] B. Coll and J. J. Ferrando, "Thermodynamic perfect fluid. Its Rainich theory," *Journal of Mathematical Physics* **30** (1989) 2918–2922.

[4] G. Dotti, "Nonmodal linear stability of the Schwarzschild black hole," *Physical Review Letters* **112** (2014) 191101, arXiv:1307.3340.

[5] J. J. Ferrando and J. A. Sáez, "On the classification of type D spacetimes," *Journal of Mathematical Physics* **45** (2002) 652–667, arXiv:gr-qc/0212086.

[6] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of the Schwarzschild metric," *Classical and Quantum Gravity* **15** (1998) 1323–1330.

[7] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of the Kerr metric," *Classical and Quantum Gravity* **26** (2009) 075013, arXiv:0812.3310.

[8] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of 2+2 warped spacetimes," *Classical and Quantum Gravity* **27** (2010) 205023, arXiv:1005.1491.

[9] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of spherically symmetric spacetimes," *Classical and Quantum Gravity* **27** no. 20, (2010) 205024, arXiv:1005.1780.

[10] J. J. Ferrando and J. A. Sáez, "Labeling spherically symmetric spacetimes with the Ricci tensor," *Classical and Quantum Gravity* **34** (2017) 045002, arXiv:1701.05023.

[11] M. B. Fröb, T.-P. Hack, and I. Khavkine, "Approaches to linear local gauge-invariant observables in inflationary cosmologies," *Classical and Quantum Gravity* **35** (2018) 115002, arXiv:1801.02632.

[12] V. P. Frolov, P. Krtouš, and D. Kubizňák, "Black holes, hidden symmetries, and complete integrability," *Living Reviews in Relativity* **20** (2017) 6, arXiv:1705.05482.

[13] A. García-Parrado Gómez-Lobo, "Bi-conformal vector fields and the local geometric characterization of conformally separable pseudo-Riemannian manifolds I," *Journal of Geometry and Physics* **56** (2006) 1069–1095, arXiv:math/0504162.

[14] A. García-Parrado Gómez-Lobo, "Local non-negative initial data scalar characterization of the Kerr solution," *Physical Review D* **92** (2015) 124053, arXiv:1510.07561.

[15] A. García-Parrado Gómez-Lobo and J. A. Valiente Kroon, "Initial data sets for the Schwarzschild spacetime," *Physical Review D* **75** (2007) 024027, arXiv:gr-qc/0609100.

[16] R. Geroch, "Limits of spacetimes," *Communications in Mathematical Physics* **13** (1969) 180–193.

[17] I. Khavkine, "The Calabi complex and Killing sheaf cohomology," *Journal of Geometry and Physics* **113** (2017) 131–169, `arXiv:1409.7212`.

[18] I. Khavkine, "Compatibility complexes of overdetermined PDEs of finite type, with applications to the Killing equation," 2018. `arXiv:1805.03751`.

[19] H. Kodama and A. Ishibashi, "A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions," *Progress of Theoretical Physics* **110** (2003) 701–722, `arXiv:hep-th/0305147`.

[20] D. S. Krongos and C. G. Torre, "Geometrization conditions for perfect fluids, scalar fields, and electromagnetic fields," *Journal of Mathematical Physics* **56** (2015) 072503, `arXiv:1503.06311`.

[21] K. Lake, "Maximally extended, explicit and regular coverings of the Schwarzschild-de Sitter vacua in arbitrary dimension," *Classical and Quantum Gravity* **23** (2006) 5883–5895, `arXiv:gr-qc/0507031`.

[22] F. S. N. Lobo and J. P. Mimoso, "Possibility of hyperbolic tunneling," *Physical Review D* **82** (2010) 044034, `arXiv:0907.3811`.

[23] M. Mars, "On local characterization results in geometry and gravitation," in *From Riemann to Differential Geometry and Relativity*, L. Ji, A. Papadopoulos, and S. Yamada, eds., ch. 18, pp. 541–570. Springer, Berlin, 2017.

[24] K. Martel and E. Poisson, "Gravitational perturbations of the Schwarzschild spacetime: A practical covariant and gauge-invariant formalism," *Physical Review D* **71** no. 10, (2005) 104003, `arXiv:gr-qc/0502028`.

[25] C. Merlin, A. Ori, L. Barack, A. Pound, and M. van de Meent, "Completion of metric reconstruction for a particle orbiting a kerr black hole," *Physical Review D* **94** (2016) 104066, `arXiv:1609.01227`.

[26] B. O'Neill, *Semi-Riemannian Geometry With Applications to Relativity*, vol. 103 of *Pure and Applied Mathematics*. Elsevier Science, 1983.

[27] G. Y. Rainich, "Electrodynamics in the general relativity theory," *Transactions of the American Mathematical Society* **27** (1925) 106–136.

[28] K. Schleich and D. M. Witt, "A simple proof of Birkhoff's theorem for cosmological constant," *Journal of Mathematical Physics* **51** no. 11, (2010) 112502, `arXiv:0908.4110`.

[29] H.-J. Schmidt, "A new proof of Birkhoff's theorem," *Gravitation and Cosmology* **3** (1997) 185–190, `arXiv:gr-qc/9709071`. `https://elibrary.ru/item.asp?id=12947329`.

[30] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*. Cambridge University Press, Cambridge, 2003. `http://dx.doi.org/10.1017/CBO9780511535185`.

[31] J. M. Stewart and M. Walker, "Perturbations of Space-Times in general relativity," *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **341** (1974) 49–74.

[32] H. Takeno, "On the spherically symmetric Space-Times in general relativity," *Progress of Theoretical Physics* **8** (1952) 317–326.

[33] F. R. Tangherlini, "Schwarzschild field in n dimensions and the dimensionality of space problem," *Il Nuovo Cimento* **27** (1963) 636–651.

[34] A. H. Taub, "Empty space-times admitting a three parameter group of motions," *The Annals of Mathematics* **53** (1951) 472–490.

[35] R. M. Wald, *General Relativity.* University of Chicago Press, Chicago, 1984.

[36] J. A. Wolf, *Spaces of Constant Curvature*, vol. 372 of *AMS-Chelsea.* American Mathematical Society, 6th ed., 2011.

Compatibility complexes of overdetermined PDEs of finite type, with applications to the Killing equation

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November 21, 2018

Abstract

In linearized gravity, two linearized metrics are considered gauge-equivalent, $h_{\mu\nu} \sim h_{\mu\nu} + K_{\mu\nu}[v]$, when they differ by the image of the Killing operator, $K_{\mu\nu}[v] = \nabla_\mu v_\nu + \nabla_\nu v_\mu$. A universal (or complete) compatibility operator for K is a differential operator K_1 such that $K_1 \circ K = 0$ and any other operator annihilating K must factor through K_1. The components of K_1 can be interpreted as a complete (or generating) set of local gauge-invariant observables in linearized gravity. By appealing to known results in the formal theory of overdetermined PDEs and basic notions from homological algebra, we solve the problem of constructing the Killing compatibility operator K_1 on an arbitrary background geometry, as well as of extending it to a full compatibility complex K_i ($i \geq 1$), meaning that for each K_i the operator K_{i+1} is its universal compatibility operator. Our solution is practical enough that we apply it explicitly in two examples, giving the first construction of full compatibility complexes for the Killing operator on these geometries. The first example consists of the cosmological FLRW spacetimes, in any dimension. The second consists of a generalization of the Schwarzschild-Tangherlini black hole spacetimes, also in any dimension. The generalization allows an arbitrary cosmological constant and the replacement of spherical symmetry by planar or pseudo-spherical symmetry.

1 Introduction

An important aspect of General Relativity is its invariance under diffeomorphisms, also called *gauge transformations* of this theory. Of course, this invariance survives linearization about some fixed background metric g and the linearized diffeomorphisms (or linearized gauge transformations) change the linearized metric as $h_{ab} \mapsto h_{ab} + K_{ab}[v]$, where $K_{ab}[v] = \nabla_a v_b + \nabla_b v_a$ is the *Killing operator* with respect to the background metric g. Solutions of the *Killing equation* $K[v] = 0$ are *Killing vectors* v_a. Because two linearized metric configurations are considered physically equivalent if they differ only by a linearized gauge transformation, an inescapable part of the study of linearized gravity (linearized General Relativity) is the need to separate gauge and physical degrees of freedom; the latter essentially parametrize equivalence classes of linearized metrics under linearized gauge transformations.

A *local gauge-invariant observable* is a differential operator $O[h]$ such that $O[K[v]] = 0$ for an arbitrary argument v_a. Clearly, such differential operators have many potential applications in linearized gravity and, not surprisingly, their study has a long history [34]. While not all useful gauge-invariant observables $O[h]$ are local (where O is *local* if it is a differential, rather than an integral, operator), the local ones are distinguished by the property that they preserve supports, $\operatorname{supp} O[h] \subseteq \operatorname{supp} h$, which helps to disentangle the gauge-invariant information contained in h from infrared or asymptotic properties of h. Further discussion of these issues, with brief surveys of previous work, can be found [14], in the context of cosmological perturbations, and in [21, 31, 1], in the context of black hole perturbations.

In this work, we are interested in the problem of explicitly constructing complete (or generating) sets of local gauge-invariant observables on spacetime backgrounds of physical interest.

Completeness refers to the ability to express any local gauge-invariant observable in terms of linear combinations of derivatives of a given set. For technical reasons [22, 23], it also becomes important to identify complete sets of differential relations between them, complete sets of differential relations between these differential relations, and so on. Phrased in mathematical terms, given a background metric g, we are interested in constructing a (full) compatibility complex for the corresponding Killing operator $K[v]$, where *full* refers to the continuation of the sequence of differential relations until it terminates (becomes identically zero), a property that is usually required implicitly.

An unfortunate aspect of the study of local gauge-invariant observables $O[h]$ is that their structure depends strongly on the background metric g, since the Killing operator $K[v]$, which determines the structure of gauge-equivalence classes, depends on g in an essential way. Thus, in principle, this problem needs to be attacked anew for each background metric of interest. Unfortunately, a full solution (a complete set of gauge-invariants, relations between them, etc.) can be found in the literature only in very few cases, even if we restrict ourselves only to the construction of complete sets of gauge-invariants (and not relations between them, etc.). To our knowledge, the full Killing compatibility complex is known only for flat (Minkowski) and constant curvature (de Sitter or anti-de Sitter) spacetimes [23]. In principle, the methods of [15, 16] could have been used to generate the compatibility complex on locally symmetric spacetimes (those with a covariantly constant Riemann tensor), but to our knowledge they have never been explicitly elaborated in the Lorentzian setting [7]. In addition, complete sets of local gauge-invariant observables are known only for cosmological (inflationary FLRW) spacetimes in any dimension, due to the recent construction in [13, 9, 14], and for the 4-dimensional Kerr black hole, as recently highlighted in [1]. Full proofs of the results announced in [1] will appear in [2] and will be based on the methods to be presented in this work.

The major obstacle to solving the problem that we have posed (the construction of a compatibility complex for the Killing operator) has so far been proving completeness (of a set of gauge-invariants, of a set of relations between them, etc.). In the flat and constant curvature cases, the proof was basically due to Calabi [8, 23], and was specific to those geometries. In the cosmological case, the proof is due to [13], but is somewhat ad-hoc and without clear generalizations.

The main innovation in this work is the application of methods from the formal theory of PDEs [32, 18, 30] and homological algebra [37] to the problem of constructing Killing compatibility complexes. In fact, a method for systematically constructing a complete compatibility operator for any overdetermined linear differential operator (under mild regularity conditions) has been known for a long time [18] (it was this method that was applied in [15, 16]). Unfortunately, it is rather cumbersome to apply directly. There do exist computer algebra implementations of this method [6], but they suffer from the problem that the input and output of this computer algebra construction must be matrices of scalar differential operators written in some explicit coordinates, which often destroys any manifest symmetry or other structure that the original linear differential operator had. This is certainly an undesirable feature when dealing with the Killing operator on a spacetime with some symmetry, product or warped product structure. However, there is a significant simplification of the general systematic construction when we restrict our attention to differential operators of *finite type*, of which the Killing operator is an example. We will take full advantage of this simplification, together with some basic notions from homological algebra, to give a practical sufficient condition (Lemma 4) for the completeness of a given set of local gauge-invariant observables in linearized gravity (or more generally, the completeness of a compatibility operator for any operator). In practice, this criterion also leads to a way to construct (Theorem 9) the full Killing compatibility complex (or more generally, the compatibility complex for any operator of finite type), which can preserve various structural properties of a given background spacetime geometry.

In Section 2, we introduce some ideas from homological algebra, applied to linear differential operators, and use it to show how to explicitly construct a compatibility complex for a PDE of finite type (the appropriateness of our definition of *finite type* operator is discussed at length in Appendix A). This technique is applied to the Killing equation in Section 3, with some examples. In particular, we treat in detail the examples of spacetimes of constant curvature (Section 3.1), cosmological FLRW spacetimes (Section 3.2) and Schwarzschild-Tangherlini black holes (Section 3.3). In each case we make some remarks about the relation or our results with the literature. Appendix B gives a helpful reference for the notation used in different subsections of Section 3. In all examples, we keep the spacetime dimension n general (that is, we allow at least $n \geq 4$). The results of Sections 3.2 and 3.3 are new. Finally, we conclude with a discussion of further work in Section 4.

Whenever speaking of differential operators, we will specifically mean a linear differential operator with smooth coefficients acting on smooth functions. More precisely, we will consider differential operators that map between sections of vector bundles, say $V_1 \to M$ and $V_2 \to M$, on some fixed manifold M, $K \colon \Gamma(V_1) \to \Gamma(V_2)$. The source and target bundle of a differential operator, $V_1 \to M$ and $V_2 \to M$ respectively in the last example, will be considered as part of its definition and will most often be omitted from the notation. We will denote the composition of two differential operators K and L by $K \circ L$, or simply by KL, if no confusion is possible. A *local section* of a vector bundle $V \to M$ is a section of the restriction bundle $V|_U \to U$ for some open $U \subset M$. A local section v that solves the differential equation $K[v] = 0$ on its domain of definition is a *local solution*.

2 Compatibility operators

We start by introducing some basic notions from *homological algebra* [37].

Definition 1. *A (possibly infinite) composable sequence K_l of linear maps, $l = l_{\min}, \dots, l_{\max}$, such that $K_{l+1} \circ K_l = 0$ when possible, is called a* (cochain) complex. *Given complexes K_l and K'_l a sequence C_l of linear maps, as in the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \bullet \arrow[r, "K_{l-1}"] \arrow[d, "C_{l-1}"] & \bullet \arrow[r, "K_l"] \arrow[d, "C_l"] & \bullet \arrow[r, "K_{l+1}"] \arrow[d, "C_{l+1}"] & \bullet \arrow[r] \arrow[d, "C_{l+2}"] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r, "K'_{l-1}"'] & \bullet \arrow[r, "K'_l"'] & \bullet \arrow[r, "K'_{l+1}"'] & \bullet \arrow[r] & \cdots
\end{tikzcd}, \tag{1}$$

such that its squares commute, that is $K'_l \circ C_l = C_{l+1} \circ K_l$ when possible, is called a cochain map *or a* morphism *between complexes. A* homotopy *between complexes K_l and K'_l (which could also be the same complex, $K_l = K'_l$) is a sequence of morphism, as the dashed arrows in the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \bullet \arrow[r, "K_{l-1}"] \arrow[d, "C_{l-1}"] & \bullet \arrow[r, "K_l"] \arrow[d, "C_l"] \arrow[dl, dashed, "H_{l-1}"] & \bullet \arrow[r, "K_{l+1}"] \arrow[d, "C_{l+1}"] \arrow[dl, dashed, "H_l"] & \bullet \arrow[r] \arrow[d, "C_{l+2}"] \arrow[dl, dashed, "H_{l+1}"] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r, "K'_{l-1}"'] & \bullet \arrow[r, "K'_l"'] & \bullet \arrow[r, "K'_{l+1}"'] & \bullet \arrow[r] & \cdots
\end{tikzcd}, \tag{2}$$

and the sequence of maps $C_l = K'_{l-1} \circ H_{l-1} + H_l \circ K_l$ is said to be a morphism induced by *the homotopy H_l. An* equivalence up to homotopy *between complexes K_l and K'_l is a pair of morphisms C_l and D_l between them, as in the diagram*

$$\begin{tikzcd}
\bullet \arrow[loop left, "\tilde{H}_{l_{\min}-1}"] \arrow[r, "K_{l_{\min}}"] \arrow[d, shift right, "C_{l_{\min}}"'] & \arrow[l, dashed, bend left, "H_{l_{\min}}"] \cdots & \bullet \arrow[r, "K_l"] \arrow[d, shift right, "C_l"'] & \bullet \arrow[l, dashed, bend left, "H_l"] \arrow[d, shift right, "C_{l+1}"'] & \cdots \arrow[r, "K_{l_{\max}}"] & \bullet \arrow[l, dashed, bend left, "H_{l_{\max}}"] \arrow[loop right, "\tilde{H}_{l_{\max}+1}"] \arrow[d, shift right, "C_{l_{\max}+1}"'] \\
\bullet \arrow[loop left, "\tilde{H}'_{l_{\min}-1}"] \arrow[r, "K'_{l_{\min}}"'] \arrow[u, shift right, "D_{l_{\min}}"'] & \arrow[l, dashed, bend right, "H'_{l_{\min}}"'] \cdots & \bullet \arrow[r, "K'_l"'] \arrow[u, shift right, "D_l"'] & \bullet \arrow[l, dashed, bend right, "H'_l"'] \arrow[u, shift right, "D_{l+1}"'] & \cdots \arrow[r, "K'_{l_{\max}}"'] & \bullet \arrow[l, dashed, bend right, "H'_{l_{\max}}"'] \arrow[loop right, "\tilde{H}'_{l_{\max}+1}"] \arrow[u, shift right, "D_{l_{\max}+1}"']
\end{tikzcd}, \tag{3}$$

such that C_l and D_l are mutual inverses up to homotopy (H_l and H'_l), that is

$$D_l \circ C_l = \mathrm{id} - K_{l-1} \circ H_{l-1} - H_l \circ K_l, \tag{4}$$
$$C_l \circ D_l = \mathrm{id} - K'_{l-1} \circ H'_{l-1} - H'_l \circ K'_l, \tag{5}$$

with the special end cases

$$D_{l_{\min}} \circ C_{l_{\min}} = \mathrm{id} - \tilde{H}_{l_{\min}-1} - H_{l_{\min}} \circ K_{l_{\min}}, \qquad K_{l_{\min}} \circ \tilde{H}_{l_{\min}-1} = 0, \tag{6}$$

$$C_{l_{\min}} \circ D_{l_{\min}} = \mathrm{id} - \tilde{H}'_{l_{\min}-1} - H'_{l_{\min}} \circ K'_{l_{\min}}, \qquad K'_{l_{\min}} \circ \tilde{H}'_{l_{\min}-1} = 0, \tag{7}$$

$$D_{l_{\max}+1} \circ C_{l_{\max}+1} = \mathrm{id} - H_{l_{\max}} \circ K_{l_{\max}} - \tilde{H}_{l_{\max}+1}, \qquad \tilde{H}_{l_{\max}+1} \circ K_{l_{\max}} = 0, \tag{8}$$

$$C_{l_{\max}+1} \circ D_{l_{\max}+1} = \mathrm{id} - H'_{l_{\max}} \circ K'_{l_{\max}} - \tilde{H}'_{l_{\max}+1}, \qquad \tilde{H}'_{l_{\max}+1} \circ K'_{l_{\max}} = 0, \tag{9}$$

where the $\tilde{H}$ maps are allowed to be arbitrary, as long as they satisfy the given identities.

Note that our definition of equivalence up to homotopy between complexes of finite length is set up in a way that allows an equivalence between longer complexes to be truncated and still remain an equivalence.

Next, we restrict our attention to the case where all maps are given by differential operators.

Definition 2. *Given a differential operator K, any composable differential operator L such that $L \circ K = 0$ is a* compatibility operator *for K. If K_1 is a compatibility operator for K, it is called* complete *or* universal *when any other compatibility operator L can be factored through $L = L' \circ K_1$ for some differential operator L. A complex of differential operators K_l, $l = 0, 1, \ldots$ is called a compatibility complex for K when $K_0 = K$ and, for each $l \geq 1$, K_l is a complete compatibility operator for K_{l-1}.*

Definition 3. *Given a (possibly infinite) complex of differential operators K_l, $l = l_{\min}, l_{\min} + 1, \ldots, l_{\max}$, we say that it is* locally exact *at a point x when, for every $l_{\min} < l < l_{\max}$, for every smooth function f_l defined on an open neighborhood $U \ni x$ such that $K_l[f_l] = 0$, there exists a smooth function g_{l-1} defined on a possibly smaller open neighborhood $V \ni x$ such that $f_l = K_{l-1}[g_{l-1}]$.* Locally exact *(without specifying a point x) means locally exact at every x.*

Note that a complete compatibility operator, say K_1, need not be unique. But, by its universal factorization property, any two compatibility operators, say K_1 and K'_1, must factor through each other, $K_1 = L_1 \circ K'_1$ and $K'_1 = L'_1 \circ K_1$ for some differential operators L_1 and L'_1.

Given two composable operators, K and K_1, the compatibility condition $K_1 \circ K = 0$ is very easy to check. On the other hand, it may be quite challenging to check completeness/universality. One way to do it is to compare K and K_1 with another pair of operators which are already known to satisfy the universality condition.

Lemma 4. *Consider two complexes of differential operators K_l and K'_l, for $l = 0, 1$. If these complexes are equivalent up to homotopy, as in the diagram*

$$\begin{tikzcd}
\bullet \arrow[r, "K_0"] \arrow[d, shift right, "C_0"'] & \bullet \arrow[r, "K_1"] \arrow[l, dashed, bend left, "H_0"] \arrow[d, shift right, "C_1"'] & \bullet \arrow[l, dashed, bend left, "H_1"] \arrow[d, shift right, "C_2"'] \\
\bullet \arrow[r, "K'_0"'] \arrow[u, shift right, "D_0"'] & \bullet \arrow[r, "K'_1"'] \arrow[l, dashed, bend right, "H'_0"'] \arrow[u, shift right, "D_1"'] & \bullet \arrow[l, dashed, bend right, "H'_1"'] \arrow[u, shift right, "D_2"']
\end{tikzcd}, \tag{10}$$

where we really only require all squares to be commutative and the identities $D_1 \circ C_1 = \mathrm{id} - K_0 \circ H_0 - H_1 \circ K_1$ and $C_1 \circ D_1 = \mathrm{id} - K'_0 \circ H'_0 - H'_1 \circ K'_1$ to hold, then K_1 is universal iff K'_1 is universal.

Furthermore, the complex K_l, $l = 0, 1$, is locally exact iff the complex K'_l, $l = 0, 1$, is locally exact.

Proof. Without loss of generality, assume that K'_1 is universal. Let $L \circ K_0 = 0$. Then $(L \circ D_1) \circ K'_0 = L \circ K_0 \circ D_0 = 0$. By universality of K'_1, there exists a differential operator L' such that $L \circ D_1 = L' \circ K'_1$. Recall that from our hypotheses that $D_1 \circ C_1 = \mathrm{id} - K_0 \circ H_0 - H_1 \circ K_1$. But then

$$\begin{aligned}
L &= L \circ (D_1 \circ C_1 + K_0 \circ H_0 + H_1 \circ K_1) \\
&= L' \circ (K'_1 \circ C_1) + (L \circ H_1) \circ K_1 \\
&= (L' \circ C_2) \circ K_1 + (L \circ H_1) \circ K_1 = L'' \circ K_1,
\end{aligned}$$

where $L'' = L' \circ C_2 + L \circ H_1$. This demonstrates the universality of K_1.

Next, without loss of generality, assume that K_l' is locally exact. Pick a point x, an open neighborhood $U \not\ni x$, and a smooth function f such that $K_1[f] = 0$. Then $K_1'[C_1[f]] = C_2[K_1[f]] = 0$. Hence, by local exactness, there exists a smooth g' defined on a possibly smaller open neighborhood $V \ni x$ such that $K_0'[g'] = C_1[f]$. Setting $g = D_0[g'] + K_0[H_0[f]]$ on $V \ni x$, direct calculation shows that

$$\begin{aligned} K_0[g] &= K_0[D_0[g']] + K_0[H_0[f]] = D_1[K_0'[g']] + K_0[H_0[f]] \\ &= D_1 \circ C_1[f] + K_0[H_0[f]] = f - H_1[K_1[f]] \\ &= f, \end{aligned} \tag{11}$$

which shows that the K_l complex is also locally exact. □

Next, we will show how to construct a universal compatibility operator for a differential operator K if it is equivalent, in the sense of a complex consisting of one operator, to some operator with a known universal compatibility operator.

Lemma 5. *Consider differential operators K_0 and K_0'. Suppose that K_0 and K_0' are equivalent up to homotopy, in the sense of the diagram*

$$\begin{tikzcd}[row sep=large, column sep=huge]
\bullet \arrow[r, "K_0"] \arrow[d, shift right, "C_0"'] \arrow[loop left, "\tilde{H}"] & \bullet \arrow[l, dashed, bend left, "H_0"] \arrow[d, shift right, "C_1"'] \\
\bullet \arrow[r, "K_0'"'] \arrow[u, shift right, "D_0"'] \arrow[loop left, "\tilde{H}'"] & \bullet \arrow[l, dashed, bend right, "H_0'"'] \arrow[u, shift right, "D_1"']
\end{tikzcd} , \tag{12}$$

where we require all squares to be commutative and the identities $D_0 \circ C_0 = \mathrm{id} - \tilde{H} - H_0 \circ K_0$, $C_0 \circ D_0 = \mathrm{id} - \tilde{H}' - H_0' \circ K_0'$ to hold, with $K_0 \circ \tilde{H} = 0$ and $K_0' \circ \tilde{H}' = 0$. Then, if a universal compatibility operator K_1' for K_0' is known, we can complete the above diagram to the following equivalence up to homotopy

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\bullet \arrow[r, "K_0"] \arrow[d, shift right, "C_0"'] & \bullet \arrow[l, dashed, bend left, "H_0"] \arrow[r, "K_1 = \begin{bmatrix} \mathrm{id} - K_0 \circ H_0 - D_1 \circ C_1 \\ K_1' \circ C_1 \end{bmatrix}"] \arrow[d, shift right, "C_1"'] & \bullet \arrow[l, dashed, bend left, "H_1 = \begin{bmatrix} \mathrm{id} & 0 \end{bmatrix}"] \arrow[d, shift right, "C_2 = \begin{bmatrix} 0 & \mathrm{id} \end{bmatrix}"'] \\
\bullet \arrow[r, "K_0'"'] \arrow[u, shift right, "D_0"'] & \bullet \arrow[l, dashed, bend right, "H_0'"'] \arrow[r, "K_1'"'] \arrow[u, shift right, "D_1"'] & \bullet \arrow[l, dashed, bend right, "H_1'"'] \arrow[u, shift right, "D_2 = \begin{bmatrix} D_2' \\ \mathrm{id} - K_1' \circ H_1' \end{bmatrix}"']
\end{tikzcd} , \tag{13}$$

with some differential operators H_1', D_2'.

Proof. From our hypotheses, C_0 and D_0 are mutual inverses, up to a homotopy correction. Our first observation is that the same property then holds for C_1 and D_1. Namely,

$$\begin{aligned} (\mathrm{id} - K_0 \circ H_0 - D_1 \circ C_1) \circ K_0 &= K_0 - K_0 \circ H_0 \circ K_0 - K_0 \circ (D_0 \circ C_0) \\ &= K_0 \circ (\mathrm{id} - H_0 \circ K_0 - D_0 \circ C_0) \\ &= K_0 \circ \tilde{H} = 0, \end{aligned} \tag{14}$$

$$(\mathrm{id} - K_0' \circ H_0' - C_1 \circ D_1) \circ K_0' = 0, \tag{15}$$

where the second identity is completely analogous to the first one. Then we also have

$$(\mathrm{id} - K_0 \circ H_0 - D_1 \circ C_1) \circ D_1 \circ K_0' = (\mathrm{id} - K_0 \circ H_0 - D_1 \circ C_1) \circ K_0 \circ D_0 = 0. \tag{16}$$

Since we know that K'_1 is a universal compatibility operator for K'_0, there must exist differential operators H'_1 and D'_2 such that

$$\mathrm{id} - K'_0 \circ H'_0 - C_1 \circ D_1 = H'_1 \circ K'_1, \tag{17}$$

$$(\mathrm{id} - K_0 \circ H_0 - D_1 \circ C_1) \circ D_1 = D'_2 \circ K'_1. \tag{18}$$

Next, defining the operators K_1, H_1, C_2 and D_2 as in the diagram (13), the remaining identities needed to show that this diagram is a homotopy equivalence are

$$D_1 \circ C_1 = \mathrm{id} - K_0 \circ H_0 - H_1 \circ K_1, \tag{19a}$$

$$C_2 \circ K_1 = K'_1 \circ C_1, \tag{19b}$$

$$D_2 \circ K'_1 = K_1 \circ D_1, \tag{19c}$$

$$(\mathrm{id} - K_1 \circ H_1 - D_2 \circ C_2) \circ K_1 = 0, \tag{19d}$$

$$(\mathrm{id} - K'_1 \circ H'_1 - C_2 \circ D_2) \circ K'_1 = 0. \tag{19e}$$

The last two, (d) and (e), follow from the same argument as in the first paragraph of this proof. The first two, (a) and (b), follow from direct calculation and the identities that we have already established earlier in the proof. To get (c), it remains to check the following identity

$$(K'_1 \circ C_1) \circ D_1 = K'_1 \circ (\mathrm{id} - K'_0 \circ H'_0 - H'_1 \circ K'_1) = (\mathrm{id} - K'_1 \circ H'_1) \circ K'_1. \tag{20}$$

This completes the proof. $\square$

Definition 6. *A* (linear) connection $\mathbb{D}$ *on a vector bundle* $V \to M$ *is a linear differential operator* $\mathbb{D}\colon \Gamma(V) \to \Gamma(T^*M \otimes_M V)$ *that in local coordinates* (x^a) *has the form* $\mathbb{D}_a v(x) = \frac{\partial}{\partial x^a} v(x) + \gamma_a(x) v(x)$*, with the* $\gamma_a(x)$ *being smooth local sections of the endomorphism bundle of* $V \to M$*. The connection is* flat *when locally its components commute,* $[\mathbb{D}_a, \mathbb{D}_b] = 0$*. The equation* $\mathbb{D} f = 0$ *is called the* ($\mathbb{D}$-)flat section equation*.*

A flat connection $\mathbb{D}$ *gives rise to a complex of differential operators* $d^{\mathbb{D}}_l \colon \Gamma(\Lambda^l T^*M \otimes_M V) \to \Gamma(\Lambda^{l+1} T^*M \otimes_M V)$*,* $l = 0, 1, \ldots, n, \ldots$*, with* $d^{\mathbb{D}} = \mathbb{D}$*, locally* $(d^{\mathbb{D}}_l w)_{a_1 \cdots a_{l+1}} = (l+1) \mathbb{D}_{[a_1} w_{a_2 \cdots a_n]}$ *for* $0 < l < n$*, and* $d^{\mathbb{D}}_l = 0$ *for* $l \geq n$*, the* ($\mathbb{D}$-)twisted de Rham complex*.*

Definition 7. *A differential operator* K *defines a* PDE of finite type $K[f] = 0$*, when* $K_0 = K$ *is equivalent to a flat connection* $K'_0 = \mathbb{D}$*, in the sense of one operator complexes (Definition 1), with the extra requirement that* $\tilde{H}_{-1} = 0$ *and* $\tilde{H}'_{-1} = 0$*.*

Remark 1. One can find in the literature different definitions for PDEs of *finite type*. Our Definition 7 is most convenient for the purposes of this work and is well-known to be equivalent to other common definitions under reasonable regularity conditions. The relation between the different definitions is discussed in the Appendix and an elementary proof of the equivalence is given in Proposition 13.

Once we know that we are faced with a PDE of finite type, we can exploit its equivalence to the equation for $\mathbb{D}$-flat sections, together with our preceding results and the following well known proposition.

Proposition 8. *Given a flat connection* $\mathbb{D}$*, the corresponding twisted de Rham complex* $d^{\mathbb{D}}_l$*,* $l = 0, 1, \ldots, n, \ldots$ *is locally exact and is also a compatibility complex for* $\mathbb{D} = d^{\mathbb{D}}_0$*.*

Theorem 9. *Let* $K[f] = 0$ *be PDE of finite type. Then, starting from an equivalence of* K *with a flat connection* $\mathbb{D}$*, we can explicitly construct a locally exact compatibility complex* K_l *for* K*,* $l = 0, 1, \ldots$*.*

Proof. The proof is by induction. Setting $K_0 = K$ and $K'_0 = \mathbb{D} = d^{\mathbb{D}}_0$, the finite type hypothesis implies that we can satisfy the hypotheses of Lemma 5 and extend the equivalence of K_0 and K'_0 to an equivalence of K_l and K'_l, $l = 0, 1$, with K_1 explicitly constructed. Suppose, inductively, that we have an equivalence up to homotopy between the complexes K_l and $K'_l = d^{\mathbb{D}}_l$, $l = 0, 1, \ldots, m$, for some $m > 0$. Iterating the previous argument, we can extend it to an equivalence up to homotopy between the complexes K_l and $K'_l = d^{\mathbb{D}}_l$, $l = 0, 1, \dots, m+1$, where K_{m+1} is an explicitly constructed as in Lemma 5.

Since the above construction of the complex K_l, $l = 0, 1, \dots$, comes with an equivalence up to homotopy with the twisted de Rham complex $K'_l = d^{\mathbb{D}}_l$, $l = 0, 1, \dots$, Lemma 4 and Proposition 8 allow us to conclude that K_l is both locally exact and is a compatibility complex for K. That is, K_1 is a universal compatibility operator for $K_0 = K$ and K_{l+1} is a universal compatibility operator for K_l, for $l > 1$. □

Note that, even though the twisted de Rham complex $d^{\mathbb{D}}_l$ terminates after $l = n$, or rather becomes trivial $d^{\mathbb{D}}_l = 0$ for $l \geq n$, the result of Theorem 9 is not guaranteed to produce a complex that eventually terminates in the same way. For instance, the simple example $K = \mathrm{id}$, produces the complex K_l, $l = 0, 1, \dots$, where $K_{2k} = \mathrm{id}$ and $K_{2k+1} = 0$, with the source and target of every operator K_l being the same as for K. Of course, in this simple example, we can force this complex to terminate by setting the target of $K_1 = 0$ to be zero dimensional, and setting $K_l = 0$ as a map between zero dimensional spaces, for each $l > 1$.

Since our goal is not a fully algorithmic construction of compatibility complexes, but rather one where human intervention is allowed along the way, we can apply similar simplifications at each step of the iterative construction given in the proof of Theorem 9. At each step, before proceeding to the next one, having obtained the operator K_{l+1}, we can replace it by a potentially simpler operator $\tilde{K}_{l+1}$ without breaking its universality property. Here, a trivial, but helpful, observation is that an operator $\tilde{K}_{l+1}$ such that $K_{l+1} = \tilde{K}'_{l+1} \circ \tilde{K}_{l+1}$ is a universal compatibility operator for K_l whenever K_{l+1} is. This way, on general principles, we expect to be able to produce a compatibility complex K_l, $l = 0, 1, \dots$, that becomes trivial $K_l = 0$ for $l > n$.

We will not try to give a rigorous proof of the fact that we can always produce a compatibility complex K_l that trivializes to $K_l = 0$ for $l > n$. Instead, in the next section, we will present examples of PDEs of finite type with compatibility complexes of finite length. In each case, we will give an explicit equivalence up to homotopy of a given complex to a twisted de Rham complex, which together with Lemma 4 and Proposition 8 serves as a witness to the fact that it is a compatibility complex. However, the reader should understand that this compatibility complex was produced by the construction given in the proof of Theorem 9, with intermediate simplifications as described above.

3 Killing equation

Consider an n-dimensional (pseudo-)Riemannian manifold (M, g), with Levi-Civita connection ∇. In Lorentzian signature, we refer to (M, g) as a *spacetime*. The *Killing equation* is an equation on sections $v \in \Gamma(TM)$, namely

$$K_{ab}[v] = \nabla_a v_b + \nabla_b v_a = 0. \tag{21}$$

The Lie derivative identity $K[v] = \mathcal{L}_v g$ implies that solutions of the Killing equations are infinitesimal isometries of (M, g). In the context of *linearized gravity* (that is, the theory of *linearized Einstein equations*), metric perturbations $h \in \Gamma(S^2 T^* M)$ are grouped into *gauge equivalence classes*, $h \sim h + K[v]$ for $v \in \Gamma(TM)$. A differential operator $L[h]$ such that $L \circ K = 0$, a compatibility operator for K in the terminology of Section 2, is interpreted as a *(local) gauge-invariant observable* or *gauge-invariant field combination* [34]. The components of a complete compatibility operator K_1 for the Killing operator K can be interpreted, by the universality property, as a generating set for all gauge-invariants, also known as a *complete set of gauge-invariant observables*.

It is well known that the Killing equation is of finite type (Definition 7), provided a regularity condition holds. The quickest way to see that is to put it into the so-called *tractor form* [11] or the form of the *Killing transport* equation [17, App.B]. Namely, we have the equivalence up to homotopy

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
v^a \arrow[r, "v^a \mapsto K_{ab}[v]"] \arrow[d, shift right, swap, "v^a \mapsto \begin{bmatrix} v_a \\ \nabla_{[b} v_{c]} \end{bmatrix}"] & h_{ab} \arrow[l, dashed, bend left, "0"] \arrow[d, shift right, swap, "h_{ab} \mapsto \begin{bmatrix} h_{ab} \\ \nabla_{[b} h_{c]a} \end{bmatrix}"] \\
\begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \arrow[u, shift right, swap, "\begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \mapsto v^a"] \arrow[r, swap, "\begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} \mapsto \mathbb{T}_{a_1} \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix}"] & \begin{bmatrix} v_{a_1:a} \\ w_{a_1:[bc]} \end{bmatrix} \arrow[u, shift right, swap, "\begin{bmatrix} v_{a_1:a} \\ w_{a_1:[bc]} \end{bmatrix} \mapsto 2v_{(a:b)}"] \arrow[l, dashed, bend right, swap, "\begin{bmatrix} v_{a_1:a} \\ w_{a_1:[bc]} \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -v_{[b:c]} \end{bmatrix}"]
\end{tikzcd} , \tag{22}
$$

where the connection operator[1] is

$$
\mathbb{T}_{a_1} \begin{bmatrix} v_a \\ w_{[bc]} \end{bmatrix} = \begin{bmatrix} \nabla_{a_1} v_a - w_{[a_1 a]} \\ \nabla_{a_1} w_{[bc]} + R_{a_1 dbc} v^d \end{bmatrix} , \tag{23}
$$

which uses the Riemann tensor[2] $R_{abc}{}^d v_d = 2\nabla_{[a}\nabla_{b]} v_c$. We should mention that the : appearing for instance in $w_{a_1:[bc]}$ or $v_{[b:c]}$ is only there to visually separate particular groups of indices. Also, we use square brackets to indicate that some of the tensors must have anti-symmetrized indices from the outset.

To check the commutativity of the square with downward arrows, first note that for $w_{bc} = \nabla_{[b} v_{c]} = \nabla_b v_c - \frac{1}{2} K_{ab}[v]$ we have

$$
2\nabla_{[a} w_{b]c} - R_{abcd} v^d = -\nabla_{[a} K_{b]c}[v]. \tag{24}
$$

Then, the antisymmetry $w_{cb} = -w_{bc}$ and the algebraic Bianchi identity $R_{[abc]d} = 0$ allow us to write

$$
\begin{aligned}
2(\nabla_a w_{bc} + R_{adbc} v^d) &= (2\nabla_{[a} w_{b]c} - R_{abcd} v^d) - (2\nabla_{[b} w_{c]a} - R_{bcad} v^d) + (2\nabla_{[c} w_{a]b} - R_{cabd} v^d) \\
&= 2\nabla_{[b} K_{c]a}[v].
\end{aligned} \tag{25}
$$

In general, the connection $\mathbb{T}_a$ is not a flat. In fact, it is flat iff (M, g) is of constant curvature, $R_{abcd} = \alpha(g_{ac} g_{bd} - g_{ad} g_{bc})$ for some constant α. However, when the local solutions of $\mathbb{T}[\begin{smallmatrix} v \\ w \end{smallmatrix}] = 0$ span a vector bundle, the restriction of $\mathbb{T}$ to this sub-bundle (which could be of rank zero) is flat and the corresponding flat section equation is equivalent to the original Killing equation (Lemma 14), hence implying its finite type. Thus, the required regularity condition is that the local solutions of the Killing equation, in tractor form, span a sub-bundle. Or, equivalently, the pointwise dimension of the span of these local solutions is constant. However, in special cases, this particular way of reducing the Killing operator to a flat connection may not be the preferred one, and a different reduction might be more convenient.

Consider a tensor $T[g]$ built covariantly out of the metric g, the Riemann tensor R, and the covariant derivatives ∇R, $\nabla\nabla R$, $\ldots$. Define its linearization $\dot{T}$ about g by the identity $T[g + \varepsilon h] = T[g] + \varepsilon \dot{T}[h] + O(\varepsilon^2)$. Recall the standard identity between the Lie derivative, T and $\dot{T}$:

$$
\mathcal{L}_v T[g] = \dot{T}[\mathcal{L}_v[g]] = \dot{T}[K[v]], \tag{26}
$$

$$
\text{where} \quad \mathcal{L}_v T_{a\cdots}^{b\cdots} = v^c \nabla_c T_{a\cdots}^{b\cdots} + T_{c\cdots}^{b\cdots} \nabla_a v^c - T_{a\cdots}^{c\cdots} \nabla_c v^b + \cdots , \tag{27}
$$

which guarantees that $\dot{T} \circ K = 0$ for the linearization $g \mapsto g + \varepsilon h$ whenever $T[g] = 0$ or some expression involving only constants and Kronecker δ's. This result is sometimes known as the

[1] The form of this connection was already derived in [17, Eq.(B.2)], though there it has a typo. The sign of $R_{a_1 dbc} v^d$ is opposite compared to ours.

[2] We follow the curvature conventions of [36].

Stewart-Walker lemma [34, Lem.2.2]. Alternatively, when $T[g] \neq 0$, this identity can be used to extract some components of v_a or $\nabla_a v_b$ by applying $\dot{T}$ to $K[v]$.

Section 3.2 and 3.3 below will involve some algebraic constructions with tensors for which it is convenient to introduce the following notation. For $A_{\mu\lambda}$ and $B_{\nu\kappa}$ symmetric tensors, we denote the *Kulkarni-Nomizu* product by

$$(A \odot B)_{\mu\nu\lambda\kappa} = A_{\mu\lambda}B_{\nu\kappa} - A_{\nu\lambda}B_{\mu\kappa} - A_{\mu\kappa}B_{\nu\lambda} + A_{\nu\kappa}B_{\mu\lambda}. \tag{28}$$

Clearly $A \odot B = B \odot A$ and the result has the symmetry type of the Riemann tensor. For tensors with two or four indices, we define the contractions

$$(A \cdot B)_{\mu\nu} = A_\mu{}^\lambda B_{\lambda\nu}, \quad \text{and} \quad (R \cdot S)_{\mu\nu\lambda\kappa} = R_{\mu\nu}{}^{\alpha\beta} S_{\alpha\beta\lambda\kappa}. \tag{29}$$

With these definitions, when A, B, C and D are symmetric, we have the useful identities

$$[(A \odot B) \cdot (C \odot D)]_{\mu\nu\lambda\kappa} = 2[(A \cdot C) \odot (B \cdot D) + (A \cdot D) \odot (B \cdot C)]_{\mu\nu\lambda\kappa}, \tag{30}$$
$$(A \odot B)_{\mu\nu}{}^\nu{}_\kappa = [A \cdot B - (\operatorname{tr} A)B - A(\operatorname{tr} B) + B \cdot A]_{\mu\kappa}. \tag{31}$$

3.1 Constant curvature spacetime

An n-dimensional *constant curvature* spacetime (M, g) of sectional curvature α is defined by a Riemann curvature tensor of the form $R_{abcd} = \alpha(g_{ac}g_{bd} - g_{ad}g_{bc})$, with α a constant. It is well-known that in this case the *Killing transport* (or *tractor*) connection defined in (23) is actually flat. Thus, we could use the methods of Section (2), in particular Theorem 9, to construct a compatibility complex for the Killing operator K on (M, g). However, in this particular situation, the compatibility complex for K is already known independently. It is sometimes called the *Calabi complex* [23]. We will denote it by C_i, $i = 0, 1, \ldots$, with $C_0 = K$ and $C_l = 0$ for $l \geq n$. The operator C_1 is essentially the linearized Riemann tensor $R_a{}^b{}_c{}^d[g]$, while C_2 is the linearized differential Bianchi identity. The remaining operators C_i, for $i > 2$, are essentially higher rank Bianchi identities. These operators have the following explicit formulas (see [23, Sec.2.2]):

$$C_0[v]_{a:b} = \nabla_a v_b + \nabla_b v_a, \tag{32a}$$
$$C_1[h]_{ab:cd} = (\nabla\nabla \odot h)_{abcd} + \alpha(g \odot h)_{abcd}, \tag{32b}$$
$$C_2[r]_{abc:de} = 3\nabla_{[a} r_{bc]de}, \tag{32c}$$
$$C_3[b]_{abcd:ef} = 4\nabla_{[a} b_{bcd]ef}, \tag{32d}$$
$$\vdots \tag{32e}$$
$$C_i[b]_{a_0\cdots a_i:bc} = (i+1)\nabla_{[a_0} b_{a_1\cdots a_i]bc} \quad (i \geq 2). \tag{32f}$$

The : notation only serves to visually separate groups of indices that are independently antisymmetric. C_0 has the symmetric index pair $a{:}b$, while C_1 has the index group $ab{:}cd$ satisfying the algebraic symmetries of the Riemann tensor. More generally, the tensor symmetry type of the target of each C_i operator is best described using Young symmetrizers (see [23, Sec.2.1] for complete details). Ignoring corresponding algebraic symmetry conditions on the tensors entering into the Calabi complex may violate its property of being a compatibility complex. Below we list the Young symmetry types and ranks of the tensor bundles serving as domains and codomains for the operators of the Calabi complex:

		Young type	$n=2$	$n=3$	$n=4$	$n \geq 2$
		(1)	2	3	4	n
	C_0					
		(2)	3	6	10	$\frac{n(n+1)}{2}$
	C_1					
		$(2,2)$	1	6	20	$\frac{n}{2}\binom{n+1}{3}$
	C_2					
		$(2,2,1)$		3	20	$\frac{n(3-1)}{2}\binom{n+1}{3+1}$
	$\vdots$					$\vdots$
		$(2,2,1^{i-2})$				$\frac{n(i-1)}{2}\binom{n+1}{i+1}$
$(1 < i)$	C_i					
		$(2,2,1^{i-1})$				$\frac{ni}{2}\binom{n+1}{i+2}$
	$\vdots$					$\vdots$
	C_{n-1}					
		$(2,2,1^{n-2})$	1	3	6	$\frac{n(n-1)}{2}$

In the diagram (22) for the equivalence up to homotopy between K and $\mathbb{T}$, the top and bottom lines can be extended to their compatibility complexes, the Calabi C_l complex for K and the twisted de Rham complex $d_l^{\mathbb{T}}$ (Definition 6) for $\mathbb{T}$. Then, using the same argument as at the beginning of the proof of Lemma 5, the vertical equivalence maps can be propagated to the rest of the complexes, thus giving a full equivalence up to homotopy between them

$$\begin{tikzcd}
\bullet \arrow[r, "C_0=K"] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left] \arrow[r, "C_1"] \arrow[d, shift right] \arrow[u, phantom] & {} \arrow[l, dashed, bend left] & \cdots & {} \arrow[r, "C_{n-1}"] & \bullet \arrow[l, dashed, bend left] \arrow[r, "0"] \arrow[d, shift right] & \cdots \\
\bullet \arrow[r, "d_0^{\mathbb{T}}=\mathbb{T}"'] \arrow[u, shift right] & \bullet \arrow[l, dashed, bend right] \arrow[r, "d_1^{\mathbb{T}}"'] \arrow[u, shift right] & {} \arrow[l, dashed, bend right] & \cdots & {} \arrow[r, "d_{n-1}^{\mathbb{T}}"'] & \bullet \arrow[l, dashed, bend right] \arrow[r, "0"'] \arrow[u, shift right] & \cdots
\end{tikzcd} \tag{33}$$

We will not discuss the explicit formulas for the vertical equivalence differential operators. For our purposes it is sufficient to know that they exist. However, if these operators were to be given explicitly, then according to Lemma 4 the equivalence diagram (33) would constitute an independent proof of the fact that the operators C_l constitute a compatibility complex for the Killing operator K on the constant curvature spacetime (M, g).

3.2 FLRW spacetimes

Consider an FLRW spacetime (M, g), where $M = I \times F$, with $I \subset \mathbb{R}$ an open interval with coordinate t and $\dim F = m$, and $g = -dt^2 + f^2 \tilde{g}^F$, where *the scale factor* $f = f(t)$ is a positive scalar function and $\tilde{g}^F_{ab} = (\pi^* g^F)_{ab}$ is the pullback of a constant curvature Riemannian metric (with sectional curvature α) on F along the standard projection $\pi\colon I \times F \to F$. Let us denote $U_a = -(dt)_a$, and note that $U_a U^a = -1$ and $f^2 \tilde{g}^F_{ab} = g_{ab} + U_a U_b$.

Below, it will be convenient to extensively rely on the product structure $M = I \times F$ and naturally decompose all tensors on M with respect to it. For instance, $v^a_b = v^t_t U^a U_b + \tilde{v}^t_b U^a + \tilde{v}^a_t U_b + \tilde{v}^a_b$, where v^t_t is a scalar and $\tilde{v}^a_t$, $\tilde{v}^a_b$ and $\tilde{v}^a_b$ are respectively sections of the pullback bundles $\pi^* TF$, $\pi^* T^* F$ and $\pi^*(T \otimes T^*)F$ over M, or equivalently simply sections TM, T^*M and $(T \otimes T^*)M$ that are annihilated by g-contraction with U_a on any index. It will also be convenient to insert extra factors of f into some tensor decompositions of this type, with the sole purpose of simplifying some forthcoming formulas (which we have unfortunately managed to do only in an ad-hoc way). If $X_a \in \Gamma(T^*F)$ then we will denote its pullback by $\tilde{X}_a = \pi^* X \in \Gamma(T^*M)$ and note that it satisfies $U^a \tilde{X}_a = 0$, where the metric g is used for contractions. On the other hand, if no such X_a was introduced previously (which will be true in most cases), we used $\tilde{X}_a \in \Gamma(T^*M)$ to denote any section that satisfies $U^a \tilde{X}_a = 0$, even if has non-trivial dependence on t. This should not generate any confusion for the reader. The same convention is extended to all purely covariant tensors.

Each of the factors has an auxiliary pseudo-Riemannian structure, $(I, -dt^2)$ and (F, g^F), with corresponding Levi-Civita connections, ∇^I and ∇^F, which we can extend to all covariant tensor fields on the product manifold $M = I \times F$. Let us denote the extensions of ∇^I and ∇^F respectively by $-U_a\partial_t$ and $\tilde{\nabla}$, which incidentally defines the convenient differential operator $\partial_t = U^a(-U_a\partial_t)$. The defining properties of these operators are the usual Leibniz rule, together with $\partial_t t = 1$, $\partial_t U_a = 0$, and $\partial_t \tilde{X} = \partial_t(\pi^* X) = 0$ for any covariant tensor field X on F, and also $\tilde{\nabla} t = 0$, $\tilde{\nabla} U_a = 0$ and $\tilde{\nabla}\tilde{X} = \widetilde{\nabla^F X} = \pi^*(\nabla^F X)$ for any covariant tensor field X on F. Since any covariant tensor on M is locally a limit of sums of products of covariant tensors pulled back from I and M, these properties are sufficient to uniquely define ∂_t and $\tilde{\nabla}$ as linear differential operators. Note that we will also frequently use the abbreviation $(-)' = \partial_t(-)$.

By the above remarks, any covector field v_a on M can be parametrized as

$$v_a = AfU_b + f^2\tilde{X}_a \tag{34}$$

where A is a scalar function on M and $\tilde{X} \in \Gamma(\pi^* T^* F) \subset \Gamma(T^* M)$, so that $U^a\tilde{X}_a = 0$. Now note that with our conventions, for any scalar A, its exterior derivative is given by $(dA)_a = -A' U_a + \tilde{\nabla}_a A$, while the Levi-Civita connection on (M, g) is given in the above parametrization by

$$\nabla_a(AfU_b + f^2\tilde{X}_b) = (dA)_a f U_b + Af' g_{ab} - 2f' f U_{[a}\tilde{X}_{b]} - f^2 U_a \tilde{X}'_b + f^2\tilde{\nabla}_a\tilde{X}_b. \tag{35}$$

Parametrizing symmetric 2-tensors h_{ab} on M as

$$h_{ab} = pU_aU_b - 2f^2 U_{(a}\tilde{Y}_{b)} + f^2\tilde{Z}_{ab}, \tag{36}$$

the Killing operator on covectors $h_{ab} = K_{ab}[v] = 2\nabla_{(a}v_{b)}$ becomes

$$\begin{bmatrix} p \\ \hline \tilde{Y} \\ \tilde{Z} \end{bmatrix} = K \begin{bmatrix} A \\ \hline \tilde{X} \end{bmatrix} = \begin{bmatrix} -2(Af)' \\ \hline \tilde{X}' - f^{-1}\tilde{\nabla} A \\ \tilde{K}[\tilde{X}] + 2Af'\tilde{g}^F \end{bmatrix} = \left[\begin{array}{c|c} -2\partial_t f & 0 \\ \hline -f^{-1}\tilde{\nabla} & \partial_t \\ 2f'\tilde{g}^F & \tilde{K} \end{array}\right] \begin{bmatrix} A \\ \hline \tilde{X} \end{bmatrix} \tag{37}$$

in block matrix operator notation, where $\tilde{K}_{bc}[\tilde{X}] = 2\tilde{\nabla}_{(b}\tilde{X}_{c)}$, which is simply our extension of the Killing operator on covectors from (F, g^F) to (M, g).

Remark 2. Each entry in our block operator matrices is a linear differential operator between some covariant tensor bundles over M. We use vertical and horizontal lines to further partition block operator matrices with respect to some special direct sum decomposition of their domain or codomain. We will use id to denote the identity endomorphism on any vector bundle and 0 to denote the zero morphism between any two vector bundles. In each case, the domains and codomains of these operators can be deduced from the context.

It is worth noting that setting the A component of v_a to zero simplifies the Killing operator to

$$K \begin{bmatrix} 0 \\ \hline \mathrm{id} \end{bmatrix} = \begin{bmatrix} 0 \\ \hline \partial_t \\ \tilde{K} \end{bmatrix}. \tag{38}$$

For a generic FLRW spacetime (see [9, Def.2.1] for a breakdown of FLRW geometries into special and generic classes, based on the properties of the scale factor f), it is well-known that the only Killing vectors are those that reduce to the Killing vectors of the spatial slices (F, g^F), appropriately propagated in time. We will see shortly that, equivalently, each Killing vector on (M, g) has the form $v_a = 0 + f^2\tilde{X}_a$, where $\tilde{K}_{ab}[\tilde{X}] = 0$ and $\partial_t\tilde{X}_a = 0$. Now, since the spatial slices (F, g^F) are of constant curvature, the spatial Killing operator $\tilde{K}$ is of the type discussed in Section 3.1. This means that $\tilde{K}$ is equivalent up to homotopy to the flat spatial Killing transport connection $\tilde{\mathbb{T}}$, so that the following two operators are also equivalent up to homotopy:

$$\begin{bmatrix} \partial_t \\ \tilde{K} \end{bmatrix} \quad \text{and} \quad \mathbb{T} = \begin{bmatrix} \partial_t \\ \tilde{\mathbb{T}} \end{bmatrix}. \tag{39}$$

Since both $[\partial_t, \tilde{K}] = 0$ and $[\partial_t, \tilde{\mathbb{T}}] = 0$, it is easy to see that $\mathbb{T}$ itself defines a flat connection. Let the operators $\tilde{C}_i$ be the extensions of the Calabi complex (32) from the constant curvature geometry

(F, g^F) to M, where we have simply replaced ∇^F by $\tilde{\nabla}$ and g^F by $\tilde{g}^F$ whenever necessary. Since such an extension preserves all operator identities, including the suitably extended equivalence up to homotopy in (33). Now, it is a straightforward exercise to check that the twisted de Rham complex $d_l^{\mathbb{T}}$ can be represented as the bottom line in the following diagram, and also to use the mentioned identities to construct the corresponding the vertical differential operators that complete this diagram to an equivalence up to homotopy:

$$\begin{tikzcd}[column sep=large, row sep=huge]
\bullet \arrow[r, "\begin{bmatrix} \partial_t \\ \tilde{C}_0 = \tilde{K} \end{bmatrix}"] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left] \arrow[r, "\begin{bmatrix} -\tilde{C}_0 & \partial_t \\ 0 & \tilde{C}_1 \end{bmatrix}"] \arrow[d, shift right] & {} \arrow[l, dashed, bend left] & \cdots & {} \arrow[r, "\begin{bmatrix} -\tilde{C}_{m-1} & \partial_t \end{bmatrix}"] & \bullet \arrow[l, dashed, bend left] \arrow[r, "0"] \arrow[d, shift right] & \cdots \\
\bullet \arrow[r, "d_0^{\mathbb{T}} = \mathbb{T} = \begin{bmatrix} \partial_t \\ \tilde{\mathbb{T}} \end{bmatrix}"'] \arrow[u, shift right] & \bullet \arrow[l, dashed, bend right] \arrow[r, "d_1^{\mathbb{T}} = \begin{bmatrix} -d_0^{\tilde{\mathbb{T}}} & \partial_t \\ 0 & d_1^{\tilde{\mathbb{T}}} \end{bmatrix}"'] \arrow[u, shift right] & {} \arrow[l, dashed, bend right] & \cdots & {} \arrow[r, "d_{m-1}^{\mathbb{T}} = \begin{bmatrix} -d_{m-1}^{\tilde{\mathbb{T}}} & \partial_t \end{bmatrix}"'] & \bullet \arrow[l, dashed, bend right] \arrow[r, "0"] \arrow[u, shift right] & \cdots
\end{tikzcd} \tag{40}$$

Hence, by Lemma 4, the top complex in (40) is also a compatibility complex.

Now, we need to examine the integrability conditions that will help us establish an explicit equivalence of the full Killing equation $K_{ab}[v] = 0$ with the system of equations $\partial_t \tilde{X}_a = 0$, $\tilde{K}_{ab}[\tilde{X}] = 0$, whose compatibility complex is given by the top line of (40). All of that crucially depends on the structure of the curvature of (M, g).

The Riemann curvature tensor, the Ricci tensor and the Ricci scalar of (M, g) are (recalling the notation from (28)) given by

$$R_{abcd} = \left(g \odot \left[\frac{1}{2} \left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) g - \left(\left(\frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) UU \right] \right)_{abcd}, \tag{41a}$$

$$\begin{aligned} R_{ab} = {} & -(m-1) \left[\left(\frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right] U_a U_b \\ & + \left[\left(\left(\frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) + m \left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right] g_{ab}, \end{aligned} \tag{41b}$$

$$\mathcal{R} = \left(\left(\frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) + (m+1) \left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right). \tag{41c}$$

We suppose that the FLRW spacetime is *non-degenerate*, that is both that $f'/f \neq 0$ and that the scalar curvature $\mathcal{R}$ is not constant,

$$\mathcal{R}' = \left[\left(\left(\frac{f'}{f} \right)' - \frac{\alpha}{f^2} \right) + (m+1) \left(\frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right]' \neq 0. \tag{42}$$

To make use of identity (26), we compute the Lie derivative

$$\mathcal{L}_v \mathcal{R} = v^a \nabla_a \mathcal{R} = A f \mathcal{R}'. \tag{43}$$

Thus, defining the operator

$$J[h] = \frac{1}{f \mathcal{R}'} \dot{\mathcal{R}}[h], \tag{44}$$

we have the identities

$$J \circ K \left[\frac{A}{\tilde{X}} \right] = A \quad \text{or} \quad J \circ K = \left[\mathrm{id} \mid 0 \right]. \tag{45}$$

The last equation also implies that

$$J \circ \left(K \left[\frac{0}{\mathrm{id}} \right] \right) = \left[\mathrm{id} \mid 0 \right] \left[\frac{0}{\mathrm{id}} \right] = 0 \quad \text{and} \quad J \circ \left(K \left[\frac{\mathrm{id}}{0} \right] \right) = \left[\mathrm{id} \mid 0 \right] \left[\frac{\mathrm{id}}{0} \right] = \mathrm{id}. \tag{46}$$

Let us denote the block matrix components of J as $J = \left[J^p \mid J^Y \quad J^Z\right]$ and also $\tilde{J} = \left[J^Y \quad J^Z\right]$. Combining the above identities with formula (38), we get

$$\tilde{J} \circ \left(K \begin{bmatrix} 0 \\ \hline \mathrm{id} \end{bmatrix} \right) = \begin{bmatrix} J^Y & J^Z \end{bmatrix} \begin{bmatrix} \partial_t \\ \tilde{K} \end{bmatrix} = 0. \tag{47}$$

Hence, knowing the first compatibility operator from the top line of (40), it must be possible to factor

$$\tilde{J} = \tilde{H}_J \begin{bmatrix} -\tilde{C}_0 & \partial_t \\ 0 & \tilde{C}_1 \end{bmatrix}. \tag{48}$$

Of course, the full operator J can be then factored as

$$J = H_J \left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right], \tag{49}$$

where $H_J = \left[J^p \mid \tilde{H}_J\right]$. For much of what follows, we will only need the fact that $\tilde{H}_J$ or H_J exists. However, a direct calculation shows that its explicit form can be deduced from the identity

$$\begin{aligned} \dot{\mathcal{R}} \begin{bmatrix} p \\ \hline \tilde{Y} \\ \tilde{Z} \end{bmatrix} &= f^{-2}\tilde{\Delta}p + m(f/f')[(f'^2/f^2)p]' + m(m+1)(f'^2/f^2)p \\ &\quad - f^{-m-1}[f^{m+1}(2\tilde{\operatorname{div}}\,\tilde{Y} - \tilde{\operatorname{tr}}\tilde{Z}')]' + f^{-2}[-\tilde{\Delta}\,\tilde{\operatorname{tr}}\,\tilde{Z} + \tilde{\operatorname{div}}\,\tilde{\operatorname{div}}\,\tilde{Z}_{ab} - (m-1)\,\alpha\,\tilde{\operatorname{tr}}\,\tilde{Z}] \\ &= f^{-2}\tilde{\Delta}p + \frac{m}{f'f^m}[f^{m+1}(f'^2/f^2)p]' + f^{-m-1}[f^{m+1}\tilde{\operatorname{tr}}(\partial_t\tilde{Z} - \tilde{C}_0[\tilde{Y}])]' - \frac{1}{2}f^{-2}\tilde{\operatorname{tr}}\,\tilde{\operatorname{tr}}\,\tilde{C}_1[\tilde{Z}], \end{aligned} \tag{50}$$

meaning that

$$J = \frac{1}{f\mathcal{R}'}\left[f^{-2}\tilde{\Delta} + \tfrac{m}{f'f^m}\partial_t f^{m+1}\tfrac{f'^2}{f^2} \;\middle|\; -\tfrac{1}{f^{m+1}}\partial_t f^{m+1}\tilde{C}_0 \quad \tfrac{1}{f^{m+1}}\partial_t f^{m+1}\tilde{\operatorname{tr}} - \tfrac{1}{2}f^{-2}\tilde{\operatorname{tr}}\,\tilde{\operatorname{tr}}\tilde{C}_1\right], \tag{51}$$

and hence

$$H_J = \frac{1}{f\mathcal{R}'}\left[f^{-2}\tilde{\Delta} + \tfrac{m}{f'f^m}\partial_t f^{m+1}\tfrac{f'^2}{f^2} \;\middle|\; \tfrac{1}{f^{m+1}}\partial_t f^{m+1}\,\tilde{\operatorname{tr}} \quad -\tfrac{1}{2}f^{-2}\,\tilde{\operatorname{tr}}\,\tilde{\operatorname{tr}}\right], \tag{52}$$

where of course we have defined $\tilde{\operatorname{tr}}$, $\tilde{\operatorname{div}}$ and $\tilde{\Delta}$ such that

$$\tilde{\Delta}\tilde{X} = \widetilde{(g^F)^{ab}\nabla_a}\nabla_b X, \quad \tilde{\operatorname{div}}\,\tilde{Y} = \widetilde{\nabla^a Y_{a\cdots}}, \quad (\tilde{\operatorname{tr}}\,\tilde{X})_{ab} = \widetilde{(g^F)^{cd}X}_{acbd}, \quad \text{and} \quad \tilde{\operatorname{tr}}\,\tilde{Z} = \widetilde{(g^F)^{ab}}Z_{ab}. \tag{53}$$

Now we are ready to follow the proof of Theorem 9 to construct a compatibility complex for the Killing operator K by lifting the compatibility complex from (40). The results of these calculations will be presented below directly in diagrammatic form, where the arrows in the diagrams satisfy the identities introduced in Section 2. All the relevant identities are easily checked by direct calculation, relying on the key identity (45), the basic commutation relations $[\partial_t, \tilde{\nabla}] = [\partial_t, \tilde{C}_i] = 0$, the compatibility identities $\tilde{C}_{i+1} \circ \tilde{C}_i = 0$ of the operators of the Calabi complex, which were introduced in Section 3.1.

We start by applying the information obtained above to give an explicit reduction of the Killing equation to the first operator from the top line of (40):

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "K=\left[\begin{array}{c|c} -2\partial_t f & 0 \\ \hline -f^{-1}\tilde{\nabla} & \partial_t \\ 2f'\tilde{g}^F & \tilde{K} \end{array}\right]"] \arrow[d, shift right, "\left[\begin{array}{c|c} 0 & \mathrm{id} \end{array}\right]"'] & \bullet \arrow[l, dashed, bend left, "\left[\begin{array}{c} J \\ \hline 0 \end{array}\right]"] \arrow[d, shift right, "\left[\begin{array}{c|cc} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{array}\right]\left(\mathrm{id}-K\left[\begin{array}{c} J \\ \hline 0 \end{array}\right]\right)"'] \\
\bullet \arrow[r, "\left[\begin{array}{c} \partial_t \\ \tilde{C}_0=\tilde{K} \end{array}\right]"'] \arrow[u, shift right, "\left[\begin{array}{c} 0 \\ \hline \mathrm{id} \end{array}\right]"'] & \bullet \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "\left[\begin{array}{cc} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{array}\right]"']
\end{tikzcd}
\tag{54}
$$

Next, we proceed by iterating the construction from Lemma 5, while simultaneously applying the simplifications discussed after the proof of Theorem 9. The following diagram should be appended on the right to (54):

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "\left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right]\left(\mathrm{id}-K\left[\begin{array}{c} J \\ \hline 0 \end{array}\right]\right)=\left(\mathrm{id}-\left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right]K\left[\begin{array}{c} H_J \\ \hline 0 \end{array}\right]\right)\left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right]"] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left, "\left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]"] \arrow[d, shift right, "\left[\begin{array}{c|cc} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{array}\right]"'] \\
\bullet \arrow[r, "\left[\begin{array}{cc} -\tilde{C}_0 & \partial_t \\ 0 & \tilde{C}_1 \end{array}\right]"'] \arrow[u, shift right] & \bullet \arrow[l, dashed, bend right, "\left[\begin{array}{c|cc} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{array}\right]K\left[\begin{array}{c} \tilde{H}_J \\ \hline 0 \end{array}\right]"'] \arrow[u, shift right, "\left(\mathrm{id}-\left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right]K\left[\begin{array}{c} H_J \\ \hline 0 \end{array}\right]\right)\left[\begin{array}{cc} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{array}\right]"']
\end{tikzcd}
\tag{55}
$$

Note that we do not repeat the labels on the left-most vertical arrows, which can be read off as the right-most vertical arrows in (54).

Two more iterations of Lemma 5 (with simultaneous simplifications) gives the following diagram, to be appended on the right to (55):

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\left[\begin{array}{c|cc} \multicolumn{3}{c}{H_J} \\ \hline 0 & -\tilde{C}_1 & \partial_t \\ 0 & 0 & \tilde{C}_2 \end{array}\right]}"] \arrow[d, shift right] & \bullet \arrow[r, "{\left[\begin{array}{c|cc} 0 & -\tilde{C}_2 & \partial_t \\ 0 & 0 & \tilde{C}_3 \end{array}\right]}"] \arrow[l, dashed, bend left, "{\left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right] K \left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]}"] \arrow[d, shift right, swap, "{\left[\begin{array}{c|cc} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{array}\right]}"] & \bullet \arrow[l, dashed, bend left, "0"] \arrow[d, shift right, swap, "{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}}"] \\
\bullet \arrow[r, swap, "{\begin{bmatrix} -\tilde{C}_1 & \partial_t \\ 0 & \tilde{C}_2 \end{bmatrix}}"] \arrow[u, shift right] & \bullet \arrow[r, swap, "{\begin{bmatrix} -\tilde{C}_2 & \partial_t \\ 0 & \tilde{C}_3 \end{bmatrix}}"] \arrow[l, dashed, bend right, swap, "0"] \arrow[u, shift right, swap, "{\left[\begin{array}{cc} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{array}\right]}"] & \bullet \arrow[l, dashed, bend right, swap, "0"] \arrow[u, shift right, swap, "{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}}"]
\end{tikzcd}
\tag{56}
$$

From this point on, the compatibility complex for K and the top line of (40) become identical.

Theorem 10. *Consider a non-degenerate FLRW spacetime (M, g), $M = I \times F$, as introduced at the top of Section 3.2, which spatially has the structure of an m-dimensional constant curvature space (F, g^F), with sectional curvature α. The full compatibility complex K_i for the Killing operator $K_0 = K$ (37) is given by*

$$K_0 = \left[\begin{array}{c|c} -2\partial_t f & 0 \\ \hline -f^{-1}\tilde{\nabla} & \partial_t \\ 2f'\tilde{g}^F & \tilde{K} \end{array}\right], \tag{57a}$$

$$K_1 = \left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right] \left(\mathrm{id} - K \left[\begin{array}{c} J \\ \hline 0 \end{array}\right]\right), \tag{57b}$$

$$K_2 = \left[\begin{array}{c|cc} \multicolumn{3}{c}{H_J} \\ \hline 0 & -\tilde{C}_1 & \partial_t \\ 0 & 0 & \tilde{C}_2 \end{array}\right], \tag{57c}$$

$$K_3 = \left[\begin{array}{c|cc} 0 & -\tilde{C}_2 & \partial_t \\ 0 & 0 & \tilde{C}_3 \end{array}\right], \tag{57d}$$

$$K_i = \begin{bmatrix} -\tilde{C}_{i-1} & \partial_t \\ 0 & \tilde{C}_i \end{bmatrix} \quad (3 < i < m), \tag{57e}$$

$$K_m = \begin{bmatrix} -\tilde{C}_{m-1} & \partial_t \end{bmatrix}, \tag{57f}$$

$$K_i = 0 \quad (m < i), \tag{57g}$$

where the operator H_J is defined in (49), ∂_t and $\tilde{\nabla}$ are the covariant derivatives pulled back along the product structure $t\colon I \times F \to I$ and $I \times F \to F$, while $\tilde{C}_i$ are the operators from the Calabi complex associated to the constant curvature space (M, g^F), as introduced in Section 3.1. (See Appendix B.2 for a more complete summary of the notation.)

Proof. The argument given around diagram (40) shows that its top line constitutes a full compatibility complex, which coincides with the bottom line of the diagram obtained by gluing (from left to right) the diagrams (54), (55) and (56), which are continued by identifying the top and bottom rows. From the preceding discussion in the current section, it is clear that each pair of consecutive squares in this glued diagram satisfies the hypotheses of Lemma 4. Thus, the top line of this glued diagram is itself a full compatibility complex, but that complex consists precisely of the operators K_i in (57). □

The non-vanishing ranks of the vector bundles in the K_i complex have the following pattern, which can be compared to a similar table for the constant curvature case at the end of Section 3.1 (where $n = m + 1$, for comparison):

		$m=2$	$m=3$	$m=4$	$m \geq 2$
		3	4	5	$m+1$
	K_0				
		6	10	15	$\frac{m(m+1)}{2} + m + 1$
	K_1				
		5	13	31	$\frac{m}{2}\binom{m+1}{3} + \frac{m(m+1)}{2} + 1$
	K_2				
		2	10	41	$m\binom{m+1}{4} + \frac{m}{2}\binom{m+1}{3} + 1$
	K_3				
			3	26	$\frac{3m}{2}\binom{m+1}{5} + m\binom{m+1}{4}$
	$\vdots$				$\vdots$
					$\frac{m(i-1)}{2}\binom{m+1}{i+1} + \frac{m(i-2)}{2}\binom{m+1}{i}$
$(3 < i)$	K_i				
					$\frac{mi}{2}\binom{m+1}{i+2} + \frac{m(i-1)}{2}\binom{m+1}{i+1}$
	$\vdots$				$\vdots$
	K_m				
		2	3	6	$\frac{m(m-1)}{2}$

Remark 3. In $n = 4$ ($m = 3$) dimensions, the common choice for gauge-invariant variables on cosmological FLRW spacetimes are the so-called *Bardeen potentials* [4]. They include two scalars components, Φ and Ψ, a divergence-free spatial vector field, $\hat{\Phi}_a$, with 3 independent components, and a divergence-free trace-free spatial symmetric 2-tensor, $\hat{E}_{ab}$, with 5 independent components. Hence, the total number of independent components (not counting the differential relations coming from the divergence-free conditions) is $1 + 1 + 3 + 5 = 10$, which is less than the 13 components of K_1 that we have counted above. The difference of course between our K_1 and the Bardeen potentials is that our expressions are all local (given only in terms of differential operators), while the Bardeen potentials are non-local (their definition involves inverted spatial Laplacians; see the Introduction to [14] for details). While we cannot claim that our construction gives the minimal number of local gauge-invariant quantities, it is not surprising that we get a larger number than a construction that allows non-local expressions.

Remark 4. It is worth noting that the K_i complex presented above is not continuously deformable through the class of generic FLRW spacetimes to the constant or zero curvature cases, which correspond to special choices of the scale factor $f(t)$. The main reason is that the operator J introduced in Equation (44) is proportional to $1/\mathcal{R}'$, which diverges when the background scalar curvature becomes constant. Since the operator J and the related operator H_J appear in several places in the formulas (57) for the operators K_i (for $i \geq 1$), it seems difficult to directly compare them with the corresponding operators C_i on constant curvature backgrounds in (32). In particular, an explicit expression for the linearized Riemann tensor C_1 on (M, g) would be rather long and unenlightening in our notation, as can already be glimpsed from the formula for the linearized scalar curvature in (50). A more fruitful comparison would be to try to express the components of our K_1 in terms of the linearized IDEAL characterization tensors that were recently constructed for the FLRW geometries [9].

However, we might gain some qualitative insight into the components of K_1, which can be interpreted as a complete set of local gauge-invariant observables, from the alternative formula that was indicated in diagram (55):

$$\begin{bmatrix} q \\ \hline \tilde{Y}_{a:b} \\ \tilde{Z}_{ab:cd} \end{bmatrix} = K_1 \begin{bmatrix} p \\ \hline \tilde{Y}_a \\ \tilde{Z}_{a:b} \end{bmatrix} = \left(\mathrm{id} - \left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right] K \begin{bmatrix} H_J \\ \hline 0 \end{bmatrix} \right) \left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \\ \hline 0 & -\tilde{C}_0 & \partial_t \\ 0 & 0 & \tilde{C}_1 \end{array}\right] \begin{bmatrix} p \\ \hline \tilde{Y}_a \\ \tilde{Z}_{ab} \end{bmatrix}. \tag{58}$$

The $\tilde{Z}_{ab:cd}$ invariants are roughly coming from the linearized Riemann operator $\tilde{C}_1$ on the spatial slice (F, g^F). While these components can be obtained from a purely spatial projection of the linearized Riemann operator on the FLRW spacetime (M, g), since the components of the linearized spacetime Riemann tensor are not by themselves invariant, they need to be deformed through the H_J and subsequent operators in the above formula to be truly invariant. The q scalar invariant comes from the difference between the p component of h_{ab} and the p component of $K[J[h]]_{ab}$. When precomposed with $h_{ab} = K[v]_{ab}$, both terms in the difference depend only on the A component of v_a and in exactly the same way, hence cancelling to give an invariant quantity. The $\tilde{Y}_{a:b}$ components are harder to interpret in familiar terms.

3.3 Schwarzschild-Tangherlini spacetimes

Consider an n-dimensional spacetime $(\bar{\mathcal{M}}, \bar{g})$ where $\bar{\mathcal{M}} = \mathcal{M} \times \mathcal{S}$, where $\dim \mathcal{M} = 2$ and $\dim \mathcal{S} = n - 2$ [26, 25, 20]. We take the second factor $(\mathcal{S}, \Omega)$ to be a maximally symmetric space, hence a constant curvature Riemannian space with sectional curvature α, where $\alpha = 1$ for a unit sphere, $\alpha = 0$ for Euclidean space, and $\alpha = -1$ for hyperbolic space (or pseudo-sphere). Let us denote local coordinates on $\mathcal{S}$ by θ^A and the Levi-Civita connection on $(\mathcal{S}, \Omega)$ by D_A; its curvature tensor is then

$$R_{ABCD} = \alpha(\Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}) = \frac{\alpha}{2}(\Omega \odot \Omega)_{ABCD}, \tag{59}$$

where we have used the Kulkarni-Nomizu product (28). The other factor $(\mathcal{M}, g)$ has signature $(-+)$, and we will presume that it has a timelike Killing vector t^a. Let us denote local coordinates on $\mathcal{M}$ by y^a and the Levi-Civita connection on $(\mathcal{M}, g)$ by ∇_a. Because $\dim \mathcal{M} = 2$, its curvature is given by

$$R_{abcd} = \frac{\mathcal{R}}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}) = \frac{\mathcal{R}}{4}(g \odot g)_{abcd}, \tag{60}$$

where $\mathcal{R} = R_{ab}{}^{ab}$ is the corresponding Ricci scalar.

We are interested in warped product [29, Ch.7] metrics of the form [26, 25, 20]

$$\bar{g} = g_{ab}dy^a dy^b + r^2 \Omega_{AB} d\theta^A d\theta^B, \tag{61}$$

where $r = r(y)$ and g_{ab} is static in the (Schwarzschild) coordinates $(y^a) = (t, r)$,

$$g_{ab} = -f dt_a dt_b + \frac{1}{f} dr_a dr_b, \tag{62}$$

with $f = f(r)$. In these coordinates, the timelike Killing vector has the form $t^a = (\partial_t)^a$. For convenience, we also introduce the notation $t_a = g_{ab}t^b = -f dt_a$ and $r_a = dr_a$. They are related as $t^a = -\varepsilon^{ab} r_b$, where $\varepsilon_{ab} = (dt \wedge dr)_{ab}$. Then, of course, $r_a r^a = f$ and $t_a t^a = -1/f$.

As we will see shortly, under our assumptions, the Einstein equations with a cosmological constant Λ,

$$\bar{R}_{ab} - \frac{1}{2}\bar{\mathcal{R}}\bar{g}_{ab} + \Lambda \bar{g}_{ab} = 0, \tag{63}$$

are solved by [25, Eq.(2.15)]

$$f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)} r^2, \tag{64}$$

where M is a constant. When $\alpha = 1$, $\Lambda = 0$ and $n \geq 4$, this metric describes the higher dimensional spherically symmetric static black holes, the so-called *Schwarzschild-Tangherlini* solutions, specializing to the Schwarzschild solution when $n = 4$. When $n = 3$, we are forced to have $\alpha = 0$

and the spacetime is actually of constant curvature. With $n = 3$ and $\Lambda < 0$, we get the BTZ metric [3]. In terms of the parameter M, the black hole mass is given by

$$\frac{(n-2)A_{n-2}}{8\pi}\frac{M}{G} = \frac{(n-2)}{2}\frac{A_{n-2}}{A_2}\frac{M}{G}, \tag{65}$$

where G is the n-dimensional Newton's constant and

$$A_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma[(n-1)/2]} \tag{66}$$

is the area of the unit $(n-2)$-sphere. When $\alpha = 0$, we get the higher dimensional version of *Taub's plane-symmetric spacetime* [35], [33, Eq.(15.29)], [5, Eq.(2.2)]. When $\alpha = -1$, we get the higher dimensional version of a *pseudo-Schwarzschild wormhole spacetime* [27].

In what follows, we restrict our attention to $n \geq 4$, which is physically reasonable, but is also forced upon us by some of our formulas, which have poles at $n = 1, 2$ or 3.

For convenience, let us introduce the notations

$$f_1(r) = rf'(r) = (n-3)\frac{2M}{r^{n-3}} - \frac{4\Lambda}{(n-1)(n-2)}r^2, \tag{67}$$

$$f_2(r) = rf_1'(r) = -(n-3)^2\frac{2M}{r^{n-3}} - \frac{8\Lambda}{(n-1)(n-2)}r^2, \tag{68}$$

as well as note that the formula (64) for f parametrized by the constants M and Λ, with α fixed, gives the general solution to the differential equation

$$f_2 + (n-5)f_1 - 2(n-3)(f-\alpha) = 0. \tag{69}$$

Any tensor on $\mathcal{M}$ decomposes as $T_a = T_t dt_a + T_r dr_a$ or $T_a \to (T_t, T_r)$, with obvious extension to higher rank tensors. With respect to this decomposition and the coordinates (t, r), the Levi-Civita connection on $(\mathcal{M}, g)$ is then [25, Eq.(2.18)]

$$\nabla_a v_b \to \begin{bmatrix} \partial_t v_t & \partial_t v_r \\ \partial_r v_t & \partial_r v_r \end{bmatrix} + \begin{bmatrix} 0 & -\frac{f_1}{2rf} \\ -\frac{f_1}{2rf} & 0 \end{bmatrix} v_t + \begin{bmatrix} -f\frac{f_1}{2r} & 0 \\ 0 & \frac{1}{f}\frac{f_1}{2r} \end{bmatrix} v_r. \tag{70}$$

Equivalently, we can summarize this information by giving the covariant derivatives of the frame (t_a, r_a),

$$\nabla_a t_b = \frac{f_1}{2r}\varepsilon_{ab} \quad \text{and} \quad \nabla_a r_b = \frac{f_1}{2r}g_{ab}. \tag{71}$$

A direct calculation gives the Ricci scalar on $(\mathcal{M}, g)$ as

$$\mathcal{R} = \frac{f_1 - f_2}{r^2}. \tag{72}$$

And finally, symmetrizing the covariant derivative as written in (70) or (71), the explicit form of the Killing operator on $(\mathcal{M}, g)$ is

$$\begin{aligned} 2\nabla_{(a}v_{b)} &= -2t_{(a}\nabla_{b)}\frac{v_t}{f} - 2t_a t_b \frac{f_1}{2rf}v_r - 2t_{(a}r_{b)}\frac{1}{f}\partial_t v_r + r_a r_b \frac{1}{f}(2f\partial_r v_r + \frac{f_1}{r}v_r) \\ &\to \begin{bmatrix} f(2\partial_t \frac{1}{f}v_t - \frac{f_1}{r}v_r) & \partial_t v_r + f\partial_r \frac{1}{f}v_t \\ \partial_t v_r + f\partial_r \frac{1}{f}v_t & \frac{1}{f}(2f\partial_r v_r + \frac{f_1}{r}v_r) \end{bmatrix}. \end{aligned} \tag{73}$$

Greek indices $\mu, \nu, \ldots$ on $\bar{\mathcal{M}}$-tensors are raised and lowered by $\bar{g}_{\mu\nu}$. Lower case Latin indices $a, b, c, \ldots$ on $\mathcal{M}$-tensors are raised and lowered by g_{ab}. And upper case Latin indices $A, B, C, \ldots$ on $\mathcal{S}$-tensors are raised and lowered by Ω_{AB}. Any $\bar{\mathcal{M}}$-tensor decomposes into sectors,[3] according to $T_\mu = T_a(dy^a)_\mu + rT_A(\mathrm{d}\theta^A)_\mu \to (T_a, rT_A)$ and $T^\mu = T^a(\partial_a)^\mu + \frac{1}{r}T^A(\partial_A)^\mu \to (T^a, \frac{1}{r}T^A)$, with

[3]While our tensor sector formalism with (pseudo-)spherical symmetry in n-dimensions is strongly inspired by [28], where it was presented for $n = 4$, our conventions differ by the introduction of r-weights in the spherical sectors.

obvious extension to higher rank tensors. With a slight departure from this convention, let us define some $\bar{\mathcal{M}}$-tensors by their sector decomposition

$$g_{\mu\nu} \to \begin{bmatrix} g_{ab} & 0 \\ 0 & 0 \end{bmatrix}, \; g^{\mu\nu} \to \begin{bmatrix} g^{ab} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \Omega_{\mu\nu} \to \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{AB} \end{bmatrix}, \; \Omega^{\mu\nu} \to \begin{bmatrix} 0 & 0 \\ 0 & \Omega^{AB} \end{bmatrix}, \tag{74}$$

so that $\bar{g}_{\mu\nu} = g_{\mu\nu} + r^2\Omega_{\mu\nu}$ and $\bar{g}^{\mu\nu} = g^{\mu\nu} + r^{-2}\Omega^{\mu\nu}$.

The pair (∇_a, D_A) defines a connection on $\bar{\mathcal{M}} = \mathcal{M} \times \mathcal{S}$, which differs [26, App.A] from the Levi-Civita connection $\bar{\nabla}_\mu$ as follows

$$\bar{\nabla}_\mu v_\nu \to \begin{bmatrix} \nabla_a v_b & r\nabla_a v_B \\ rD_A \frac{v_b}{r} & r^2 D_A \frac{v_B}{r} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & r^2\Omega_{AB}\frac{r^c}{r} \end{bmatrix} v_c + \begin{bmatrix} 0 & 0 \\ -r_b\delta_A^C & 0 \end{bmatrix} v_C. \tag{75}$$

Remark 5. In giving the formula for $\bar{\nabla}_\mu$, we have essentially extended the action of ∇_a and D_A as linear differential operators to tensors defined on $\bar{\mathcal{M}} = \mathcal{M} \times \mathcal{S}$. The extension is done in exact analogy with the procedure described at the start of Section 3.2. Recall that the covariant derivatives ∇_a and D_A simply act as coordinate derivatives on scalars on $\mathcal{M}$ and $\mathcal{S}$. Suitably extending these coordinate derivatives to $\bar{\mathcal{M}}$, the same can be said for $(\mathcal{M}, \mathcal{S})$-mixed tensors on $\bar{\mathcal{M}}$. But for the sake of uniformity in notation, we continue to use the notation ∇_a and D_A even when they act on $\mathcal{M}$- and $\mathcal{S}$-scalars respectively.

Next, we need to carefully study the structure of the curvature tensor. The spacetime Riemann curvature tensor on $(\bar{\mathcal{M}}, \bar{g})$ is [26, App.A]

$$\begin{aligned} \bar{R}_{\mu\nu\lambda\kappa} &= \frac{\mathcal{R}}{2}(g \odot g)_{\mu\nu\lambda\kappa} + \frac{1}{2r^2}(\alpha - r_a r^a)(r^2\Omega \odot r^2\Omega)_{\mu\nu\lambda\kappa} - \left(\frac{\nabla\nabla r}{r} \odot r^2\Omega\right)_{\mu\nu\lambda\kappa} \\ &= \frac{f_1 - f_2}{4r^2}(g \odot g)_{\mu\nu\lambda\kappa} + \frac{(\alpha - f)}{2r^2}(r^2\Omega \odot r^2\Omega)_{\mu\nu\lambda\kappa} - \frac{f_1}{2r^2}(g \odot r^2\Omega)_{\mu\nu\lambda\kappa}, \end{aligned} \tag{76}$$

with the corresponding Ricci tensor

$$\begin{aligned} \bar{R}_{\mu\lambda} &= \frac{f_1 - f_2}{4r^2} 2g_{\mu\lambda} + \frac{(\alpha - f)}{2r^2} 2(n-3)r^2\Omega_{\mu\lambda} - \frac{f_1}{2r^2}(2r^2\Omega + (n-2)g)_{\mu\lambda} \\ &= -\frac{f_2 + (n-3)f_1}{2r^2} g_{\mu\lambda} - \frac{f_1 + (n-3)(f-\alpha)}{r^2} r^2\Omega_{\mu\lambda}. \end{aligned} \tag{77}$$

To satisfy Einstein's equations in the presence of a cosmological constant (63), we must have

$$\bar{R}_{\mu\lambda} = \frac{2\Lambda}{(n-2)}\bar{g}_{\mu\lambda} = \frac{2\Lambda}{(n-2)}(g + r^2\Omega)_{\mu\lambda}, \tag{78}$$

which implies

$$f_1 = -(n-3)(f-\alpha) - \frac{2\Lambda}{(n-2)}r^2, \tag{79}$$

$$f_2 = -(n-3)f_1 - \frac{4\Lambda}{(n-2)}r^2. \tag{80}$$

Eliminating the explicit dependence on Λ, we obtain precisely the second order ODE (69) whose general solution is given by $f(r)$ in (64).

Recalling the definition of the Kulkarni-Nomizu and contraction products (28) and (29), we get the following useful identities, where we used $\bar{g}_{\mu\nu}$ for contractions:

$$(g \odot g) \cdot (g \odot g) = 4(g \odot g), \tag{81}$$
$$(r^2\Omega \odot r^2\Omega) \cdot (r^2\Omega \odot r^2\Omega) = 4(r^2\Omega \odot r^2\Omega), \tag{82}$$
$$(g \odot r^2\Omega) \cdot (g \odot r^2\Omega) = 2(g \odot r^2\Omega), \tag{83}$$
$$(g \odot r^2\Omega) \cdot (g \odot g) = 0, \tag{84}$$
$$(g \odot r^2\Omega) \cdot (r^2\Omega \odot r^2\Omega) = 0, \tag{85}$$
$$(g \odot g) \cdot (r^2\Omega \odot r^2\Omega) = 0, \tag{86}$$
$$(g \odot g)_{\lambda\nu}{}^{\nu}{}_{\kappa} = -2g_{\lambda\kappa}, \tag{87}$$
$$(r^2\Omega \odot r^2\Omega)_{\lambda\nu}{}^{\nu}{}_{\kappa} = -2(n-3)(r^2\Omega)_{\lambda\kappa}, \tag{88}$$
$$(g \odot r^2\Omega)_{\lambda\nu}{}^{\nu}{}_{\kappa} = -2(r^2\Omega)_{\lambda\kappa} - (n-2)g_{\lambda\kappa}, \tag{89}$$
$$(g \odot g)_{\mu\lambda\nu\kappa} r^{\lambda} r^{\kappa} = 2(r^{\lambda} r_{\lambda}) g_{\mu\nu} - 2r_{\mu} r_{\nu}, \tag{90}$$
$$(g \odot r^2\Omega)_{\mu\lambda\nu\kappa} r^{\lambda} r^{\kappa} = (r^{\lambda} r_{\lambda})(r^2\Omega)_{\mu\nu}, \tag{91}$$
$$(r^2\Omega \odot r^2\Omega)_{\mu\lambda\nu\kappa} r^{\lambda} r^{\kappa} = 0. \tag{92}$$

Defining

$$\begin{aligned}
\bar{T}_{\mu\nu\lambda\kappa} &= \bar{R}_{\mu\nu\lambda\kappa} - \frac{\Lambda}{(n-1)(n-2)}(\bar{g} \odot \bar{g})_{\mu\nu\lambda\kappa} \\
&= \frac{M}{r^{n-1}}\left[\frac{(n-2)(n-3)}{2}(g \odot g)_{\mu\nu\lambda\kappa} + (r^2\Omega \odot r^2\Omega)_{\mu\nu\lambda\kappa} - (n-3)(g \odot r^2\Omega)_{\mu\nu\lambda\kappa}\right],
\end{aligned} \tag{93}$$

we also get the identities

$$\begin{aligned}
\bar{T} \cdot \bar{T} &= \left(\frac{M}{r^{n-1}}\right)^2 \left[(n-2)^2(n-3)^2(g \odot g) + 4(r^2\Omega \odot r^2\Omega) + 2(n-3)^2(g \odot r^2\Omega)\right] \\
\bar{T} \cdot \bar{T} \cdot \bar{T} \cdot \bar{T} &= \left(\frac{M}{r^{n-1}}\right)^4 \left[4(n-2)^4(n-3)^4(g \odot g) + 64(r^2\Omega \odot r^2\Omega) + 8(n-3)^4(g \odot r^2\Omega)\right] \\
(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\nu}{}_{\kappa} &= -\left(\frac{M}{r^{n-1}}\right)^2 \left[2(n-2)^2(n-3)^2 g + 8(n-3)r^2\Omega + 2(n-3)^2(2r^2\Omega + (n-2)g)\right]_{\mu\kappa} \\
&= -\left(\frac{M}{r^{n-1}}\right)^2 \left[2(n-1)(n-2)(n-3)^2 g + 4(n-1)(n-3)r^2\Omega\right]_{\mu\kappa} \\
(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu} &= \left(\frac{M}{r^{n-1}}\right)^2 \left[4(n-1)(n-2)(n-3)^2 + 4(n-1)(n-2)(n-3)\right] \\
&= 4(n-1)(n-2)^2(n-3)\left(\frac{M}{r^{n-1}}\right)^2 \\
\frac{\bar{\nabla}_{\lambda}(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu}}{(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu}} &= -2(n-1)\frac{r_{\lambda}}{r}
\end{aligned}$$

We would like to use these identities to write $(r^2\Omega)_{\lambda\kappa}$ and a simple r-dependent scalar as covariant expression in the curvature. For the latter, the simplest choice seems to be

$$\bar{T}^{(1)}[\bar{g}] := \frac{(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu}}{4(n-1)(n-2)^2(n-3)} = \left(\frac{M}{r^{n-1}}\right)^2. \tag{94}$$

Next, we encounter a slight dimension dependence in the expression for $(r^2\Omega)_{\lambda\kappa}$. When $n > 4$, we can use

$$(r^2\Omega)_{\lambda\kappa} = \frac{2(n-2)^2}{(n-1)(n-4)} \frac{(\bar{T} \cdot \bar{T})_{\lambda\nu}{}^{\nu}{}_{\kappa}}{(\bar{T} \cdot \bar{T})_{\mu\nu}{}^{\mu\nu}} + \frac{(n-2)(n-3)}{(n-1)(n-4)}\bar{g}_{\lambda\kappa} =: \bar{T}^{(2)}_{\lambda\kappa}[\bar{g}], \tag{95}$$

while for $n = 4$ the simplest expression we could find is

$$(r^2\Omega)_{\lambda\kappa} = -\frac{2(n-1)(n-2)^4}{(n-3)^3[\bar\nabla(\bar T\cdot\bar T)_{\mu\nu}{}^{\mu\nu}]^2} \left(\bar T\cdot\bar T\cdot\bar T\cdot\bar T - \frac{(n-3)}{(n-1)}(\bar T\cdot\bar T)_{\mu\nu}{}^{\mu\nu}\,\bar T\cdot\bar T\right)_{\lambda\rho\kappa\sigma} \frac{\bar\nabla^\rho(\bar T\cdot\bar T)_{\mu\nu}{}^{\mu\nu}}{(\bar T\cdot\bar T)_{\mu\nu}{}^{\mu\nu}}\frac{\bar\nabla^\sigma(\bar T\cdot\bar T)_{\mu\nu}{}^{\mu\nu}}{(\bar T\cdot\bar T)_{\mu\nu}{}^{\mu\nu}} =: \bar T^{(3)}_{\lambda\kappa}[\bar g]. \quad (96)$$

Although, since it also works for $n > 4$, if desired, the more complicated expression $\bar T^{(3)}[\bar g]$ could actually be used in higher dimensions too.

To make use of identity (26), we compute the Lie derivatives

$$\mathcal{L}_v\left(\frac{M}{r^{n-1}}\right)^2 = -2(n-1)\frac{r^c v_c}{r}\left(\frac{M}{r^{n-1}}\right)^2, \quad (97)$$

$$\mathcal{L}_v(r^2\Omega)_{\mu\nu} \to \begin{bmatrix} 0 & r(r\nabla_a\frac{v_B}{r}) \\ r(r\nabla_b\frac{v_A}{r}) & 2r^2(D_{(A}\frac{1}{r}v_{B)} + \Omega_{AB}\frac{r^c v_c}{r}) \end{bmatrix}. \quad (98)$$

Hence, defining the linear operators

$$J_1[h] := -\frac{1}{2(n-1)}\frac{r}{f}\left(\frac{M}{r^{n-1}}\right)^{-2}\dot{\bar T}^{(1)}[h], \quad (99)$$

$$J_2[h]_{aB} := \frac{1}{r^2}\begin{cases}\dot{\bar T}^{(2)}_{aB}[h] & (n>4) \\ \dot{\bar T}^{(3)}_{aB}[h] & (n=4)\end{cases}, \quad (100)$$

the Lie derivative formula (26) implies the compositional identities

$$J_1\circ\bar K[v] = \frac{r^c v_c}{f}, \quad (101)$$

$$J_2\circ\bar K[v]_{aB} = \nabla_a\frac{v_B}{r}. \quad (102)$$

Based on Equation (75), the explicit expression for the Killing operator is

$$\bar K_{\mu\nu}[v] = 2\bar\nabla_{(\mu}v_{\nu)} \to \begin{bmatrix} 2\nabla_{(a}v_{b)} & r^2\nabla_a\frac{1}{r}v_B + rD_B\frac{v_a}{r} \\ r^2\nabla_b\frac{1}{r}v_A + rD_A\frac{v_b}{r} & 2r^2D_{(A}\frac{1}{r}v_{B)} + 2r^2\frac{r^c v_c}{r}\Omega_{AB}\end{bmatrix}. \quad (103)$$

For further convenience, we parametrize

$$v_\mu \to \begin{bmatrix} u_t\,f dt_a + u_r dr_a \\ r(rX_A)\end{bmatrix} \quad\text{and}\quad h_{\mu\nu} \to \begin{bmatrix} p\,r_a r_b - 2t_{(a}w_{b)} & r(rY_{aB}) \\ r(rY_{bA}) & r^2 Z_{AB}\end{bmatrix}. \quad (104)$$

The Killing equation $h = \bar K[v]$ then becomes

$$\begin{bmatrix} p \\ \hline w \\ Y \\ Z \end{bmatrix} = \bar K\begin{bmatrix} u_r \\ \hline u_t \\ X\end{bmatrix} = \left[\begin{array}{c|cc} \frac{1}{f}(2f\partial_r + \frac{f_1}{r}) & 0 & 0 \\ \hline dr\,\frac{1}{f}\partial_t - dt\,\frac{f_1}{2r} & \nabla & 0 \\ dr\,\frac{f}{r^2}D\frac{1}{f} & dt\,\frac{f}{r^2}D & \nabla \\ 2\Omega\frac{f}{r} & 0 & C_0 \end{array}\right]\begin{bmatrix} u_r \\ \hline u_t \\ X\end{bmatrix}, \quad (105)$$

where $C_0[X]_{AB} = D_A X_B + D_B X_A$ is the Killing operator on the constant curvature factor $(\mathcal{S},\Omega)$, and hence the first operator of the Calabi complex C_i, $i \geq 0$, which constitutes a compatibility complex for C_0 (Section 3.1).

Remark 6. Note that we are continuing here to use the block matrix notation for differential operators, as discussed previously in Remark 2.

With the above parametrizations for v and h, the compositional identities for the operators J_1 (99) and J_2 (100) simplify to

$$J_1 \circ \bar{K} = \begin{bmatrix} J_1^p \mid J_1^w & J_1^Y & J_1^Z \end{bmatrix} \circ \bar{K} = \begin{bmatrix} \mathrm{id} \mid 0 & 0 \end{bmatrix}, \tag{106a}$$

$$J_2 \circ \bar{K} = \begin{bmatrix} J_2^p \mid J_2^w & J_2^Y & J_2^Z \end{bmatrix} \circ \bar{K} = \begin{bmatrix} 0 \mid 0 & \nabla \end{bmatrix}. \tag{106b}$$

Now we have all the information that we need to use the methods of Section 2 to construct a compatibility complex for the Killing operator $\bar{K}$. We will follow roughly the same outline as we did in the Section 3.2 on cosmological FLRW geometries.

From now on, our strategy will be to show that our Killing operator $\bar{K}_0 = \bar{K}$ is equivalent to each of the operators

$$\tilde{K}_0 \begin{bmatrix} u_t \\ X \end{bmatrix} = \begin{bmatrix} \bar{d}_0 = \bar{\nabla} \to \begin{bmatrix} \nabla \\ D \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ dt\, \frac{f}{r^2}\, D & d_0 = \nabla \\ 0 & C_0 \end{bmatrix} \begin{bmatrix} u_t \\ X \end{bmatrix} \quad \text{and} \quad K_0 \begin{bmatrix} u_t \\ X \end{bmatrix} = \begin{bmatrix} \bar{d}_0 & 0 \\ 0 & d_0 \\ 0 & C_0 \end{bmatrix} \begin{bmatrix} u_t \\ X \end{bmatrix}, \tag{107}$$

where we have introduced the notation $\bar{d}_i$ and d_i, $i \geq 0$, for the usual exterior derivatives acting on i-forms on $\bar{\mathcal{M}}$ and $\mathcal{M}$ respectively (hence the corresponding de Rham complexes). In the sequel, we will use the notations $\bar{d}_0$ and $[\begin{smallmatrix} \nabla \\ D \end{smallmatrix}]$ completely interchangeably. Then, we will lift the known compatibility complex for K_0 first to $\tilde{K}_0$ and finally to $\bar{K}_0$. This known compatibility complex has the form

$$K_0 = \begin{bmatrix} \bar{d}_0 & 0 \\ 0 & d_0 \\ 0 & C_0 \end{bmatrix}, \tag{108a}$$

$$K_1 = \begin{bmatrix} \bar{d}_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & -C_0 & d_0 \\ 0 & 0 & C_1 \end{bmatrix}, \tag{108b}$$

$$K_2 = \begin{bmatrix} \bar{d}_2 & 0 & 0 & 0 \\ 0 & C_0 & d_1 & 0 \\ 0 & 0 & -C_1 & d_0 \\ 0 & 0 & 0 & C_2 \end{bmatrix}, \tag{108c}$$

$$K_i = \begin{bmatrix} \bar{d}_i & 0 & 0 & 0 \\ 0 & C_{i-2} & d_1 & 0 \\ 0 & 0 & -C_{i-1} & d_0 \\ 0 & 0 & 0 & C_i \end{bmatrix} \quad (2 < i < n-2), \tag{108d}$$

$$K_{n-2} = \begin{bmatrix} \bar{d}_{n-2} & 0 & 0 & 0 \\ 0 & C_{n-4} & d_1 & 0 \\ 0 & 0 & -C_{n-3} & d_0 \end{bmatrix}, \tag{108e}$$

$$K_{n-1} = \begin{bmatrix} \bar{d}_{n-1} & 0 & 0 \\ 0 & C_{n-3} & d_1 \end{bmatrix}, \tag{108f}$$

$$K_i = 0 \quad (n \leq i). \tag{108g}$$

It is straightforward to construct an equivalence between this complex and a twisted de Rham complex, similar to how it was done in (40), thus showing that each of the above compatibility operators is complete.

We start with the explicit reduction of $\bar{K}_0$ to $\tilde{K}_0$ and then to K_0. Here and in each subsequent step, we give pairs of diagrams, which could be concatenated vertically, illustrating the passage from the $\bar{K}_i$ to the $\tilde{K}_i$ and to the K_i sequences. All the diagrams below illustrate equivalences up to homotopy, as discussed in Section 2. All the required identities can be checked by direct calculation, making careful use of the known identities $d_{i+1} \circ d_i = 0$, $\bar{d}_{i+1} \circ \bar{d}_i = 0$, $C_{i+1} \circ C_i = 0$, as well as the compositional identities (106).

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "\bar{K}_0=\bar{K}"] \arrow[d, shift right, "{\left[\begin{array}{c|cc} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{array}\right]}"'] & \bullet \arrow[l, dashed, bend left, "{\begin{bmatrix} J_1 \\ \hline 0 \\ 0 \end{bmatrix}}"] \arrow[d, shift right, "{\left[\begin{array}{c|ccc} 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{array}\right] \left(\mathrm{id} - \bar{K} \begin{bmatrix} J_1 \\ \hline 0 \\ 0 \end{bmatrix} \right)}"'] \\
\bullet \arrow[r, "\tilde{K}_0=\begin{bmatrix} \nabla & 0 \\ dt\,\frac{f}{r^2}D & \nabla \\ 0 & C_0 \end{bmatrix}"'] \arrow[u, shift right, "{\begin{bmatrix} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}}"'] & \bullet \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "{\begin{bmatrix} 0 & 0 & 0 \\ \hline \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{bmatrix}}"']
\end{tikzcd}
\tag{109a}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "\tilde{K}_0=\begin{bmatrix} \nabla & 0 \\ dt\,\frac{f}{r^2}D & \nabla \\ 0 & C_0 \end{bmatrix}"] \arrow[d, shift right, "{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}}"'] & \bullet \arrow[l, dashed, bend left, "0"] \arrow[d, shift right, "{\begin{bmatrix} \begin{bmatrix} \mathrm{id} \\ r^2 dt\cdot J_2^w \end{bmatrix} & \begin{bmatrix} 0 \\ r^2 dt\cdot (J_2^Y - \mathrm{id}) \end{bmatrix} & \begin{bmatrix} 0 \\ r^2 dt\cdot J_2^Z \end{bmatrix} \\ J_2^w & J_2^Y & J_2^Z \\ 0 & 0 & \mathrm{id} \end{bmatrix}}"'] \\
\bullet \arrow[r, "K_0=\begin{bmatrix} \bar{d}_0 & 0 \\ 0 & d_0 \\ 0 & C_0 \end{bmatrix}"'] \arrow[u, shift right, "{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}}"'] & \bullet \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "{\begin{bmatrix} [\mathrm{id} & 0\,] & 0 & 0 \\ [0 & dt\,\frac{f}{r^2}] & \mathrm{id} & 0 \\ [0 & 0\,] & 0 & \mathrm{id} \end{bmatrix}}"']
\end{tikzcd}
\tag{109b}
$$

Next, as we did previously in Section 3.2, we iterate the construction from Lemma 5, while applying the simplifications discussed after the proof of Theorem 9. As before, the $\bar{K}_i$ and $\tilde{K}_i$ complexes are built up by appending the following diagrams to the right of the diagrams in (109). Also as before, we do not repeat the labels on the vertical arrows if they can be read off a preceding diagram.

The resulting operators $\bar{K}_1$ and $\tilde{K}_1$ will be, respectively, compatibility operators for $\bar{K}_0$ and $\tilde{K}_0$. Some of the auxiliary arrows in these diagrams use the operators $\tilde{H}_{J_1}$ and H_{J_1}, which are defined as follows. Noting that

$$
\begin{aligned}
J_1 \bar{K} \begin{bmatrix} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix} &= J_1 \begin{bmatrix} 0 & 0 \\ \hline \nabla & 0 \\ dt\,\frac{f}{r^2}D & \nabla \\ 0 & C_0 \end{bmatrix} \\
&= \begin{bmatrix} J_1^w & J_1^Y & J_1^Z \end{bmatrix} \begin{bmatrix} \nabla & 0 \\ dt\,\frac{f}{r^2}D & \nabla \\ 0 & C_0 \end{bmatrix} = \left[\begin{array}{c|cc} \mathrm{id} & 0 & 0 \end{array}\right] \begin{bmatrix} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix} = 0,
\end{aligned}
$$

we must be able to factor

$$\begin{bmatrix} J_1^w & J_1^Y & J_1^Z \end{bmatrix} = \tilde{H}_{J_1} \tilde{K}_1, \quad \text{and} \quad \left[J_1^p \;\middle|\; J_1^w \quad J_1^Y \quad J_1^Z \right] = H_{J_1} \left[\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & \tilde{K}_1 \end{array}\right] \tag{110}$$

through some operators $\tilde{H}_{J_1}$ and H_{J_1}. A bit more precisely, $H_{J_1} = \left[J_1^p \;\middle|\; \tilde{H}_{J_1} \right]$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\bar{K}_1 = \left[\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & \tilde{K}_1 \end{array}\right] \left(\mathrm{id} - \bar{K} \left[\begin{array}{c} J_1 \\ \hline 0 \\ 0 \end{array}\right] \right)"}] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left, "{\left[\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & 0 \end{array}\right]}"] \arrow[d, shift right, swap, "{\left[0 \;\middle|\; \mathrm{id} \right]}"] \\
\bullet \arrow[u, shift right] \arrow[r, swap, "\tilde{K}_1"] & \bullet \arrow[l, dashed, bend right, swap, "{\left[\begin{array}{c|ccc} 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{array}\right] \bar{K} \left[\begin{array}{c} \tilde{H}_{J_1} \\ \hline 0 \\ 0 \end{array}\right]"}] \arrow[u, shift right, swap, "{\left(\mathrm{id} - \left[\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & \tilde{K}_1 \end{array}\right] \bar{K} \left[\begin{array}{c} H_{J_1} \\ \hline 0 \\ 0 \end{array}\right] \right) \left[\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline \mathrm{id} & 0 & 0 & 0 & 0 \\ \hline 0 & \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{array}\right]"}]
\end{tikzcd} \tag{111a}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\tilde{K}_1 = \left[\begin{array}{ccc} \begin{bmatrix} 0 & -dr\cdot(-) \end{bmatrix} & dr\cdot(-) & 0 \\ \hline \bar{d}_1 \begin{bmatrix} \mathrm{id} & 0 \\ 0 & -r^2 dt\cdot(-) \end{bmatrix} & \bar{d}_1 \begin{bmatrix} 0 \\ r^2 dt\cdot(-) \end{bmatrix} & 0 \\ 0 & d_1 & 0 \\ 0 & -C_0 & d_0 \\ 0 & 0 & C_1 \end{array}\right] \begin{bmatrix} \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \mathrm{id} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ J_2^w & J_2^Y & J_2^Z \\ 0 & 0 & \mathrm{id} \end{bmatrix}"}] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left, "{\begin{bmatrix} 0 \\ -\frac{1}{f}\, dr \\ 0 \end{bmatrix} \left[\mathrm{id} \;\middle|\; 0 \quad 0 \quad 0 \quad 0 \right]"}] \arrow[d, shift right, swap, "{\left[\begin{array}{c|cccc} 0 & \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{array}\right]"}] \\
\bullet \arrow[u, shift right] \arrow[r, swap, "{K_1 = \begin{bmatrix} \bar{d}_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & -C_0 & d_0 \\ 0 & 0 & C_1 \end{bmatrix}"}] & \bullet \arrow[l, dashed, bend right, swap, "{\begin{bmatrix} \begin{bmatrix} 0 \\ -r^2 dt\cdot(-) \end{bmatrix} \\ -\mathrm{id} \\ 0 \end{bmatrix} \tilde{H}_{J_2}"}] \arrow[u, shift right, swap, "{\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{array}\right] + \left[\begin{array}{c} dr\cdot(-) \\ \hline \bar{d}_1 \begin{bmatrix} 0 \\ r^2 dt\cdot(-) \end{bmatrix} \\ d_1 \\ -C_0 \\ 0 \end{array}\right] \tilde{H}_{J_2}"}]
\end{tikzcd} \tag{111b}$$

Above, we have used the notations $dt\cdot(-) = dt^a(-)_a$ and $dr\cdot(-) = dr^a(-)_a$. Also, the operator

$\tilde{H}_{J_2}$ is defined as follows. Noting that

$$J_2\bar{K}\begin{bmatrix}0\\\hline \mathrm{id}\\0\end{bmatrix} = J_2\begin{bmatrix}0\\\hline \nabla\\ dt\,\frac{f}{r^2}D\\0\end{bmatrix} = \begin{bmatrix}J_2^w & J_2^Y\,dt\,\frac{f}{r^2}\end{bmatrix}\left(\bar{d}_0 = \begin{bmatrix}\nabla\\D\end{bmatrix}\right) = \begin{bmatrix}0 & \vert & 0 & \nabla\end{bmatrix}\begin{bmatrix}0\\\hline \mathrm{id}\\0\end{bmatrix} = 0,$$

$$J_2\bar{K}\begin{bmatrix}0\\\hline 0\\\mathrm{id}\end{bmatrix} = J_2\begin{bmatrix}0\\\hline 0\\\nabla\\C_0\end{bmatrix} = \begin{bmatrix}J_2^Y & J_2^Z\end{bmatrix}\begin{bmatrix}d_0 = \nabla\\C_0\end{bmatrix} = \begin{bmatrix}0 & \vert & 0 & \nabla\end{bmatrix}\begin{bmatrix}0\\\hline 0\\\mathrm{id}\end{bmatrix} = \nabla = \begin{bmatrix}\mathrm{id} & 0\end{bmatrix}\begin{bmatrix}\nabla\\C_0\end{bmatrix},$$

we must be able to factor

$$\begin{bmatrix}\begin{bmatrix}J_2^w & J_2^Y\,dt\,\frac{f}{r^2}\end{bmatrix} & J_2^Y - \mathrm{id} & J_2^Z\end{bmatrix} = \tilde{H}_{J_2}\begin{bmatrix}\bar{d}_1 & 0 & 0\\0 & d_1 & 0\\0 & -C_0 & d_0\\0 & 0 & C_1\end{bmatrix} \tag{112}$$

through some operator $\tilde{H}_{J_2}$.

For convenience, we note that

$$\begin{bmatrix}\mathrm{id} & \vert & 0 & 0 & 0 & 0\end{bmatrix}\tilde{K}_1 = \begin{bmatrix}dr\cdot J_2^w & dr\cdot(J_2^Y - \mathrm{id}) & dr\cdot J_2^Z\end{bmatrix},$$

while on the other hand

$$\begin{bmatrix}0 & \vert & \tilde{H}_{J_2}\end{bmatrix}\tilde{K}_1 = J_2^Y\frac{dr}{f}\begin{bmatrix}dr\cdot J_2^w & dr\cdot(J_2^Y - \mathrm{id}) & dr\cdot J_2^Z\end{bmatrix}.$$

Then, defining

$$H_{J_2} = \begin{bmatrix}-J_2^Y\frac{dr}{f} & \vert & \tilde{H}_{J_2}\end{bmatrix}, \quad \text{we have} \quad H_{J_2}\tilde{K}_1 = 0. \tag{113}$$

With the next iteration of Lemma 5, we construct the compatibility operators $\bar{K}_2$ and $\tilde{K}_2$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\bar{K}_2 = \begin{bmatrix}H_{J_1}\\\hline 0 \mid \tilde{K}_2\end{bmatrix}}"] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left, "{\begin{bmatrix}\mathrm{id} & \vert & 0\\\hline 0 & \vert & \tilde{K}_1\end{bmatrix}\bar{K}\begin{bmatrix}\mathrm{id} & \vert & 0\\\hline 0 & \vert & 0\end{bmatrix}}"] \arrow[d, shift right, "{\begin{bmatrix}0 \mid \mathrm{id}\end{bmatrix}}"'] \\
\bullet \arrow[u, shift right] \arrow[r, "\tilde{K}_2"'] & \bullet \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "{\begin{bmatrix}0\\\hline \mathrm{id}\end{bmatrix}}"']
\end{tikzcd} \tag{114a}$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\tilde{K}_2=\left[\begin{array}{c|cccc} \multicolumn{5}{c}{H_{J_2}} \\ \hline 0 & \bar{d}_2 & 0 & 0 & 0 \\ 0 & 0 & C_0 & d_1 & 0 \\ 0 & 0 & 0 & -C_1 & d_0 \\ 0 & 0 & 0 & 0 & C_2 \end{array}\right]"}] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left, "{-\left[\begin{array}{c} dr\cdot(-) \\ \hline \bar{d}_1 \begin{bmatrix} 0 \\ r^2 dt\cdot(-) \end{bmatrix} \\ d_1 \\ -C_0 \\ 0 \end{array}\right]\left[\begin{array}{c|cccc} \mathrm{id} & 0 & 0 & 0 & 0 \end{array}\right]"] \arrow[d, shift right, swap, "{\left[\begin{array}{c|cccc} 0 & \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{array}\right]"] \\
\bullet \arrow[r, swap, "{K_2=\begin{bmatrix} \bar{d}_2 & 0 & 0 & 0 \\ 0 & C_0 & d_1 & 0 \\ 0 & 0 & -C_1 & d_0 \\ 0 & 0 & 0 & C_2 \end{bmatrix}"}] \arrow[u, shift right] & \bullet \arrow[l, dashed, bend right, swap, "0"] \arrow[u, shift right, swap, "{\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{array}\right]"}]
\end{tikzcd}
\tag{114b}
$$

With two more iterations of Lemma (5), we construct the compatibility operators $\bar{K}_3$, $\bar{K}_4$ and $\tilde{K}_3$, $\tilde{K}_4$.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\bar{K}_3=\left[\begin{array}{c|c|cccc} 0 & 0 & \bar{d}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & d_1 & 0 \\ 0 & 0 & 0 & 0 & -C_2 & d_0 \\ 0 & 0 & 0 & 0 & 0 & C_3 \end{array}\right]"}] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend left, "0"] \arrow[r, "{\bar{K}_4=\begin{bmatrix} \bar{d}_4 & 0 & 0 & 0 \\ 0 & C_2 & d_1 & 0 \\ 0 & 0 & -C_3 & d_0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}"}] \arrow[d, shift right, swap, "{\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"}] & \bullet \arrow[l, dashed, bend left, "0"] \arrow[d, shift right, swap, "{\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"}] \\
\bullet \arrow[r, swap, "{\tilde{K}_3=\left[\begin{array}{c|cccc} 0 & \bar{d}_3 & 0 & 0 & 0 \\ 0 & 0 & C_1 & d_1 & 0 \\ 0 & 0 & 0 & -C_2 & d_0 \\ 0 & 0 & 0 & 0 & C_3 \end{array}\right]"}] \arrow[u, shift right] & \bullet \arrow[l, dashed, bend right, swap, "0"] \arrow[r, swap, "{\tilde{K}_4=\begin{bmatrix} \bar{d}_4 & 0 & 0 & 0 \\ 0 & C_2 & d_1 & 0 \\ 0 & 0 & -C_3 & d_0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}"}] \arrow[u, shift right, swap, "{\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"}] & \bullet \arrow[l, dashed, bend right, swap, "0"] \arrow[u, shift right, swap, "{\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"}]
\end{tikzcd}
\tag{115a}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "\tilde{K}_3 = \left[\begin{array}{c|cccc} 0 & \bar{d}_3 & 0 & 0 & 0 \\ 0 & 0 & C_1 & d_1 & 0 \\ 0 & 0 & 0 & -C_2 & d_0 \\ 0 & 0 & 0 & 0 & C_3 \end{array}\right]"] \arrow[d, shift right] & \bullet \arrow[r, "\tilde{K}_4 = \begin{bmatrix} \bar{d}_4 & 0 & 0 & 0 \\ 0 & C_2 & d_1 & 0 \\ 0 & 0 & -C_3 & d_0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}"] \arrow[l, dashed, bend left, "0"] \arrow[d, shift right, "\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"'] & \bullet \arrow[l, dashed, bend left, "0"] \arrow[d, shift right, "\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"'] \\
\bullet \arrow[r, "K_3 = \begin{bmatrix} \bar{d}_3 & 0 & 0 & 0 \\ 0 & C_1 & d_1 & 0 \\ 0 & 0 & -C_2 & d_0 \\ 0 & 0 & 0 & C_3 \end{bmatrix}"'] \arrow[u, shift right] & \bullet \arrow[r, "K_4 = \begin{bmatrix} \bar{d}_4 & 0 & 0 & 0 \\ 0 & C_2 & d_1 & 0 \\ 0 & 0 & -C_3 & d_0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}"'] \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"'] & \bullet \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "\begin{bmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{bmatrix}"']
\end{tikzcd}
\tag{115b}
$$

From this point on, the complexes $\bar{K}_i$, $\tilde{K}_i$ become identical with K_i from (108).

Theorem 11. *Consider the family of n-dimensional ($n \geq 4$) spacetimes $(\bar{\mathcal{M}}, \bar{g})$ introduced at the top of Section 3.3, warped products of a static 2-dimensional factor $(\mathcal{M}, g)$ and a constant curvature factor $(\mathcal{S}, \Omega)$ with sectional curvature α, which includes the higher dimensional Schwarzschild (Schwarzschild-Tangherlini), Taub and pseudo-Schwarzschild solutions, possibly with a nonzero cosmological constant. The full compatibility complex $\bar{K}_i$ for the Killing operator $\bar{K}_0 = \bar{K}$ (105) is given by*

$$
\bar{K}_0 = \left[\begin{array}{c|cc} \frac{1}{f}(2f\partial_r + \frac{f_1}{r}) & 0 & 0 \\ \hline dr\,\frac{1}{f}\partial_t - dt\,\frac{f_1}{2r} & \nabla & 0 \\ dr\,\frac{f}{r^2}D\frac{1}{f} & dt\,\frac{f}{r^2}D & \nabla \\ 2\Omega\frac{f}{r} & 0 & C_0 \end{array}\right], \tag{116a}
$$

$$
\begin{aligned}
\bar{K}_1 = {} & \left[\begin{array}{c|ccc} \mathrm{id} & 0 & 0 & 0 \\ \hline 0 & \begin{bmatrix} 0 & -dr\cdot(-) \end{bmatrix} & dr\cdot(-) & 0 \\ \hline 0 & \bar{d}_1 \begin{bmatrix} \mathrm{id} & 0 \\ 0 & -r^2 dt\cdot(-) \end{bmatrix} & \bar{d}_1 \begin{bmatrix} 0 \\ r^2 dt\cdot(-) \end{bmatrix} & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & 0 & -C_0 & d_0 \\ 0 & 0 & 0 & C_1 \end{array}\right] \left[\begin{array}{c|ccc} \mathrm{id} & 0 & 0 & 0 \\ \hline 0 & \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \mathrm{id} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & J_2^w & J_2^Y & J_2^Z \\ 0 & 0 & 0 & \mathrm{id} \end{array}\right] \\
& \left(\left[\begin{array}{c|ccc} \mathrm{id} & 0 & 0 & 0 \\ \hline 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{array}\right] - \bar{K} \left[\begin{array}{c} J_1 \\ \hline 0 \\ 0 \end{array}\right] \right),
\end{aligned}
\tag{116b}
$$

$$\bar{K}_2 = \left[\begin{array}{c|c|cccc} \multicolumn{6}{c}{H_{J_1}} \\ \hline 0 & \multicolumn{5}{c}{H_{J_2}} \\ \hline 0 & 0 & \bar{d}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_0 & d_1 & 0 \\ 0 & 0 & 0 & 0 & -C_1 & d_0 \\ 0 & 0 & 0 & 0 & 0 & C_2 \end{array}\right], \tag{116c}$$

$$\bar{K}_3 = \left[\begin{array}{c|c|cccc} 0 & 0 & \bar{d}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & d_1 & 0 \\ 0 & 0 & 0 & 0 & -C_2 & d_0 \\ 0 & 0 & 0 & 0 & 0 & C_3 \end{array}\right], \tag{116d}$$

$$\bar{K}_i = \begin{bmatrix} \bar{d}_i & 0 & 0 & 0 \\ 0 & C_{i-2} & d_1 & 0 \\ 0 & 0 & -C_{i-1} & d_0 \\ 0 & 0 & 0 & C_i \end{bmatrix} \quad (3 < i < n-2), \tag{116e}$$

$$\bar{K}_{n-2} = \begin{bmatrix} \bar{d}_{n-2} & 0 & 0 & 0 \\ 0 & C_{n-4} & d_1 & 0 \\ 0 & 0 & -C_{n-3} & d_0 \end{bmatrix}, \tag{116f}$$

$$\bar{K}_{n-1} = \begin{bmatrix} \bar{d}_{n-1} & 0 & 0 \\ 0 & C_{n-3} & d_1 \end{bmatrix}, \tag{116g}$$

$$\bar{K}_i = 0 \quad (n \le i). \tag{116h}$$

where $f(r)$ is defined in (64) and $f_1 = rf'(r)$, $\bar{d}_i$ and d_i denote the exterior derivatives on i-forms, on $\bar{\mathcal{M}}$ and $\mathcal{M}$ respectively, while D and C_i are the covariant derivative and the Calabi complex operators (32) on $(\mathcal{S}, \Omega)$, and we have also used the operators J_1 (99), J_2 (100), H_{J_1} (110), H_{J_2} (113). (See Appendix B.3 for a more complete summary of the notation.)

While we have unambiguously defined the operators J_1, J_2, H_{J_1}, and H_{J_2}, we have not computed them explicitly. For our purposes here, it is sufficient that they exist and satisfy a few defining properties. Of course, in individual cases, they could be easily computed using computer algebra.

Proof. The proof is very much parallel to the proof of Theorem 10. We start with the knowledge that the complex (108) is a full compatibility complex. Then, gluing together (from left to right) the diagrams (109), (111), (114) and (115), we observe that the glued diagrams satisfy the hypotheses of Lemma 4. This implies, that $\tilde{K}_i$ is a full compatibility complex as well, which in turn implies that so is $\bar{K}_i$, whose operators we have explicitly listed in (116). □

The non-vanishing ranks of the vector bundles in the $\bar{K}_i$ complex have the following pattern, which can be compared to similar table for the constant curvature (Section 3.1) and FLRW cases (Section 3.2, where $m = n - 1$, for comparison):

		$n=4$	$n=5$	$n=6$	$n \geq 4$
		4	5	6	$(n-2)+1+1$
	$\bar{K}_0$				
		10	15	21	$\frac{(n-1)(n-2)}{2} + 2(n-2) + 3$
	$\bar{K}_1$				
		18	35	64	$\frac{n-2}{2}\binom{n-1}{3} + (n-1)(n-2) + (n-2) + \binom{n}{2} + (n-2) + 1$
	$\bar{K}_2$				
		12	35	95	$(n-2)\binom{n-1}{4} + (n-2)\binom{n-1}{3} + \frac{(n-1)(n-2)}{2} + \binom{n}{3} + (n-2) + 1$
	$\bar{K}_3$				
		2	17	81	$\frac{3(n-2)}{2}\binom{n-1}{5} + 2(n-2)\binom{n-1}{4} + \frac{(n-2)}{2}\binom{n-1}{3} + \binom{n}{4}$
	$\vdots$				$\vdots$
					$\frac{(n-2)(i-1)}{2}\binom{n-1}{i+1} + (n-2)(i-2)\binom{n-1}{i} + \frac{(n-2)(i-3)}{2}\binom{n-1}{i-1} + \binom{n}{i}$
$(3 < i)$	$\bar{K}_i$				
					$\frac{(n-2)i}{2}\binom{n-1}{i+2} + (n-2)(i-1)\binom{n-1}{i+1} + \frac{(n-2)(i-2)}{2}\binom{n-1}{i} + \binom{n}{i+1}$
	$\vdots$				$\vdots$
	$\bar{K}_{n-1}$				
		2	4	7	$\frac{(n-2)(n-3)}{2} + 1$

Remark 7. In $n = 4$ dimensions, it is well-known [21, 31] that, for practical purposes, taking the linearized Einstein equations into account, the gauge invariant degrees of freedom for linear perturbations on the Schwarzschild background reduce to the Regge-Wheeler (axial) and Zerilli (polar) scalars, or equivalently the complex Teukolsky scalar. It is even possible to give the Regge-Wheeler and Zerilli scalars local and manifestly gauge-invariant definitions, based on the linearization of curvature tensors vanishing on the Schwarzschild background [10]. However, it is also known that there exist so-called *algebraically special* modes that are not pure gauge but lie in the kernel of these gauge-invariants [38]. Hence, this small set of invariants cannot be considered complete in our sense. In our construction, $\bar{K}_1$ has 18 independent components (without taking the linearized Einstein equations into account, though). But our construction proves that they form a complete set of local gauge-invariants.

Remark 8. In analogy with Remark 4 about FLRW geometries, it is worth noting that the $\bar{K}_i$ complex presented above is not continuously deformable through the class of gST spacetimes to the $M = 0$ case, which corresponds to the constant curvature limit. The main reason, again, is that the operators J_1 and J_2, introduced in Equations (99) and (100), are proportional to $1/M$ and hence diverge in that limit. These operators, together with their factorizations H_{J_1} and H_{J_2} appear in several places in the formulas (116) for the operators $\bar{K}_i$ (for $i \geq 1$). Thus, also in this case, it would be difficult to compare the local gauge-invariant components of our $\bar{K}_1$ operator to the components of the linearized Riemann operator, which would also be given by long and unenlightening expressions. A more fruitful comparison would be to try to express the components of our $\bar{K}_1$ in terms of the linearized IDEAL characterization tensors that were recently constructed for the gST geometries [24].

However, the intuition proposed in the second paragraph of Remark 4 still largely applies to the components of our $\bar{K}_1$. In particular, our J_1 operator is directly analogous to the J operator introduced for FLRW geometries. On the other hand, the J_2 operator did not have a direct analogy, so the way it induces gauge invariant components of $\bar{K}_1$ is slightly different.

4 Discussion

In this work, we have studied the construction of the compatibility complex (Definition 2) K_l, $l = 0, 1, 2, \ldots$, for a linear differential operator K_0 of finite type (Definition 7). The construction proceeds by putting the operator K_0 into a canonical form of a flat connection and then lifting the resulting twisted de Rham complex to a compatibility complex for K_0 (Theorem 9). Our primary and motivating example of an operator of finite type is the Killing operator $K_{ab}[v] = \nabla_a v_b + \nabla_b v_a$ on a Lorentzian (or even pseudo-Riemannian) manifold (M, g). Once known, the components of the first compatibility operator K_1 can be interpreted (as discussed in the Introduction) as a complete set of local gauge-invariant observables in linearized gravity on (M, g).

We have applied the abstract construction of Section 2 to several physically motivated examples: flat (Minkowski) and constant curvature (de Sitter or anti-de Sitter) spacetimes in Section 3.1, cosmological (FLRW) spacetimes in Section 3.2, (Schwarzschild-Tangherlini) spherically symmetric black hole spacetimes[4] in Section 3.3. In each case, we have kept the dimension $n = \dim M$ general, allowing at least $n \geq 4$. While the contents of Section 3.1 are well-known (they were previously reviewed in more detail in [23]), the Killing compatibility complexes constructed in Sections 3.2 and 3.3 are new.

One may wish to compare the main result for FLRW geometries, Theorem 10, with the recent works [13, 9, 14], which were the first to (a) construct, (b) give a geometric interpretation to and (c) prove completeness for the first compatibility operator K_1 in a context very similar the one considered in Section 3.2 (the difference is that here we do not include the presence of a dynamical scalar inflaton field on an cosmological FLRW geometry). The systematic approach developed in this work can also be easily applied in the presence of an inflaton field. Then, the systematically constructed compatibility operator K_1 would be necessarily equivalent to what was obtained in [13, 9, 14]. The difference is that our systematic construction automatically comes with a proof of completeness, while the previous proof of completeness given in [13] relied very heavily on parallels with known results for the flat and constant curvature cases [19, 23], without an obvious way to generalize it. On the other hand, our systematic construction does not give a Stewart-Walker-like (cf. the introduction to Section 3) geometric interpretation to K_1 as a linear local gauge-invariant observable. On the other hand, the approach put forward in [9, 14], of constructing a *candidate* K_1 by linearizing an *IDEAL characterization* of the background geometry, automatically gives K_1 a Stewart-Walker-like geometric interpretation, but does not automatically prove completeness.[5] Thus, we see great potential in joining the methods of the current work with those of [9, 14] to construct universal Killing compatibility operators (equivalently, complete sets of linear local gauge-invariant observables) on a variety of backgrounds, while getting the benefits of straightforward geometric interpretation and of a systematic way to prove completeness.

For the Schwarzschild black hole (and its higher dimensional generalizations), the Regge-Wheeler and Zerilli local gauge-invariants have been known for a long time [25]. Other local gauge-invariants have also been proposed (see [21, 31, 1] for a brief review). However, to our knowledge, no claim of completeness has ever been made for an explicit set of local gauge-invariants on Schwarzschild. Thus, even our construction of the first compatibility K_1 operator in Section 3.3 appears to be new. On the other hand, the 4-dimensional Schwarzschild black hole does have a known IDEAL characterization [12], recently extended to higher dimensions [24], so as was argued in the previous paragraph its linearization would have provided a good candidate for K_1. To our knowledge, this has not been done explicitly in the literature. Again, comparing that heuristic construction with our systematic approach would be very interesting.

The next logical step is to apply our methods to the Kerr black hole and higher dimensional (Myers-Perry) generalizations. As a first step, we intend to construct a Killing compatibility complex for the Kerr geometry [2], thus providing a proof of completeness for the list of local gauge-invariants recently proposed in [1].

Once the Killing compatibility complex is known on a given geometry, this information has interesting applications to the symplectic and Poisson structures on the space of solutions of linearized gravity [23, Sec.5].

Acknowledgments. The author thanks S. Aksteiner, L. Andersson and T. Bäckdahl for many useful discussions on gauge-invariant observables, as well as B. Kruglikov and W. Seiler for pertinent feedback at the early stages of this work. Also, the author was partially supported by the GAČR project 18-07776S and RVO: 67985840.

[4]The family of spacetimes considered Section 3.3 is actually richer than just asymptotically flat spherically symmetric black holes (the Schwarzschild-Tangherlini ones). More generally, it allows for a non-zero cosmological constant and also allows to substitute spherical symmetry for planar or pseudo-spherical symmetry, which respectively give rise Taub's plane symmetric spacetimes or to pseudo-Schwarzschild solutions.

[5]Although, the only possibility we know in which completeness might fail is when the IDEAL characterization tensors vanish at *quadratic* or *higher* order when approaching the isometry class of the characterized geometry in the space of metrics. Then their *linearization* might fail to capture all of the linear invariants.

A Flat connection form for PDEs of finite type

We have chosen to express our definition of a PDE of *finite type* (Definition 7) directly in terms of a *flat connection* (Definition 6). Elsewhere in the literature, the definition is given in different terms, but the equivalence with the form of a flat connection is well-known, even for nonlinear equations. For instance, the contents of Rmk.2.3.3, Rmk.2.3.6, and Ex.2.3.17 of [30] concern precisely the equivalence between these two possible definitions.

For the convenience of the reader, we give an elementary proof of this equivalence for linear equations, which is all that we will need. When we speak of *equivalence* below, we mean in the sense of one operator complexes of Definition 1. The length of our proof mostly reflects the amount of notation that we have needed to introduce along the way to make the argument as explicit as possible.

For the following definition, we need to quickly introduce the notion of jets and jet bundles [30, Secs.2.1–2]. Given a vector bundle $V \to M$, the *N-jet* $j^N_x v$ at $x \in M$ of a section $v \in \Gamma(V)$ is the equivalence class of sections that have the same Taylor expansion about x up to order N in local adapted coordinates on $V \to M$. The definition clearly does not depend on the choice of adapted coordinates. Denote by $J^N_x V$ the vector space of all N-jets at x and let $J^N V = \bigsqcup_{x\in M} J^N_x V$ be the *N-jet bundle* of V, which can naturally be given the structure of a smooth vector bundle $J^N V \to M$, with $J^0 V = V$. By throwing away higher terms of Taylor series expansions, we can define natural projections $\pi^N_{N'} \colon J^N V \to J^{N'} V$ when $N \geq N'$. By assigning to a section $v \in \Gamma(V)$ its own N-jet at each point of M, we can define the natural *N-jet extension* differential operator $j^N \colon \Gamma(V) \to \Gamma(J^N V)$, which is universal in the sense that for any differential operator $D \colon \Gamma(V) \to \Gamma(W)$ of order at most N, there exists a unique vector bundle map $p^0(D) \colon J^N V \to W$ such that $D[v] = d(j^N v)$. Extending this notation, we denote by $p^l(D) = p^0(j^l \circ D)$ the *l-th prolongation* of D.

Definition 12. *Let $V \to M$ and $W \to M$ be vector bundles and $K \colon \Gamma(V) \to \Gamma(W)$ a linear differential operator. The PDE $K[v] = 0$ is said to be of* finite type *when (a) locally there exists an integer $N < \infty$, a vector bundle morphism $\kappa \colon J^N V \to J^{N+1} V$ and a differential operator $\lambda \colon \Gamma(W) \to \Gamma(J^{N+1} V)$ such that $j^{N+1} v - \kappa(j^N v) = \lambda[K[v]]$ for any $v \in \Gamma(V)$, as well as (b) locally the dimension of the solution space is finite and constant.*

Proposition 13. *If a differential operator $K \colon \Gamma(V) \to \Gamma(W)$ between vector bundles $V \to M$, $W \to M$ defines a PDE of finite type $K[v] = 0$, then K is equivalent to a flat connection operator $\bar{\mathbb{D}}$ on some vector bundle $\bar{U} \to M$.*

Proof. Our proof will consist of three steps: (a) equivalence of the original differential operator K to a connection $\mathbb{D}$ on $J^N V$ together with a non-differential constraint $E \colon J^N V \to W'$, (b) equivalence to the restriction $\tilde{\mathbb{D}}$ of $\mathbb{D}$ to the sub-bundle $\tilde{U} \hookrightarrow J^N V \to M$ satisfying the non-differential constraint $E(\tilde{u}) = 0$, and (c) equivalence to the restriction $\bar{\mathbb{D}}$ of $\tilde{\mathbb{D}}$ to the sub-bundle $\bar{U} \hookrightarrow \tilde{U} \to M$ spanned by flat sections.

Since all of our definitions and claims are local, we might as well work in local adapted coordinates on V, W and any other vector bundles. For instance, we will use coordinates (x^a) on M, (x^a, v^α) on V and $(x^a, v^\alpha, v^{\alpha,1}, \ldots, v^{\alpha,N})$ on $J^N V$, such that $v_{\alpha,k}(j^N \phi(x)) = \partial^k v^\alpha(\phi(x))$, where ∂^k stands for all possible independent partial derivatives of order k with respect to the (x^a) coordinates, with similar notations used for other bundles. Also, when there is no confusion, we will denote a general section of a vector bundle V by v, a general section of W by w, and so on. We will denote a general section of $J^N V$ by $v^{(N)} = (v, v^1, \ldots, v^N)$.

Before proceeding, let us establish some notation. Namely, supposing that K is a differential operator of order k, there is a unique non-differential bundle map that factors K through k-jets, which we denote by $p^0(K)(j^k v) = K[v]$ and similarly $p^l(K)(j^{k+l} v) = j^l K[v]$. Also, we will need the algebraic operators ι_k defined by the identity[6] $\partial^k(\partial^l v) = \iota_k(\partial^{k+l} v)$, as well as the differential operators Δ^N_k defined by the identity

$$\Delta^N_k[\partial v - \iota_1(v^1), \ldots, \partial v^{N-1} - \iota_1(v^N), \partial v^N - \iota_1(v^{N+1})] = \partial^k v - \iota_k(v^k), \tag{117}$$

[6] The non-triviality of the identity stems from the fact that $\partial^k(\partial^l v)$ has more components than $\partial^{k+l} v$, if we do not symmetrize the partial derivatives between the ∂^k and ∂^l operators.

for $k \le N$. These operators basically encode the identities $\partial^0 v - v = 0$, $\partial^1 v - \iota_1(v^1) = \partial v - \iota_1(v^1)$, $\partial^2 v - \iota_2(v^2) = \partial(\partial v - \iota_1(v^1)) + (\partial v^1 - \iota_1(v^2))$, ...

(a) Essentially, all the information that we will need to establish the first equivalence is contained in the bundle map κ and the differential operator λ from Definition 7. The non-trivial information is contained in the highest order components, $\partial^{N+1} v - \kappa^{N+1}(v, \partial v, \ldots, \partial^N v) = \lambda^{N+1}[K[v]]$, which implies the identity

$$\partial(\partial^N v) - \iota_1(\kappa^{N+1}(v, \partial v, \ldots, \partial^N v)) = \iota_1(\lambda^{N+1}[K[v]]). \tag{118}$$

That last identity will be a crucial piece of our definition of a connection on $J^N V$.

To complete the necessary definitions, we will need the differential operator $P_1 \colon \Gamma(W) \to \Gamma(T^*M \otimes_M J^N V)$ given by

$$P_1[w] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \iota_1(\lambda^{N+1}[w]) \end{bmatrix}. \tag{119}$$

Next, the bundle W' and the bundle map $E \colon J^N \to W'$ are chosen so that $\ker E = \pi_0^{N'}(\ker p^{N'}(K))$ for some N'. Since we are allowed to specify N and N' as we like, we can pick them so that $N' > 0$, $N > k$. The meaning is that E takes into account all integrability conditions of order N that can be obtained by prolonging the equation $K[v] = 0$ by N' differentiations. By construction, there must exist a differential operator $P_0 \colon \Gamma(W) \to \Gamma(W')$ such that $P_0[K[v]] = E(j^N v)$. Also, since we have presumed that $N > k$, we have $\pi_k^N(\ker E) \subset \ker p^0(K)$ and there must exist a bundle map $\bar{P}_0 \colon W' \to W$ such that $P_0 E = p^0(K)\pi_k^N$. It remains to define the connection on $J^N V$, which we give by the formula

$$\mathbb{D} \begin{bmatrix} v \\ v^1 \\ \vdots \\ v^{N-1} \\ v^N \end{bmatrix} = \begin{bmatrix} \partial v - \iota_1(v^1) \\ \partial v^1 - \iota_1(v^2) \\ \vdots \\ \partial v^{N-1} - \iota_1(v^N) \\ \partial v^N - \iota_1(\kappa^{N+1}(v, v^1, \ldots, v^N)) \end{bmatrix}. \tag{120}$$

Having introduced all the necessary notation. The desired equivalence is explicitly exhibited by the diagram

$$\begin{tikzcd}
v \arrow[rr, "K"] \arrow[d, shift right, "j^N"'] & & w \arrow[ll, dashed, bend left, "0"] \arrow[d, shift right, "\begin{bmatrix} P_1 \\ P_0 \end{bmatrix}"'] \\
v^{(N)} \arrow[u, shift right, "\pi_0^N"'] \arrow[rr, "\begin{bmatrix} \mathbb{D} \\ E \end{bmatrix}"'] & & \begin{bmatrix} v_1^{(N)} \\ w' \end{bmatrix} \arrow[u, shift right, "\begin{bmatrix} p^0(K)\Delta_k^N & \bar{P}_0 \end{bmatrix}"'] \arrow[ll, dashed, bend right, "\begin{bmatrix} -\Delta_N^N & 0 \end{bmatrix}"']
\end{tikzcd}, \tag{121}$$

where v_1^N denotes a general section of $T^*M \otimes_M J^N V \to M$. To prove that we have an equivalence, we must verify all the conditions required by Definition 1. The commutativity of the squares formed by solid arrows follows from direct computations. One is involves only the defining properties of P_0 and P_1, while the other uses the definitions of Δ_k^N and $\bar{P}_0$:

$$\begin{aligned} \begin{bmatrix} p^0(K)\Delta_k^N & \bar{P}_0 \end{bmatrix} \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} - K\pi_0^N &= p^0(K)(\Delta_k^N \mathbb{D}) + (\bar{P}_0 E) - p^0(K)\pi_k^N(j^N \pi_0^N) \\ &= p^0(K)\pi_k^N(j^N \pi_0^N - \mathrm{id}) + p^0(K)\pi_k^N(\mathrm{id} - j^N \pi_0^N) = 0. \end{aligned}$$

The remaining checks involve the homotopy corrections:

$$\begin{aligned}
\pi_0^N j^N &= \mathrm{id} - 0, \\
j^N \pi_0^N &= \mathrm{id} + (j^N \pi_0^N - \mathrm{id}) \\
&= \mathrm{id} + \Delta_N^N \mathbb{D},
\end{aligned}$$

$$\begin{aligned}
\left(\mathrm{id} - \begin{bmatrix} p^0(K)\Delta_k^N & \bar{P}_0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_0 \end{bmatrix} - 0\right) K &= K - p^0(K)\Delta_k^N (P_1 K) - \bar{P}_0(P_0 K) \\
&= K - p^0(K)(\Delta_k^N \mathbb{D}) j^N - (\bar{P}_0 E) j^N \\
&= K - p^0(K)(j^k \pi_0^N - \pi_k^N) j^N - p^0(K) j^k = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix} - \begin{bmatrix} P_1 \\ P_0 \end{bmatrix} \begin{bmatrix} p^0(K)\Delta_k^N & \bar{P}_0 \end{bmatrix} - \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} \begin{bmatrix} -\Delta_N^N & 0 \end{bmatrix}\right) \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} \\
&\qquad = \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} - \begin{bmatrix} P_1 \\ P_0 \end{bmatrix} p^0(K) \left((j^k \pi_0^N - \pi_k^N) + \pi_k^N\right) - \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} (\mathrm{id} - j^N \pi_0^N) \\
&\qquad = \left(\begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} j^N - \begin{bmatrix} P_1 \\ P_0 \end{bmatrix} K\right) \pi_0^N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

(b) The next step is to eliminate the non-differential $E(v, v^1, \ldots, v^N) = 0$ constraint. By introducing a sub-bundle $\iota\colon \tilde{U} \hookrightarrow J^N V \to M$ such that $\tilde{U} = \ker E$. In general $\ker E$ need not be a vector bundle (the fiber ranks may be non-constant over M), hence $\tilde{U}$ might not exist as a bundle. However, from requirement in Definition 12(b), we know that the dimension of the solution space of $K[v] = 0$ is finite and locally constant, which by part (a) of our proof also applies to the solution space of $\mathbb{D}[v^{(N)}] = 0$ under the E-constraint. In fact, the dimension of the solution space on a neighborhood of $x \in M$ is bounded from above by $\dim \ker_x E$, simply because a $\mathbb{D}$-flat section is uniquely determined by its value at any one point. The only reason that an element $\tilde{u}_x \in \ker_x E$ might not correspond to a local solution is that there might exist some higher order differential consequence of $\mathbb{D}[v^{(N)}] = 0$ (or equivalently of $K[v] = 0$) that imposes further integrability conditions on $j^N v$, which $\tilde{u}_x$ may not satisfy. However, from the theory of formal integrability of PDEs (*Cartan-Kuranishi* theorem [30, Sec.7.4]), it is well known that past a certain finite differential order N', no further constraints on $j^N v$ will appear from considering $\partial^{N'} K[v] = 0$ or higher order differential consequences. Let us use this order N' (or any higher one) to influence the definition of the E-constraint that we introduced in part (a) of the proof. That is, we are free to assume that E has been chosen such that every element $\tilde{u}_x \in \ker_x E$ defines a unique local solution[7] of $K[v] = 0$ with $j^N v(x) = \tilde{u}_x$. In other words, $\dim \ker_x E$ is equal to the local dimension of the solution space about $x \in M$. But then, by the finite type hypothesis on $K[v] = 0$, we know that $\dim \ker_x E$ is locally constant, meaning that $\ker E$ is indeed a vector bundle, which we can denote by $\iota\colon \tilde{U} \to J^N V \to M$.

Since we are working locally, we are free to presume that there also exists a projection bundle map $q\colon J^N V \to \tilde{U}$ such that $q\iota = \mathrm{id}$. In the other direction, we have the identity $(\mathrm{id} - \iota q)\iota = 0$, which means that there must exist a bundle map $h\colon W' \to J^N V$ such that $\iota q = \mathrm{id} - hE$. Further, we can define a connection operator $\tilde{\mathbb{D}}$ on $\tilde{U}$ by the formula

$$\tilde{\mathbb{D}}\tilde{u} = q_1 \mathbb{D}\iota(\tilde{u}), \tag{122}$$

where we have introduced the convenient notation $q_1 = \mathrm{id} \otimes q\colon T^*M \otimes_M J^N V \to T^*M \otimes_M \tilde{U}$. We will use the same convention also for $E_1 = \mathrm{id} \otimes E$ and $\iota_1 = \mathrm{id} \otimes \iota$. We finally have all the

[7] The existence of such a solution is guaranteed by applying $\mathbb{D}$-parallel transport.

ingredients to exhibit the next equivalence

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
v^{(N)} \arrow[r, "\begin{bmatrix} \mathbb{D} \\ E \end{bmatrix}"] \arrow[d, shift right, "q"'] & \begin{bmatrix} v_1^{(N)} \\ w' \end{bmatrix} \arrow[l, dashed, bend left, "\begin{bmatrix} 0 & h \end{bmatrix}"] \arrow[d, shift right, "\begin{bmatrix} q_1 & -q_1\mathbb{D}h \end{bmatrix}"'] \\
\tilde{u} \arrow[r, "\tilde{\mathbb{D}}"'] \arrow[u, shift right, "\iota"'] & \tilde{u}_1 \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "\begin{bmatrix} \iota_1 \\ 0 \end{bmatrix}"']
\end{tikzcd} . \tag{123}
$$

The commutativity of the solid arrow squares we first need one more identity. Let us define

$$E' = h_1 E_1 \mathbb{D} - \mathbb{D} h E \tag{124}$$

and note that it is a non-differential operator (as follows from the basic properties of the connection operator). From its definition, $E'(v^{(N)}) = 0$ is also an integrability condition. But by the discussion from the preceding paragraph, $E(V^{(N)}) = 0$ already takes into account all possible integrability conditions. Hence, we must be able to factor $E' = h'E$ for some bundle map $h' \colon W' \to T^*M \otimes_M W'$, from which follows the identity

$$h_1 E_1 \mathbb{D} = \mathbb{D} h E + h' E = (\mathbb{D} h + h') E. \tag{125}$$

Hence,

$$
\begin{aligned}
\begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} \iota - \begin{bmatrix} \iota_1 \\ 0 \end{bmatrix} \tilde{\mathbb{D}} &= \begin{bmatrix} \mathbb{D}\iota - (\iota_1 q_1)\mathbb{D}\iota \\ E\iota \end{bmatrix} = \begin{bmatrix} (h_1 E_1 \mathbb{D})\iota \\ 0 \end{bmatrix} = \begin{bmatrix} (\mathbb{D}h + h')(E\iota) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\tilde{\mathbb{D}} q - \begin{bmatrix} q_1 & -q_1 \mathbb{D} h \end{bmatrix} \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} &= q_1 \mathbb{D}(\iota q) - q_1 \mathbb{D}(\mathrm{id} - hE) = 0.
\end{aligned}
$$

The first set of identities involving the homotopy corrections also easily follows from the definition of the ι and q bundle maps. We check the remaining ones by direct computation:

$$\mathrm{id} - \begin{bmatrix} q_1 & -q_1 \mathbb{D} h \end{bmatrix} \begin{bmatrix} \iota_1 \\ 0 \end{bmatrix} = \mathrm{id} - q_1 \iota_1 = 0,$$

$$
\begin{aligned}
\left(\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix} - \begin{bmatrix} \iota_1 \\ 0 \end{bmatrix} \begin{bmatrix} q_1 & -q_1 \mathbb{D} h \end{bmatrix} - \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} \begin{bmatrix} 0 & h \end{bmatrix} \right) &= \begin{bmatrix} \mathrm{id} - \iota_1 q_1 & \iota_1 q_1 \mathbb{D} h - \mathbb{D} h \\ 0 & \mathrm{id} - Eh \end{bmatrix} \\
&= \begin{bmatrix} h_1 E_1 & -(h_1 E_1 \mathbb{D}) h \\ 0 & \mathrm{id} - Eh \end{bmatrix} = \begin{bmatrix} h_1 E_1 & -(\mathbb{D} h + h') E h \\ 0 & \mathrm{id} - Eh \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{id} & (\mathbb{D} h + h') \\ 0 & \mathrm{id} \end{bmatrix} \begin{bmatrix} h_1 E_1 & -(\mathbb{D} h + h') \\ 0 & \mathrm{id} - Eh \end{bmatrix},
\end{aligned}
$$

where the last factor has the property

$$\begin{bmatrix} h_1 E_1 & -(\mathbb{D} h + h') \\ 0 & \mathrm{id} - Eh \end{bmatrix} \begin{bmatrix} \mathbb{D} \\ E \end{bmatrix} = \begin{bmatrix} (h_1 E_1 \mathbb{D} - \mathbb{D} h E - h' E) \\ E(\mathrm{id} - hE) \end{bmatrix} = \begin{bmatrix} 0 \\ (E\iota) q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(c) At this point, we know that the local solutions of $K[v] = 0$ are in bijection with the $\tilde{\mathbb{D}}$-flat local sections of $\tilde{U} \to M$. In principle, it is now sufficient to check that $\mathbb{D}$ is flat (if it were not flat, then the rank of $\tilde{U}$ could not coincide with the dimension of the local solution space of $K[v] = 0$, though the two do coincide by construction from part (b) of our proof). However, we will take a slightly indirect route and show a more general result, that will also be referred to in our discussion of the Killing equation in Section 3. Namely, provided the local solutions of $\tilde{\mathbb{D}}\tilde{u} = 0$ span a sub-bundle $\bar{\iota}\colon \bar{U} \hookrightarrow \tilde{U} \to M$, we will show that the restriction of $\tilde{D}$ to $\bar{U} \to M$ is flat and the original $\tilde{\mathbb{D}}\tilde{u} = 0$ equation is equivalent to the new $\bar{\mathbb{D}}\bar{u} = 0$ equation.

In our case, from the finite type assumption on $K[v] = 0$, we know that the local solution space has locally constant (finite) dimension, which is easily seen to be equivalent to the local solutions of $\tilde{\mathbb{D}}\tilde{u} = 0$ spanning a sub-bundle.

Now, under our hypotheses and since we are working locally, we can choose a frame on $\bar{U} \to M$ which corresponds to flat sections $\tilde{\boldsymbol{u}}_\beta$ on $\tilde{U}$. Namely, we define $\bar{\iota}(\bar{u}) = \bar{u}^\beta \tilde{\boldsymbol{u}}_\beta$. Locally, there also exists the projection bundle map $\bar{q}\colon \tilde{U} \to \bar{U} \to M$, which satisfies $\bar{q}\bar{\iota} = \mathrm{id}$ and hence acts as $\bar{q}\left(\bar{u}^\beta \tilde{\boldsymbol{u}}_\beta\right) = \bar{u}$. Hence, using again the notation $\bar{\iota}_1 = 1 \otimes \bar{\iota}$, we get the identity

$$\tilde{\mathbb{D}}[\bar{\iota}(\bar{u})] = d\bar{u}^\beta \otimes \tilde{\boldsymbol{u}}_\beta + \bar{u}^\beta(\tilde{\mathbb{D}}\tilde{\boldsymbol{u}}_\beta) = d\bar{u}^\beta \otimes \tilde{\boldsymbol{u}}_\beta = \iota_1\left(\bar{\mathbb{D}}\bar{u}\right), \tag{126}$$

where d is simply the exterior derivative acting on the scalars $\bar{u}^\beta$ and we have defined $\bar{\mathbb{D}}$ to act on the frame components of $\bar{u}$ as $(\bar{\mathbb{D}}\bar{u})^\beta = d\bar{u}^\beta$. The operator $\bar{\mathbb{D}}$ is clearly a flat connection on the bundle $\bar{U} \to M$. It remains only to exhibit the equivalence between the equations $\tilde{\mathbb{D}}\tilde{u} = 0$ and $\bar{\mathbb{D}}\bar{u} = 0$.

As we already discussed in part (b) of our proof, when $\bar{\iota}$ is not surjective, there must be some integrability conditions that follow from the differential consequences of $\tilde{\mathbb{D}}\tilde{u} = 0$. In other words, there exists a differential operator $\bar{\lambda}$ such that $\bar{\lambda}\tilde{\mathbb{D}}$ is a non-differential operator, satisfying $(\bar{\lambda}\tilde{\mathbb{D}})\iota = 0$, with $\iota\colon \bar{U} \hookrightarrow \ker \bar{\lambda}\tilde{\mathbb{D}}$ actually being an isomorphism. Again, as before, this means that there exists an operator $\bar{h}$ such that $\bar{\iota}\bar{q} = \bar{h}(\bar{\lambda}\tilde{\mathbb{D}})$. Recalling again the notation, $\tilde{U}_1 = T^*M \otimes_M \tilde{U}$ and $\bar{U}_1 = T^*M \otimes_M \bar{U}$, as well as $\bar{q}_1 = \mathrm{id} \otimes \bar{q}$ and $\bar{\iota}_1 = \mathrm{id} \otimes \bar{\iota}$. With that in mind, the desired equivalence is explicitly given by the diagram

$$\begin{tikzcd}
\tilde{u} \arrow[r, "\tilde{\mathbb{D}}"] \arrow[d, shift right, "\bar{q}"'] & \tilde{u}_1 \arrow[l, dashed, bend left, "\bar{h}\bar{\lambda}"] \arrow[d, shift right, "\bar{q}_1(\mathrm{id}-\tilde{\mathbb{D}}\bar{h}\bar{\lambda})"'] \\
\bar{u} \arrow[r, "\bar{\mathbb{D}}"'] \arrow[u, shift right, "\bar{\iota}"'] & \bar{u}_1 \arrow[l, dashed, bend right, "0"'] \arrow[u, shift right, "\bar{\iota}_1"']
\end{tikzcd} . \tag{127}$$

The arguments to check all the required identities are similar to those in part (b). We check that the solid arrows form commutative squares by direct computation:

$$\begin{aligned}
&\bar{\mathbb{D}}\bar{q} = \bar{q}_1\tilde{\mathbb{D}}(\bar{\iota}\bar{q}) = \bar{q}_1\tilde{\mathbb{D}} - \bar{q}_1\tilde{\mathbb{D}}\bar{h}\bar{\lambda}\tilde{\mathbb{D}} = \bar{q}_1(\mathrm{id} - \tilde{\mathbb{D}}\bar{h}\bar{\lambda})\tilde{\mathbb{D}}, \\
&\tilde{\mathbb{D}}\bar{\iota} = \bar{\iota}_1\bar{\mathbb{D}}.
\end{aligned}$$

To check some of the identities with the homotopy corrections, we need one more identity. For ease of notation, define $\bar{E} = \bar{\lambda}\tilde{\mathbb{D}}$, which by assumption is a non-differential operator which incorporates all the integrability conditions of the equation $\tilde{\mathbb{D}}\tilde{u} = 0$. Just as in part (b), since all integrability conditions must factor through $\bar{E}$, there must exist an operator $\bar{h}'$ such that $\bar{h}_1\bar{E}_1\tilde{\mathbb{D}} - \tilde{\mathbb{D}}\bar{h}\bar{E} = \bar{h}'\bar{E}$, which implies the identity

$$[\bar{h}_1\bar{E}_1 - (\tilde{\mathbb{D}}\bar{h} + \bar{h}')\bar{\lambda}]\tilde{\mathbb{D}} = 0. \tag{128}$$

Hence, we can verify that

$$
\begin{aligned}
\mathrm{id} - \overline{q\iota} &= 0, \\
\mathrm{id} - \overline{\iota q} - \bar{h}\bar{\lambda}\tilde{\mathbb{D}} &= 0, \\
(\mathrm{id} - \bar{q}_1(\mathrm{id} - \tilde{\mathbb{D}}\bar{h}\bar{\lambda})\bar{\iota}_1)\bar{\mathbb{D}} &= (\mathrm{id} - (\bar{q}_1\bar{\iota}_1))\bar{\mathbb{D}} + \bar{q}_1\tilde{\mathbb{D}}\bar{h}\bar{\lambda}(\bar{\iota}_1\bar{\mathbb{D}}) \\
&= \bar{q}_1\tilde{\mathbb{D}}\bar{h}((\bar{\lambda}\tilde{\mathbb{D}})\bar{\iota}) = 0, \\
(\mathrm{id} - \bar{\iota}_1\bar{q}_1(\mathrm{id} - \tilde{\mathbb{D}}\bar{h}\bar{\lambda}) - \tilde{\mathbb{D}}\bar{h}\bar{\lambda})\tilde{\mathbb{D}} &= (\mathrm{id} - \bar{\iota}_1\bar{q}_1)(\mathrm{id} - \tilde{\mathbb{D}}\bar{h}\bar{\lambda})\tilde{\mathbb{D}} \\
&= \bar{h}_1\bar{E}_1\tilde{\mathbb{D}} - \bar{h}_1\bar{E}_1\tilde{\mathbb{D}}(\bar{h}\bar{\lambda}\tilde{\mathbb{D}}) \\
&= \bar{h}_1\bar{E}_1\tilde{\mathbb{D}} - (\bar{h}_1\bar{E}_1\tilde{\mathbb{D}})(\mathrm{id} - \overline{\iota q}) \\
&= (\tilde{\mathbb{D}}\bar{h} + \bar{h}')(\bar{E}\bar{\iota})\bar{q} = 0.
\end{aligned}
$$

This concludes the proof. □

Since according to Definition 12 the flat section equation $\mathbb{D}v = 0$ for a flat connection $\mathbb{D}$ is itself of finite type (with $N = 0$), Proposition 13 shows that Definitions 7 and 12 are clearly equivalent and can be used interchangeably.

The reader might notice that the structure of parts (b) and (c) in the proof of Proposition 13 is rather similar. The reason that we have included both of them in detail is that part (c) can basically be read independently and establishes the following (of course also well-know) more specific result:

Lemma 14. *Let $\mathbb{D}$ be a connection on a vector bundle V. If the local solutions of the flat section equation $\mathbb{D}v = 0$ span a sub-bundle $W \hookrightarrow V$, then the restriction $\bar{\mathbb{D}} = \mathbb{D}|_W$ of $\mathbb{D}$ to W is a flat connection on W. Moreover, $\mathbb{D}$ is equivalent to $\bar{\mathbb{D}}$ in the sense of one operator complexes (Definition 1).*

B Notation reference

B.1 Constant curvature spacetime

α	curvature constant	Section 3.1
C_i, $C_0 = K$	Calabi compatibility complex for Killing operator K	(32)
$S \odot T$	Kulkarni-Nomizu product	(28)

B.2 FLRW spacetimes

$(M, g) = (I \times F, -dt^2 + f^2\tilde{g}^F)$	FLRW geometry, with scale factor f	Section 3.2
α	curvature constant	Section 3.2
U_a	unit covector normal to F factor	Section 3.2
$R_{abcd}, R_{ab}, \mathcal{R}$	background Riemann, Ricci and scalar curvatures on (M, g)	(41)
$\partial_t, \tilde{\nabla}$	derivative operators extended from I and F to M	Section 3.2
$\tilde{\Delta}, \tilde{\operatorname{div}}, \tilde{\operatorname{tr}}$	Laplacian, divergence and trace extended from F to M	(53)
$v_a = AfU_b + f^2\tilde{X}_a$	covector parametrization, $U^a\tilde{X}_a = 0$	(34)
$h_{ab} = pU_aU_b - 2f^2U_{(a}\tilde{Y}_{b)} + f^2\tilde{Z}_{ab}$	symmetric 2-tensor parametrization, $U^a\tilde{Y}_a = 0 = U^a\tilde{Z}_{ab}$	(36)
K	Killing operator on (M, g)	(37)
$\tilde{C}_i$, $\tilde{C}_0 = \tilde{K}$	extension of Calabi and Killing operators from (F, g^F) to M	(40)
$J, \tilde{J}$	operator to extract $A = J \circ K[v]$, subcomponent $\tilde{J}$	(44), (47), (51)
$H_J, \tilde{H}_J$	factorization of $J, \tilde{J}$	(49), (48), (52)

B.3 Schwarzschild-Tangherlini spacetimes

$(\bar{\mathcal{M}},\bar{g}) = (\mathcal{M}\times\mathcal{S}, g + r^2\Omega), \bar{\nabla}_\mu$	generalized Schwarzschild-Tangherlini (gST) spacetime, covariant derivative	Section 3.3
$\bar{R}_{abcd}, \bar{R}_{ab}, \bar{\mathcal{R}}$	background Riemann, Ricci and scalar curvatures on $(\bar{\mathcal{M}},\bar{g})$	(76), (77)
$\bar{T}_{abcd}$	Λ-shifted background Riemann curvature on $(\bar{\mathcal{M}},\bar{g})$	(93)
$(\mathcal{S},\Omega), D_A$	constant curvature factor, covariant derivative extended to $\bar{\mathcal{M}}$	Section 3.3
$(g,\Omega), \nabla_a$	radio-temporal factor, covariant derivative extended to $\bar{\mathcal{M}}$	Section 3.3
$R_{abcd}, R_{ab}, \mathcal{R}$	background Riemann, Ricci and scalar curvatures on $(\mathcal{M},g)$	(60)
$-f(r)$	the $\bar{g}_{tt}$ metric component, its derivatives	(62), (64)
$f_1(r) = rf'(r), f_2(r) = r'f_1(r)$	derivatives of $f(r)$	(67), (68)
M, Λ, α	mass, cosmological, curvature constants	(64)
$t_a = -f dt_a$	timelike Killing covector on $(\mathcal{M},g)$	Section 3.3
$r, r_a = dr_a$	radial coordinate and covector on $(\mathcal{M},g)$	Section 3.3
$v_\mu \to \begin{bmatrix} u_t f dt_a + u_r dr_a \\ r(rX_A) \end{bmatrix}$	covector parametrization on $\bar{\mathcal{M}}$	(104)
$h_{\mu\nu} \to \begin{bmatrix} p\, r_a r_b - 2t_{(a}w_{b)} & r(rY_{aB}) \\ r(rY_{bA}) & r^2 Z_{AB} \end{bmatrix}$	symmetric 2-tensor parametrization on $\bar{\mathcal{M}}$	(104)
$\bar{K}$	Killing operator on $(\bar{\mathcal{M}},\bar{g})$	(105)
C_i, $C_0 = K$	extension of Calabi and Killing operators from $(\mathcal{S},\Omega)$ to $\bar{\mathcal{M}}$	(105)
J_1	operator to extract $u_r = J_1 \circ \bar{K}[v]$	(99)
J_2	operator to extract $\nabla_a \frac{v_B}{r} = J_2 \circ \bar{K}[v]_{aB}$	(100)
$H_{J_1}, \tilde{H}_{J_1}$	factorization of J_1	(110)
$H_{J_2}, \tilde{H}_{J_2}$	factorization of parts of J_2	(113), (112)

References

[1] S. Aksteiner and T. Bäckdahl, "All local gauge invariants for perturbations of the Kerr spacetime," *Physical Review Letters* **121** (2018) 051104, `arXiv:1803.05341`.

[2] L. Andersson, S. Aksteiner, T. Bäckdahl, I. Khavkine, and B. Whiting, "Compatibility complex for black hole spacetimes," 2018. In preparation.

[3] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, "Geometry of the 2+1 black hole," *Physical Review D* **48** (1993) 1506–1525, `arXiv:gr-qc/9302012`.

[4] J. M. Bardeen, "Gauge-invariant cosmological perturbations," *Physical Review D* **22** (1980) 1882–1905.

[5] M. L. Bedran, M. O. Calvão, F. M. Paiva, and I. Damião Soares, "Taub's plane-symmetric vacuum spacetime reexamined," *Physical Review D* **55** (1997) 3431–3439, arXiv:gr-qc/9608058.

[6] Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, and D. Robertz, "The MAPLE package Janet: I. polynomial systems. II. linear partial differential equations," in *Proceedings of the 6th International Workshop on Computer Algebra in Scientific Computing, Passau (Germany)*, V. G. Ganzha, E. W. Mayr, and E. V. Vorozhtsov, eds., pp. 31–54. Institut für Informatik, Technische Universität München, Garching, 2003. https://wwwb.math.rwth-aachen.de/Janet/.

[7] M. Cahen and N. Wallach, "Lorentzian symmetric spaces," *Bulletin of the American Mathematical Society* **76** (1970) 585–591.

[8] E. Calabi, "On compact, Riemannian manifolds with constant curvature. I," in *Differential Geometry*, C. B. Allendoerfer, ed., vol. 3 of *Proceedings of Symposia in Pure Mathematics*, pp. 155–180. AMS, Providence, RI, 1961.

[9] G. Canepa, C. Dappiaggi, and I. Khavkine, "IDEAL characterization of isometry classes of FLRW and inflationary spacetimes," *Classical and Quantum Gravity* **35** (2018) 035013, arXiv:1704.05542.

[10] G. Dotti, "Nonmodal linear stability of the Schwarzschild black hole," *Physical Review Letters* **112** (2014) 191101, arXiv:1307.3340.

[11] M. Eastwood, "Notes on projective differential geometry," in *Symmetries and Overdetermined Systems of Partial Differential Equations*, M. Eastwood and W. Miller, eds., vol. 144 of *The IMA Volumes in Mathematics and its Applications*, ch. 3, pp. 41–60. Springer, New York, NY, 2008.

[12] J. J. Ferrando and J. A. Sáez, "An intrinsic characterization of the Schwarzschild metric," *Classical and Quantum Gravity* **15** (1998) 1323–1330.

[13] M. B. Fröb, T.-P. Hack, and A. Higuchi, "Compactly supported linearised observables in single-field inflation," *Journal of Cosmology and Astroparticle Physics* **2017** (2017) 043, arXiv:1703.01158.

[14] M. B. Fröb, T.-P. Hack, and I. Khavkine, "Approaches to linear local gauge-invariant observables in inflationary cosmologies," *Classical and Quantum Gravity* **35** (2018) 115002, arXiv:1801.02632.

[15] J. Gasqui and H. Goldschmidt, "Déformations infinitésimales des espaces riemanniens localement symétriques. I," *Advances in Mathematics* **48** (1983) 205–285.

[16] J. Gasqui and H. Goldschmidt, "Complexes of differential operators and symmetric spaces," in *Deformation Theory of Algebras and Structures and Applications*, M. Hazewinkel and M. Gerstenhaber, eds., vol. 247 of *NATO ASI Series*, pp. 797–827. Kluwer, Dordrecht, 1988.

[17] R. Geroch, "Limits of spacetimes," *Communications in Mathematical Physics* **13** (1969) 180–193.

[18] H. Goldschmidt, "Existence theorems for analytic linear partial differential equations," *The Annals of Mathematics* **86** (1967) 246–270.

[19] A. Higuchi, "Equivalence between the Weyl-tensor and gauge-invariant graviton two-point functions in Minkowski and de Sitter spaces," 2013. arXiv:1204.1684.

[20] A. Ishibashi and H. Kodama, "Stability of higher-dimensional Schwarzschild black holes," *Progress of Theoretical Physics* **110** (2003) 901–919, arXiv:hep-th/0305185.

[21] J. Jezierski, "'Peeling property' for linearized gravity in null coordinates," *Classical and Quantum Gravity* **19** (2002) 2463–2490, arXiv:gr-qc/0111030.

[22] I. Khavkine, "Covariant phase space, constraints, gauge and the Peierls formula," *International Journal of Modern Physics A* **29** (2014) 1430009, arXiv:1402.1282.

[23] I. Khavkine, "The Calabi complex and Killing sheaf cohomology," *Journal of Geometry and Physics* **113** (2017) 131–169, arXiv:1409.7212.

[24] I. Khavkine, "IDEAL characterization of higher dimensional spherically symmetric black holes," 2018. arXiv:1807.09699.

[25] H. Kodama and A. Ishibashi, "A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions," *Progress of Theoretical Physics* **110** (2003) 701–722, arXiv:hep-th/0305147.

[26] H. Kodama, A. Ishibashi, and O. Seto, "Brane world cosmology: Gauge-invariant formalism for perturbation," *Physical Review D* **62** (2000) 064022, arXiv:hep-th/0004160.

[27] F. S. N. Lobo and J. P. Mimoso, "Possibility of hyperbolic tunneling," *Physical Review D* **82** (2010) 044034, arXiv:0907.3811.

[28] K. Martel and E. Poisson, "Gravitational perturbations of the Schwarzschild spacetime: A practical covariant and gauge-invariant formalism," *Physical Review D* **71** no. 10, (2005) 104003, arXiv:gr-qc/0502028.

[29] B. O'Neill, *Semi-Riemannian Geometry With Applications to Relativity*, vol. 103 of *Pure and Applied Mathematics*. Elsevier Science, 1983.

[30] W. M. Seiler, *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*, vol. 24 of *Algorithms and Computation in Mathematics*. Springer, 2010.

[31] A. G. Shah, B. F. Whiting, S. Aksteiner, L. Andersson, and T. Bäckdahl, "Gauge-invariant perturbations of Schwarzschild spacetime," 2016. arXiv:1611.08291.

[32] D. C. Spencer, "Overdetermined systems of linear partial differential equations," *Bulletin of the American Mathematical Society* **75** (1969) 179–240.

[33] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*. Cambridge University Press, Cambridge, 2003.

[34] J. M. Stewart and M. Walker, "Perturbations of Space-Times in general relativity," *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **341** (1974) 49–74.

[35] A. H. Taub, "Empty space-times admitting a three parameter group of motions," *The Annals of Mathematics* **53** (1951) 472–490.

[36] R. M. Wald, *General Relativity*. University of Chicago Press, Chicago, 1984.

[37] C. A. Weibel, *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[38] B. F. Whiting and L. R. Price, "Metric reconstruction from Weyl scalars," *Classical and Quantum Gravity* **22** (2005) S589–S604.