

ON GLOBAL LEPAGEAN EQUIVALENTS

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In this note, the sheaf of lepagean equivalents of the zero lagrangian on a fibered manifold $\pi^r: j^r Y \rightarrow X$ is shown to be fine so that the existence of a global first variation formula is established by a sheaftheoretical method.

For a detailed information about lepagean equivalents (also called Poincaré-Cartan equivalents) and their meaning for the first variation formula see Krupka [4], [5]. Below we only sketch the facts. Sheaf theory may be consulted in [2].

The structure we work on is a fibered manifold, say $\pi: Y \rightarrow X$, considered with its r -jet prolongations $\pi^r: j^r Y \rightarrow X$, $r \in \mathbb{N}$. Recall that $j^0 Y \cong Y$, and denote by $\Gamma(\pi)$ the set of sections of π . The projections $j_x^r \gamma \mapsto j_x^s \gamma$ are denoted by $\pi_s^r: j^r Y \rightarrow j^s Y$.

Fibered coordinate systems on Y are denoted $(U, (x^i, y^k))$, $i = 1, \dots, \dim X$, $k = 1, \dots, \dim Y - n$ and give rise to fibered coordinate systems $(j^r U, (x^i, y^k, y_{i_1 \dots i_r}^k, \dots, y_{i_1 \dots i_{r-1} i_r}^k))$ on $j^r Y$. For a function $f: j^r Y \rightarrow \mathbb{R}$, $\partial f / \partial x_i, \partial f / \partial y^k, \dots, \partial f / \partial y_{i_1 \dots i_r}^k$ are the usual partial derivatives whilst

$$dx_j = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial y^k} y_j^k + \dots + \frac{\partial f}{\partial y_{i_1 \dots i_r}^k} y_{i_1 \dots i_r j}^k$$

(summation) is the so called total derivative.

On Y there is the sheaf $\Omega_q j^r$ of r -th order q -forms and a well-defined map $h: \Omega_q j^r \rightarrow \Omega_q j^{r+1}$

$$(h\rho)_{j_x^{r+1}\gamma} = ((j^r\gamma \circ \pi^{r+1})^*\rho)_{j_x^{r+1}\gamma}.$$

The form $h\rho$ is always π^r -horizontal, i.e. $i_{\Xi}\rho = 0$ whenever Ξ is a vector vanishing under the projection π^r . A form ρ is π^r -horizontal if and only if $h\rho = (\pi_r^{r+1})^*\rho$ and is contact (i.e., $(j^r\gamma)^*\rho = 0$ for every $\gamma \in \Gamma(\pi)$) if and only if $h\rho = 0$. Obviously, h is a $\wedge$ -homomorphism $f \mapsto f \circ \pi_r^{r+1}$ (for functions), $dx^i \mapsto dx^i$, $dy_{i_1 \dots i_s}^k \mapsto y_{i_1 \dots i_s}^k dx^j$ (summation)

The subsheaf of $\Omega_q j^r$ of horizontal forms we denote by $\Omega_q^0 j^r$.

Similarly, $p = (\pi^{r+1})^* - h : \Omega_q j^r \rightarrow \Omega_q j^{r+1}$ satisfies $p(\rho \wedge \sigma) = p\rho \wedge p\sigma + p\rho \wedge h\sigma + h\rho \wedge p\sigma$, $p\rho$ is always contact, ρ is contact if and only if $p\rho = (\pi_r^{r+1})^*\rho$ and ρ is π^r -horizontal if and only if $p\rho = 0$. In fibered coordinates p is defined as the mapping $f \mapsto f \circ \pi_r^{r+1}$ (for functions), $dx^i \mapsto 0$, $dy_{i_1 \dots i_s}^k \mapsto \omega_{i_1 \dots i_s}^k := dy_{i_1 \dots i_s}^k - y_{i_1 \dots i_s}^k dx^j$.

The subsheaf of $\Omega_q j^r$ of contact forms we denote by $\Omega_q^{\geq 1} j^r$.

Let $\rho \in \Omega_q j^r U$ be a form. Let us apply p to its coordinate expression. Obviously, at least one of the forms $\omega_{i_1 \dots i_s}^k$ occurs in every summand of $p\rho$ and it is possible to decompose $p\rho$ as follows:

$$p\rho = p^1\rho + p^2\rho + \dots + p^n\rho$$

where $n = \dim X$ and $p^l\rho$ contains just terms of the form

$$f \omega_{i_1 \dots i_1}^{k_1} \wedge \omega_{i_2 \dots i_2}^{k_2} \wedge \dots \wedge \omega_{i_l \dots i_l}^{k_l} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{q-1}}$$

i.e. containing some $\omega_{i_1 \dots i_s}^k$ just l -times.

Essential is that this decomposition does not depend on a fibered coordinate system chosen. So, there arise maps $p^l : \Omega_q j^r \rightarrow \Omega_q j^{r+1}$ satisfying $p^l(\rho \wedge \sigma) = p^l\rho \wedge h\sigma + p^{l-1}\rho \wedge p^1\sigma + \dots + p^1\rho \wedge p^{l-1}\sigma + h\rho \wedge p^l\sigma$, and we can define a q -form ρ to be l -contact if $p^l\rho = (\pi_r^{r+1})^*\rho$. The sheaf of l -contact forms we denote by $\Omega_q^l j^r$.

According to [4], [5] a sheaf of r -th order lagrangians on a fibered manifold $\pi : Y \rightarrow X$ over an n -dimensional orientable compact base manifold X is just $\Omega_n^0 j^r$, a sheaf $\text{Lep}^r j^s$ of s -th order lepagean equivalents of r -th order lagrangians is the subsheaf of $\Omega_n^r j^s$ consisting of forms ρ such that $h\rho \in \Omega_n^0 j^r U$, and $p^1 d\rho$ is π_0^s -horizontal (i.e. $i_{\Xi} p^1 d\rho = 0$ whenever Ξ vanishes under π_0^s).

From [4], [5] we also know that the sheaf map $h : \text{Lep}^r j^{2r-1} \rightarrow \Omega_n^0 j^r$ is "epi". Our purpose in this note is to show that the induced map of global sections is surjective.

Any form $\vartheta \in \text{Lep}^r j^{2r-1} U$ with $h\vartheta = \lambda$ is called a (local) *lepagean equivalent* of the lagrangian λ and then $E_\lambda := p^1 d\vartheta$ is the *Euler-Lagrange form* of λ .

In fibered coordinates, the procedure of [4] gives for a lagrangian $\lambda = \wedge dx^1 \wedge \dots \wedge dx^n$

$$\vartheta = \wedge dx^1 \wedge \dots \wedge dx^n + \sum_{s=1}^r \wedge_{j_1 \dots j_{s-1} j_s}^k \omega_{j_1 \dots j_{s-1}}^k \wedge dx^1 \wedge \dots \wedge (dx^{j_s}) \dots \wedge dx^n +$$

+ (≥ 2)-contact terms which are unessential, hence usually omitted, and

$$\begin{aligned} \wedge_{j_1 \dots i_{s-1} j_s}^k &= (-1)^{j_s-1} \frac{\partial \Lambda}{\partial y_s^k} \frac{d}{dx_i} \frac{\partial \Lambda}{\partial y_{j_1 \dots j_s}^k} + \dots \\ &\dots + (-1)^{r-s} \frac{d^{r-s}}{dx^{i_1} \dots dx^{i_{r-s}}} \frac{\partial \Lambda}{\partial y_{j_1 \dots j_s i_1 \dots i_{r-s}}^k} + \dots \end{aligned}$$

+, the facultative terms which do not affect $p^1 dp$ but can make ρ invariant.

The facultative terms may occur when $\dim X > 1$ and $r > 2$ so that in this case the local lepagean equivalent is not unique and its global existence, although guaranteed, is not selfevident.

This problem is equivalent to that of an existence of the global first variation formula and as such was first solved by Kupershmidt [6]. A direct proof is first due to Krupka [4] who found additional geometric conditions under which the local lepagean equivalent comes out uniquely i.e. is global. The problem was solved independently by Horák, Kolář [3]. Also Garcia, Muñoz [1] in accordance with their approach to the calculus of variations construct the form ρ depending on a pair of connections on the underlying fibered manifold.

Easily can be the existence of global lepagean equivalents deduced as follows: The homomorphism

$$\text{Lep}^r j^{2r-1} \xrightarrow{h} \Omega_n^0 j^r$$

has as its kernel the sheaf $\text{Lep}_0 j^{2r-1}$ of lepagean equivalents of the zero lagrangian and there arises a short exact sequence of sheaves of abelian groups

$$\text{Lep}_0 j^{2r-1} \hookrightarrow \text{Lep}^r j^{2r-1} \xrightarrow{h} \Omega_n^0 j^r$$

and a long exact sequence of abelian groups

$$\text{Lep}_0 j^{2r-1} Y \hookrightarrow \text{Lep}^r j^{2r-1} Y \xrightarrow{h} \Omega_n^0 j^r Y \xrightarrow{\partial} H^1(Y, \text{Lep}_0 j^{2r-1}) \rightarrow \dots$$

is induced.

Lemma: *The sheaf $\text{Lep}_0 j^r$ is fine, hence acyclic.*

Proof: Recall from [2] that being fine is an acyclicity criterion which amounts to saying that to every two closed disjoint subsets A, B of $j^r Y$ there exists a sheaf homomorphism $\varphi: \text{Lep}_0 j^r \rightarrow \text{Lep}_0 j^r$ such that $\varphi|_A = \text{id}$, $\varphi|_B = 0$ on

some neighbourhoods A', B' of A, B respectively. Recall also that the fineness is a local property which enables us to confine ourselves on some coordinate neighbourhood $j^r U \rightarrow \pi U$.

Thus, let A, B be two disjoint closed subsets of $j^r U$, let $\eta: j^r U \rightarrow \mathbf{R}$ be a smooth function equal to 1 on some neighbourhood A' of A and zero on some neighbourhood B' of B .

Recall from [4] that for a form

$$\vartheta = \sum_{s=0}^r \Theta_{i_1 \dots i_s j}^k \omega_{i_1 \dots i_s}^k \wedge \Omega_j$$

where $\omega_{i_1 \dots i_s}^k = dy_{i_1 \dots i_s}^k - y_{i_1 \dots i_s j}^k dx^j$ and $\Omega_j = (-1)^{j-1} dx^1 \wedge \dots \wedge (dx^j) \wedge \dots \wedge dx^n$, $\vartheta \in \text{Lep}_0 j^r$ if and only if

$$\Theta_{(i_1 \dots i_r, j)}^k = 0,$$

$$\Theta_{(i_1 \dots i_s, j)}^k = \frac{d\Theta_{i_1 \dots i_s j, l}^k}{dx_l}, \quad 0 \leq s < r,$$

where $(i_1 \dots i_s j)$ stands for the symmetrization.

We can construct the desired homomorphism $\varphi: \text{Lep}_0 j^r \rightarrow \text{Lep}_0 j^r$ as follows: Put for $\rho \in \text{Lep}_0 j^r U$

$$\psi\rho = \sum_{s=0}^r \Psi_{i_1 \dots i_s j}^k \omega_{i_1 \dots i_s}^k \wedge \Omega_j$$

where

$$\Psi_{i_1 \dots i_r, j}^k = 0,$$

$$\Psi_{i_1 \dots i_s, j}^k = \Theta_{i_1 \dots i_s j, p}^k \frac{d\eta}{dx^p} + \frac{d\Psi_{i_1 \dots i_s j, p}^k}{dx^p}, \quad 0 \leq s \leq r.$$

Then put $\varphi\rho = \eta \cdot \rho + \psi\rho$, and

$$\Phi_{i_1 \dots i_s, j}^k = \eta \cdot \Theta_{i_1 \dots i_s j}^k + \Psi_{i_1 \dots i_s, j}^k.$$

Obviously,

$$\Phi_{(i_1 \dots i_r, j)}^k = \eta \cdot \Theta_{(i_1 \dots i_r, j)}^k$$

and

$$\begin{aligned} \Phi_{(i_1 \dots i_s, j)}^k - \frac{d}{dx^p} \Phi_{i_1 \dots i_s j, p}^k &= \eta \cdot \Theta_{(i_1 \dots i_s, j)}^k + \Theta_{i_1 \dots i_s j, p}^k \frac{d\eta}{dx^p} + \\ &+ \frac{d\Psi_{i_1 \dots i_s j, p}^k}{dx^p} - \frac{d}{dx^p} \Phi_{i_1 \dots i_s j, p}^k = \eta \cdot \Theta_{(i_1 \dots i_s, j)}^k + \end{aligned}$$

$$\begin{aligned}
& + \Theta_{i_1 \dots i_s, p}^k \frac{d\eta}{dx^p} \frac{d}{dx^p} (\eta \cdot \Theta_{i_1 \dots i_s, p}^k) = \\
& = \eta \cdot (\Theta_{i_1 \dots i_s, j}^k \frac{d}{dx^p} \Theta_{i_1 \dots i_s, p}^k)
\end{aligned}$$

whence actually $\varphi: \text{Lep}_0 j^r \rightarrow \text{Lep}_0 j^r$. Besides that, ψ obviously vanishes on every open set on which η is constant, particularly on A' , B' , so that

$$\varphi_{iA'} = id, \quad \varphi_{iB'} = 0$$

as desired. $\square$

Moreover, it can be shown that $p^1 d(\varphi\rho) = p^1 d(\eta \cdot \rho)$.

Consequently, $H^1(Y, \text{Lep}_0 j^{2r-1}) = 0$ and $h: \text{Lep}^r j^{2r-1} Y \rightarrow \Omega_n^0 j^r Y$ is surjective.

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