A NEW NUCLEOLUS-LIKE METHOD TO FIND THE PRIORITY VECTOR OF A PAIRWISE COMPARISON MATRIX

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Abstract
Given n objects, such as variants from which the best one with respect to some criterion is to be selected or which are to be ranked from the best one to the worst one, a pairwise comparison matrix is obtained when the decision-maker considers various pairs of the objects and judges the objects in the pair relatively with respect to the criterion. In this paper, we follow the general unified framework by Cavallo and D'Apuzzo (2009) and assume that the relative importance of the objects is evaluated by using elements of a given divisible Abelian linearly ordered group (alo-group). To obtain the priority vector of the resulting pairwise comparison matrix, we get inspired by the notion of nucleolus (Schmeidler, 1969), which is a well-known solution concept of a cooperative game with transferable utility, and propose a new nucleolus-like method to find the priority vector of the given pairwise comparison matrix. The method utilizes the theory of linear programming in abstract spaces (Bartl, 2007).

Keywords: *pairwise comparison matrix, priority vector, alo-group, nucleolus, cooperative TU-game*
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1. Introduction

Let a decision-maker (DM) consider a set $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ of objects and let the DM wish to assess the given objects with respect to some criterion: The outcome of this assessment consists in *priorities* $w_1, w_2, \dots, w_n$ which reflect the importance of the objects $c_1, c_2, \dots, c_n$, respectively, with respect to the criterion. The priorities are traditionally positive real numbers, often normalized so that they sum up to one (i.e. $w_1, w_2, \dots, w_n > 0$ and $w_1 + w_2 + \cdots + w_n = 1$). It is, however, a difficult task for a human to assess the objects and assign the priorities to them at once, while it is easier to judge two objects (i.e. a pair of them) with respect to the criterion (Thurstone, 1927, 1994; see Saaty, 1977, and the references therein for further details). Using this idea, the multiplicative scale is prominently used in the pairwise comparison (Saaty, 1977, 1980): for each pair c_i and c_j of the objects, let the DM judge *how many times* the object c_i is better than c_j with respect to the given criterion for $i, j = 1, 2, \dots, n$. We thus obtain an $n \times n$ pairwise comparison (PC) matrix $\boldsymbol{A}$ of positive numbers whose entry a_{ij} is the DM's aforementioned judgement how many times c_i is better than c_j with respect to the criterion for $i, j = 1, 2, \dots, n$. Moreover, we assume that the PC matrix A is *reciprocal*; that is, it holds $a_{ji} = 1/a_{ij}$, whence it follows that $a_{ii} = 1$, i.e. the object c_i is of the same importance when compared with itself, for $i, j = 1, 2, \dots, n$.

Given an $n \times n$ reciprocal pairwise comparison matrix $\boldsymbol{A}$, the purpose is to find the priorities $w_1, w_2, \dots, w_n$ so that $a_{ij} \approx w_i/w_j$ for $i, j = 1, 2, \dots, n$. This task is central, e.g., in the Analytic Hierarchy Process (Saaty, 1980), which makes it possible to solve large multi-criteria decision-making problems in an organized way. There are many methods to derive the *priority vector* $\boldsymbol{w} = (w_1, w_2, \dots, w_n)^{\mathrm{T}}$, consisting of the priorities $w_1, w_2, \dots, w_n$, of the given reciprocal matrix $\boldsymbol{A}$ if it consists of positive real numbers (Choo & Wedley, 2004; Lin, 2007; Ramík, 2020), including the arithmetic mean method (AMM), least squares method (LSM), chi-square method (CSM), logarithmic least squares method (LLSM) or geometric mean method (GMM), weighted least squares method (WLSM), and Saaty's eigenvector method (EVM). Letting $\boldsymbol{e} = (1, 1, \dots, 1)^{\mathrm{T}}$ be the vector of n ones and denoting by

$\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ and by $\mathbb{R}_{++}$ the columns of $\boldsymbol{A}$ and the set of the positive real numbers, respectively, recall that $\boldsymbol{w} = \sum_{j=1}^{n} \boldsymbol{a}_j / n\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a}_j$, i.e. the arithmetic mean of the additively normalized columns of $\boldsymbol{A}$, in AMM, while the priority vector $\boldsymbol{w}$ is obtained as an optimal solution to the optimization problem

$$\text{minimize } \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - w_i/w_j)^2 \qquad \text{subject to } w_1, w_2, \ldots, w_n \in \mathbb{R}_{++}, \tag{1}$$

$$\text{minimize } \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - w_i/w_j)^2/(w_i/w_j) \qquad \text{subject to } w_1, w_2, \ldots, w_n \in \mathbb{R}_{++}, \tag{2}$$

$$\text{minimize } \sum_{i=1}^{n}\sum_{j=1}^{n}(\log a_{ij} - \log(w_i/w_j))^2 \qquad \text{subject to } w_1, w_2, \ldots, w_n \in \mathbb{R}_{++}, \tag{3}$$

$$\text{minimize } \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}w_j - w_i)^2 \qquad \text{subject to } w_1, w_2, \ldots, w_n \in \mathbb{R}_{++}, \tag{4}$$

$$\text{maximize } \lambda_{\max} \qquad \text{subject to } \boldsymbol{A}\boldsymbol{w} = \lambda_{\max}\boldsymbol{w}, \quad \lambda_{\max} \in \mathbb{R}_{++} \quad \text{and} \quad \boldsymbol{w} \in \mathbb{R}_{++}^{n} \tag{5}$$

in LSM, CSM, LLSM, WLSM, and EVM, respectively, see Bozóki (2008), Jensen (1983), Rabinowitz (1976), Chu et al. (1979), and Saaty and Vargas (1984). While the optimal solution to (2)–(5) can be made unique by requiring the priority vector to be additively normalized, i.e. by adding the constraint $\sum_{i=1}^{n} w_i = 1$ to each of the optimization problems (2)–(5), problem (1) can have several optimal solutions, even if normalization is imposed, in general (Jensen, 1983). Recall that

$$w_i = \sqrt[n]{a_{i1}a_{i2}\ldots a_{in}} \qquad \text{for} \quad i = 1, 2, \ldots, n \tag{6}$$

and

$$\boldsymbol{w} = \lim_{k\to\infty} \boldsymbol{A}^k\boldsymbol{e}/\|\boldsymbol{A}^k\boldsymbol{e}\| \qquad \text{with} \qquad \lambda_{\max} = \lim_{k\to\infty} \left\|\boldsymbol{A}^{k+1}\boldsymbol{e}/\|\boldsymbol{A}^k\boldsymbol{e}\|_{\infty}\right\|_{\infty} \tag{7}$$

is an optimal solution to (3) and (5), respectively, where $\|\boldsymbol{v}\|$ is any vector norm of a vector $\boldsymbol{v} \in \mathbb{R}^n$, such as the additive one ($\|\boldsymbol{v}\| = |v_1| + |v_2| + \cdots + |w_n|$, so $\|\boldsymbol{A}^k\boldsymbol{e}\| = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^k\boldsymbol{e}$), and $\|\boldsymbol{v}\|_{\infty}$ is the Chebyshev vector norm ($\|\boldsymbol{v}\|_{\infty} = \max\{|v_1|, |v_2|, \ldots, |w_n|\}$) of the vector $\boldsymbol{v} \in \mathbb{R}^n$. Out of the aforementioned methods, EVM and LLSM, whose alternative name, GMM, is justified by equation (6), are perhaps the most popular methods of deriving the priority vector of a reciprocal pairwise comparison matrix $\boldsymbol{A}$.

2. The general unified framework for pairwise comparison matrices

Cavallo and D'Apuzzo (2009) note that various meanings can be given to the entry a_{ij} of the pairwise comparison matrix $\boldsymbol{A}$ when comparing two objects c_i and c_j pairwise, including:

- multiplicative meaning, when $a_{ij} \in \mathbb{R}_{++}$ represents the preference ratio of c_i over c_j with respect to the criterion; the value $a_{ij} > 1$ and $a_{ij} = 1$ means strict preference of c_i to c_j and indifference, respectively; and reciprocity means $a_{ji} = 1/a_{ij}$;
- additive meaning, when $a_{ij} \in \mathbb{R}$ represents the difference of preferences between c_i and c_j with respect to the criterion; the value $a_{ij} > 0$ and $a_{ij} = 0$ means strict preference of c_i to c_j and indifference, respectively; and reciprocity means $a_{ji} = 0 - a_{ij}$;
- fuzzy meaning, when $a_{ij} \in (0, 1)$ represents the truth value of the statement that c_i is preferred to c_j with respect to the criterion; the value $a_{ij} > 0.5$ and $a_{ij} = 0.5$ means strict preference of c_i to c_j and indifference, respectively; and reciprocity means $a_{ji} = 1 - a_{ij}$.

To unify these various approaches, Cavallo and D'Apuzzo (2009) recall the notion of a *linearly ordered commutative group*, i.e. *Abelian linearly ordered group* (*alo-group* for short), which is a quintuple $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ where G is a set, $\odot$ and $^{-1}$ is a binary and unary, respectively, operation on G, $e \in G$ is a neutral element, and $\leq$ is a binary relation of linear ordering of G such that $(G, \odot, ^{-1}, e)$ is a commutative (i.e. Abelian) group and $\leq$ is compatible with $\odot$; that is, associativity and commutativity holds ($a \odot (b \odot c) = (a \odot b) \odot c$ and $a \odot b = b \odot a$ for every $a, b, c \in G$), existence of inverse element is provided and e is the neutral element of the group ($a \odot a^{-1} = e$ and $a \odot e = a$ for every

$a \in G$), and $a \leq b$ implies $a \odot c \leq b \odot c$ for every $a, b, c \in G$. Let us denote $a \div b = a \odot b^{-1}$ for convenience for every $a, b \in G$. Moreover, for any integer $n \in \mathbb{Z}$ and for any $x \in G$, let $x^n = x \odot x \odot \cdots \odot x$ if $n > 0$, where the element x is multiplied $(n-1)$-times with itself, let $x^n = e$ if $n = 0$, and let $x^n = (x^{-n})^{-1} = e \div (x^{-n})$ if $n < 0$. Recall that the alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ is *divisible* iff, for any $n \in \mathbb{Z}$ and for any $a \in G$, there exists an $x \in G$ such that $x^n = a$, which intuitively means that $x = \sqrt[n]{a}$. Now, the popular examples of divisible alo-groups include:

- the multiplicative group of the positive real numbers $\mathcal{R}_{++} = (\mathbb{R}_{++}, \cdot, ^{-1}, 1, \leq)$, where $\cdot$, $^{-1}$, and $\leq$ is the usual multiplication, reciprocation ($a^{-1} = 1/a$), and usual linear ordering of the real numbers, respectively, and the number 1 is neutral with respect to multiplication;
- the additive group of the real numbers $\mathcal{R} = (\mathbb{R}, +, -, 0, \leq)$, where $+$, $-$, and $\leq$ is the usual addition, additive inverse ($-a = 0 - a$), and usual linear ordering of the real numbers, respectively, and the number 0 is neutral with respect to addition;
- the fuzzy multiplicative alo-group $\mathcal{F} = ((0,1), \odot, ^{-1}, 0.5, \leq)$, where $a \odot b = ab/(ab + (1-a)(1-b))$ for any a and b of the open interval $(0,1)$, and $a^{-1} = 1 - a$ for every $a \in (0,1)$, the number 0.5 is neutral with respect to $\odot$, and $\leq$ is the usual ordering of the real numbers.

Clearly, these examples of divisible alo-groups correspond, respectively, to the aforegiven interpretations (meanings) of the entry a_{ij} of a pairwise comparison matrix $\boldsymbol{A}$. In this paper, we follow the general unified framework introduced by Cavallo and D'Apuzzo (2009): from now on, we assume that the relative importance of the objects c_i and c_j is evaluated by using elements of the given divisible alo-group $\mathcal{G}$; that is, we have $a_{ij} \in G$ for every $i, j = 1, 2, \ldots, n$, or $\boldsymbol{A} \in G^{n \times n}$.

Now, given the reciprocal pairwise comparison matrix $\boldsymbol{A} \in G^{n \times n}$, the task is to find priorities $w_1, w_2, \ldots, w_n \in G$ so that $a_{ij} \approx w_i \div w_j$ for $i, j = 1, 2, \ldots, n$.

However, reviewing the aforementioned traditional methods of finding the priority vector $\boldsymbol{w}$ of a pairwise comparison matrix $\boldsymbol{A}$, we realize, that AMM, LSM, CSM, LLSM, WLSM, and EVM *cannot be applied* due to intrinsic reasons of these methods: each of them needs two binary operations (both addition $+$ and multiplication $\cdot$) to compute the priority vector, but only *one* binary operation (multiplication $\odot$) is available within the divisible alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$. Since $\mathcal{G}$ is divisible, we can let $w_i = \sqrt[n]{a_{i1} \odot a_{i2} \odot \cdots \odot a_{in}}$ for $i = 1, 2, \ldots, n$; that is, we can apply GMM, but we lack its LLSM interpretation (3) is in the present setting of the divisible alo-group $\mathcal{G}$.

To conclude, GMM – see Csató (2019) for its characterization – appears to be the only known method that can be applied to compute a priority vector $\boldsymbol{w}$ of the given pairwise comparison matrix $\boldsymbol{A}$ in the setting of the divisible alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$. This fact justifies the purpose of this paper, which is to propose another – lexicographic or nucleolus-like – method to compute the priority vector of a pairwise comparison matrix $\boldsymbol{A}$.

3. The lexicographic ordering

Let $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ be a (divisible) alo-group and let n be a natural number. We now recall the notion of lexicographic ordering of the space G^{n^2}, which will be used in the formulation of optimization problem (8) below. Let $\boldsymbol{\theta}', \boldsymbol{\theta}'' \in G^{n^2}$ be two n^2-component vectors consisting of the entries $\theta'_1, \theta'_2, \ldots, \theta'_{n^2} \in G$ and $\theta''_1, \theta''_2, \ldots, \theta''_{n^2} \in G$, respectively. We say that vector $\boldsymbol{\theta}'$ *is lexicographically less than or equal to* vector $\boldsymbol{\theta}''$ and we write $\boldsymbol{\theta}' \preccurlyeq_{\text{lex}} \boldsymbol{\theta}''$ iff there exists an index $k_0 \in \{1, 2, \ldots, n^2, n^2 + 1\}$ such that $\theta'_k = \theta''_k$ for all $k = 1, 2, \ldots, k_0 - 1$ and $\theta'_{k_0} < \theta''_{k_0}$ if $k_0 \leq n^2$. It is easy to see that, as the ordering $\leq$ of the alo-group $\mathcal{G}$ is linear, the lexicographic ordering $\preccurlyeq_{\text{lex}}$ of the space G^{n^2} is also linear.

4. A new lexicographic (*nucleolus-like*) method to compute the priority vector of a PC matrix

Given the divisible alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ and having the information on the relative importance of the n objects $c_1, c_2, \ldots, c_n$ with respect to the criterion in the form of a pairwise comparison matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{with} \quad a_{ij} \in G \quad \text{and} \quad a_{ji} = a_{ij}^{-1} \quad \text{for} \quad i,j = 1,2,\dots,n\,,$$

our goal is to find a priority vector $\boldsymbol{w} \in G^n$, i.e. the weights $w_1, w_2, \dots, w_n \in G$ of the objects, in such a way that $a_{ij} \approx w_i \div w_j$ for $i,j = 1,2,\dots,n$.

Inspired by the concept of nucleolus from the cooperative game theory (Schmeidler, 1969), for any reciprocal matrix $\boldsymbol{A} \in G^{n\times n}$ and for any vector $\boldsymbol{w} \in G^n$, we make up the vector $\boldsymbol{\theta}(\boldsymbol{w}) \in G^{n^2}$ as follows: find pairs of indices (i_1,j_1), (i_2,j_2), …, (i_{n^2},j_{n^2}) so that

$$\{\ (i_1,j_1),\ (i_2,j_2),\ \dots,\ (i_{n^2},j_{n^2})\ \} \quad = \quad \begin{Bmatrix} (1,1), & (1,2), & \dots, & (1,n), \\ (2,1), & (2,2), & \dots, & (2,n), \\ \vdots & \vdots & \ddots & \vdots \\ (n,1), & (n,2), & \dots, & (n,n) \end{Bmatrix}$$

and

$$\left[a_{i_1j_1} \div (w_{i_1} \div w_{j_1})\right] \geq \left[a_{i_2j_2} \div (w_{i_2} \div w_{j_2})\right] \geq \dots \geq \left[a_{i_{n^2}j_{n^2}} \div (w_{i_{n^2}} \div w_{j_{n^2}})\right],$$

and let

$$\boldsymbol{\theta}(\boldsymbol{w}) = \begin{pmatrix} a_{i_1j_1} \div (w_{i_1} \div w_{j_1}) \\ a_{i_2j_2} \div (w_{i_2} \div w_{j_2}) \\ \vdots \\ a_{i_{n^2}j_{n^2}} \div (w_{i_{n^2}} \div w_{j_{n^2}}) \end{pmatrix}.$$

In other words, we sort all the deviations $[a_{ij} \div (w_i \div w_j)]$ of the ratio $(w_i \div w_j)$ of the weights w_i and w_j from the value a_{ij} provided by the DM in the descending order, and make up a single n^2-component column vector $\boldsymbol{\theta}(\boldsymbol{w})$ out of these deviations.

Finally, solve the unconstrained problem

$$\text{lexicographically minimize } \boldsymbol{\theta}(\boldsymbol{w}) \qquad \text{subject to} \quad \boldsymbol{w} \in G^n\,. \tag{8}$$

The idea behind (8) is to minimize the largest deviation $[a_{i_1j_1} \div (w_{i_1} \div w_{j_1})]$ first; when it cannot be made smaller, we minimize the second largest deviation $[a_{i_2j_2} \div (w_{i_2} \div w_{j_2})]$, and so on, cf. the main idea behind the concept of nucleolus (Schmeidler, 1969). In this sense, the optimal value of (8) can also be seen as another (new) measure of the inconsistency of the given pairwise comparison matrix $\boldsymbol{A}$; see Brunelli (2018) and Mazurek (2023) who review other inconsistency indices of pairwise comparison matrices.

Notice that, if $\boldsymbol{w}^*, \boldsymbol{w}^{**} \in G^n$ are such that $\boldsymbol{w}^* = \boldsymbol{w}^{**} \odot \boldsymbol{c}$, where $\boldsymbol{c} = (c,c,\dots,c)^{\mathrm{T}}$ is the vector of n elements c, i.e. $w_i^* = w_i^{**} \odot c$ for $i = 1,2,\dots,n$, for some constant $c \in G$ – in words: the vector $\boldsymbol{w}^*$ is the vector $\boldsymbol{w}^{**}$ multiplied by the constant c in each of its components – then $\theta(\boldsymbol{w}^*) = \theta(\boldsymbol{w}^{**})$. Consequently, to obtain a *unique* optimal solution to (8), we have to add an additional constraint to the optimization problem, such as either require the vector to be multiplicatively normalized ($\bigodot_{i=1}^n w_i = e$) or fix a coordinate of the vector (e.g. let $w_1 \coloneqq e$).

The following observation is crucial: The divisible alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ constitutes a linearly ordered vector space over the linearly ordered field $\mathbb{Q}$ of the rational numbers. — To see this, it is enough to define the multiplication of a vector $\vec{v} \in G$ by a scalar $\lambda = p/q \in \mathbb{Q}$, where $p \in \mathbb{Z}$ is an integer number and $q \in \mathbb{Z}_{++}$ is a positive integer number. We then let the result $\vec{w} \in G$ of the multiplication of the vector $\vec{v} \in G$ by the scalar $\lambda \in \mathbb{Q}$ be $\vec{w} = \vec{v}^{\lambda}$, where $\vec{w} = \vec{v}^{\lambda} \in G$ is defined as follows: Let $\vec{u} = \vec{v}^{p}$, see Section 2, and then, by using the divisibility, let $\vec{w} \in G$ be the unique element such that $\vec{w}^{q} = \vec{u}$; that is, we have $\vec{w}^{q} = \vec{u} = \vec{v}^{p}$, hence $\vec{w} = \vec{v}^{p/q} = \vec{v}^{\lambda}$, which was to be defined. — This observation allows us to revive the theory of linear programming in abstract vector spaces due to Bartl (2007) and, subsequently, by following Schmeidler (1969), we obtain the following results:

Theorem 1. *If the alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ is divisible, then problem (8) has at least one optimal solution.*

Theorem 2. *If the alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ is divisible, then problem (8) possesses no more than one optimal solution (up to the scaling by a constant).*

We omit the proofs of Theorem 1 and Theorem 2 in this paper due to lack of space.

5. A computational procedure to solve problem (8)

Given a divisible alo-group $\mathcal{G} = (G, \odot, ^{-1}, e, \leq)$ and a reciprocal pairwise comparison matrix $\boldsymbol{A} \in G^{n \times n}$, the theory of linear programming (Bartl, 2007) is used in an algebraic proof of Theorem 1. As the main idea behind (8) is to minimize the largest deviation $[a_{i_1 j_1} \div (w_{i_1} \div w_{j_1})]$ first, to minimize the second largest deviation $[a_{i_2 j_2} \div (w_{i_2} \div w_{j_2})]$ then, and so on, it follows that an optimal solution to problem (8) can be found by solving a sequence of linear programming problems. In particular, the (unique) solution $\boldsymbol{w}^* \in G^n$ to (8) can be found by using the "lexicographic centre" approach of Maschler et al. (1979), which we recall below.

Assume for a while that the minimum values of the first $k-1$ largest deviations $[a_{i_1 j_1} \div (w_{i_1} \div w_{j_1})] \geq [a_{i_2 j_2} \div (w_{i_2} \div w_{j_2})] \geq \cdots \geq [a_{i_{k-1} j_{k-1}} \div (w_{i_{k-1}} \div w_{j_{k-1}})]$ have already been found and are fixed. Then our task is to minimize the k-th largest deviation $[a_{i_k j_k} \div (w_{i_k} \div w_{j_k})]$, see Steps 1–4 of the following procedure. To this end, we maintain the index set $\mathcal{I} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_{k-1}, j_{k-1})\}$ of the pairs whose largest deviation has already been minimized and we also keep the values Δ_{ij} of these deviations, which must not be enlarged any more; that is, we always require $a_{ij} \div (w_i \div w_j) \leq \Delta_{ij}$ for $(i,j) \in \mathcal{I}$; the set $\mathcal{I}$ is initially empty (see Step 0). In addition, we also maintain the index set $\mathcal{N} \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \setminus \mathcal{I}$ of the pairs, within which we find the k-th largest deviation $[a_{i_k j_k} \div (w_{i_k} \div w_{j_k})]$ and minimize it; that is, we minimize ε subject to $a_{ij} \div (w_i \div w_j) \leq \varepsilon$ for $(i,j) \in \mathcal{N}$ and the aforegiven constraints (see Step 1). Notice that, due to the reciprocity of $\boldsymbol{A}$, it holds $a_{ji} \div (w_j \div w_i) = (e \div a_{ij}) \odot (w_i \div w_j) = e \div [a_{ij} \div (w_i \div w_j)]$. Therefore, if both (i,j), $(j,i) \in \mathcal{N}$, then $e \leq \varepsilon$ follows. Consequently, if $i = j$, then $a_{ii} = e$ due to the reciprocity of $\boldsymbol{A}$, and $a_{ii} \div (w_i \div w_i) = e$, which means we can omit the "diagonal" indices (i,i) from $\mathcal{N}$ for efficiency (see Step 0).

Let ε' be the optimal value of the above optimization problem (see (9) in Step 1), i.e. the minimum value of the k-th largest deviation $[a_{i_k j_k} \div (w_{i_k} \div w_{j_k})]$ whose index pair $(i_k, j_k) \in \mathcal{N}$ is to be found, and let $w'_1, w'_2, \ldots, w'_n$ be an optimal solution to the above problem. Then $\mathcal{I}' := \{ (i,j) \in \mathcal{N} : a_{ij} \div (w'_i \div w'_j) = \varepsilon' \}$ is the index set of the "candidates" within which the index pair (i_k, j_k) of the k-th largest deviation $[a_{i_k j_k} \div (w_{i_k} \div w_{j_k})]$ can be found. Let $\Delta' = \max_{(i,j) \in \mathcal{N} \setminus \mathcal{I}'} a_{ij} \div (w'_i \div w'_j)$ be the next smaller deviation, observe $\Delta' < \varepsilon'$, and take a value in between, such as the mean $\Delta'' = \sqrt{\Delta' \odot \varepsilon^*}$, so that $\Delta' < \Delta'' < \varepsilon'$. Finally, minimize $\odot_{(i,j) \in \mathcal{I}'} a_{ij} \div (w_i \div w_j)$, i.e. all the deviations that attained the value of the k-th largest one, subject to $a_{ij} \div (w_i \div w_j) \leq \varepsilon'$ for $(i,j) \in \mathcal{I}'$, i.e. we do not allow these deviations to grow, and $a_{ij} \div (w_i \div w_j) \leq \Delta_{ij}$ for $(i,j) \in \mathcal{I}$, i.e. the previous deviations whose values were fixed already are kept, and also $a_{ij} \div (w_i \div w_j) \leq \Delta''$ for $(i,j) \in \mathcal{N} \setminus \mathcal{I}'$, i.e. the remaining constraints must be relaxed (i.e. their right-hand sides increased) slightly so that some of the minimized deviations $a_{ij} \div (w_i \div w_j) \leq \varepsilon'$ for $(i,j) \in \mathcal{I}'$ can decrease in order that only the k-th largest one, which cannot be decreased, remains (see problem (10) in Step 3), and we repeat this step as follows.

Let $w_1^*, w_2^*, \ldots, w_n^*$ be an optimal solution to the last optimization problem (see (10) in Step 3) and let $\mathcal{I}^* := \{ (i,j) \in \mathcal{I}' : a_{ij} \div (w_i^* \div w_j^*) = \varepsilon' \}$ be the index set of the deviations that have not decreased. If $\mathcal{I}^* \subset \mathcal{I}'$ then increase Δ'' ~~slightly~~, but keep it less than ε', i.e. let ~~$\Delta'' := \sqrt{\Delta'' \odot \varepsilon'}$~~, reduce the index set of the "candidates", i.e. let $\mathcal{I}' := \mathcal{I}^*$, and repeat, i.e. solve the last optimization problem again (go to Step 3). Otherwise, if $\mathcal{I}^* = \mathcal{I}'$, then none of the deviations with index pair in $\mathcal{I}'$ can be decreased; that is, all of them have the minimum value of the k-th largest deviation. Notice that the values $[a_{ji} \div (w_j \div w_i)]$ are the reciprocal ones of the optimum values $[a_{ij} \div (w_i \div w_j)]$ for $(i,j) \in \mathcal{I}^*$, i.e. are of the same absolute value, which is the reason why they can be omitted from further calculations,

Corrigenda:

* sufficiently

** $\Delta'' := \sqrt{\varepsilon' \odot \max(\Delta', \Delta'')}$ where Δ' is the maximum of $a_{ij} \div (w_i \div w_j)$ taken over $(i,j) \in \mathcal{I}'$ such that $a_{ij} \div (w_i \div w_j) < \varepsilon'$.

too. That is, let $\mathcal{N} \coloneqq \mathcal{N} \setminus (\mathcal{I}^* \cup \mathcal{I}^{*-1})$, where $\mathcal{I}^{*-1} = \{\,(j,i) : (i,j) \in \mathcal{I}^*\,\}$ is the inverse of the index set $\mathcal{I}^*$. If $\mathcal{N} \neq \emptyset$, then fix the upper bound for the new active constraints by setting $\Delta_{ij} \coloneqq \varepsilon'$ for $(i,j) \in \mathcal{I}^*$, let $\mathcal{I} \coloneqq \mathcal{I} \cup \mathcal{I}^*$, and repeat the entire procedure (go to Step 1). Otherwise, if $\mathcal{N} = \emptyset$, then we are done: the vector $\boldsymbol{w}^* = (w_1^*, w_2^*, \dots, w_n^*)$ is an optimal solution to (8), therefore it is a priority vector the PC matrix $\boldsymbol{A}$ found by the new lexicographic method.

The entire procedure is as follows:

STEP 0 (initialization). Set $\mathcal{N} \coloneqq \{1,2,\dots,n\} \times \{1,2,\dots,n\} \setminus \{\,(i,i) : i \in \{1,2,\dots,n\}\,\}$ and $\mathcal{I} \coloneqq \emptyset$.

STEP 1. Solve the following linear programming problem:

$$\text{minimize} \quad \varepsilon \tag{9}$$

subject to

$$\begin{aligned} a_{ij} \div (w_i \div w_j) &\leq \varepsilon \qquad \text{for} \quad (i,j) \in \mathcal{N}, \\ a_{ij} \div (w_i \div w_j) &\leq \Delta_{ij} \qquad \text{for} \quad (i,j) \in \mathcal{I} \end{aligned}$$

where $\varepsilon, w_1, w_2, \dots, w_n \in G$ are variables.

STEP 2. Let $\varepsilon', w_1', w_2', \dots, w_n'$ be an optimal solution to (9), let

$$\mathcal{I}' \coloneqq \{\,(i,j) \in \mathcal{N} : a_{ij} \div (w_i' \div w_j') = \varepsilon'\,\}$$

be the indices of the constraints active at the optimum, and let

$$\Delta'' \coloneqq \sqrt{\varepsilon' \odot \max_{(i,j) \in \mathcal{N} \setminus \mathcal{I}'} a_{ij} \div (w_i' \div w_j')}\,.$$

STEP 3. Solve the following linear programming problem:

$$\text{minimize} \quad \odot_{(i,j) \in \mathcal{I}'} a_{ij} \div (w_i \div w_j) \tag{10}$$

subject to

$$\begin{aligned} a_{ij} \div (w_i \div w_j) &\leq \varepsilon' \qquad \text{for} \quad (i,j) \in \mathcal{I}', \\ a_{ij} \div (w_i \div w_j) &\leq \Delta_{ij} \qquad \text{for} \quad (i,j) \in \mathcal{I}, \\ a_{ij} \div (w_i \div w_j) &\leq \Delta'' \qquad \text{for} \quad (i,j) \in \mathcal{N} \setminus \mathcal{I}' \end{aligned}$$

where $w_1, w_2, \dots, w_n \in G$ are variables.

STEP 4. Let $w_1^*, w_2^*, \dots, w_n^*$ be an optimal solution to (10) and let

$$\mathcal{I}^* \coloneqq \{\,(i,j) \in \mathcal{I}' : a_{ij} \div (w_i^* \div w_j^*) = \varepsilon'\,\}$$

be the indices of the constraints still active at the optimum. If $\mathcal{I}^* \subset \mathcal{I}'$, then let ~~$\Delta'' \coloneqq \sqrt{\Delta'' \odot \varepsilon'}$~~, let * $\mathcal{I}' \coloneqq \mathcal{I}^*$, and go to Step 3. Otherwise, if $\mathcal{I}' = \mathcal{I}^*$, then let $\mathcal{N} \coloneqq \mathcal{N} \setminus (\mathcal{I}^* \cup \mathcal{I}^{*-1})$. If $\mathcal{N} \neq \emptyset$, then set $\Delta_{ij} \coloneqq \varepsilon'$ for $(i,j) \in \mathcal{I}^*$, let $\mathcal{I} \coloneqq \mathcal{I} \cup \mathcal{I}^*$, and go to Step 1. Otherwise, if $\mathcal{N} = \emptyset$, then stop as we have computed a priority vector $\boldsymbol{w}^* = (w_1^*, w_2^*, \dots, w_n^*)$ of the PC matrix $\boldsymbol{A}$.

6. Example

Given four cities denoted by A, B, C, D, a decision maker wishes to find the city that is the best one to live in. By using Saaty's scale, which is a subset of the usual multiplicative divisible alo-group $\mathcal{R}_{++} = (\mathbb{R}_{++}, \cdot, ^{-1}, 1, \leq)$, and considering all unordered pairs {A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D} of the cities, the decision maker compares and rates the cities as follows:

- city A is 3× better to live in than city B;
- city C is 2× better to live in than city A;
- city A is 4× better to live in than city D;
- city C is 8× better to live in than city B;
- city B is 2× better to live in than city D;
- city C is 3× better to live in than city D.

These comparisons and ratings are summarized by the following pairwise comparison matrix:

Corrigendum:

* let $\Delta' \coloneqq \max_{(i,j) \in \mathcal{I}' \setminus \mathcal{I}^*} a_{ij} \div (w_i' \div w_j')$,
let $\Delta'' \coloneqq \sqrt{\varepsilon' \odot \max(\Delta', \Delta'')}$,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1/2 & 4 \\ 1/3 & 1 & 1/8 & 2 \\ 2 & 8 & 1 & 3 \\ 1/4 & 1/2 & 1/3 & 1 \end{pmatrix}.$$

We calculate the priority vector of this PC matrix $\boldsymbol{A}$ by the new nucleolus-like method (NLM) and by the geometric mean method (GMM), both of which can be applied within any divisible alo-group. Notice that, in our example, the multiplicative alo-group $\mathcal{R}_{++} = (\mathbb{R}_{++}, \cdot, ^{-1}, 1, \leq)$ is a subgroup of the multiplicative group of the entire field of the real numbers $\mathbb{R}$. Therefore, we can apply Saaty's eigenvector method (EVM) to calculate the priority vector of the given PC matrix $\boldsymbol{A}$ too. The priority vectors calculated by these three methods, rounded to an accuracy of 9 decimal places, are as follows:

$$\boldsymbol{w}_{\mathrm{NLM}} \doteq \begin{pmatrix} 1.542210825 \\ 0.545253866 \\ 2.496639467 \\ 0.476323126 \end{pmatrix}, \qquad \boldsymbol{w}_{\mathrm{GMM}} \doteq \begin{pmatrix} 1.565084580 \\ 0.537284966 \\ 2.632148026 \\ 0.451801002 \end{pmatrix}, \qquad \boldsymbol{w}_{\mathrm{EVM}} \doteq \begin{pmatrix} 1.480530077 \\ 0.540767565 \\ 2.664987449 \\ 0.468680584 \end{pmatrix}.$$

Notice that all three vectors are multiplicatively normalized; that is, the product of their components is equal to 1, i.e. the neutral element of the multiplicative alo-group $\mathcal{R}_{++}$. (We do not normalize the vectors additively, i.e. so that their components sum up to 1, because only multiplication and no addition is available in the alo-group $\mathcal{R}_{++}$.)

We interpret the results as follows: By the new nucleolus-like method (NLM), city C is the best one to live in and is about 1.619× better than A, which is about 2.828× better than B, which is about 1.145× better than D. By the geometric mean method (GMM), city C is the best one to live in and is about 1.682× better than A, which is about 2.913× better than B, which is about 1.189× better than D. By Saaty's eigenvector method (EVM), city C is the best one to live in and is about 1.800× better than A, which is about 2.738× better than B, which is about 1.154× better than D.

7. Conclusion

Given a pairwise comparison matrix $\boldsymbol{A}$, the purpose is to find the priorities $w_1, w_2, \ldots, w_n$ so that $a_{ij} \approx w_i/w_j$ for $i, j = 1, 2, \ldots, n$. Many methods (see, e.g., (1)–(4) above) consider the additive deviation of the priority ratio (w_i/w_j) from the respective pairwise comparison element (a_{ij}), i.e. $a_{ij} - w_i/w_j$, which cannot be used in the unified setting of an alo-group because only one of the two operations of addition and multiplication (or subtraction and division, respectively) is available in the alo-group. Under the additional assumption that the alo-group is divisible, the geometric mean method (GMM) seems to be the only method, out of the traditionally known ones, that can be applied in this general setting, but its (logarithmic) least squares interpretation (LLSM) is lost here.

The contribution of this paper consists in that we have proposed a new lexicographic or nucleolus-like method (NLM) to find the priority vector of the given pairwise comparison matrix $\boldsymbol{A}$, which can also be applied in the general setting of a divisible alo-group, because we consider the multiplicative deviation of the weight ratio from the respective pairwise comparison element, i.e. $a_{ij}/(w_i/w_j)$ or $a_{ij} \div (w_i \div w_j)$, cf. Cavallo and D'Apuzzo (2009). The method is based on the concept of nucleolus (Schmeidler, 1969), which is known in the setting of cooperative games with transferable utility: here, the vector of the deviations sorted in the decreasing order is lexicographically minimized; that is, the largest deviation is minimized, then the second largest deviation is minimized, and so on.

Finally, using the "lexicographic centre" approach (Maschler et al., 1979), we have proposed a computational procedure to find the priority vector by the NLM; that is, to solve optimization problem (8), which possesses a unique solution (see Theorems 1 and 2). In the example, which we have provided, it seems that the priority vector provided by the new NLM is somewhat similar to that found by the GMM, but relatively different from that found by Saaty's EVM. The properties of the new NLM, and its comparison with other methods, are to be investigated in further research.

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